2.1) a) Let B is a event for the neighbour having boys. then $P(B>=1)$ represents event neighbour having atleast one boy.

$$P(B>=1) = P(B=1) + P(B=2)$$

$$= P(\{x \in \{BG, GB\}\}) + P(x \in \{BB\})$$

$$= 2/4 + 1/4 = 3/4$$

let G represents ~~event~~ event of having girl, then we need to find

$$P(G=1/B>=1) = \frac{P(G=1 \cap B>=1)}{P(B>=1)}$$

$$= \frac{P(x \in \{BG, GB\})}{3/4}$$

$$= \frac{2/4}{3/4} = 2/3.$$

b) Here, we already know the ~~posterior~~ prior event, so probability becomes

P(probability that neighbour having one girl and one boy)

$$= P(x \in \{BG, GB\}) = 2/4 = 1/2.$$

2.4) Let D be the event of patient having disease,

P be the event of patient test getting Positive.

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Given that $P(P/D) = P($Test showing positive when Patient infact having disease$)$

$= 0.99$

and also given that $P(P^c/D^c) = P($Test showing negative when disease is not there$)$

$= 0.99$

So, $P(P/D^c) = 1 - P(P^c/D^c) = 1 - 0.99 = 0.01$

also given that $P(D) = 1/10{,}000$, so $P(D^c) = 1 - \frac{1}{10{,}000}$

$= \frac{9999}{10{,}000}$

Now, we need to find out

$P(D/P) = P($patient having disease given test is positive$)$

By Bayes theorm

$$P(D/P) = \frac{P(DP)}{P(P)} = \frac{P(P/D)\,P(D)}{P(P/D)\,P(D) + P(P/D^c)\,P(D^c)}$$

$$= \frac{\frac{99}{100} \times \frac{1}{10{,}000}}{\frac{99}{100} \times \frac{1}{10{,}000} + \frac{1}{100} \times \frac{9999}{10{,}000}} = \frac{99}{10{,}098} = 0.0098$$

2.7) To prove pairwise independence doesn't imply mutual independence, we can show one example.

Consider a sample space for taking ~~2 balls~~ one ball out of urn containing four balls and let events

$E = \{1, 2\}$, $F = \{1, 3\}$, $G = \{1, 4\}$ represents taking ~~balls~~ ~~and~~ any of there numbered balls ~~(with replacement)~~

$P(\cancel{EF}) = P(E) = P(F) = P(G) = 2/4 = 1/2$

$P(EF) = P(\{1\}) = 1/4 = P(E)P(F)$

likewise $P(EG) = P(FG) = 1/4$ which is

Same for $P(E)P(G) = P(F)P(G) = 1/4$

However, $P(EFG) = P(\{1\}) = P(\text{taking '1' ball}) = 1/4$

which is not same as $P(E)P(F)P(G) = 1/8$

2.10) Gamma variable $X \sim Ga(a, b) = \dfrac{b^a}{\Gamma(a)} x^{a-1} e^{-xb}$.

Y is inverse Gamma variable of $X \sim Ga(a, b)$

i.e $Y = 1/X. \Rightarrow X = 1/y$

By using change of variable formula

$$P_y(y) = P_x(x)\left|\frac{dx}{dy}\right|$$

$$= \frac{b^a}{\Gamma(a)} \cdot (1/y)^{a-1} \cdot e^{-\cancel{x}b/y} \left|\frac{d}{dy}(1/y)\right|$$

$$= \frac{b^a}{\Gamma(a)} \cdot y^{-a+1} \cdot e^{-b/y} \left|(-1/y^2)\right|$$

$$= \frac{b^a}{\Gamma(a)} \cdot y^{-(a+1)} \cdot e^{-b/y}$$

1(b) $x \sim B(a,b) = \frac{1}{B(a,b)}\left(x^{a-1}(1-x)^{b-1}\right)$

Where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$

$$E(x) = \int_{-\infty}^{\infty} x \cdot p(x)\, dx = \int_{-\infty}^{\infty} x \cdot \frac{1}{B(a,b)} \cdot x^{a-1}(1-x)^{b-1}\, dx$$

$$= \frac{1}{B(a,b)} \int_0^1 x^{(a+1)-1}(1-x)^{b-1}\, dx.$$ [$\because$ Integral is from $(0,1)$ since X support is from $(0,1)$]

$$\boxed{\begin{array}{l} \text{Here pdf of r.v } X \text{ should sum to } 1. \\ \text{i.e } \int_{-\infty}^{\infty} p(x)\, dx = 1 \\ \int_0^1 \frac{1}{B(a,b)} x^{a-1} \cdot (1-x)^{b-1}\, dx = 1 \\ \Rightarrow \int_0^1 x^{a-1} \cdot (1-x)^{b-1}\, dx = B(a,b) \end{array}}$$

