取後不放回的抽樣方式

在單群落之情況中，假設目標區域包含S種共同物種，並將該地區分為T個大致相等的區塊。若在取樣區塊中發現該物種，則被紀錄為存在，反之則為不存在，針對群落進行取後不放回之隨機抽樣，分別抽取 t 的區塊數，僅記錄每個採樣樣本中物種的發生率。令ψ_i與是物種i在樣本群落中所佔的區塊數量$(i = 1, 2, \ldots, S)$。在此，ψ_i 可作為衡量物種i在該區域中分佈程度使用。且為了確保所調查區域中的所有區塊都被調查時，所有的物種都被考慮在內，機率模型中的參數必須大於零。綜上所述，可對ψ_i進行假設：遵循參數 T 和π_i的零截尾 Beta 二項分佈 (zero-truncated beta-binomial distribution) (Shen and He, 2008)：

$$P(\psi_i \mid \pi_i) = \binom{T}{\psi_i} \frac{\pi_i^{\psi_i}(1-\pi_i)^{T-\psi_i}}{1-(1-\pi_i)^T}, i = 1, 2, \ldots, S$$

其中，π_i表示物種i的樣本中物種出現與否的檢測率。當取後不放回之隨機抽樣從 T 個區塊中抽出 t 個區塊，並且每個樣本區塊中僅記錄物種的存在與否，以形成逐種樣本發生矩陣X_i，應遵循超幾何分佈 (hypergeometric distribution)：

$$P(X_i = k \mid \psi_i) = \frac{\binom{\psi_i}{k}\binom{T-\psi_i}{t-k}}{\binom{T}{t}}, i = 1, 2, \ldots, S$$

又$P(X_i = k \mid \psi_i)$與ψ_i有關，X_i來自於π_i，因此可推導出：

$$\begin{aligned} P(X_i = k \mid \pi_i) &= \sum_{\psi_i} P(X_i = k \mid \psi_i) P(\psi_i \mid \pi_i) \\ &= \binom{t}{k} \frac{\pi_i^k (1-\pi_i)^{t-k}}{1-(1-\pi_i)^T} \frac{(1-\pi_i)^T I(k=0)}{1-(1-\pi_i)^T} \end{aligned}$$

其中，*I(A)*為指標函數，表示若出現A情況時，則該式為 1，反之則即為 0。並假設π_i為一來自 beta 分佈的隨機樣本，可將式式子表示為：

$$f(\pi_i) = G(\alpha, \beta, T)(1-(1-\pi_i)^T)\pi_i^{\alpha-1}(1-\pi_i)^{\beta-1}, 0 < \pi_i < 1$$

其中$G(\alpha,\beta,T)=\left[\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}-\frac{\Gamma(\alpha)\Gamma(\beta+T)}{\Gamma(\alpha+\beta+T)}\right]^{-1}$。

後隨邊際分佈X_i可通過$P(X_i=k|\,\psi_i)$獲得，故：

$$p_k=P(X_i=k)=\int_0^1 P(X_i=k\,|\,\pi_i)\,f(\pi_i)d\pi_i$$

$$=\begin{cases} G(\alpha,\beta,T)\left[\frac{\Gamma(\alpha)\Gamma(t+\beta)}{\Gamma(t+\alpha+\beta)}-\frac{\Gamma(\alpha)\Gamma(\beta+T)}{\Gamma(\alpha+\beta+T)}\right], if\ k=0 \\ G(\alpha,\beta,T)\binom{t}{k}\frac{\Gamma(\alpha+k)\Gamma(t+\beta-k)}{\Gamma(t+\alpha+\beta)}, if\ k>0 \end{cases} \quad (4)$$

