Sampling

Uniform sampling

Let x be a function periodic by $a > 0$.

For our convenience we assume that x is "nice", i.e. differentiable.

We will take N samples uniformly within one period, i.e at points

$$k\frac{a}{N} \qquad (k = 0, \ldots, N-1).$$

Then we obtain the discrete function:

$$x : \left\{ k\frac{a}{N} \ : \ k = 0, \ldots N-1 \right\} \to \mathbb{C}.$$

This generates the vector:

$$x = (x[0], \ldots, x[N-1]) \text{, where } x[k] = x\left(k\frac{a}{N}\right) \quad (k = 0, \ldots, N-1).$$

The QUESTION

How to avoid loss of data?

In other words: How to perform sampling so that the original signal can be recovered from the samples?

Answer: Find relation between the Fourier coefficients

Original signal: periodic Fourier transform.

Sampled signal: discrete Fourier transform.

The values $x[k] = x\left(k\frac{a}{N}\right)$ will be calculated from Fourier transforms in two different ways.

The discrete case

$$x[k] = \sum_{n=0}^{N-1} \widehat{x}[n] e^{2\pi i k \frac{n}{N}} .$$

The periodic case

$$x\left(k\frac{a}{N}\right) = \sum_{n=-\infty}^{\infty} \widehat{x}_{\frac{n}{a}} e_{\frac{n}{a}}\left(k\frac{a}{N}\right) = \sum_{n=-\infty}^{\infty} \widehat{x}_{\frac{n}{a}} e^{2\pi i \frac{n}{a} k \frac{a}{N}} = \sum_{n=-\infty}^{\infty} \widehat{x}_{\frac{n}{a}} e^{2\pi i k \frac{n}{N}} .$$

Remark: The assumption made for x guarantees the convergence.

Reformulation of the periodic Fourier transform

If $n_2 = n_1 + jN$, then $e^{2\pi ik\frac{n_2}{N}} = e^{2\pi ik\frac{n_1+jN}{N}} = e^{2\pi ik\frac{n_1}{N}}$.

Consequently

$$x\left(k\frac{a}{N}\right) = \sum_{n=-\infty}^{\infty} \widehat{x}_{\frac{n}{a}} e^{2\pi ik\frac{n}{N}} = \sum_{n=0}^{N-1} \left(\sum_{j=-\infty}^{\infty} \widehat{x}_{\frac{n}{a}+\frac{jN}{a}} e^{2\pi ik\frac{n+jN}{N}} \right)$$

$$= \sum_{n=0}^{N-1} \left(\sum_{j=-\infty}^{\infty} \widehat{x}_{\frac{n}{a}+\frac{jN}{a}} \right) e^{2\pi ik\frac{n}{N}} .$$

Comparison

$$\sum_{n=0}^{N-1} \widehat{x}[n] e^{2\pi ik\frac{n}{N}} = x[k] = x\left(\frac{k}{N}\right) = \sum_{n=0}^{N-1} \left(\sum_{j=-\infty}^{\infty} \widehat{x}_{\frac{n}{a}+\frac{jN}{a}} \right) e^{2\pi ik\frac{n}{N}}$$

Decomposition of the sample vector in the trigonometric basis in two different ways. Conclusion: same coefficients.

Alias effect

$$\widehat{x}[n] = \sum_{j=-\infty}^{\infty} \widehat{x}_{\frac{n}{a}+\frac{jN}{a}} = \cdots + \widehat{x}_{\frac{n}{a}-\frac{N}{a}} + \widehat{x}_{\frac{n}{a}} + \widehat{x}_{\frac{n}{a}+\frac{N}{a}} + \ldots \qquad (n = 0, \ldots, N-1)\ .$$

Natural periodization: $\widehat{x}[n+\ell N] = \widehat{x}[n]$ $(n,, \ell \in \mathbb{Z})$.

Sampling: periodization of the spectrum.

Error formula

$$\widehat{x}[n] - \widehat{x}_{\frac{n}{a}} = \sum_{j\neq 0} \widehat{x}_{\frac{n}{a}+\frac{jN}{a}}\,.$$

Symmetric form

Take the index set from $-N/2$ to $N/2$ instead of from 0 to $N-1$

$$|\widehat{x}[n] - \widehat{x}_{\frac{n}{a}}| \leq \sum_{j\neq 0} |\widehat{x}_{\frac{n}{a}+\frac{jN}{a}}| \leq \sum_{|k|\geq \frac{N}{2}} |\widehat{x}_{\frac{k}{a}}| \qquad (|n| \leq \frac{N}{2}),$$

Conclusion

$$\lim_{N\to\infty} |\widehat{x}[n] - \widehat{x}_{\frac{n}{a}}| = 0.$$

Recovery

Recall

$$\widehat{x}[n] = \sum_{j=-\infty}^{\infty} \widehat{x}_{\frac{n}{a}+\frac{jN}{a}} \qquad (n = 0, \dots, N-1)\ .$$

The values $\widehat{x}_{\frac{n}{a}+\frac{jN}{a}}$ are compressed into one value $\widehat{x}[n]$.

Reconstruction is possible if there is only one nonzero term in the sum.

Condition: $\widehat{x}_{\frac{n}{a}+\frac{jN}{a}} = 0$ if $j \neq 0$ $(n = -N/2, \dots n/2)$.

In other form: $\widehat{x}_{\frac{n}{a}} = 0$, whenever $|n| > N/2$.

Frequencies

Recall: the meaning of $\frac{n}{a}$ in $e_{\frac{n}{a}}$ is frequency.

Reconstruction is possible:

$$\text{the spectrum is zero outside the interval } \left(-\frac{N}{2a}, \frac{N}{2a}\right).$$

Conditions:

1. The spectrum must be bounded, i.e.

$$\text{there is an } f_0 \text{ such that } \widehat{x}_{\frac{n}{a}} = 0 \text{ whenever } \frac{n}{a} \notin [-f_0, f_0].$$

2. Moreover $f_0 \leq \frac{N}{2a}$.

Nyquist frequency

The second condition in another form:

the sampling time is $\frac{a}{N}$. Consequently the sampling frequency is $f = \frac{N}{a}$.

Then the condition is

$$f \geq 2f_0\,.$$

Example for Alias effect

Let

$$x(t) = \sum_{n=-6}^{6} c_n e^{2\pi i n \frac{t}{a}}.$$

Then $\widehat{x}_{\frac{n}{a}} = c_n$.

Let $N = 4$. Then

$$\begin{aligned}
\widehat{x}[-2] &= c_{-6} + c_{-2} + c_2 + c_6 \\
\widehat{x}[-1] &= c_{-5} + c_{-1} + c_3 \\
\widehat{x}[0] &= c_{-4} + c_0 + c_4 \\
\widehat{x}[1] &= c_{-3} + c_1 + c_5
\end{aligned}$$