# Filtering in the Frequency Domain

November 26, 2018

## 2D Fourier-transform

- Original image
  $f : \{0, \dots, M-1\} \times \{0, \dots, N-1\} \to \{0, \dots, L-1\}$.
  Méret: $M \times N$. Gray scale levels: $L$.
- Fourier-transform:

$$\mathcal{F}f(m,\ n) = \frac{1}{MN} \sum_{j=0}^{M-1} \sum_{k=0}^{N-1} f(j,k) \cdot e^{-2\pi i(mj/M + nk/N)}$$

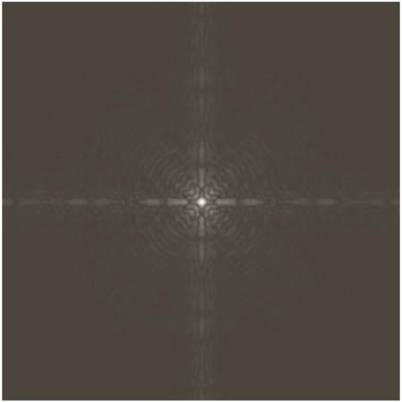
$(m = 0, \dots, M-1,\ \ n = 0, \dots, N-1)$

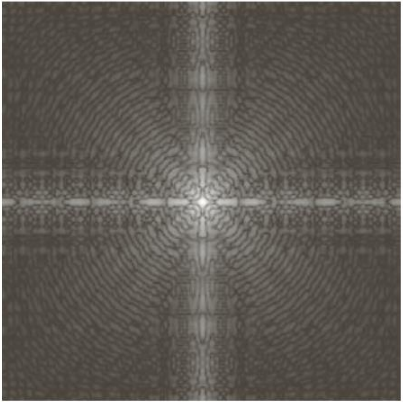
Remark:

- Replacing $1/MN$ by another factor $\mathcal{F}f$ can be scaled according to the needs.
- Complex Fourier-coefficients:
  $\mathcal{F}f(m,n) = |\mathcal{F}f(m,n)| \cdot e^{i\phi(m,n)}$
  Spectrum (matrix): $|\mathcal{F}f(m,n)|$.

The dynamics of Fourier-coefficients is much larger than the number of gray levels. Reason: there is a sum with $M \times N$ terms in the formula. It is often as much as $10^6$ in practice.

Representation of the spectrum: $\log(1 + |\mathcal{F}f|)$ (intensity transform).





## Image with real values

"Symmetry" around the origin. Since

$$\mathcal{F}f(m,n) = \overline{\mathcal{F}f(m,n)}$$

we have

$$|\mathcal{F}f(m,n)| = |\mathcal{F}f(-m,-n)|\,,\ \phi(m,n) = -\phi(-m,-n)\,.$$

Remark: $\mathcal{F}f$ is periodic by $(M,N)$.

## Visualization problem

The left bottom corner in the matrix $\mathcal{F}f$, the neighborhood of the origin is in relation with right upper corner.

Let the center of symmetry be in the center of the image. In other words perform shifting:$(0,0) \to (M/2, N/2)$. (We may suppose: both $M$ and $N$ are even).
See previous figure.

## Implementation

- translation of $\mathcal{F}f$ means modulation of $f$ :

$$\tau_{(-M/2,-N/2)}\mathcal{F}f \quad \Leftrightarrow \quad \mu_{(M/2,N/2)}f\,.$$

- Modified original image:

$$\begin{aligned} g(m,n) &= \mu_{(M/2,N/2)}f(m,n) \\ &= e^{2\pi i\left(m(M/2)/M+n(N/2)/N\right)}f(m,n) \\ &= (-1)^{m+n}f(m,n)\,. \end{aligned}$$

## Spectrum

- Translation of the image $\Rightarrow$ modulation of the Fourier-transform $\Rightarrow$ the spectrum doesn't change.
- Rotation: $f,\ g : \mathbb{R}^2 \to \mathbb{R}$, polar transform. $f,\ g : \mathbb{R}^+ \times [0, 2\pi) \to \mathbb{R}$,

$$g(r,\Theta) = f(r,\Theta - \Theta_0) \ \Rightarrow\ \mathcal{F}g(\rho,\omega) = \mathcal{F}f(\rho,\omega - \Theta_0)$$

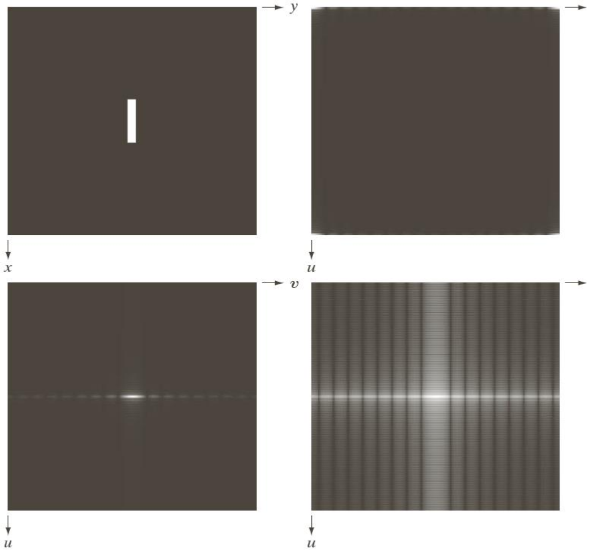
$$x = r\cos\Theta,\ \ y = r\sin\Theta,\ \ \lambda = \rho\cos\omega,\ \ \mu = \rho\sin\omega$$

$$\mathcal{F}g(\lambda,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-2\pi i(x\lambda + y\mu)}dx\,dy$$