將其推廣至兩群落：

$$p_{kl}=P(X_i=k)P(Y_i=l)=$$

$$\begin{cases} G_1\left[\frac{\Gamma(\alpha_1)\Gamma(t_1+\beta_1)}{\Gamma(t_1+\alpha_1+\beta_1)}-\frac{\Gamma(\alpha_1)\Gamma(\beta_1+T_1)}{\Gamma(\alpha_1+\beta_1+T_1)}\right]G_2\left[\frac{\Gamma(\alpha_1)\Gamma(t_1+\beta_1)}{\Gamma(t_1+\alpha_1+\beta_1)}-\frac{\Gamma(\alpha_1)\Gamma(\beta_1+T_1)}{\Gamma(\alpha_1+\beta_1+T_1)}\right], if\ k=0\ and\ l=0 \\ G_1\binom{t_1}{k}\frac{\Gamma(\alpha_1+k)\Gamma(t_1+\beta_1-k)}{\Gamma(t_1+\alpha_1+\beta_1)}G_2\binom{t_2}{l}\frac{\Gamma(\alpha_2+l)\Gamma(t_2+\beta_2-l)}{\Gamma(t_2+\alpha_2+\beta_2)}, if\ k>0\ and\ l>0 \\ G_1\left[\frac{\Gamma(\alpha_1)\Gamma(t_1+\beta_1)}{\Gamma(t_1+\alpha_1+\beta_1)}-\frac{\Gamma(\alpha_1)\Gamma(\beta_1+T_1)}{\Gamma(\alpha_1+\beta_1+T_1)}\right]G_2\binom{t_2}{l}\frac{\Gamma(\alpha_2+l)\Gamma(t_2+\beta_2-l)}{\Gamma(t_2+\alpha_2+\beta_2)}, if\ k=0\ and\ l>0 \end{cases}$$

其中p_{kl}為量樣本的物種豐富度正好分別為k和l的平均機率。令$Q_{kl}=\sum_{i=1}^{S}I(X_i=k\ and\ Y_i=l)$式在樣本中第一群落出現$k$次且第二群即出現$l$次的區塊數，則$D_{12}$為樣本中觀測到的共同物種數量，$D_{12}=\sum_{k=1}^{T_1}\sum_{l=1}^{T_2}Q_{kl}$。

Chiu (2022) 基於 Good-Turing 頻率公式與柯西不等式 (Cauchy-Schwarz inequality) 之概念，針對單一群落的估計得出近似式：$\frac{E(Q_0)}{E(Q_1)}\geq\frac{E(Q_1)}{2E(Q_2)}\geq\frac{E(Q_2)}{3E(Q_3)}\dots$，其中$Q_k$為出現$k$個區塊的物種數。從中可以得知，在物種估計時，採取出現較少次的物種，可以更多提供未出現物種的資訊，有助於縮小物種豐富度的估計結果。根據等式 (4)，給出未觀測到的豐富度的期望值、唯一值和重複值：

$$E(Q_0)=P(X_i=0)=S_{12}\times G(\alpha,\beta,T)\left[\frac{\Gamma(\alpha)\Gamma(t+\beta)}{\Gamma(t+\alpha+\beta)}-\frac{\Gamma(\alpha)\Gamma(\beta+T)}{\Gamma(\alpha+\beta+T)}\right]$$

$$E(Q_1)=P(X_i=1)=S_{12}\times\ G(\alpha,\beta,T)\binom{t}{1}\frac{\Gamma(\alpha+1)\Gamma(t+\beta-1)}{\Gamma(t+\alpha+\beta)}$$

$$E(Q_2)=P(X_i=2)=S_{12}\times\ G(\alpha,\beta,T)\binom{t}{2}\frac{\Gamma(\alpha+2)\Gamma(t+\beta-2)}{\Gamma(t+\alpha+\beta)}$$

並將其推廣至兩群落：

$$E(Q_{0+}) = S_{12} \times P(X_i = 0) \times \big(1 - P(Y_i = 0)\big)$$

$$= S_{12} \times G(\alpha_1, \beta_1, T_1)$$

$$\times \left[\frac{\Gamma(\alpha_1)\Gamma(t_1 + \beta_1)}{\Gamma(t_1 + \alpha_1 + \beta_1)} - \frac{\Gamma(\alpha_1)\Gamma(\beta_1 + T_1)}{\Gamma(\alpha_1 + \beta_1 + T_1)}\right]$$

$$\times \left\{1 - G(\alpha_2, \beta_2, T_2)\left[\frac{\Gamma(\alpha_2)\Gamma(t_2 + \beta_2)}{\Gamma(t_2 + \alpha_2 + \beta_2)} - \frac{\Gamma(\alpha_2)\Gamma(\beta_2 + T_2)}{\Gamma(\alpha_2 + \beta_2 + T_2)}\right]\right\}$$

