Windowing

Why windowing is needed

What is windowing?

- We can process finite many samples at a time.
- How to get finite many samples?
- Two equivalent models:
  a) First determine a finite interval, and then take samples
  b) First generate the discrete signal, then take the samples that are within a predetermined interval.
- Let us take model a).
- Signal: f. Finite interval: I. Naive approach: $f \cdot \chi_I$.
- We will investigate the effect of restriction of a signal to a finite interval

Window functions

- Fix a finite interval: I.
- Window functions: $h : \mathbb{R} \to \mathbb{R}$, $h(t) = 0 \ \ (t \notin I)$.
- Windowing in general: $f \cdot h$.
- Rectangular window: χ_I.
- Rectangular window seems to be a very logical choice.

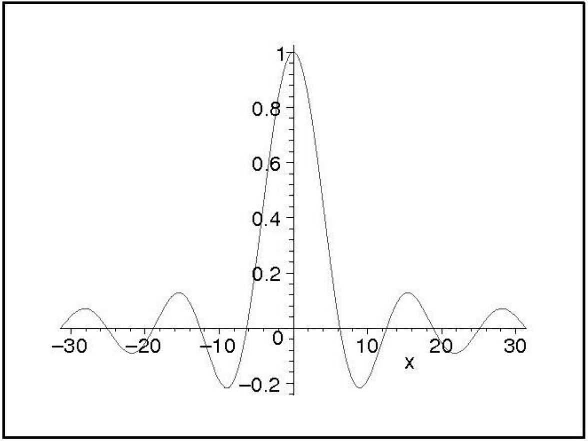
The effect of windowing

- Windowing in the time domain: multiplication by the window function.
- Windowing in the frequency domain: convolution by the Fourier transform of the window function.

The case of the rectangular window

- First we will investigate the effect of the rectangular window in the frequency domain.
- Let us take the normalized case: $I = [-1/2, 1/2]$.
  Then the window function is the well known rect function: $h = \mathrm{rect}$.
  It is known that $\widehat{\mathrm{rect}} = \mathrm{sinc}$.
- Let $f(t) = e^{2\pi i \lambda_0 t}$. Then

$$\widehat{x \cdot \mathrm{rect}}(\lambda) = \int_{-\infty}^{\infty} \mathrm{rect}\,(t) e^{2\pi i \lambda_0 t} e^{-2\pi i \lambda t} dtdt = \int_{-0.5}^{0.5} e^{-2\pi i (\lambda - \lambda_0) t} dt$$
$$= \frac{\sin \pi(\lambda - \lambda_0)}{\pi(\lambda - \lambda_0)} = \mathrm{sinc}\,(\lambda - \lambda_0)\,.$$



Sinc function

The problem with the rectangular window

- The Fourier transform of a very nice signal, a monochromatic signal, becomes a shifted sinc function, which is complicated one. The specrum of the original signal is concentrated on one point λ_0, while the spectrum of the windowed signal is spread.
- The problems
  The width of the main lobe of sinc
  The side lobes tend to 0 slowly. The rate of convergence is $1/\lambda$.
- A linear combination of a few monochromatic signal:
  The spectrum will be combination of sinc type functions.
  Complicated spectrum.
  Sinc terms corresponding to frequencies that are close to each other interfere. Hard to separate them.
  Small side lobes may add up.
- Consequence: after windowing a nice signal may become a complicated one, which is undesirable. It makes the following steps in signal processing cumbersome.

Improving the rectangular window: triangular window

- How to improve the rectangular window?
  Construct a window function with more concentrated Fourier transform.
  Make the side lobes tend to 0 faster.
- The idea: let us take sinc^2 instead of sinc. The side lobes will tend to 0 by $1/\lambda^2$.
- The corresponding window function is the convolution of rect by itself.

$$g(t) = \int_{-\infty}^{\infty} \mathrm{rect}\,(t-u)\mathrm{rect}\,(u)\,du = \big|[t-0.5, t+0.5] \cap [-0.5, 0.5]\big|$$
$$= \begin{cases} t+1, & -1 \leq t < 0 \\ 1-t, & 0 \leq t \leq 1 \end{cases}$$

- Unfortunately, the width of the main lobe has doubled. It can be corrected by dilation, doubling the size of the window function.

Improving the rectangular window: triangular window. Contd.

- Problem with the triangular window function at the end points: $g(\pm 1) = 0$. We loose information in the neighborhoods of ± 1.
- Solution: overlapping windows. Triangles supported on the intervals $[k, k+2]$ ($k \in \mathbb{Z}$).
- The sum of the overlapped triangular window functions is const ant 1.

