PS_MA2102_ Tutorial_ 1

Problem-1: Let $S = \{1,2,3,...n\}$. Determine the number of towers of the form $\boxed{\Phi \subseteq A \subseteq B \subseteq S}$.

$$C = \{(B,A) \mid \phi \subseteq A \subseteq B \subseteq S\} \qquad |C| = ?$$

$$S \supseteq B \supseteq A \supseteq \phi$$

$$C_k = \{(B,A) \mid \phi \subseteq A \subseteq B \subseteq S,\ |B| = k\}$$

$$k = 0,1,2,\dots n$$

$$C = \bigcup_{k=0}^{n} C_k \qquad C_i \cap C_j = \phi \quad i \neq j$$

$$|C| = \sum_{k=0}^{n} |C_k|$$

$$|C_k| = \binom{n}{k} \times 2^k, \quad k = 0,1,2,\dots n$$

$$|C| = \sum_{k=0}^{n} \binom{n}{k} 2^k = (1+2)^n = 3^n$$

$(\because$ Binomial theorem $)$

Problem-2: There is even number $2n$ of people at party, and they talk together in pairs with everyone(*so n pairs*). In how many different ways can the $2n$ people be talking like this?

Sol:

$$\frac{(2n)!}{n!\,(\underbrace{2!\,2!\cdots 2!}_{n\text{ time}})} = \frac{(2n)!}{n!\,(2!)^n}$$

$$\frac{n!}{(n_1!\,n_2!\cdots n_k!)}$$

Problem-3: Now suppose there is odd number of $2n+1$ of people at the party with everyone but one person talking with someone. How many different pairings are possible?

Sol

$$\frac{(2n+1)!}{n!\,(\underbrace{2!\,2!\cdots 2!}_{n\text{ time}}\ 1!)} = \frac{(2n+1)!}{n!\,(2!)^n}$$

Problem-4: In how many ways can 12 indistinguishable apples and 1 orange be distributed among three children in such a way that each child gets at least one fruit?

$x_1+x_2+\dots+x_r=n$

$>0 \quad \binom{n-1}{r-1}$

$\geq 0 \quad \binom{n+r-1}{r-1}$

Sol.

	A	B	C
# apples	x_1	x_2	x_3

$x_1+x_2+x_3=12, \quad x_1 \geq 0, \quad x_2>0, \quad x_3>0$

$y_1=x_1+1>0 \qquad y_2=x_2, \quad y_3=x_3$

$\Rightarrow x_1=y_1-1$

$(y_1-1)+y_2+y_3=12$

$y_1+y_2+y_3=13, \qquad y_1>0, y_2>0, y_3>0$

$$\binom{13-1}{3-1}=\binom{12}{2}$$

$$\binom{12}{2}+\binom{12}{2}+\binom{12}{2}=3\times\binom{12}{2}$$

Problem-5: There are $2n+1$ identical books to be put in a book case with three shelves. In how many ways can this be done if each pair of shelves together contain more than the other shelf.

$x_1+x_2+x_3=2n+1, \quad x_1+x_2>x_3$ and $x_1+x_3>x_2$ and $x_2+x_3>x_1$

$\sim$

$x_1+x_2<x_3$ (or)

$x_1+x_3<x_2$ (or)

$x_2+x_3<x_1$

$x_1+x_2+x_3=2n+1$

$2n+1<x_3+x_3$

$2n+1<2x_3 \Rightarrow x_3 \geq n+1$

Total way distribute books $\quad x_1+x_2+x_3=2n+1, \quad x_i \geq 0, \; i=1,2,3$

$$\binom{2n+1+3-1}{3-1}=\binom{2n+3}{2}$$

distribution we dont like

$x_1+x_2+x_3=2n+1, \quad x_1 \geq n+1$ (or) $x_2 \geq n+1$ (or) $x_3 \geq n+1$

(ii) $x_1+x_2+x_3=2n+1$, $x_1\geq n+1$, $x_2\geq 0$, $x_3\geq 0$

$y_1=x_1-(n+1)\geq 0$, $y_2=x_2$, $y_3=x_3$

$x_1=y_1+(n+1)$,

$y_1+(n+1)+y_2+y_3=2n+1$

$\underline{y_1+y_2+y_3=n}$, $y_1\geq 0$, $y_2\geq 0$, $y_3\geq 0$

$\binom{n+3-1}{3-1}=\binom{n+2}{2}$

$x_1+x_2+x_3=2n+1$, $x_1\geq n+1$ (or) $x_2\geq n+1$ (or) $x_3\geq n+1$

Solution $=3\times\binom{n+2}{2}$

integer solution $x_1+x_2+x_3=2n+1$, $x_1+x_2>x_3$ and $x_1+x_3>x_2$ and $x_2+x_3>x_1$

$$=\binom{2n+3}{2}-3\times\binom{n+2}{2}$$

Problem-6: Determine a general formula for the number of permutations of the set $\{1,2,3,\ldots,n\}$ in which exactly k $(0\leq k\leq n)$ are in their natural position.

sol $\binom{n}{k}\times 1\times D_{n-k}=\binom{n}{k}D_{n-k}$

Problem-7: In how many ways a teacher can distribute the 30 response sheets to 30 students so that at least one student receives his/her own response sheet.

sol: $\boxed{30!-D_{30}}$

Problem-8: Use the combinatorial reasoning to derive the identity

$$n! = \binom{n}{0}D_n + \binom{n}{1}D_{n-1} + \binom{n}{2}D_{n-2} + \dots + \binom{n}{n-1}D_1 + \binom{n}{n}D_0$$

