MA2102: PS (Assignment-3)

1. A total of 4 buses carrying 148 students from the same school arrives at football stadium. The buses carry, respectively 40,33,25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying this randomly selected. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on his bus.
   a. Which of $E(X)$ or $E(Y)$ do you think larger? Why?
   b. Compute $E(X)$ and $E(Y)$.
2. Let X be the nonnegative integer-valued discrete random variable, with CDF F_X and PMF p_X then show that $\sum_{x=0}^{\infty} x\left(1-F_X(x)\right)=\frac{1}{2}\left(E(X^2)-E(X)\right)$
3. Let X be a random variable with mean μ and $f_X(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{-(x-\mu)^2}{2\sigma^2}}$, $-\infty<x<\infty$
   Then find μ_i' and μ_i for $i=1,2,3,4$ (Hint: First find central moments)
4. Find $E(X), Var(X)$ and MGF for the following discrete distributions
   a. $p_X(x)=\frac{1}{N}$ $x=1,2\ldots.N$ (Discrete Uniform distribution)
   b. $p_X(x)=\binom{x+r-1}{r-1}p^r(1-p)^x$, $x=0,1,2,\ldots..$ and $r\geq 1,\ 0\leq p\leq 1$
   (Negative Binomial distribution)
5. Find $E(X), Var(X)$ of random variable X with PMF $p_X(x)=\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$, $x=0,1,2\ldots.n$ and $max(0,n-(N-M))\leq x\leq \min(n,M)$, here N,M and n are fixed
   (Hyper Geometric distribution)

6. Find $E(X), Var(X)$ and MGF for the following continuous distributions
   a. $f_X(x)=\begin{cases}\frac{1}{b-a} & ,a<x<b\\ 0 & otherwise\end{cases}$ (Continuous Uniform distribution)
   b. $f_X(x)=\begin{cases}\frac{\lambda^\alpha x^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)} & ,x\geq 0\\ 0 & x<0\end{cases}$ (Gamma distribution)
   c. $f_X(x)=\frac{1}{\sqrt{2\pi}}e^{\frac{-x^2}{2}}$, $-\infty<x<\infty$ (Standard Normal distribution)
7. Find $E(X), Var(X)$ of random variable X with PDF $f_X(x)=\begin{cases}\frac{x^{a-1}(1-x)^{b-1}}{\mathrm{B(a,b)}} & ,0<x<1\\ 0 & otherwise\end{cases}$
   where $B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ (Beta distribution)
8. Let $g:[0\ \infty)\to[0\ \infty)$ be strictly increasing and non- negative. Show that
   $P(|X|\geq a)\leq\frac{E(g(|X|))}{g(a)}$, for $a>0$
9. Let X be random variable with MGF $M_X(t), -h<t<h$. Prove that
   $P(X\geq a)\leq e^{-at}M_X(t), 0<t<h$, and that $P(X\leq a)\leq e^{-at}M_X(t), -h<t<0$,
10. Let X be a random variable with $E(X)=3$ and $E(X^2)=13$, determine the lower bound for $P(-2<X<8)$
11. From past experience a professor knows that the test score of a student taking his final examination is a random variable with mean 75.
    a. Give an upper bound for the probability that a student's test score will exceed 85.
    b. Suppose in addition, the professor knows that the variance of a student's test score is equal to 25. What can be said about the probability that a student will score between 65 and 85?

12. Find all the non-central moments of the distribution that has MGF $M_X(t) = \frac{1}{(1-t)^3}$, for $-1 < t < 1$
13. The MGF of random variable is given by $M_X(t) = \frac{1}{2}e^{-5t} + \frac{1}{6}e^{4t} + \frac{1}{8}e^{5t} + \frac{5}{24}e^{25t}$. Find the PMF of X

14. Let $\psi(t) = logM_X(t)$, Prove that $\psi'(0) = \mu$ and $\psi''(0) = \sigma^2$. The function $\psi(t)$ is called cumulant generating function of X.

15. Let $K(t) = E(t^X)$ exist in an interval around 1. Show that $K^{(m)}(1) = \alpha_m$, where $\alpha_m = E(X(X-1)(X-2)\ldots.(X-m+1)$
16. For continuous random variable X, with PDF $f_X(x) = 4x^3$, $0 < x < 1$, zero elsewhere. Find $Q_{\frac{1}{10}}, Q_{\frac{1}{20}}$.
17. For continuous random variable X, with PDF $f_X(x) = \frac{1}{x^2}$, $1 < x < \infty$, zero elsewhere. Find $Q_{\frac{1}{4}}, Q_{\frac{1}{2}}$.
18. Let X be a positive random variable show that
    a. $E(\frac{1}{X}) \geq \frac{1}{E(X)}$
    b. $E(X^3) \geq (E(X))^3$
    c. $E(\log\left(\frac{1}{X}\right)) \geq \log\left(\frac{1}{E(X)}\right)$
19. For continuous random variable X, with PDF $f_X(x) = \frac{1}{2}x^2e^{-x}$, zero elsewhere. Find the mode of the distribution X. (Mode of the distribution of a random variable X is a value of x that maximizes the PDF/PMF)
20. Let X be a random variable with mean μ and variance σ^2 such that the $\mu_3 = E(X-\mu)^3$ exists. The value $\frac{\mu_3}{\sigma^3}$ is used as measure of $skewness$. Graph each of the following $PDF's$ and show that this measure is negative, zero, and positive for these respective distributions (which are said to be skewed to left, not skewed, and skewed to the right, respectively)
    i. $f_X(x) = \begin{cases} \frac{x+1}{2}, & -1 < x < 1 \\ 0 & elsewhere \end{cases}$
    ii. $f_X(x) = \begin{cases} \frac{1}{2}, & -1 < x < 1 \\ 0 & elsewhere \end{cases}$
    iii. $f_X(x) = \begin{cases} \frac{1-x}{2}, & -1 < x < 1 \\ 0 & elsewhere \end{cases}$

*****GOOD LUCK******