$$E(Q_{1+}) = S_{12} \times P(X_i = 1) \times \big(1 - P(Y_i = 0)\big)$$

$$= S_{12} \times G(\alpha_1, \beta_1, T_1)\binom{t}{1}\left(\frac{\Gamma(\alpha_1 + 1)\Gamma(t_1 + \beta_1 - 1)}{\Gamma(t_1 + \alpha + \beta_1)}\right)$$

$$\times \left\{1 - G(\alpha_2, \beta_2, T_2)\left[\frac{\Gamma(\alpha_2)\Gamma(t_2 + \beta_2)}{\Gamma(t_2 + \alpha_2 + \beta_2)} - \frac{\Gamma(\alpha_2)\Gamma(\beta_2 + T_2)}{\Gamma(\alpha_2 + \beta_2 + T_2)}\right]\right\}$$

$$E(Q_{2+}) = S_{12} \times P(X_i = 2) \times \big(1 - P(Y_i = 0)\big)$$

$$= S_{12} \times G(\alpha_1, \beta_1, T_1) \times \binom{t_1}{2}\left(\frac{\Gamma(\alpha_1 + 2)\Gamma(t_1 + \beta_1 - 2)}{\Gamma(t_1 + \alpha + \beta_1)}\right)$$

$$\times \left\{1 - G(\alpha_2, \beta_2, T_2)\left[\frac{\Gamma(\alpha_2)\Gamma(t_2 + \beta_2)}{\Gamma(t_2 + \alpha_2 + \beta_2)} - \frac{\Gamma(\alpha_2)\Gamma(\beta_2 + T_2)}{\Gamma(\alpha_2 + \beta_2 + T_2)}\right]\right\}$$

將 α 設定為 1，且$T \gg \beta$。成立以下近似值：

$$\frac{E(Q_{0+})}{E(Q_{1+})} = \left(\frac{\beta_1 + t_1 - 1}{t_1}\frac{T_1 - t_1}{T_1 + \beta_1}\right) \approx \left[\left(\frac{\beta_1}{t_1} + 1\right)\frac{(T_1 - t_1)}{T_1 + \beta_1}\right]$$

$$\frac{E(Q_{1+})}{E(Q_{2+})} = \frac{t_1 + \beta_1 - 2}{t_1 - 1} \approx \left(\frac{\beta_1}{t_1} + 1\right)$$

得 $\beta_1 = \left(\frac{E(Q_{1+}) - E(Q_{2+})}{E(Q_{2+})}\right)t_1$ ，代入 $E(Q_{0+}) = \frac{E(Q_{1+})^2}{E(Q_{2+})}\left(\frac{(T_1 - t_1)}{T_1 + \beta_1}\right)$。

得：

$$E(Q_{0+}) = \begin{cases} \dfrac{t_1 - 1}{t_1}\dfrac{(1 - q)E(Q_{1+})}{E(Q_{2+}) + q\big(E(Q_{1+}) - E(Q_{2+})\big)^{+}}, if\ E(Q_{2+}) > 0 \\ \dfrac{t_1 - 1}{t_1}\dfrac{1 - q}{q}\big(E(Q_{1+}) - 1\big), if\ E(Q_{2+}) = 0 \end{cases}$$

$(A)^{+}$表示：若$A > 0$時，則等於$(A)^{+} = A$；若$A \le 0$，則等於$(A)^{+} = 1$。

同理 $E(Q_{0+})$ 也依此證明，得$\beta_2 = \left(\frac{E(Q_{+1}) - E(Q_{+2})}{E(Q_{+2})}\right)t_2$：

$$E(Q_{+0}) = \begin{cases} \frac{t_2-1}{t_2}\frac{(1-q)E(Q_{+1})}{E(Q_{+2})+q\left(E(Q_{+1})-E(Q_{+2})\right)^{+}}, if\ E(Q_{+2})>0 \\ \frac{t_2-1}{t_2}\frac{1-q}{q}\left(E(Q_{+1})-1\right), if\ E(Q_{+2})=0 \end{cases}$$

