# Review of MHD Basics

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SHIELD Summer School

June 2, 2025

# Content

- Ideal MHD equations and assumptions
- Magnetic pressure and tension
- Conservation laws and discontinuities

# Hierarchy of plasma modeling

N-body interaction → Averaging → Kinetic description → Moments → Fluid description.

**N-body interaction**

- Describe the motion of all charged particles under the effect of external and internal fields.
- Newtonian or Hamiltonian mechanics.

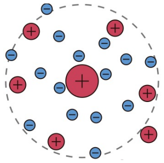
**Kinetic description**

- Describe the velocity distribution function based on statistical mechanics.
- Boltzmann or Vlasov equation.

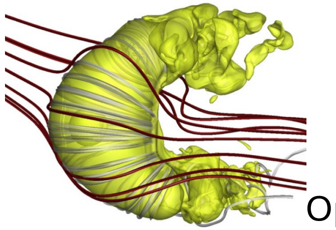
**Fluid description.**

- Describe the macroscopic quantities (density, velocity, temperature, etc.).
- Hydrodynamic or MHD equations.

Debye scale ~ km — Kinetic scale $\sim 10 - 10^4$ km — Global scale $\gtrsim 10^8$ km



Wikipedia



Opher et al. (2015)

Small scale, microscopic physics.

Large scale, macroscopic physics.

# Ideal Magnetohydrodynamic (MHD) equations

- Continuity: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$
- Equation of motion: $\rho \frac{\partial \boldsymbol{u}}{\partial t} + \rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \frac{1}{4\pi} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}$
- Faraday's law: $\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B})$
- Adiabatic equation of state: $\frac{d}{dt} \left( \frac{p}{\rho^{\gamma}} \right) = 0$

Notes: Mass density $\rho$, bulk velocity $\boldsymbol{u}$, thermal pressure $p$, magnetic field $\boldsymbol{B}$, current density $\boldsymbol{j} = \frac{c}{4\pi} (\nabla \times \boldsymbol{B})$, electric field $\boldsymbol{E} = -\frac{1}{c} \boldsymbol{u} \times \boldsymbol{B}$, convective derivative $\frac{d}{dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$, adiabatic index $\gamma$.

# Assumptions and approximations

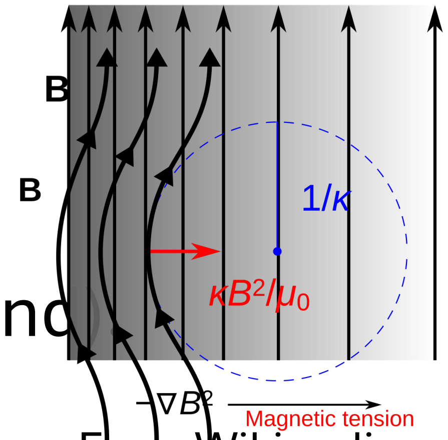
- Hydrodynamic assumptions: ignoring moments higher than second order, isotropic pressure, ideal gas, zero viscosity.
- MHD assumptions: single fluid, charge neutrality, high electric conductivity, convective electric field only, no ionization sources.
- Extensions to ideal MHD: viscous-resistive (useful for magnetic reconnection & turbulence), Hall (reconnection), multi-fluid (pickup ions), etc.

# Magnetic pressure and magnetic tension

$$\frac{1}{4\pi}(\nabla\times\boldsymbol{B})\times\boldsymbol{B} = \frac{B^2}{4\pi}\boldsymbol{b}\cdot\nabla\boldsymbol{b} - \nabla_\perp\left(\frac{B^2}{8\pi}\right)$$

Notes: magnetic unit vector $\boldsymbol{b} = \boldsymbol{B}/B$, curvature of magnetic field $\boldsymbol{\kappa} = \boldsymbol{b}\cdot\nabla\boldsymbol{b}$, perpendicular gradient $\nabla_\perp = \nabla - \boldsymbol{b}(\boldsymbol{b}\cdot\nabla)$

