Introduction to Filter Construction

December 30, 2018

The ideal low pass filter in the frequency range $[-\lambda_0, \lambda_0]$ $(\lambda_0 > 0)$

- In the frequency domain: multiplication by $\chi_{[-\lambda_0,\lambda_0]}$ (the characteristic function of the interval $[-\lambda_0, \lambda_0]$)
- Signal f. The Fourier transform of the filtered signal: $\widehat{f} \cdot \chi_{[-\lambda_0,\lambda_0]}$.
- Note that the characteristic function can be received from the rectangle function by dilation : $\chi_{[-\lambda_0,\lambda_0]} = \delta_{1/2\lambda_0}\text{rect}$.
- In time domain: convolution by the inverse Fourier transform of $\chi_{[-\lambda_0,\lambda_0]}$
- Since the inverse Fourier transform of rect is sinc we have by the dilation rule that the inverse Fourier transform of $\delta_{1/2\lambda_0}\text{rect}$ is $2\lambda_0\delta_{2\lambda_0}\text{sinc}$.
- Filtered signal in the time domain: $\int_{-\infty}^{\infty} f(x-t)\frac{\sin 2\pi\lambda_0 t}{\pi t}\, dt$
- The kernel function does not vanish on $(-\infty, 0]$, therefore the filter is not realizable.

Differential equations. Why?

- The technical realization of filters: input-output electric circuit. The electric circuits can be can be represented by differential equations.
- Example: RC circuit. Laws: Ohm's, Krichoff's etc.. Generally: electrodynamics-Maxwell equations.
- The model: linear ordinary differential equation with constant coefficients.

$$\sum_{k=0}^{q} b_k g^{(k)} = \sum_{j=0}^{p} a_j f^{(j)}$$

  $p, q \in \mathbb{N}$, a_j, $b_k \in \mathbb{R}$, $(j = 0 \ldots, p,\ k = 0, \ldots q)$, $a_p b_q \neq 0$.
  Input: f. Output g.
- Example: RC circuit. $p = 0$, $a_0 = 1$, and $q = 1$, $b_1 = RC$, $b_0 = 1$.

$$RCf' + f = g$$

Transfer function

- Filters are designed in the frequency domain but often used in the time domain.
- It is easy to check that the system given by the equation $\sum_{k=0}^{q} b_k g^{(k)} = \sum_{j=0}^{p} a_j f^{(j)}$ is linear and time invariant.
- Filters in the frequency domain: characterized by the transfer function of the system. Namely, filtering is multiplication by the transfer function.
- Input $f = e_\lambda$. Transfer function: H. Output $g = H(\lambda) \cdot e_\lambda$. (Recall: $e_\lambda(t) = e^{2\pi i \lambda t}$ $\lambda, t \in \mathbb{R}$)
- Since $f^{(k)}(t) = \left(e_\lambda(t)\right)^{(k)} = (2\pi\lambda i)^k e_\lambda(t)$, and $g^{(k)}(t) = \left(H(\lambda) \cdot e_\lambda(t)\right)^{(k)} = H(\lambda) \cdot (2\pi\lambda i)^k e_\lambda(t)$. Substituting it in the differential equation we have

$$\sum_{k=0}^{q} b_k H(\lambda)(2\pi i\lambda)^k e_\lambda = \sum_{j=0}^{p} a_j (2\pi i\lambda)^j e_\lambda$$

Transfer function contd.

- Considering that $e_\lambda \neq 0$ we obtain

$$H(\lambda) = \frac{\sum_{j=0}^{p} a_j (2\pi i\lambda)^j}{\sum_{k=0}^{q} b_k (2\pi i\lambda)^k}$$

- Set: $P(z) = \sum_{j=0}^{p} a_j z^j,\ Q(z) = \sum_{k=0}^{q} b_k z^k \quad (z \in \mathbb{C})$.
  P and Q are complex polynomials, and

$$H(\lambda) = \frac{P(2\pi i\lambda)}{Q(2\pi i\lambda)} \quad (\lambda \in \mathbb{R})$$

  We will suppose that Q has no root on the imaginary axis.
- The transfer function is a rational function.
  (H can be extended, and viewed as a complex-complex function)

