An analog and a discrete filter

The RC circuit and its equation

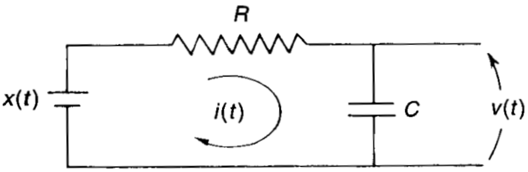

Figure: The so called RC circuit

Input voltage: $x(t)$. Output voltage (on the capacitor): $v(t)$.
The electric current in the circuit: $i(t)$.
Kirchhoff's voltage law: The directed sum of the potential differences (voltages) around any closed loop is zero.

$$Ri(t) + v(t) = x(t)$$

Since $v = Q/C$ and $i(t) = Q'(t)$ we have $Q'(t) = C \cdot v'(t)$. Thus

Solution of the de.

The homogeneous equation

The equation: $RCv'(t) + v(t) = 0$. The solution: $v_h(t) = Ke^{-\frac{t}{RC}}$.

A particular solution

By variation of constants:

$$v_p(t) = \frac{1}{RC}\int_{-\infty}^{t} e^{-\frac{t-s}{RC}} x(s)\,ds.$$

The complete solution

$$v(t) = Ke^{-\frac{t}{RC}} + \frac{1}{RC}\int_{-\infty}^{t} e^{-\frac{t-s}{RC}} x(s)\,ds.$$

Assume: constant 0 input generates constant 0 output.
It can only happen if $K = 0$. Consequently:

$$Ax(t) := v(t) = \frac{1}{RC}\int_{-\infty}^{t} e^{-\frac{t-s}{RC}} x(s)\,ds.$$

A is linear

If $Ax_1 = v_1$ and $Ax_2 = v_2$ then

$$\begin{aligned}
A(\lambda_1 x_1 + \lambda_2 x_2)(t) &= \frac{1}{RC}\int_{-\infty}^{t} e^{-\frac{t-s}{RC}} \left(\lambda_1 x_1(s) + \lambda_2 x_2(s)\right) ds \\
&= \lambda_1 \frac{1}{RC}\int_{-\infty}^{t} e^{-\frac{t-s}{RC}} x_1(s)\,ds \\
&+ \lambda_2 \frac{1}{RC}\int_{-\infty}^{t} e^{-\frac{t-s}{RC}} x_2(s)\,ds \\
&= \lambda_1 Ax_1(t) + \lambda_2 Ax_2(t)
\end{aligned}$$

Properties (cont.)

A is time invariant

Let $y(t) = x(t - t_0)$ and $Ax = v$. Then

$$
\begin{aligned}
Ay(t) &= \frac{1}{RC}\int_{-\infty}^{t} e^{-\dfrac{t-s}{RC}} x(s - t_0)\, ds \\
&\text{substituting } u = s - t_0 \\
&= \frac{1}{RC}\int_{-\infty}^{t-t_0} te^{-\dfrac{t-(u+t_0)}{RC}} x(u)\, du \\
&= \frac{1}{RC}\int_{-\infty}^{t-t_0} e^{-\dfrac{(t-t_0)-u}{RC}} x(u)\, du \\
&= v(t - t_0)
\end{aligned}
$$

Properties (cont.)

A is realizable (causal)

$$Ax(t) := v(t) = \frac{1}{RC}\int_{-\infty}^{t} e^{-\dfrac{t-s}{RC}} x(s)\, ds.$$

The upper limit of the integration is t. This means that $v(t) = Ax(t)$ determined by the past values of the input signal x only.

A is stable, continuous

Taking the norm $\|x\|_\infty := \sup_{t\in\mathbb{R}} |x(t)|$ we obtain:

$$|Ax(t)| \leq \frac{1}{RC}\int_{-\infty}^{t} e^{-\dfrac{t-s}{RC}} \|x(s)\|_\infty\, ds \leq \|x\|_\infty \frac{1}{RC}\int_{-\infty}^{t} e^{-\dfrac{t-s}{RC}}\, ds$$
$$= \|x\|_\infty,$$

i.e. $\|Ax\|_\infty \leq \|x\|_\infty$. The system is continuous with respect to this norm.

