Histogram equalization

Discrete case

Image

$$X_d : J \times K \mapsto \{0, \ldots, L-1\}$$

Histogram

$$M_{X_d} : \{0, \ldots, L-1\} \to \mathbb{N} \,,$$

where $M_{X_d}(k)$ the number of (i, j) $(1 \leq i \leq J, \leq j \leq K)$ coordinates, pixels, for which the value of X_d equals k :

$$M_{X_d}(k) = \operatorname{card}\{(i, j) \,:\, X_d(i, j) = k\}$$

Normalized version

$$\mu_{X_d} : \left\{0, \frac{1}{L}, \frac{2}{L}, \ldots, \frac{L-1}{L}\right\} \to [0, 1] \,,$$

$$\mu_{X_d}(k) = \frac{\operatorname{card}\{(i, j) \,:\, X_d(i, j) = k\}}{J \cdot K}$$

Analogue (continuous) model

Image

$$X : [0,1]^2 \mapsto [0,1]\,.$$

Distribution function:

$$m_X : [0,1] \to [0,1] \quad m_X(x) = |\{(t,u) \in [0,1]^2 \,:\, X(t,u) \le x\}|\,.$$

Probability density function function (hisztogram):

$$\mu_X : [0,1] \to \mathbb{R}^+\,, \quad \text{where} \quad m_X(x) = \int_0^x \mu(s)\,ds\,.$$

Equalized histogram:

$$m_X(x) = x\,, \qquad \text{azaz} \quad \mu_X(s) = 1\,.$$

Histogram equalization

Transformation

$$T : [0,1] \to [0,1]\,, \nearrow$$

New image

$$Y = T \circ X : [0,1]^2 \to [0,1]\,, \quad Y(t,s) = T(X(t,s))\,.$$

We need

$$m_Y(x) = x \qquad (x \in [0,1]), \quad \text{azaz} \quad \mu_Y \equiv 1\,.$$

Obviously

$$\begin{aligned} m_X(x) &= |\{(t,s) \in [0,1]^2 \,:\, X(t,s) \leq x\}| \\ &= \\ m_Y(T(x)) &= |\{(t,s) \in [0,1]^2 \,:\, Y(t,s) \leq T(x)\}| \end{aligned}$$

Cont.

Expressing this equality by histograms

$$\int_0^x \mu_X(t)\,dt = \int_0^{T(x)} \mu_Y(s)\,ds\,.$$

After differentiation we obtain

$$\mu_X(x) = \mu_Y(T(x)) \cdot T'(x)\ .$$

Hence $\mu_Y \equiv 1$ we have

$$T(x) = \int_0^x \mu_X(t)\,dt = m_X(x)\,.$$

Example

Let : $\mu_X(t) = 2 - 2t \;\; (0 \le t \le 1)$.
Then

$$T(x) = m_X(x) = \int_0^x \mu_X(t)\, dt = 2x - x^2 \,,$$

i.e.

$$T^{-1}(x) = 1 - \sqrt{1-x}$$

It follows

$$\begin{aligned} m_Y(x) &= m_X(T^{-1}(x)) = 2(1-\sqrt{1-x}) - (1-\sqrt{1-x})^2 \\ &= 2 - 2\sqrt{1-x} - (1 - 2\sqrt{1-x} + 1 - x) \\ &= x \end{aligned}$$

Consequently: $\mu_Y(x) = 1$.

Discrete case

$$X_d : J \times K \mapsto \{0, \ldots, L-1\},$$

$$M_{X_d}(k) = \text{card}\,\{(i,j) \,:\, X_d(i,j) = k\},$$

$$\mu_{X_d}(k) = \frac{\text{card}\,\{(i,j) \,:\, X_d(i,j) = k\}}{J \cdot K}$$

Ekkor

$$T : \left\{0, 1/(L-1), 2/(L-1), \ldots 1\right\} \to \left\{0, 1/(L-1), 2/(L-1), \ldots 1\right\},$$

$$T\left(\frac{k}{L-1}\right) \approx \sum_{j=0}^{k} \mu_{X_d}(k)$$

64×64-es, 8-level image: intensity histogram equalization.

k	M(k)	μ_X	μ_Y	μ_{Y_d}
0	523	0.13	0.13	1/7
1/7	780	0.19	0.32	2/7
2/7	1053	0.26	0.58	4/7
3/7	818	0.20	0.78	5/7
4/7	470	0.11	0.89	6/7
5/7	222	0.05	0.94	1
6/7	164	0.04	0.98	1
1	66	0.02	1.00	1