Filter: stability-transfer function

- Note that the differential system is easy to reconstruct from the transfer function: the coefficients in the differential systems are the coefficients of the polynomials P, Q.
- Stability in roughly speaking: small change in the input results in small change in the output.
- Stability in mathematical terminology: continuity.
- Filters are linear operators.
  Therefore continuity is equivalent with boundedness.
  That is: there exists a constant $C > 0$ such that $\|f\| \leq C\|g\|$.
  ($\| \, . \, \|$ stands for the norm by which we measure the signals)
- BIBO stability: bounded input bounded output.
- Recall that if $f = e_\lambda$ then $g = H(\lambda)e_\lambda$. Since $|e_\lambda| \equiv 1$ we have that the consequence of BIBO stability is that the transfer function H must be bounded.

Filter: stability-transfer function contd.

- H is a rational function. Therefore it is bounded if and only if the degree of the numerator is not greater than the degree of the denominator.
- The filter is stable if and only if $\deg P \leq \deg Q$.
- What if if $\deg P = \deg Q$? By polynomial division we obtain

$$\frac{P}{Q} = K + \frac{P^*}{Q},$$

  where K is a constant, and $\deg P^* < \deg Q$.
- The term K corresponds to simple amplification, and so it is irrelevant in filter construction.
  Consequence: we may suppose $\deg P < \deg Q$.

Filter: causality-transfer function

- Let us suppose that Q has q distinct roots: $z_1, \ldots, z_q$. (The multiplicity of every root is 1.)
- The transfer function can then be decomposed as sum of partial fractions of the form

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^{q} \frac{\beta_k}{z - z_k}$$

- Turning back to the original form of the transfer function

$$H(\lambda) = \frac{P(2\pi i\lambda)}{Q(2\pi i\lambda)} = \sum_{k=1}^{q} \frac{\beta_k}{2\pi i\lambda - z_k} = \sum_{k=0}^{q} \frac{\beta_k}{-z_k^{(1)} + 2\pi i(\lambda - \frac{z_k^{(2)}}{2\pi})},$$

where $z_k^{(1)}$ is the real, and $z_k^{(2)}$ is the imaginary part of z_k.

Filter: causality-transfer function

- The individual terms

$$\frac{\beta_k}{-z_k^{(1)} + 2\pi i(\lambda - \frac{z_k^{(2)}}{2\pi})} \quad (k = 1, \ldots, q)$$

  in the decomposition can be considered as transfer functions.
- How can these individual elementary rational transfer functions be interpreted in the time domain? Composition by the inverse Fourier transform, i.e. impulse response function.

$$g = f * h_k, \text{ where } \widehat{h}_k = \frac{\beta_k}{-z_k^{(1)} + 2\pi i(\lambda - \frac{z_k^{(2)}}{2\pi})} \quad (k = 1, \ldots, q)$$

Filter: causality-transfer function contd.

- Note that the imaginary part $z_k^{(2)}$ has nothing to do with causality. Its effect is translation in the transfer function (frequency domain). This means modulation for the impulse response function (time domain), which is a realizable operation.
- Consequently, it is enough to consider the simplified form
$$\frac{1}{-z_k^{(1)} + 2\pi i\lambda}$$
- Let $a > 0$. The two cases:

  a) if $h = e^{-at}\chi_{[0,+\infty)}$ then $\widehat{f}(\lambda) = \dfrac{1}{a + 2\pi i\lambda}$

  b) if $h = e^{at}\chi_{(-\infty,0]}$ then $\widehat{f}(\lambda) = \dfrac{1}{a - 2\pi i\lambda}$
- Case a): $h = e^{-at}\chi_{[0,+\infty)}$, realizable. The impulse response function is zero on the negative half line. Output does not depend on the future values of the input. (See RC circuit)

Filter: causality-transfer function contd.

