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Q1) Find solution using Simplex method

A) MAX Z = 5x1 + 2x2
subject to
3x1 + 5x2 <= 15
5x1 + 2x2 <= 10
and x1,x2 >= 0

Solution:
Problem is

Max $Z = 5x1 + 2x2$

subject to

$3x1 + 5x2 \leq 15$

$5x1 + 2x2 \leq 10$

and $x1, x2 \geq 0;$

The problem is converted to canonical form by adding slack variables as appropiate

1. As the constraint-1 is of type '$\leq$' we should add slack variable $S1$

2. As the constraint-2 is of type '$\leq$' we should add slack variable $S2$

After introducing slack variables

Max $Z = 5x1 + 2x2 + 0S1 + 0S2$

subject to

$3x1 + 5x2 + S1 \quad = 15$

$5x1 + 2x2 \quad + S2 = 10$

and $x1, x2, S1, S2 \geq 0$

Iteration-1		Cj	5	2	0	0	

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B	***CB***	***XB***	***x*1**	***x*2**	***S*1**	***S*2**	**MinRatio** ***XB/x*1**
*S*1	0	15	3	5	1	0	15/3=5
***S*2**	0	10	5	2	0	1	10/5=2→
***Z*=0**		***Zj***	**0**	**0**	**0**	**0**	
		Zj-Cj	-5↑	-2	0	0	

Negative minimum $Zj\text{-}Cj$ is -5 and its column index is 1. So, the entering variable is $x1$.

Minimum ratio is 2 and its row index is 2. So, the leaving basis variable is $S2$.

∴ The pivot element is 5.

Entering $=x1$, Departing $=S2$, Key Element $=5$

$R2(\text{new})=R2(\text{old}) \div 5$

$R1(\text{new})=R1(\text{old}) - 3R2(\text{new})$

Iteration-2		*Cj*	5	2	0	0	
B	***CB***	***XB***	***x*1**	***x*2**	***S*1**	***S*2**	**MinRatio**
*S*1	0	9	0	19/5	1	-3/5	
*x*1	5	2	1	2/5	0	1/5	
***Z*=10**		***Zj***	**5**	**2**	**0**	**1**	
		Zj-Cj	0	0	0	1	

Since all $Zj\text{-}Cj \geq 0$

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Hence, optimal solution is arrived with value of variables as:
$x1=2, x2=0$

Max $Z=10$

B) **MIN Z = x1 - 3x2 + 2x3**
subject to
3x1 - x2 + 2x3 <= 7
-2x1 + 4x2 + 4x3 <= 12
-4x1 + 3x2 + 8x3 <= 10
and x1,x2,x3 >= 0

Solution:
Problem is

Min $Z = x1 - 3x2 + 2x3$
subject to
$3x1 - x2 + 2x3 \leq 7$
$-2x1 + 4x2 + 4x3 \leq 12$
$-4x1 + 3x2 + 8x3 \leq 10$
and $x1, x2, x3 \geq 0$;
$\therefore$ Max $Z = -x1 + 3x2 - 2x3$

The problem is converted to canonical form by adding slack, surplus and artificial variables as appropiate

1. As the constraint-1 is of type '$\leq$' we should add slack variable $S1$

2. As the constraint-2 is of type '$\leq$' we should add slack variable $S2$

3. As the constraint-3 is of type '$\leq$' we should add slack variable $S3$

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After introducing slack variables

Max $Z = -x1 + 3x2 - 2x3 + 0S1 + 0S2 + 0S3$

subject to

$$
\begin{aligned}
3x1 - x2 + 2x3 + S1 &= 7 \\
-2x1 + 4x2 + 4x3 + S2 &= 12 \\
-4x1 + 3x2 + 8x3 + S3 &= 10
\end{aligned}
$$

and $x1, x2, x3, S1, S2, S3 \geq 0$

Iteration-1		Cj	-1	3	-2	0	0	0	
B	***CB***	***XB***	***x*1**	***x*2**	***x*3**	***S*1**	***S*2**	***S*3**	**MinRatio** ***XBx*2**
$S1$	0	7	3	-1	2	1	0	0	---
$S2$	0	12	-2	**4**	4	0	1	0	12/4=3→
$S3$	0	10	-4	3	8	0	0	1	10.3=3.3333
***Z*=0**		***Zj***	**0**	**0**	**0**	**0**	**0**	**0**	
		Zj-Cj	1	-3↑	2	0	0	0	

Negative minimum Zj-Cj is -3 and its column index is 2. So, the entering variable is $x2$.

