Discrete Fourier transform

Complex vector space

The space: $\mathbb{C}^N$, where $N \in \mathbb{N}$ is fixed. N-dimensional linear space.

The elements: $(x[0], \ldots, x[N-1]) \in \mathbb{C}^N$. Complex N-dimensional vectors.

Canonical basis vectors: $\delta_n[k] = \begin{cases} 0, & k \neq n; \\ 1, & k = n. \end{cases}$ $(k, n = 0, \ldots, N-1)$.

The δ_n's are linearly independent. Canonical basis:

$$\{\delta_n \,:\, n = 0, \ldots, N-1\}.$$

The decompositions of elements in the canonical basis:

$$x = \sum_{n=0}^{N-1} x[n] \cdot \delta_n.$$

Equivalent interpretations: discrete functions, periodic sequences.

The trigonometric basis

The discrete trigonometric vectors (functions) in $\mathbb{C}^N$:

$$e_n : \{0, \ldots, N-1\} \to \mathbb{C}, \quad e_n[k] = e^{2\pi i k \frac{n}{N}} \quad (k, n = 0, \ldots, N-1).$$

Scalar product: $< x, y >:= \sum_{k=0}^{N-1} x[k]\overline{y[k]}$.

The corresponding norm: $\left(< x, x > \right)^{1/2} = \|x\| = \left(\sum_{k=0}^{N-1} |x[k]|^2 \right)^{1/2}$.

The vector x, and y are orthogonal if $< x, y >= 0$.

The e_n trigonometric vectors form an orthogonal basis:

$$< e_n, e_j >= \sum_{k=0}^{N-1} e^{2\pi i (n-j) \frac{k}{N}} = \begin{cases} 0 & \text{, if } j \neq n \\ N & \text{, if } j = n. \end{cases}$$

Normalization factor: $\frac{1}{\sqrt{N}}$

Discrete Fourier transform

Decomposition of x in the discrete trigonometric basis:

$$x = \sum_{n=0}^{N-1} c[n] e_n.$$

How to obtain the coefficients $c[n]$? (The scalar product is linear.)

$$< x, e_j >= \sum_{n=0}^{N-1} c[n] < e_n, e_j >= (\text{orthogonality}) = N\text{\c{d}}otc[k]$$

Discrete Fourier coefficients:

$$\widehat{x}[n] = c[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{2\pi i (n-j) \frac{k}{N}} \quad (n = 0, \dots, N-1).$$

Discrete Fourier transform: $\mathcal{F}x = \widehat{x} = (\widehat{x}[0], \dots, \widehat{x}[N-1]) \in \mathbb{C}^N$.

The matrix representation of the discrete Fourier transform

Let us introduce the notation: $\omega_N = e^{2\pi i \frac{1}{N}}$.
Set

$$\Omega_N := \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega_N & \omega_N^2 & \dots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \dots & \omega_N^{2(N-1)} \\ 1 & & & & \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \dots & \omega_N^{(N-1)^2} \end{pmatrix}$$

Then

$$\mathcal{F}x = \widehat{x} = \frac{1}{N}\overline{\Omega}_N x.$$

Since $\left(\frac{1}{N}\overline{\Omega}_N\right)^{-1} = \Omega_N$ we have that the inverse transform is

$$\mathcal{F}^{-1}y = \Omega_N y\ , \quad \text{therefore} \quad x = \Omega_N \widehat{x}.$$

Both $\mathcal{F}$, and $\mathcal{F}^{-1}$ are linear transforms.

Properties of the discrete Fourier transform

Linearity

$$\widehat{\alpha x + \beta y} = \alpha \widehat{x} + \beta \widehat{y}.$$

Time shift

$$y[k] = x[k - k_0] \quad \Longrightarrow \quad \widehat{y}[n] = e^{-2\pi i n \frac{k_0}{N}} \widehat{x}[n]$$

Substitute the index $j = k - k_0$ to obtain

$$\widehat{y}[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[k - k_0] e^{-2\pi i n \frac{k}{N}} = \frac{1}{N} \sum_{j=-k_0}^{N-1-k_0} x[j] e^{-2\pi i n \frac{j+k_0}{N}}$$

$$= e^{-2\pi i n \frac{k_0}{N}} \widehat{x}[n] .$$

Frequency shift (modulation)

$$y[k] = e^{2\pi i k \frac{n_0}{N}} x[k] \quad \Longrightarrow \quad \widehat{y}[n] = \widehat{x}[n - n_0]\,.$$

Similarly to the case of time shift use index shifting

$$\widehat{y}[n] = \frac{1}{N} \sum_{k=0}^{N-1} e^{2\pi i k \frac{n_0}{N}} x[k] e^{-2\pi i n \frac{k}{N}} = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{-2\pi i (n-n_0) \frac{k}{N}} = \widehat{x}[n - n_0]\,.$$

Discrete differentiation

$$y[k] = x[k] - x[k-1] \quad \Longrightarrow \quad \widehat{y}[n] = (1 - e^{-2\pi i \frac{n}{N}})\widehat{x}[n]\,.$$

It is a simple consequence of linearity and time shift properties.

