Fourier transform on the real line

The treatment is based on analogies: recall the discrete and the periodic models.

The function space

The signal: $x : \mathbb{R} \to \mathbb{C}$

Proper function space: $L^1(\mathbb{R})$, $L^2(\mathbb{R})$, etc. (Lebesgue–integral).

Instead: a more simple mathematical model.

We assume (improper Riemann–integral)

$$\exists \quad \int_{-\infty}^{\infty} |x(u)|^k \, du = \lim_{t\to\infty} \int_{-t}^{t} |x(u)|^k \, du < \infty \,,$$

where $k = 1$ or 2. Notation: $x \in R^k(-\infty, +\infty)$.

The trigonometric system

$$e_\lambda(t) = e^{2\pi i \lambda t} \qquad (\lambda \in \mathbb{R}) \,.$$

Fourier transform

$$\widehat{x} : \mathbb{R} \to \mathbb{C}\,, \qquad x \in R^1(-\infty, +\infty), \qquad \widehat{x}(\lambda) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi\lambda t}\, dt\,.$$

The spectrum is "continuous".

Inversion

The Fourier transform is invertible.

Inversion formula: If x, $\widehat{x} \in R(-\infty, \infty)$, then

$$x(t) = \int_{-\infty}^{\infty} \widehat{x}(\lambda) e^{i2\pi\lambda t}\, d\lambda\,.$$

Remark: The inversion formula holds in the case if $x \in R^1(-\infty, \infty)$ piecewise (finite many pieces) continuously differentiable and $x' \in R^1(-\infty, \infty)$.

Examples

1. $$x(t) = e^{-a|t|}, \qquad \widehat{x}(\lambda) = \frac{2a}{a^2 + 4\pi^2\lambda^2};$$

2. $$x(t) = \begin{cases} e^{-at} & , \ t > 0 \\ 0 & , \ t < 0 \end{cases}, \qquad \widehat{x}(\lambda) = \frac{1}{a + 2i\pi\lambda};$$

3. $$\operatorname{rect} t := \chi_{[-0.5,0.5]}, \quad \operatorname{sinc} \lambda := \begin{cases} \dfrac{\sin \pi\lambda}{\pi\lambda} & , \ \lambda \neq 0 \\ 1 & , \ \lambda = 0 \end{cases}$$

$$x(t) = \operatorname{rect} t, \qquad \widehat{x}(\lambda) = \operatorname{sinc} \lambda;$$

Note that $\operatorname{sinc} \notin R^1(-\infty, \infty)$.

4. $$x(t) = e^{-\pi t^2}, \qquad \widehat{x}(\lambda) = e^{-\pi\lambda^2}.$$

The Gauss–function is eigenfunction of the Fourier transform.

Remark: The inversion formula can be generalized: $\widehat{\operatorname{sinc}} = \operatorname{rect}$.

Properties

1. Translation: $y(t) = x(t - \tau)$ $\qquad \widehat{y}(u) = e^{-2\pi i \tau} \widehat{x}(u)$.
2. Modulation: $y(t) = e^{2\pi i \lambda} x(t)$ $\qquad \widehat{y}(u) = \widehat{x}(u - \lambda)$.

   These are similar to the discrete and the periodic cases.
3. Dilation: $y(t) = x(at)$ $\qquad \widehat{y}(u) = \frac{1}{|a|} \widehat{x}(\frac{u}{a})$.

   This is new. "Uncertainty relations"
4. Convolution: $z(t) = (x * y)(t)$ $\qquad \widehat{z}(u) = \widehat{x}(u)\widehat{y}(u)$.
5. Multiplication: $z(t) = x(t)y(t)$ $\qquad \widehat{z}(u) = (\widehat{x} * \widehat{y})(u)$.

   In both cases convolution on the real line:

$$(x * y)(t) = \int_{-\infty}^{\infty} x(t - u)y(u)\, du\,.$$