Spatial Filtering

Images are taken from Gonzalez–Woods

December 4, 2019

General model

- $f : [0,1]^2 \to [0,1]$ original image (input).
- T image-image transformation.
- $g = T(f)$ transformed image (output)

T can be linear, nonlinear transform.

Let us start with the linear case, in particular with filters.

Filters

Mathematical model

$$g(x,y) = \int_{\mathbb{R}^2} f(x-t, y-u) h(t,u)\, du\, dt$$

Convolution: h kernel function.

Remark: translation invariance.

Problem

f is defined only on the unit square $[0,1]^2$.

One option: extension $f(x,y) = 0 \ \ ((x,y) \notin [0,1)^2)$.

Consequence: it is enough to take $h\big|_{[-1,1]^2}$. h is supported on a compact set.

The discrete case

- Images f, g are $M \times N$ matrices.
- h is an $m \times n$ matrix. Let $m = 2i+1$, $n = 2j+1$. Typically: $m << M$, $n << N$. Pl. $m, n : 3, 5, 7$.
- Then

$$g(x,y) = \sum_{k=-i}^{i} \sum_{\ell=-j}^{j} f(x-k, y-\ell) h(k,\ell)$$

$(x = 1, \ldots, M,\ y = 1 \ldots, N)$.

Jargon for h : mask, window, kernel.

Local operation: The new pixel value $g(x, y)$ depends only on the pixel values in the original image f that are in the neighborhood of (x, y). Translation invariance!

How about the margins?

No generally good method. "Overextending" mask.

The bad area, stripe increases by repeating filtering.

Possibilities

- 0 extension. Highly artificial.
- Extension by using the average. Less artificial.
- Extension by using some kind of local average. More complicated then those above.
- Periodic extension. No edge. Complicated. The two opposite margins of the image have usually nothing in common. Even or odd extension.

How to generate discrete filters from continuous (analogue) ones.

MethodM: sampling, discrete approximation.

Example: Gauss filters

$$h(x,y) = e^{-\frac{x^2+y^2}{2\sigma^2}}$$

3×3 discrete filter

$$\widetilde{h}(k,\ell) = h(k,\ell) \qquad (k = -1, 0, 1,\ \ell = -1, 0, 1).$$

Smoothing by local averages

Low pass filters.

Examples for weighted average masks:

$$\frac{1}{9} \times \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \quad , \quad \frac{1}{16} \times \begin{array}{|c|c|c|}\hline 1 & 2 & 1 \\ \hline 2 & 4 & 2 \\ \hline 1 & 2 & 1 \\ \hline \end{array}$$

What is behind averaging?

- Reduction of irrelevant details. Irrelevant details: their dimension is less then the dimension of the mask.
- Smoothing of fast intensity change.

The effect of averaging

- Advantage: noise reduction.
- Disadvantage: blurring edges.
- Disadvantage: new intensity values may come up.

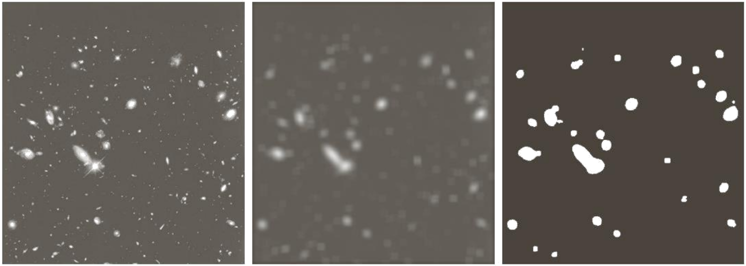
Enhanced smoothing

- Repeated application.
- Increasing the dimension of the mask.

Application

Assimilation of small object into the background, highlighting important objects (patchy forms).

1. Image: taken by the Hubble telescope, 528X485 pixel.
2. Image: 15X15 averaging filter applied to image 1.
3. Image: 2. image after thresholding.



Nonlinear smoothing filters

Median filter

- Mask dimension: $(2i+1)(2j+1)$.
- $g(x,y)$ is defined as the median of the intensity values $\{f(x+k, y+\ell) : -i \le k \le i, -j \le \ell \le j\}$.

The idea behind using the median filter

Make the intensity values of the neighbors closer to each other.

The effect of the median filter

- Extremal values, for instance noise, are erased.
- Small groups of pixels that are lighter or darker then the background disappear. The dimension of the group is smaller than the dimension of the mask.