Discrete integration

We assume that $\sum_{k=0}^{N-1} x[k] = 0$.

Let $y[k] = \sum_{j=0}^{k} x[j]$.

Then

$$\widehat{y}[n] = \frac{1}{1 - e^{-2\pi i \frac{n}{N}}} \widehat{x}[n].$$

By definition

$$\widehat{y}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{k} x[j] e^{-2\pi i n \frac{k}{N}} = \frac{1}{N} \sum_{j=0}^{N-1} x[j] \sum_{k=j}^{N-1} e^{-2\pi i n \frac{k}{N}}.$$

Properties of the discrete Fourier transform (cont.)

Discrete integration (cont.)

By the formula for partial sums of geometric series we have

$$\sum_{k=j}^{N-1} e^{-2\pi i n \frac{k}{N}} = \sum_{k=0}^{j-1} e^{-2\pi i n \frac{k}{N}} = \frac{e^{-2\pi i n \frac{j}{N}} - 1}{1 - e^{-2\pi i \frac{n}{N}}}.$$

Hence

$$\begin{aligned}\widehat{y}[n] &= \frac{1}{N}\sum_{j=0}^{N-1} x[j] \sum_{k=j}^{N-1} e^{-2\pi i n \frac{k}{N}} = \frac{1}{1 - e^{-2\pi i \frac{n}{N}}}\, \frac{1}{N}\sum_{j=0}^{N-1} x[j](e^{-2\pi i n \frac{j}{N}} - 1) \\ &= \frac{1}{1 - e^{2\pi i \frac{n}{N}}}\widehat{x}[n]\,.\end{aligned}$$

Properties of the discrete Fourier transform (cont.)

Discrete convolution

Definition: $(y * x)[k] = \sum_{j=0}^{N-1} y[k-j]x[j] \qquad (k = 0, \ldots, N-1).$

Remark: x is extended periodically.

Discrete Fourier transform of convolution

$$z = y * x \quad \Longrightarrow \quad \widehat{z} = N\widehat{y} \cdot \widehat{x}$$

Start from the definition of the Fourier transform

$$\begin{aligned}
\widehat{z}[n] &= \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} y[k-j]x[j]e^{-2\pi i n \frac{k}{N}} \\
&= \frac{1}{N} \sum_{j=0}^{N-1} x[j]e^{-2\pi i n \frac{j}{N}} \sum_{k=0}^{N-1} y[k-j]e^{-2\pi i n \frac{k-j}{N}} = N\,\widehat{x}[n] \cdot \widehat{y}[n].
\end{aligned}$$

Properties of the discrete Fourier transform (cont.)

Multiplication

$$z = y \cdot x \quad \Longrightarrow \quad \widehat{z} = \widehat{y} * \widehat{x}.$$

Start from the right:

$$\begin{aligned}
(\widehat{y} * \widehat{x})[n] &= \sum_{j=0}^{N-1} \widehat{y}[n-j]\,\widehat{x}[j] \\
&= \frac{1}{N} \sum_{j=0}^{N-1} \Big(\sum_{k=0}^{N-1} y[k] e^{-2\pi i (n-j) \frac{k}{N}} \Big) \widehat{x}[j] \\
&= \frac{1}{N} \sum_{k=0}^{N-1} y[k] e^{-2\pi i n \frac{k}{N}} \sum_{j=0}^{N-1} \widehat{x}[j] e^{2\pi i j \frac{k}{N}}
\end{aligned}$$

Properties of the discrete Fourier transform (cont.)

Multiplication (cont.)

Since

$$\sum_{j=0}^{N-1} \widehat{x}[j] e^{2\pi i j \frac{k}{N}} = x[k]$$

we have

$$(\widehat{y} * \widehat{x})[n] = \frac{1}{N} \sum_{k=0}^{N-1} y[k] x[k] e^{-2\pi i n \frac{k}{N}} = \widehat{y \cdot x}[n].$$