So, $E(x) = \frac{1}{B(a,b)}\left[B(a+1,b)\right] \Rightarrow \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+1+b)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$

The definition of $\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x}\, dx$

by integrating into parts $\boxed{\int u\, dv = uv - \int du \cdot v}$

$$\Gamma(a) = \left(x^{a-1} \cdot -e^{-x}\right)_0^{\infty} + (a-1)\int_0^{\infty} x^{a-2} \cdot -e^{-x}\, dx$$

$$= 0 + (a-1)\Gamma(a-1) \Rightarrow (a-1)\Gamma(a-1) = \Gamma(a)$$

It leads to

$$E(x) = \frac{(a)\Gamma(\cancel{a})\Gamma(\cancel{b})}{(a+b)\Gamma(\cancel{a+b})} \cdot \frac{\Gamma(\cancel{a+b})}{\Gamma(\cancel{a})\Gamma(\cancel{b})} = \frac{a}{a+b}$$

Mode of distribution is where its gets maximum probability. i.e Mode $(x) = \{ x ; \forall x \in R_x \; \exists \max pdf(x) \}$

To get maximum/ minimum we can do differentiation and equal it to zero.

$$\frac{d}{dx} \text{Beta}(a,b) = 0 \Rightarrow \frac{d}{dx} \frac{1}{B(a,b)} \cdot x^{a-1} \cdot (1-x)^{b-1} = 0$$

$$\frac{d}{dx} \cdot \left(x^{a-1} \cdot (1-x)^{b-1} \right) = 0$$

By differentiating in parts $\int uv = u dv + v du$.

$$(a-1)\, x^{a-2} (1-x)^{b-1} + (b-1)(-1) \cdot (1-x)^{b-2} \cdot x^{a-1} = 0$$

$$(a-1)\, x^{a-2} \cdot (1-x)^{b-1} = (b-1)(1-x)^{b-2} \cdot x^{a-1}$$

$$(a-1) \cdot (1-x) = (b-1) \cdot x$$

$$a-1-ax+x = bx-x$$

$$a-1 = x(a+b-2)$$

$$x = \frac{a-1}{a+b-2}$$

To see, if x is minimum or maximum we can substitute this in second degree of differentiation

i.e $\frac{d^2}{d^2x}$ Beta(a,b). and get the sign of it

at $x = \frac{a-1}{a+b-2}$. which will be < 0

so, mode of Beta$(a,b) = \frac{a-1}{a+b-2}$

Variance of the distribution

$$Var(X) = E(X^2) - (E(X))^2$$

$$E(X^2) = \frac{1}{B(a,b)} \int_0^1 x^2 \cdot x^{a-1} (1-x)^{b-1} dx$$

$$= \frac{1}{B(a,b)} \cdot B(a+2, b) = \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+b+2)} \cdot \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$$

$$= \frac{(a+1)a}{(a+b)(a+b+1)} \quad \text{and} \quad \{E(X)\}^2 = \frac{a^2}{(a+b)^2}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$= \frac{(a+1)a}{(a+b+1)(a+b)} - \frac{a^2}{(a+b)^2} = \frac{(a+b)(a+1)a - a^2(a+b+1)}{(a+b+1)(a+b)}$$

$$= \frac{a^2\cancel{(a+b)} + a(a+b) - a^2\cancel{(a+b)} - a^2}{(a+b+1)(a+b)} = \frac{ab}{(a+b+1)(a+b)}$$

2.17) X, Y are two independent uniform distributions in support [0,1]