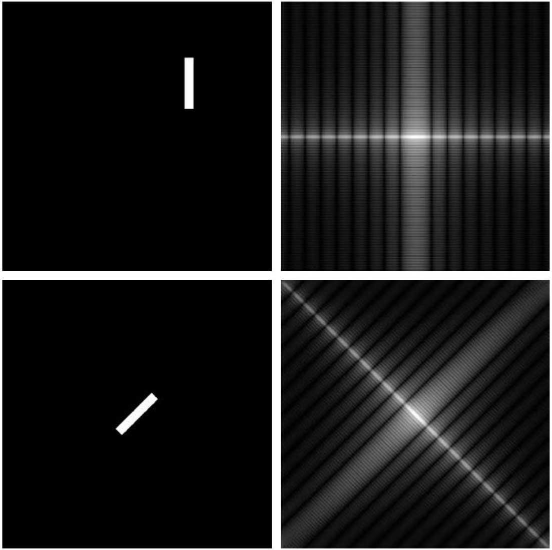
$$= \int_0^{\infty}\int_0^{2\pi} g(r,\Theta)e^{-2\pi i r\rho(\cos\Theta\cos\omega + \sin\Theta\sin\omega)}rd\Theta\,dr$$

$$= \mathcal{F}g(\rho,\omega) = \int_0^{\infty}\int_0^{2\pi} f(r,\Theta - \Theta_0)e^{-2\pi i r\rho\cos(\Theta - \omega)}rd\Theta\,dr$$

$$= \int_0^{\infty}\int_0^{2\pi} f(r,\Psi)e^{-2\pi i r\rho\cos(\Psi - (\omega - \Theta_0))}rd\Theta\,dr = \mathcal{F}f(r,\omega - \Theta_0)$$



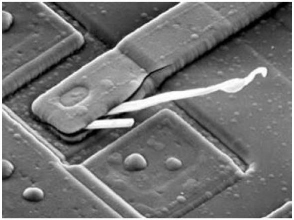

a) Kép  b) Fourier-tr.  c) Centrált spektrum  d) Log tr. utáni ábrázolás



a) Eltolt kép b) Eltolt kép spektruma c) Elforgatott kép d) Elforgatott kép spektruma

Information contained in the spectrum

- Slow intensity changes: the points close to the origin in the a spectrum.
- Fast intensity changes (edges, corners): the point far from the origin in the spectrum.



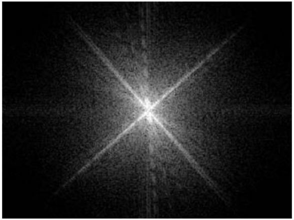
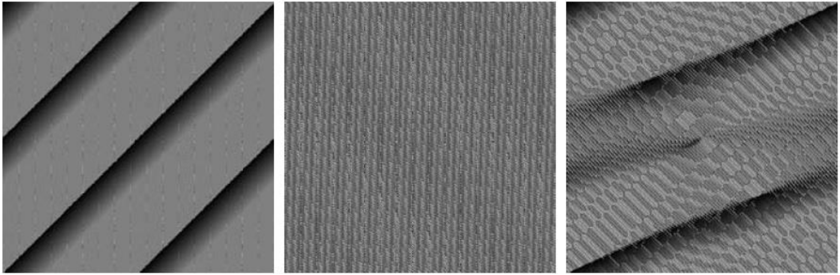


Image of an integrated circuit (magnification 2500)
On the image: edges in the direction $\pm 45^\circ$ and failure.
In the spectrum: edges in the direction $\pm 45^\circ$ and vertical line leaning left.

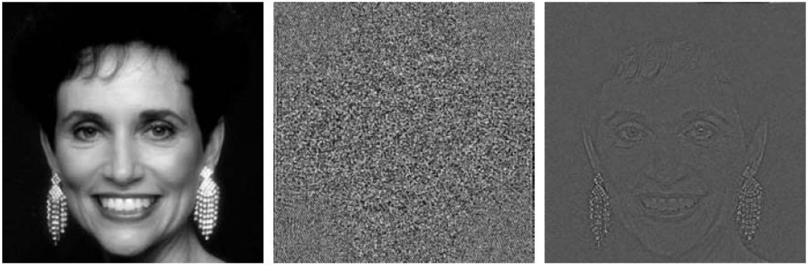
## Phase

Little direct information is visible.