Rectangular window-triangular window

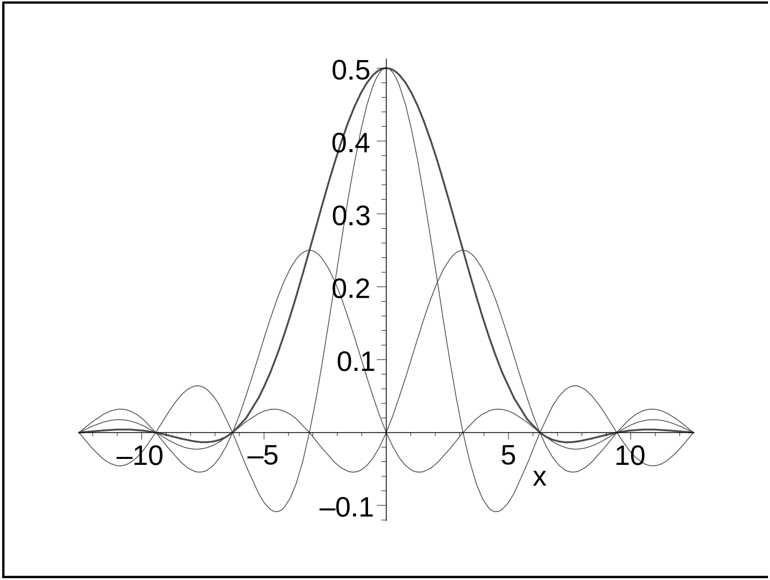
- The triangular window is better than the rectangular one. What is the reason?
- Windowed function by rectangular window will be discontinues at the endpoints of the window.
- Triangular windows are equal to 0 at the endpoints. No discontinuity is generated.

Improving the rectangular window: Hahn window

- Improving the rectangular window by another idea.
- Use shifts of sinc functions in such a way that the side lobes compensate each other. This makes the side lobes tend to 0 faster: $1/\lambda^3$.

$$
\begin{aligned}
\widehat{h}(\lambda) =& \frac{1}{4}\text{sinc}\,(\lambda-1) + \frac{1}{2}\text{sinc}\;\lambda + \frac{1}{4}\text{sinc}\;(\lambda+1) \\
=& \frac{1}{4}\frac{\sin\pi(\lambda-1)}{\pi(\lambda-1)} + \frac{1}{2}\frac{\sin\pi\lambda}{\pi\lambda} + \frac{1}{4}\frac{\sin\pi(\lambda+1)}{\pi(\lambda+1)} \\
=& \frac{\sin\pi\lambda}{\pi}\Big(\frac{1}{2}\frac{1}{\lambda} - \frac{1}{4}\frac{1}{\lambda-1} - \frac{1}{4}\frac{1}{\lambda+1}\Big) \\
=& -\frac{1}{2\pi}\frac{1}{\lambda(\lambda^2-1)}\sin\pi\lambda
\end{aligned}
$$

- The spectrum tends to zero by the rate of $1/\lambda^3$.

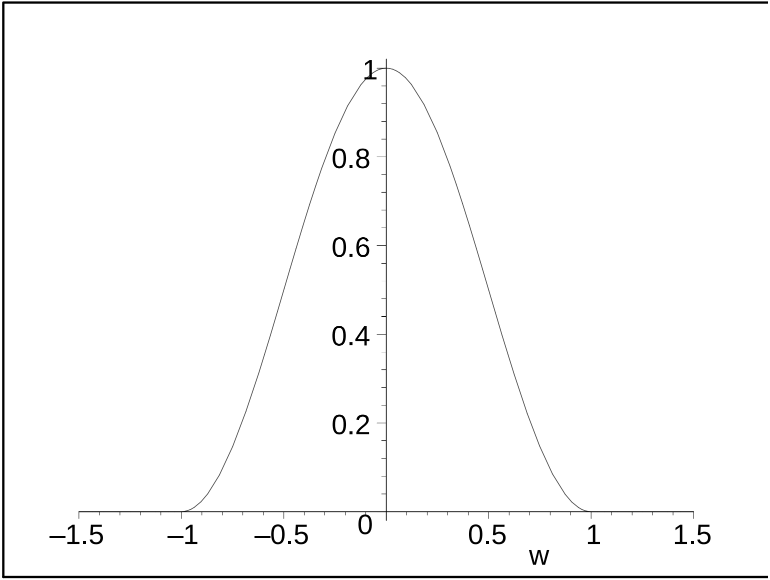


The Fourier transform of the Hahn window

Improving the rectangular window: Hahn window

- We have calculated the Hahn window in the frequency domain.
- Hahn window in the time domain.
  In the construction we used shifts of sinc function in the frequency domain.
  This means modulation of the rectangular window in the time domain.

$$h(t) = \frac{1}{2}\operatorname{rect} t \left(1 + \frac{1}{2}e^{-2\pi it} + \frac{1}{2}e^{2\pi it}\right)$$
$$= \frac{1+\cos 2\pi t}{2}\operatorname{rect} t$$



The Hahn window