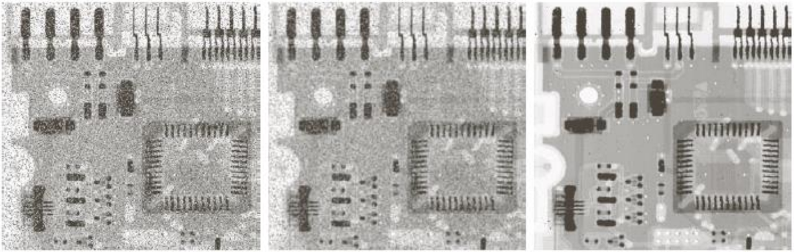
The advantage of the median filter

- Especially effective for certain noise: salt and pepper.
- Less blurring than in averaging.
- No new intensity values come up.

Remarks

- Iterated application is possible.
- Scalable: minimum-median-maximum.

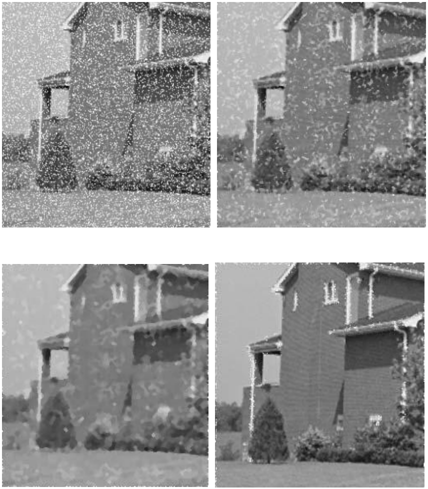
Example



a b c

FIGURE 3.35 (a) X-ray image of circuit board corrupted by salt-and-pepper noise. (b) Noise reduction with a 3×3 averaging mask. (c) Noise reduction with a 3×3 median filter. (Original image courtesy of Mr. Joseph E. Pascente, Lixi, Inc.)

1. 15% white noise, 2. 3X3 median filter,
3. 10-fold iteration, 4. 5x5 median filter and tresholding



Heuristics

Smoothing: averaging, integration, low pass filter.

Sharpening: high pass filter, differentiation.

Effects: highlighting the fast intensity changes: edges, noise. The domains with slow intensity changing will be represented by small weights.

Discrete version

- $\frac{\partial f}{\partial x}(x,y) =\approx f(x+1,y) - f(x,y)$
- $\frac{\partial^2 f}{\partial x^2}(x,y) \approx f(x+1,y) + f(x-1,y) - 2f(x,y)$

Laplace operator

- $\triangle f(x,y) = \dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2}, \quad \triangle f = \partial_{11} f + \partial_{22} f$
- Isotropic : $(\triangle f) \circ R = \triangle(f \circ R)$. R : rotation.
- Noise sensitive.
- Discrete version:
  $\triangle f(x,y) = f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)$
- Mask:

a)

0	1	0
1	-4	1
0	1	0

,

b)

0	-1	0
-1	4	-1
0	-1	0

,

c)

1	1	1
1	-8	1
1	1	1

,

d)

-1	-1	-1
-1	8	-1
-1	-1	-1

The total sum of the entries is 0.

Application of Laplace transform

$$g(x, y) = f(x, y) + c\triangle f(x, y)$$

the sign of c depends on which variant we use.

Highlighting sudden intensity changes.

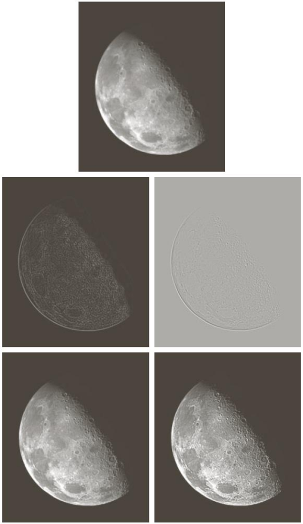
Problem

$\triangle f$ takes on negative values as well. Those will be 0, i.e. black in the image.

Rescaling

$$\triangle f - \min \triangle f$$

Examples for image sharpening by using Laplace masks.



- 1. Blurred image about the North Pole of the Moon.
- 2. The result of Laplace mask with no scaling
- 3. The result of Laplace mask with scaling
- 4. Image sharpening with the a) Laplace mask
- 5. Image sharpening with the c) Laplace mask

Unsharp masking, highboost filtering

Steps

1. Smoothing the original image: $\bar{f}$.
2. Subtract the smooth image from the original one: $h = f - \bar{f}$.
   Unsharp mask.
3. New image: $g(x, y) = f(x, y) + k \cdot h(x, y)$.

$k > 1$ case: highboost filtering.

- 1. Original image
- 2. Smoothing by Gauss filter
- 3. Unsharp mask
- 4. Unsharp masking: $k = 1$
- 5. The result of highboost filtering