$(A)^{+}$表示：若$A > 0$時，則等於$(A)^{+}=A$；若$A \leq 0$，則等於$(A)^{+}=1$。

又：

$$\begin{aligned} E(Q_{00}) &= S_{12}\times P(X_i=0)\times P(Y_i=0) \\ &= S_{12}G(\alpha_1,\beta_1,T_1)G(\alpha_2,\beta_2,T_2) \\ &\times\left[\frac{\Gamma(\alpha_1)\Gamma(t_1+\beta_1)}{\Gamma(t_1+\alpha_1+\beta_1)}-\frac{\Gamma(\alpha_1)\Gamma(\beta_1+T_1)}{\Gamma(\alpha_1+\beta_1+T_1)}\right] \\ &\times\left[\frac{\Gamma(\alpha_2)\Gamma(t_2+\beta_2)}{\Gamma(t_2+\alpha_2+\beta_2)}-\frac{\Gamma(\alpha_2)\Gamma(\beta_2+T_2)}{\Gamma(\alpha_2+\beta_2+T_2)}\right] \end{aligned}$$

$$E(Q_{11}) = S_{12}\times P(X_i=1)\times P(Y_i=1)$$

$$= S_{12}\times G(\alpha_1,\beta_1,T_1)G(\alpha_2,\beta_2,T_2)$$

$$\times\binom{t_1}{1}\binom{t_2}{1}\left(\frac{\Gamma(\alpha_1+1)\Gamma(t_1+\beta_1-1)}{\Gamma(t_1+\alpha_1+\beta_1)}\right)\left(\frac{\Gamma(\alpha_2+1)\Gamma(t_2+\beta_2-1)}{\Gamma(t_2+\alpha_2+\beta_2)}\right)$$

$$E(Q_{22}) = S_{12}\times P(X_i=2)\times P(Y_i=2)$$

$$= S_{12}\times G(\alpha_1,\beta_1,T_1)G(\alpha_2,\beta_2,T_2)$$

$$\times\binom{t_1}{2}\binom{t_2}{2}\left(\frac{\Gamma(\alpha_1+2)\Gamma(t_1+\beta_1-2)}{\Gamma(t_1+\alpha_1+\beta_1)}\right)\left(\frac{\Gamma(\alpha_2+2)\Gamma(t_2+\beta_2-2)}{\Gamma(t_2+\alpha_2+\beta_2)}\right)$$

並成立以下近似值：

$$\begin{aligned} \frac{E(Q_{00})}{E(Q_{11})} &= \left(\frac{\beta_1+t_1-1}{t_1}\frac{T_1-t}{T_1+\beta_1}\right)\left(\frac{\beta_2+t_2-1}{t_2}\frac{T_2-t_2}{T_2+\beta_2}\right) \\ &\approx \left[\left(\frac{\beta_1}{t_1}+1\right)\frac{(T_1-t_1)}{T_1+\beta_1}\right]\left[\left(\frac{\beta_2}{t_2}+1\right)\frac{(T_2-t_2)}{T_2+\beta_2}\right] \end{aligned}$$

將$\begin{cases} \beta_1 = \left(\frac{E(Q_{1+})-E(Q_{2+})}{E(Q_{2+})}\right)t_1 \\ \beta_2 = \left(\frac{E(Q_{+1})-E(Q_{+2})}{E(Q_{+2})}\right)t_2 \end{cases}$，代入 $E(Q_{00}) \approx \left[\left(\frac{\beta_1}{t_1}+1\right)\frac{(T_1-t_1)}{T_1+\beta_1}\right]\left[\left(\frac{\beta_2}{t_2}+1\right)\frac{(T_2-t_2)}{T_2+\beta_2}\right]E(Q_{11})$

後，並加入$\left(\frac{t_1-1}{t_1}\frac{t_2-1}{t_2}\right)$對估計式進行調整，最終得估計式：

$$S_{wBB1} = D_{12} + E(Q_{00}) + E(Q_{0+}) + E(Q_{+0})$$

其中：

$$E(Q_{00}) = \left(\frac{t_1 - 1}{t_1}\frac{t_2 - 1}{t_2}\right) \times \left[\left(\frac{\beta_1}{t_1} + 1\right)\frac{(T_1 - t_1)}{T_1 + \beta_1}\right] \times \left[\left(\frac{\beta_2}{t_2} + 1\right)\frac{(T_2 - t_2)}{T_2 + \beta_2}\right] E(Q_{11})$$