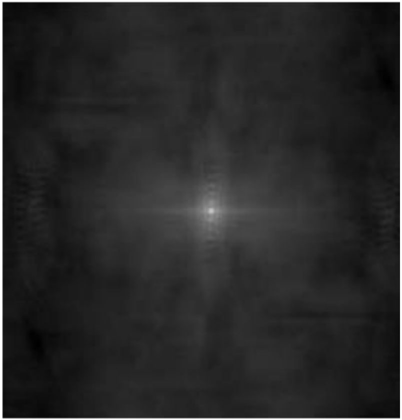
Phase: a) original image (rectangle) b) shifted image c) image rotated by 45 degree

Reconstruction from the phase only. The spectrum is chosen to be constant 1 : $|\mathcal{F}f \equiv 1|$.



a) original image b) phase c) reconstruction from the phase only

Reconstruction from the spectrum only. The phase is set to be constant 0.



## Filtering in the frequency domain: the basic formula

Convolution in the space domain

$$(f * g)(j,k) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) \cdot g(m-j, n-k)$$

Multiplication in the frequency domain

$$(f * g)(m,n) = \mathcal{F}f(m,n)\mathcal{F}g(m,n)$$

## Problem

- In the space domain: during the calculation we do not stay within the domain $\{0, \ldots, M-1\} \times \{0, \ldots, N-1\}$.
- In the formula above Fourier-transform is the discrete Fourier-transform: periodic. It is the Fourier-transform of the periodization of the image.

## Solution: zero padding

The image is extended by $0$ values.

Generally: the dimensions of $f$ and $g$ are $M_1 \times N_1$, $M_2 \times N_2$ respectively.

The extension: $A \geq M_1 + M_2 - 1$, $B \geq N_1 + N_2 - 1$

$$f(m,n) = \begin{cases} f(m,n), & ha \quad (m,n) \in \{0,\ldots,M_1-1\} \times \{0,\ldots,N_1-1\}; \\ 0, & ha \quad M_1 \leq m \leq A \text{ vagy } N_1 \leq n \leq B. \end{cases}$$

$$g(m,n) = \begin{cases} g(m,n), & ha \quad (m,n) \in \{0,\ldots,M_2-1\} \times \{0,\ldots,N_2-1\}; \\ 0, & ha \quad M_1 \leq m \leq A \text{ vagy } N_2 \leq n \leq B. \end{cases}$$

Then the periodization will not cause distortion.

Further problem: smooth intensity transition is advised if the values at the margins are not $0$.

Remark: In case of an image with dimension $M \times N$ we have $A = 2M - 1$, $B = 2N - 1$

## Summary

1. The dimension of the original image $f$, is $M \times N$.
2. Padding to an $2M \times 2N$ image: $f_p$.
3. In order to centralize the Fourier-transform: $f_{p,c}(m,n) = (-1)^{m+n} f_p(m,n)$.
4. Calculation of $\mathcal{F} f_{p,c}$.
5. Choosing a symmetric filter with of proper size: $H$.
6. Filtering in the frequency domain: $\mathcal{F} f_{p,c} \cdot H$.
7. Calculation of the filtered image: $g_{p,c} = \mathcal{F}^{-1}(\mathcal{F} f_{p,c} \cdot H)$.
8. "Decentralization": $g_p(m,n) = g_{p,c}(m,n) \cdot (-1)^{m+n}$.
9. Choosing the $M \times N$-es block: $g$. (Inverse padding)

Remark: because of errors in the calculations $\mathcal{F}^{-1}(\mathcal{F} f_{p,c} \cdot H)(m,n)$ in the 7th step is not necessarily real. Take the real part of it.

## The effect of the filters in the phase

Symmetric real valued functions.

If $\mathcal{F}f = R(\mathcal{F}f) + i \cdot I(\mathcal{F}f)$, then

$$\mathcal{F}f \cdot H = H \cdot R(\mathcal{F}f) + iH \cdot I(\mathcal{F}f)\,.$$

Consequence:

$$\phi = \text{arctg}\,\frac{R(\mathcal{F}f)}{I(\mathcal{F}f)} = \text{arctg}\,\frac{H \cdot R(\mathcal{F}f)}{H \cdot I(\mathcal{F}f)}$$

which means that filters do not change the phase.

Remark: in the calculation of the phase by means of arctg we extend the function arctg from $[-\pi/2, \pi/2]$ to $[-\pi, \pi]$, by taking account the sign of $R(\mathcal{F}f)$.

## Filters in the frequency domain. Why?

Filtering in the space and the frequency domains correspond to each other (convolution-multiplication)

Comparison

- In the space domain: convolution. Convolution by masks with small dimensions ($(3 \times 3)$, $(5 \times 5)$ etc.) Simple implementation.
- In the frequency domain: multiplication. The transforms are periodic. Multiplication: the filter is of the same size as the original image.

Advantage: The effect of filtering is more visible in the frequency domain. The construction is more simple.

How to construct a filter in the space domain?

- Construct a filter in the frequency domain.
- Take the inverse Fourier-transform of the filter.
- Consider this whole size space it as a starting point for the