Minimum ratio is 3 and its row index is 2. So, the leaving basis variable is $S2$.

∴ The pivot element is 4.

Entering $=x2$, Departing $=S2$, Key Element $=4$

$R2(\text{new}) = R2(\text{old}) \div 4$

$R1(\text{new}) = R1(\text{old}) + R2(\text{new})$

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$R3(\text{new})=R3(\text{old}) - 3R2(\text{new})$

Iteration-2		Cj	-1	3	-2	0	0	0	
B	***CB***	***XB***	***x*1**	***x*2**	***x*3**	***S*1**	***S*2**	***S*3**	**MinRatio** ***XBx*1**
***S*1**	0	10	52	0	3	1	14	0	10/5/2=4→
$x2$	3	3	-12	1	1	0	14	0	---
$S3$	0	1	-52	0	5	0	-34	1	---
***Z*=9**		***Zj***	**-32**	**3**	**3**	**0**	**34**	**0**	
		Zj-Cj	-12↑	0	5	0	34	0	

Negative minimum Zj-Cj is -12 and its column index is 1. So, the entering variable is $x1$.

Minimum ratio is 4 and its row index is 1. So, the leaving basis variable is $S1$.

∴ The pivot element is 52.

Entering $=x1$, Departing $=S1$, Key Element $=52$

$R1(\text{new})=R1(\text{old}) \times 25$

$R2(\text{new})=R2(\text{old}) + 12R1(\text{new})$

$R3(\text{new})=R3(\text{old}) + 52R1(\text{new})$

Iteration-3		Cj	-1	3	-2	0	0	0	
B	***CB***	***XB***	***x*1**	***x*2**	***x*3**	***S*1**	***S*2**	***S*3**	**MinRatio**
$x1$	-1	4	1	0	6/5	2/5	1/10	0	
$x2$	3	5	0	1	8/5	1/5	3/10	0	

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$S3$	0	11	0	0	8	1	-1/2	1	
Z=11		**Zj**	**-1**	**3**	**18/5**	**1/5**	**4/5**	**0**	
		Zj-Cj	0	0	28/5	1/5	4/5	0	

Since all $Zj-Cj \geq 0$

Hence, optimal solution is arrived with value of variables as :
$x1=4, x2=5, x3=0$

Max $Z=11$

$\therefore$ Min $Z=-11$

Q2)

Mathematical Proof:

Let's denote the given linear programming problem as follows:

Maximize: Z = c1x1 + c2x2 + ... + cnxn

Subject to:

a11x1 + a12x2 + ... + a1nxn ≤ b1

a21x1 + a22x2 + ... + a2nxn ≤ b2

...

am1x1 + am2x2 + ... + amnxn ≤ bm

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Where $c_1, c_2, \ldots, c_n$ are the coefficients of the variables in the objective function, a_{ij} are the coefficients in the constraints, and $b_1, b_2, \ldots, b_m$ are the right-hand side values of the constraints.

Suppose there is a basic feasible solution with basic variables $x_B = (x_{B1}, x_{B2}, \ldots, x_{Bm})$ and non-basic variables $x_N = (x_{N1}, x_{N2}, \ldots, x_{Nn})$.

According to the Unbounded Solution Theorem:

For at least one j, $y_{ij} \geq 0$ for all $i = 1, 2, \ldots, m$, and $z_j - c_j < 0$.

This implies that for a particular index j:

$y_{1j}, y_{2j}, \ldots, y_{mj} \geq 0$

$z_j - c_j < 0$

Let's assume that such an index j exists.