並在 (14) 中代入 $\begin{cases}\beta_1 = \left(\frac{E(Q_{1+}) - E(Q_{2+})}{E(Q_{2+})}\right) t_1 \\ \beta_2 = \left(\frac{E(Q_{+1}) - E(Q_{+2})}{E(Q_{+2})}\right) t_2\end{cases}$。

$$E(Q_{0+}) = \begin{cases}\dfrac{t_1 - 1}{t_1}\dfrac{(1-q)E(Q_{1+})^2}{E(Q_{2+}) + q\big(E(Q_{1+}) - E(Q_{2+})\big)^+}, if\ E(Q_{2+}) > 0 \\ \dfrac{t_1 - 1}{t_1}\dfrac{1-q}{q}(E(Q_{1+}) - 1), if\ E(Q_{2+}) = 0\end{cases}$$

$$E(Q_{+0}) = \begin{cases}\dfrac{t_2 - 1}{t_2}\dfrac{(1-q)E(Q_{+1})^2}{E(Q_{+2}) + w_2\big(E(Q_{+1}) - E(Q_{+2})\big)^+}, if\ E(Q_{+2}) > 0 \\ \dfrac{t_2 - 1}{t_2}\dfrac{1-q}{q}(E(Q_{+1}) - 1), if\ E(Q_{+2}) = 0\end{cases}$$

$(A)^+$表示：若$A > 0$時，則等於$(A)^+ = A$；若$A \leq 0$，則等於$(A)^+ = 1$。

並在S_{wBB1}的基礎上，加入Q_{12}、Q_{21} 對β_1、β_2的估計進行修正，成立以下近似值：

$$\frac{E(Q_{11})}{E(Q_{22})} \approx \left(\frac{\beta_1}{t_1} + 1\right)\left(\frac{\beta_2}{t_2} + 1\right)$$

經由式 (12) 與 式 (17) 推得出：

$$\begin{cases}\beta_1 = \dfrac{E(Q_{11})}{E(Q_{22})}\left(\dfrac{t_2}{\beta_2 + t_2}\right) t_1 - t_1 \\ \beta_2 = \dfrac{E(Q_{11})}{E(Q_{22})}\left(\dfrac{t_1}{\beta_1 + t_1}\right) t_2 - t_2\end{cases}$$

又，

$$E(Q_{12}) = S_{12} \times P(X_i = 1) \times P(Y_i = 2)$$

$$= S_{12} \times G(\alpha_1, \beta_1, T_1) G(\alpha_2, \beta_2, T_2)$$

$$\times \binom{t_1}{1}\binom{t_2}{2}\left(\frac{\Gamma(\alpha_1 + 1)\Gamma(t_1 + \beta_1 - 1)}{\Gamma(t_1 + \alpha_1 + \beta_1)}\right)\left(\frac{\Gamma(\alpha_2 + 2)\Gamma(t_2 + \beta_2 - 2)}{\Gamma(t_2 + \alpha_2 + \beta_2)}\right)$$

$$E(Q_{21}) = S_{12} \times P(X_i = 2) \times P(Y_i = 1)$$

$$= S_{12} \times G(\alpha_1, \beta_1, T_1) G(\alpha_2, \beta_2, T_2)$$

$$\times \binom{t_1}{2}\binom{t_2}{1}\left(\frac{\Gamma(\alpha_1+2)\Gamma(t_1+\beta_1-2)}{\Gamma(t_1+\alpha_1+\beta_1)}\right)\left(\frac{\Gamma(\alpha_2+1)\Gamma(t_2+\beta_2-1)}{\Gamma(t_2+\alpha_2+\beta_2)}\right)$$

並成立以下近似式：

$$\frac{E(Q_{12})}{E(Q_{21})} = \left(\frac{t_2-1}{t_1-1}\right)\left(\frac{t_1+\beta_1-2}{t_2+\beta_2-2}\right) \approx \left(\frac{t_2}{t_1}\right)\left(\frac{t_1+\beta_1}{t_2+\beta_2}\right)$$

由上推得：

$$\begin{cases} \beta_1 = \dfrac{E(Q_{12})}{E(Q_{21})}\left(\dfrac{t_1}{t_2}\right)(t_2+\beta_2) - t_1 \\ \beta_2 = \dfrac{E(Q_{21})}{E(Q_{12})}\left(\dfrac{t_2}{t_1}\right)(t_1+\beta_1) - t_2 \end{cases}$$