$X \sim U(0,1)$ and $Y \sim U(0,1)$

Consider min of these two is Z. $\Rightarrow Z = \min\{X, Y\}$

Distribution function of Z

$$F_Z(z) = P(\{Z \leq z\})$$

$$= P(\{\min(X,Y) \leq z\})$$

we can write $F_Z(z) = 1 - P(\{Z > z\})$

$$F_Z(z) \Rightarrow 1 - P(\{\min(X,Y) > z\})$$

$$\Rightarrow 1 - P(\{X > z\})P(\{Y > z\})$$ [Because, $\min(x,y)$ is ~~greater~~ smaller than x, y]

$$= 1 - \left[(1 - P(\{X \le z\}))(1 - P\{Y \le z\})\right]$$

$$= 1 - \left[\left(1 - \int_0^z 1 \cdot dx\right)\left(1 - \int_0^z 1 \cdot dy\right)\right]$$

$$= 1 - [(1-z)(1-z)] = 1 - (1-z)^2$$

Expectation of ~~$F_Z(z)$~~ in r.v Z is

$$E(Z) = \int_{-\infty}^{\infty} z \cdot f_Z(z)\, dz$$

$$= \int_0^1 z \cdot \frac{d}{dz}\left(1 - (1-z)^2\right) \cdot dz$$

$$= \int_0^1 z \cdot 2(1-z)\, dz$$

$$= 2\left[\left(\frac{z^2}{2}\right)_0^1 - \left(\frac{z^3}{3}\right)_0^1\right]$$

$$= 2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{3}.$$

So, the expected value of minimum of X, Y is $E(\min(X,Y)) = \frac{1}{3}$.

* question answers:

27) Let A throw coin K times and B throw n-k times. probability of A getting same heads as B is summation over all $i \le K$.

$P\{\text{same number of heads}\}$

$$= \sum_i P[(A=i)\cdot(B=i)]$$

Both of these events follow Binomial r.v and are independent.

$$P(A=i) = Bi(i|(K,1/2)) = {}^K C_i \cdot (1/2)^i \cdot (1/2)^{k-i}$$

$$= {}^K C_i (1/2)^K$$

$$P(B=i) = Bi(i|(n-K,1/2)) = {}^{n-k} C_i (1/2)^i \cdot (1/2)^{n-k-i}$$

$$= {}^{n-k} C_i (1/2)^{n-k}.$$

$P\{\text{Same number of heads}\}$

$$= \sum_i P[A=i]\cdot P[B=i] = \sum_i {}^k C_i (1/2)^k \cdot {}^{n-k} C_i \cdot (1/2)^{n-K}$$

$$= \sum_i \cdot {}^k C_i \cdot {}^{n-k} C_i \cdot (1/2)^n$$

$$= (1/2)^n \cdot {}^n C_K \quad (\because \sum_i {}^{n-k} C_i \cdot {}^k C_i = {}^n C_k)$$

which same as

$$P\{K \text{ heads in } n\} = Bi(K|(n,1/2)) = {}^n C_K \cdot (1/2)^k (1/2)^{n-k}$$

$$= {}^n C_k \cdot (1/2)^n.$$

~~28~~) 37) $X_1, X_2 \dots X_n$ are independent uniform r.v.s.

$M = \max\{X_1, X_2 \dots X_n\}$

$F_M(x) = P(M \le x) = P(\max\{X_1, X_2 \dots X_n\} \le x)$

$= P(\{X_1 \le x, X_2 \le x \dots X_n \le x\})$ ($\because$ since $x_1 \dots x_n$ $\max\{x_1 \dots x_n\}$

$= P(X_1 \le x) P(X_2 \le x) \dots P(X_n \le x)$ ($\because$ since there are independent)

$= \int_0^x 1 \cdot dx \cdot \int_0^x 1 \cdot dx \dots \int_0^1 1 \cdot dx$

$= x^n$.

density function of M is $f_X(x) = \frac{d}{dx} F_M(x)$

$= \frac{d}{dx} x^n = n \cdot x^{n-1}$

38) $f(x) = \begin{cases} ce^{-2x} & 0 < x < \infty \\ 0 & x < 0 \end{cases}$

Since integral of probability density function must equal to 1.

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} ce^{-2x}\,dx = 1$

$\Rightarrow \int_0^{\infty} ce^{-2x}\,dx = 1$

$\frac{c}{-2} \cdot \left(e^{-2x}\right)_0^{\infty} = 1$

$\frac{c}{-2}(-1) = 1 \Rightarrow c = 2.$

$P(X>2) = \int_2^{\infty} 2 \cdot e^{-2x}\,dx = \frac{2}{-2}\left(e^{-2x}\right)_2^{\infty} = -1 \cdot (0 - e^{-4}) = 1/e^4$