- Magnetic pressure: $p_B = \frac{B^2}{8\pi}$, plasma beta: $\beta = \frac{p}{p_B}$
- Magnetic tension: $\frac{B^2}{4\pi}\boldsymbol{\kappa}$
- Important for confining plasmas (including the solar wind)




From Wikipedia

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# MHD Conservation Laws

- Mass: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0$
- Momentum: $\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla \cdot \left( \rho \boldsymbol{u}\boldsymbol{u} + \left( p + \frac{B^2}{8\pi} \right) \boldsymbol{I} - \frac{\boldsymbol{B}\boldsymbol{B}}{4\pi} \right) = 0$
- Magnetic field: $\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{u}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{u}) = 0$
- Total energy: $\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 + \frac{p}{\gamma - 1} + \frac{B^2}{8\pi} \right) + \nabla \cdot \left[ \left( \frac{1}{2} \rho u^2 + \frac{\gamma p}{\gamma - 1} \right) \boldsymbol{u} - \frac{(\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B}}{4\pi} \right] = 0$

(Kinetic energy flux $\frac{1}{2} \rho u^2 \boldsymbol{u}$; enthalpy flux $\frac{\gamma p}{\gamma - 1} \boldsymbol{u}$; Poynting flux $\frac{c\boldsymbol{E} \times \boldsymbol{B}}{4\pi}$)

# MHD jump conditions

$$[\rho u_n] = 0$$

$$\left[\rho u_n^2 + \left(p + \frac{B^2}{8\pi}\right) - \frac{B_n^2}{4\pi}\right] = 0$$

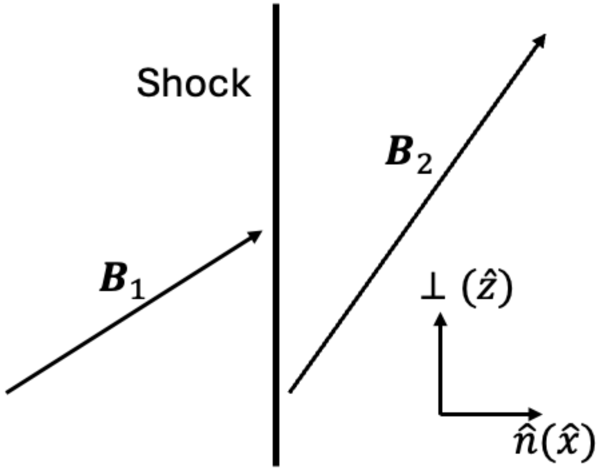
$$\left[\rho u_n \boldsymbol{u}_\perp - \frac{B_n \boldsymbol{B}_\perp}{4\pi}\right] = 0$$

$$[B_n] = 0$$

$$[B_n \boldsymbol{u}_\perp - u_n \boldsymbol{B}_\perp] = 0$$

$$\left[\left(\frac{1}{2}\rho u^2 + \frac{\gamma p}{\gamma - 1} + \frac{B_\perp^2}{4\pi}\right) u_n - \frac{(\boldsymbol{B}_\perp \cdot \boldsymbol{u}_\perp)}{4\pi} B_n\right] = 0$$

# MHD Shock




- Co-planarity of magnetic field.
- Parallel shock: reduced to hydro or "switch on."
- Perpendicular shock: $\boldsymbol{B}_\perp$ compressed like the fluid, i.e., compression ratio $\frac{B_2}{B_1} = \frac{\rho_2}{\rho_1} = \frac{u_{n1}}{u_{n2}} = \delta$.

  Perpendicular shock adiabatic: $u_{n1}^2 = \frac{2\delta\left[C_{s1}^2 + V_{A1}^2\left(\delta - \frac{1}{2}\gamma(\delta-1)\right)\right]}{\gamma+1-(\gamma-1)\delta}$.

  Upstream sound speed & Alfvén speed $C_{s1}^2 = \frac{\gamma p_1}{\rho_1}, V_{A1}^2 = \frac{B_1^2}{4\pi\rho_1}$.