又可從式 (18) = 式 (22) 得：

$$(t_1+\beta_1)^2 = \frac{E(Q_{11})}{E(Q_{22})}\frac{E(Q_{12})}{E(Q_{21})}t_1^2 \Rightarrow \beta_2^2 + 2t_2\beta_2 + t_2^2\left(1 - \frac{E(Q_{11})}{E(Q_{22})}\frac{E(Q_{12})}{E(Q_{21})}\right) = 0$$

$$(t_2+\beta_2)^2 = \frac{E(Q_{11})}{E(Q_{22})}\frac{E(Q_{21})}{E(Q_{12})}t_2^2 \Rightarrow \beta_2^2 + 2t_2\beta_2 + t_2^2\left(1 - \frac{E(Q_{11})}{E(Q_{22})}\frac{E(Q_{21})}{E(Q_{12})}\right) = 0$$

並依公式解$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$，得：

$$\begin{cases} \beta_1 = -t_1 + t_1\sqrt{\dfrac{E(Q_{11})}{E(Q_{22})}\dfrac{E(Q_{12})}{E(Q_{21})}} = \left(-1 + \sqrt{\dfrac{E(Q_{11})}{E(Q_{22})}\dfrac{E(Q_{12})}{E(Q_{21})}}\right)t_1 \\ \beta_2 = -t_2 + t_2\sqrt{\dfrac{E(Q_{11})}{E(Q_{22})}\dfrac{E(Q_{21})}{E(Q_{12})}} = \left(-1 + \sqrt{\dfrac{E(Q_{11})}{E(Q_{22})}\dfrac{E(Q_{21})}{E(Q_{12})}}\right)t_2 \end{cases}$$

最終得：

$$S_{wBB2} = D_{12} + E(Q_{00}) + E(Q_{0+}) + E(Q_{+0})$$

其中$E(Q_{00})$等於 (14)、$E(Q_{0+})$等於 (15) 且 $E(Q_{0+})$等於 (16)，且在 (14) 中的β_1與β_2使用帶入 (25) 計算。

此外，在 Chao and Lin (2012) 中提出兩群集的取後不放回的共同種估計方式：

$$S_{12,wor2} = D_{12} + \frac{Q_{1+}^2}{2k_1 Q_{2+} + rQ_{1+}} + \frac{Q_{+1}^2}{2k_2 Q_{+2} + rQ_{+1}} + \frac{Q_{11}^2}{4k_1 k_2 Q_{22} + 2k_1 r_2 Q_{21} + 2k_2 r_1 Q_{12} + r_1 r_2 Q_{11}}$$

其中 $k_i = \frac{t_i}{t_i - 1}$ 與 $r_j = \frac{\frac{t_j}{T}}{(1-\frac{t_j}{T})}$ 。

標準差估計

根據$(Q_0, Q_1, \ldots, Q_t)$的漸近分布，其服從大小為S以及機率為$\left(\frac{E[Q_0]}{S}, \frac{E[Q_1]}{S}, \ldots, \frac{E[Q_t]}{S}\right)$的多項分布 (multinomial distribution)。所提出的物種豐富度估計量的變異數估計量可以使用 delta 方法導出，表示為

$$\widehat{var}(\widehat{S_{12}}) = \sum_{i=1}^{t} \sum_{j=1}^{t} \frac{\partial \widehat{S_{12}}}{\partial Q_i} \frac{\partial \widehat{S_{12}}}{\partial Q_j} \widehat{cov}(Q_i, Q_j) \tag{37}$$

其中 $\widehat{cov}(Q_i, Q_j) = \begin{cases} Q_i\left(1 - \frac{Q_i}{\hat{S}}\right), if\ i = j \\ -\frac{Q_i Q_j}{\hat{S}}, if\ i \neq j \end{cases}$

95%信賴區間

在此，物種豐富的信賴區間通過假$\hat{S}_{12} - D_{12}$符合對數常態分佈 (log normal distribution) (Chiu et al., 2014)，為此確保了信賴區間之下限值大於觀察到的物種豐富度。故，物種豐富度之95%信賴區間為：

$$[\, D_{12} + \frac{\widehat{S_{12}} - D_{12}}{R}, D_{12} + \left(\hat{S}_{12} - D_{12}\right) \times R]$$

其中 $R = \left\{1.96\left[\log\left(1 + \frac{\widehat{var}(\widehat{S_{12}})}{(\widehat{S_{12}} - D_{12})^2}\right)\right]^{\frac{1}{2}}\right\}$以此計算 95%信賴區間的樣本涵蓋率 (95% confidence interval coverage rate, 95% CI Coverage) 。