The RC circuit is filter

The output in form of convolution

$$Ax(t) = \frac{1}{RC}\int_{-\infty}^{t} e^{-\dfrac{t-s}{RC}} x(s)\,ds = \int_{-\infty}^{\infty} h(t-s)x(s)\,ds =: (h*x)(t),$$

where

$$h(t) = \begin{cases} 0, & ha \;\; t < 0; \\ \frac{1}{RC}e^{-\frac{t}{RC}}, & ha \;\; t \geq 0. \end{cases}$$

Impulse Response Function

h : the output generated by the unit impulse input.
It will be more transparent in the discrete case.

The transfer function

Recall

Transfer function: $H(\lambda)$

Input: $x(t) = e_\lambda(t) = e^{2\pi i \lambda t}$.

Output $v = Ae_\lambda = H(\lambda)e_\lambda$.

The calculation of the transfer function

The equation: $RCv'(t) + v(t) = x(t)$.

Substituting $v'(t) = H(\lambda)2\pi i \lambda e_\lambda(t)$ we have

$$(2i\pi\lambda RC + 1)H(\lambda)e_\lambda = e_\lambda\,.$$

$e_\lambda \neq 0$, therefore

$$H(\lambda) = \frac{1}{1 + 2i\pi\lambda RC}.$$

Low pass filter

Energy spcetrum: $|H(\lambda)|^2 = \dfrac{1}{1 + 4\pi^2\lambda^2 R^2 C^2}$.

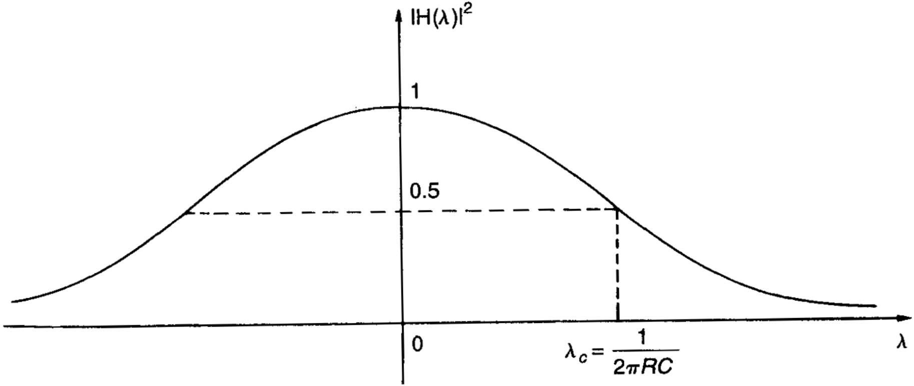

Figure: The energy spectrum of the RC circuit

Generalization

Butterworth filters

Butterworth filter of order n

$$|H(\lambda)|^2 = \frac{1}{1 + \left(\frac{\lambda}{\lambda_0}\right)^{2n}}.$$

RC circuit: $n = 1$.

Taking the limit $n \to \infty$: ideal filter.

A discrete filter

Input signal (sequence): x. Output signal: y.

$$y[k] = ay[k-1] + x[k] \quad (a > 0, k \in \mathbb{Z})$$

Remark: Using the notation $\Delta y_k = y_k - y_{k-1}$ the equation can be written in the form of difference equation. Formal similarity with the differential equation of the RC circuit.

The solution of the equation

Set $y[k] = a^k v[k]$. Then

$$v[k] - v[k-1] = a^{-k} x[k].$$

Summing them up:

$$v[k] - v[k-j] = \sum_{\ell=k-j+1}^{k} a^{-\ell} x[\ell] \qquad (j \in \mathbb{N}).$$

Suppose that

$$\sum_{n=-\infty}^{0} |a^{-n}x[n]| = \sum_{n=0}^{\infty} |a^{n}x[-n]| < \infty.$$

holds for the input signal.
Sufficient conditions: $a < 1$ and x is bounded.

Natural assumptions:

- Boundedness needs no explanations.
- If $a \geq 1$ then the system is not stable. To this order take the unit impulse input

$$x[n] = \begin{cases} 1, & n = 0; \\ 0, & n \neq 0. \end{cases}$$

The output is $y[n] = a^n \;\; n \geq 0$.