Since y_{ij} represents the coefficient of the j-th variable in the i-th constraint, and z_j is the coefficient of the j-th variable in the objective function, the difference $z_j - c_j$ represents the potential decrease in the objective function if the j-th variable were to enter the basis.

However, if $z_j - c_j < 0$, this indicates that the objective function can be decreased by increasing the value of the j-th variable, making it more negative. Since the coefficients $y_{ij} \geq 0$, increasing any of the basic variables corresponding to these coefficients will not violate the constraints. As a result, the objective function can be decreased infinitely (unbounded) without violating the constraints, which contradicts the existence of an optimal solution.

Therefore, the assumption that there exists a basic feasible solution where for at least one j, $y_{ij} \geq 0$ and $z_j - c_j < 0$ leads to a contradiction. Hence, there can be no optimal solution to the given linear programming problem.

This completes the proof of the Unbounded Solution Theorem.

Q3) Statement: A sufficient condition for a basic feasible solution to an L.P.P. to be an optimum (maximum) is that $z_j - c_j \geq 0$ for all j for which the column vector $a_j \in A$ is not in the basis B. Proof by Contradiction:

1. Assume that there exists a basic feasible solution to the given linear programming problem where $z_j - c_j \geq 0$ for all j for which $a_j \in A$ is not in the basis B.

2. Let's denote this basic feasible solution as $(x_1, x_2, \ldots, x_m)$, where x_j represents the values of the basic variables and non-basic variables.

3. Now, consider the following: - The condition $z_j - c_j \geq 0$ for all non-basic variables implies that the reduced costs for all non-basic variables are non-negative. In other words, increasing the value of any non-basic variable from zero would either improve the objective function value or leave it unchanged. - Since this condition holds for all non-basic variables, there is no incentive to enter any of the non-basic variables into the basis. Adding any of the non-basic variables would not lead to an improvement in the objective function value.

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4. Therefore, if $z_j - c_j \geq 0$ for all non-basic variables, it implies that the current basic feasible solution is optimal because there is no need to pivot on any non-basic variable to reach a better solution.

5. Since this contradicts the assumption that there might be a better solution, we conclude that the current basic feasible solution is indeed the optimum (maximum) solution. 1 2 This completes the proof by contradiction, demonstrating that if for a basic feasible solution in an L.P.P., the condition $z_j - c_j \geq 0$ holds for all non-basic variables x_j (i.e., for all j for which the column vector $a_j \in A$ is not in the basis B), then that basic feasible solution is indeed an optimal (maximum) solution

Q4) Theorem: Conditions for Alternative Optimal Solutions

(a) If there is an optimum basic feasible solution to a linear programming problem and for some a_j not belonging to B, $z_j - c_j = 0$ with all $y_{ij} < 0$ ($i = 1, 2, . . . , m$), then a non-basic alternative optimum will exist. (b) If $z_j - c_j = 0$ for some a_j not belonging to B and at least one $y_{ij} > 0$ ($i = 1, 2, . . . , m$), then an alternative optimum will exist. Proof (a):

1. Assume there is an optimum basic feasible solution, and for some a_j not belonging to the basis B, $z_j - c_j = 0$ with all $y_{ij} < 0$ ($i = 1, 2, ..., m$).

2. This condition indicates that the objective function coefficient c_j for the variable x_j is equal to the reduced cost z_j for that variable. Moreover, all dual variables y_{ij} for the constraints are negative.

3. If $z_j - c_j = 0$, it means that increasing the value of x_j by one unit would not change the objective function value. This suggests that there is room to increase x_j while keeping the objective function value constant.

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4. Since all yij are negative, it implies that increasing xj would lead to a decrease in the constraint values, creating the possibility to pivot that variable into the basis.

5. This implies that there exists an alternative basic feasible solution where xj is in the basis and some basic variable corresponding to yij < 0 exits the basis. Since this new solution has the same objective function value as the optimum solution, it is an alternative optimum.