Proof

$A = \{1, 2, 3, \dots n\}$

S = ~~set of~~ set of permutation of A $\qquad |S| = n!$

S_k = set of permutation of A in which exactly k objects are fixed,

$|S_k| = \binom{n}{k} \times 1 \times D_{n-k} = \binom{n}{k} D_{n-k}$ $\qquad k = 0, 1, 2 \dots n$

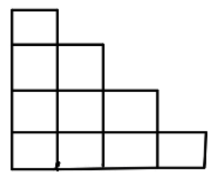
$S = \bigcup_{k=0}^{n} S_k$, $\quad S_i \cap S_j = \phi, \ i \neq j$

$|S| = \sum_{k=0}^{n} |S_k|$

$n! = \sum_{k=0}^{n} \binom{n}{k} D_{n-k}$

$$\boxed{n! = \binom{n}{0} D_n + \binom{n}{1} D_{n-1} + \binom{n}{2} D_{n-2} + \dots + \binom{n}{n-1} D_1 + \binom{n}{n} D_0}$$

Problem-9: How many ways can we arrange the letters of the word 'PEPPER' on the grid shown right. So that no row is empty.



Sol

$S \leftarrow$ set of all arrangements of letters of the word PEPPER

$|S| = \binom{10}{6} \times \frac{6!}{3!\,2!\,1!}$

Let $A_i \leftarrow$ set of all arrangement in grid so that ith row is empty $\quad i = 1, 2, 3, 4$

$|A_1^c \cap A_2^c \cap A_3^c \cap A_4^c| = ?$

$|A_1^c \cap A_2^c \cap A_3^c \cap A_4^c| = |S| - (\underbrace{A_1 \cup A_2 \cup A_3 \cup A_4}|$

$= |S| - \Big[\sum_{1 \le i \le 4} |A_i| - \sum_{1 \le i < j \le 4} |A_i \cap A_j| + \sum_{1 \le i < j < k \le 4} |A_i \cap A_j \cap A_k|$

$- |A_1 \cap A_2 \cap A_3 \cap A_4| \Big]$

$|A_1| = \binom{9}{6}\frac{6!}{3!2!1!}$, $|A_2| = \binom{8}{6}\frac{6!}{3!2!1!}$, $|A_3| = \binom{7}{6}\frac{6!}{3!2!1!}$

$|A_4| = \binom{6}{6}\frac{6!}{3!2!1!}$, $|A_1 \cap A_2| = \binom{7}{6}\frac{6!}{3!2!1!}$, $|A_1 \cap A_3| = \binom{6}{6}\frac{6!}{3!2!1!}$

$|A_1 \cap A_4| = 0$, $|A_2 \cap A_3| = 0$, $|A_2 \cap A_4| = 0$, $|A_3 \cap A_4| = 0$

$|A_1 \cap A_2 \cap A_3 \cap A_4| = 0$, $|A_i \cap A_j \cap A_k| = 0$ $1 \le i < j < k \le 4$

$$|A_1^c \cap A_2^c \cap A_3^c \cap A_4^c| = \binom{10}{6}\frac{6!}{3!2!1!} - \Big[\binom{9}{6}\frac{6!}{3!2!1!} + \binom{8}{6}\frac{6!}{3!2!1!} + \binom{7}{6}\frac{6!}{3!2!1!}$$

$$+ \binom{6}{6}\frac{6!}{3!2!1!} - \binom{7}{6}\frac{6!}{3!2!1!} - \binom{6}{6}\frac{6!}{3!2!1!} - 0 - 0 - 0 + 0$$

Problem-10: Give a combinatorial proof of the identity $\sum_{k=1}^{n}\binom{n}{k}k^2 = 2^{n-2}n(n+1)$

$$\sum_{k=1}^{n}\binom{n}{k}\binom{k}{1}\binom{k}{1}$$

$$\sum_{k=1}^{n}\binom{n}{k}k^2 = \binom{n}{1}\times 2^{n-1} + \binom{n}{1}\binom{n-1}{1}\times 2^{n-2}$$

$$= n2^{n-1} + n(n-1)\,2^{n-2}$$

$$= n\,2^{n-2}(2+n-1)$$

$$= 2^{n-2}\,n\,(n+1)$$

$$\therefore \boxed{\sum_{k=1}^{n}\binom{n}{k}k^2 = 2^{n-2}\,n\,(n+1)}$$

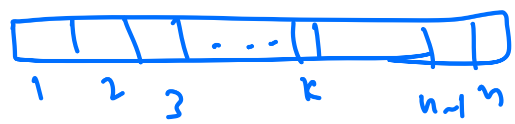
Problem-11: Give a combinatorial proof of the identity $\sum_{k=1}^{n} \binom{n}{k} k^3 = 2^{n-3} n^2 (n+3)$

Exercise

Problem-12: (Fermat's identity) Prove $\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}$, $n \geq k$

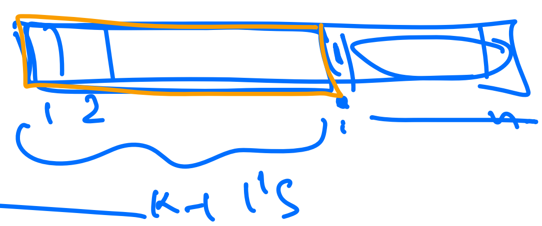
S : set of binary string of length n with exactly k 1's

$|S| = \binom{n}{k}$



S_i : set of binary string of length n with exactly k 1's and last 1 occur at position i, $k \leq i \leq n$

$|S_i| = \binom{i-1}{k-1}$ $\quad k \leq i \leq n$



$S = \bigcup_{i=k}^{n} S_i$, $\quad S_i \cap S_j = \phi \quad i \neq j$

$|S| = \sum_{i=k}^{n} |S_i| \Rightarrow \boxed{\binom{n}{k} = \sum_{i=k}^{n} \binom{i-1}{k-1}}$