If $a < 1$ and x is bounded then $\sum_{n=0}^{\infty} |a^n x[-n]| < \infty$. Then the right side of

$$v[k] - v[k-j] = \sum_{\ell=k-j+1}^{k} a^{-\ell} x[\ell]$$

is convergent as $j \to \infty$.
This implies the convergence of the left side: the limit $\lim_{j\to-\infty} v[j] = b \in \mathbb{R}$ exists.

$$v[k] = b + \sum_{\ell=-\infty}^{k} a^{-\ell} x[\ell]\,.$$

Hence

$$y[k] = ba^k + \sum_{n=-\infty}^{k} a^{k-n} x[n] \qquad (k \in \mathbb{Z}).$$

The output of the constant 0 input is 0 : $b = 0$.

$$y[k] = \sum^{k} a^{k-n} x[n].$$

The properties of the discrete system

If can be proved similarly to the analog case that the system is :

- linear,
- time invariant,
- causal,
- stable.

Discrete filter.

Discrete convolution

Let

$$h[n] = \begin{cases} a^n & \text{, if } \; n \geq 0 \\ 0 & \text{, if } \; n < 0. \end{cases}$$

Then

$$y[k] = \sum_{n=-\infty}^{\infty} h[k-n]x[n] =: h * x.$$

Impulse response

The unit impulse: $x[k] = \begin{cases} 1 & \text{, if } k = 0 \\ 0 & \text{, if } k \neq 0. \end{cases}$
The impulse response: the output of the unit impulse is

$$y[k] = h[k].$$

Transfer function

Set $z \in \mathbb{C}$, $x[k] = z^k$. The convergence condition above holds only if $|z| > |a|$.
Then

$$y[k] = \sum_{n=-\infty}^{k} a^{k-n} x[n] = a^k \sum_{n=-\infty}^{k} \left(\frac{z}{a}\right)^n = a^k \sum_{n=-k}^{\infty} \left(\frac{a}{z}\right)^n$$
$$= a^k \left(\frac{a}{z}\right)^{-k} \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = z^k \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a} x[k].$$

Transfer function (cont.)

The transfer function of the system: $H(z) = \dfrac{z}{z-a}$.

Analytic on the domain $|z| > |a|$.

Let: $|z| = 1$, which can be written in the form $e^{2\pi i\lambda}$.

Then z^k is a discrete trigonometric function.

The transfer function with variable λ

$$H(\lambda) = \frac{e^{2\pi i\lambda}}{e^{2\pi i\lambda} - a}.$$

The energy spectrum

$$|H(\lambda)|^2 = H(\lambda) \cdot \overline{H(\lambda)} = \frac{1}{(\cos 2\pi\lambda t - a)^2 + \sin^2 2\pi\lambda t}$$
$$= \frac{1}{1 + a^2 - 2a\cos 2\pi\lambda} = \frac{1}{(1-a)^2 + 4a\sin^2 \pi\lambda}.$$

The energy spectrum of the discrete system for $a = \frac{1}{2}$.

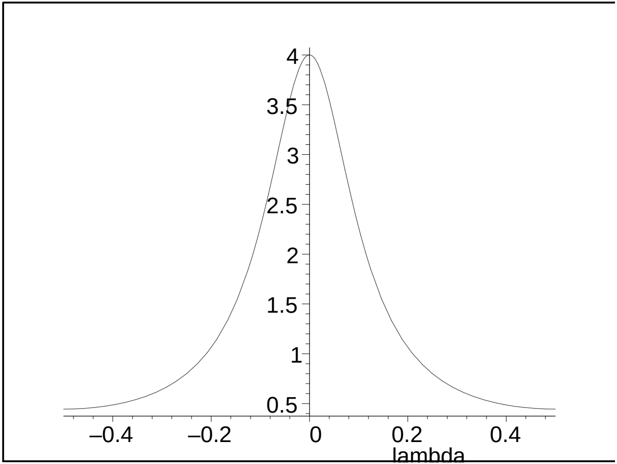

Figure: The energy spectrum of the discrete system