- Case b): $h = e^{at}\chi_{(-\infty,0]}$, not realizable. In fact it is anti-realizable. Output depends only the future values of the input. The impulse response function is zero on the positive half line. Output depends only on the future values of the input.
- The term $\dfrac{1}{-z_k^{(1)} + 2\pi i\lambda}$ represents a realizable filter if and only if $z_k^{(1)} < 0$.
- The impulse response functions corresponding to the terms in the partial fraction decomposition are linearly independent on the negative half line. Therefore, the sum vanishes on the negative half line if and only if every term does so.
- Consequently: The filter is realizable if and only if the real part of every root of the denominator Q is negative. In other words all of the roots of Q are in the negative complex half plane.

Filter: stability and causality-transfer function. Summary

- The filter with transfer function $H = \dfrac{P}{Q}$, where P, and Q are polynomials is stable if and only if $\deg p \leq \deg Q$. Moreover, we may suppose$\deg p < \deg Q$.
- The filter with transfer function $H = \dfrac{P}{Q}$, where P, and Q are polynomials is realizable if and only if the roots of Q are in the negative complex half plane.
  (We have assumed that the multiplicity of every root is 1.)
  Problem: finding the roots of a polynomial.
  Instead: Routh criterion.
- Filter design: find rational function that satisfies the given filter conditions. Rational approximation.
  Let the rational function be as simple as possible.
  Take care of the conditions assuring stability and causality.

Mathematical model

- Linear equation of discrete filters

$$\sum_{k=0}^{q} b_k y_{n-k} = \sum_{j=0}^{p} a_j x_{n-j} \qquad (q,\, p \in \mathbb{N},\; b_0 = 1,\; a_0 \neq 0)\,,$$

  Input sequence: x. Output sequence: y.

- Examples

  a) $y_n - y_{n-1} = x_n$ b) $y_n = \frac{1}{4}(x_n + x_{n-1} + x_{n-2} + x_{n-3})$.

- Impulse function: $x_n = \begin{cases} 1 & n = 0, \\ 0 & n \neq 0. \end{cases}$

- Impulse response

  a) $y_n = 1 \;\; (n \geq 0)$, b) $y_n = \begin{cases} \frac{1}{4} & n = 0, 1, 2, 3, \\ 0 & \text{egyébként.} \end{cases}$

FIR, IIR filters

- Impulse response
  a) IIR: infinite impulse response
  b) FIR: finite impulse response.
- Two categories: IIR, FIR.
- How to see it from the equation.
  a) IIR: $q > 0$ b) FIR: $q = 0$

Properties

- It is easy to check that the discrete system defined by the equation
$$\sum_{k=0}^{q} b_k y_{n-k} = \sum_{j=0}^{p} a_j x_{n-j}$$
is linear, time invariant, and realizable.
- The only real question is stability.

Transfer function

- If $x_n = z^n$ ($z \in \mathbb{C}$) then there exists a constant $H(z) \in \mathbb{C}$ such that $y_n = H(z)x_n$.
- The calculation of $H(z)$ by substitution

$$H(z) = \frac{\sum_{j=0}^{p} a_j z^{-j}}{\sum_{k=0}^{q} b_k z^{-k}} = \frac{a_0 + a_1 z^{-1} + \cdots + a_p z^{-p}}{1 + b_1 z^{-1} + \cdots + b_q z^{-q}}.$$

- H : the transfer function of the discrete filter.

Stability

BIBO stability: there exists a $K > 0$ such that $\|y\|_\infty \leq K\|x\|_\infty$

$(\|y\|_\infty = \sup\{|y_n| \ : \ n \in \mathbb{N}\})$

Stability contd.

- Stability criterion for the transfer function:

  the poles of H are within the unit disc.

  Poles of H : the transfer function is a rational function. The poles are the zeros of the denominator. In other words the points at which H is not defined.

  Unit disc $D = \{z \in \mathbb{C} \,:\, |z| < 1\}$.
- Checking the criterion without calculating the poles: Schur-Cohn test, or Jury test.

Using analog technique for IIR filter construction

- One of most commonly used method of transforming analog filters into appropriate IIR filters is known as bilinear transformation.
- Bilinear transformation
  $$T(z) = \frac{1 - z^{-1}}{1 + z^{-1}} \quad (z \in \mathbb{C},\ z \neq -1)$$
  takes the unit disc onto the negative half plane.