Fourier transform of periodic functions

The function space

Functions periodic by a ($a > 0$) :

$$x : \mathbb{R} \to \mathbb{C},\ x(t+a) = x(t) \quad (t \in \mathbb{R})$$

We assume: x is Riemann-integrable, i.e. $x \in R[0, a]$.

Scalar product

$$< x, y >:= \int_0^a x(t)\overline{y}(t)\, dt.$$

Norm generated by the scalar product

$$\|x\|_2 := \sqrt{< x, x >} = \Big(\int_0^a |x(t)|^2\, dt \Big)^{1/2}.$$

Trigonometric functions periodic by a :

$$e_{\frac{n}{a}}(t) = e^{2\pi i n \frac{t}{a}} \qquad (n \in \mathbb{Z}).$$

Trigonometric system

The trigonometric system $(e_{\frac{n}{a}},\ n \in \mathbb{Z})$ is orthogonal

$$< e_{\frac{n}{a}}, e_{\frac{m}{a}} >= \begin{cases} a & \text{, if } n = m \\ 0 & \text{, if } n \neq m, \end{cases}$$

$\|e_{\frac{n}{a}}\|_2 = \sqrt{a}$, therefore the normalization factor is $\sqrt{\frac{1}{a}}$.

Fourier coefficients

$$\widehat{x}_{\frac{n}{a}} := \frac{1}{a} < x, e_{\frac{n}{a}} >= \frac{1}{a} \int_0^a x(t) e^{-2\pi i n \frac{t}{a}} dt \qquad (n \in \mathbb{Z}).$$

Fourier transformation

$$x \longrightarrow \widehat{x} = (\widehat{x}_{\frac{n}{a}}\ ,\ n \in \mathbb{Z}).$$

Fourier transform of x : $\widehat{x}$.

Fourier series, partial sums

$$Sx(t) = \sum_{n=-\infty}^{\infty} \widehat{x}_{\frac{n}{a}} e^{2\pi i n \frac{t}{a}}, \quad S_N x(t) = \sum_{n=-N}^{N} \widehat{x}_{\frac{n}{a}} e^{2\pi i n \frac{t}{a}} \qquad (N \in \mathbb{N}).$$

Under proper conditions the Fourier series is convergent, and

$$x(t) = \lim_{N \to \infty} S_N x(t) = \sum_{n=-\infty}^{\infty} \widehat{x}_{\frac{n}{a}} e^{2\pi i n \frac{t}{a}}.$$

The sequence of Fourier coefficients

Important property

$$\lim_{|n| \to \infty} \widehat{x}_{\frac{n}{a}} = 0.$$

Properties of the Fourier transform

- Linearity: $\widehat{\alpha x + \beta y} = \alpha \widehat{x} + \beta \widehat{y}$.
- Translation, time shift:

$$y(t) = x(t - t_0)\,, \qquad \widehat{y}_{\frac{n}{a}} = e^{-2\pi i n \frac{t_0}{N}} \widehat{x}_{\frac{n}{a}}\,.$$

  Hint: Integration by substitution ($u = t - t_0$).
- Frequency shift (modulation):

$$y(t) = e^{2\pi i n_0 \frac{t}{a}} x(t)\,, \qquad \widehat{y}_{\frac{n}{a}} = \widehat{x}_{\frac{n}{a} - \frac{n_0}{a}}\,.$$

  Hint: simple direct calculation.
- Differentiation

$$y(t) = x'(t)\,, \qquad \widehat{y}_{\frac{n}{a}} = \frac{2\pi i}{a} n \widehat{x}_{\frac{n}{a}}\,.$$

  Hint: integration by part

Properties of the Fourier transform

- Integration

$$y(t) = \int_0^t x(u)\,du\,, \qquad \widehat{y}_{\frac{n}{a}} = \frac{a}{2\pi i}\frac{1}{n}\widehat{x}_{\frac{n}{a}}\,.$$

Hint:integration by part.

- Convolution: $\widehat{y * x} = a\widehat{y} \cdot \widehat{x}$, where

$$(y * x)(t) = \int_0^a y(t-u)x(u)\,du\,.$$

Hint: Directly by definition an then by separation of the integrals.

- Multiplication: $\widehat{(x \cdot y)} = \widehat{y} * \widehat{x}$.

The convolution of two sequences is defined as

$$(u * v)[n] = \sum_{k=-\infty}^{\infty} u[n-k] \cdot v[k]\,.$$

Hint: Start from the right side and interchange integration and summation.