Projection Properties of de la Vallée Poussin type operators

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Trigonometric Fourier series

Complex trigonometric system:

$$\varepsilon_j(x) := e^{ijx} \qquad (x \in \mathbb{R},\ j \in \mathbb{Z})$$

Trigonometric Fourier coefficients ($f \in L_1(0, 2\pi)$ complex valued function):

$$\hat{f}(j) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ijt} dt \qquad (j \in \mathbb{Z}).$$

Trigonometric Fourier series of f:

$$S[f] := \sum_{j \in \mathbb{Z}} \hat{f}(j) \varepsilon_j.$$

The n-th partial sum:

$$(S_n f)(x) := \sum_{j=-n}^{n} \hat{f}(j) \varepsilon_j(x) \qquad (x \in \mathbb{R}).$$

First we consider the Banach space $(C_{2\pi}, \|\cdot\|_\infty)$.
The set of trigonometric polynomials of degree at most $n \in \mathbb{N}$:

$$\mathcal{T}_n := \text{span}\,\{\varepsilon_j \;:\; -n \le j \le n\}.$$

Now $S_n : \; C_{2\pi} \to \mathcal{T}_n$ is a (bounded) linear operator with the projection property

$$(S_n g)(x) = g(x) \qquad (g \in \mathcal{T}_n, x \in \mathbb{R}).$$

Theorem (Faber–Marcinkiewicz–Berman)

Let $T_n : \; C_{2\pi} \to \mathcal{T}_n$ denote a linear (trigonometric) projection, i.e. suppose that $T_n g = g$, $(g \in \mathcal{T}_n)$. Then we have

$$\|T_n\| \ge \|S_n\|.$$

We remark that $\|S_n\| = \frac{4}{\pi^2} \log n + O(1)$.

(Generalized) de la Vallée Poussin means

The de la Vallée Poussin means of (trigonometric) Fourier series:

$$V_{n,m}f := \frac{1}{m+1}\sum_{k=0}^{m} S_{n+k}f.$$

Special cases:

1. partial sum operator S_n $(m = 0)$;
2. Fejér means $(n = 0)$;
3. classic de la Vallée Poussin means $(n = m + 1)$.

Now we have $V_{n,m}: \ C_{2\pi} \to \mathcal{T}_{n+m}$ and $(V_{n,m}g)(x) = g(x)$, where $g \in \mathcal{T}_n, x \in \mathbb{R}$.
In this talk we deal with this type of projection property.

Theorem (Nikolaev)

Let $n, m \in \mathbb{N}$, $n \geq 1$ and let $T_{n,m} : \ C_{2\pi} \to \mathcal{T}_{n+m}$ denote a de la Vallée Poussin type projection, i.e. a linear operator for which $T_{n,m} g = g$, $(g \in \mathcal{T}_n)$. Then there exists a positive constant $c \in \mathbb{R}$, independent of n, m, such that

$$\| T_{n,m} \| \geq c \log \frac{n+m}{m+1}.$$

We remark that for de la Vallée Poussin means,

$$\| V_{n,m} \| = \frac{4}{\pi^2} \log \frac{n+m}{m+1} + O(1).$$

Question/problem: Is the norm of $V_{n,m}$ minimal among these projections?

Theorem (Bernstein; Deregowska,Lewandowska)

If $n = m + 1$ and $n > 1$, then we have $\|T_{n,n-1}\| \geq \|V_{n,n-1}\|$, i.e. the minimality of the classical DLVP means.

Theorem (Nikolaev)

If $n|m+1$ and $n > 1$, then the corresponding DLVP operator $V_{n,m}$ is minimal.

Theorem (Bernstein)

If $m+1$ is even, then the corresponding DLVP operator $V_{n,m}$ is minimal.

Theorem (NZs)

Let $n, m \in \mathbb{N}$, $n \geq 1$ and let $T_{n,m} : \ C_{2\pi} \to \mathcal{T}_{n+m}$ denote a de la Vallée Poussin type projection. Then, $\|T_{n,m}\| \geq \|V_{n,m}\|$ holds if and only if $(m+1, 2n+m+1) > 1$.

Ingredients

1. Simple kernel with explicit roots
   The n-th partial sum can be expressed as

$$(S_n f)(x) = (D_n * f)(x) = \frac{1}{2\pi} \int_0^{2\pi} f(y) D_n(x-y)\, dy,$$

   where

$$D_n(x) = \frac{\sin \frac{2n+1}{2} x}{\sin \frac{x}{2}}, \qquad (x \in \mathbb{R}).$$

   For the de la Vallée Poussin means: $V_{n,m} = G_{n,m} * f$, where

$$G_{n,m}(x) = \frac{1}{m+1} \sum_{k=0}^{m} D_{n+k}(x) = \frac{\sin \frac{m+1}{2} x \;\; \sin \frac{2n+m+1}{2} x}{(m+1) \sin^2 \frac{x}{2}}.$$

2. Mimicking Cheney *et al.*: $\|T_{n,m}\| = \frac{1}{2\pi} \|G_{n,m} + y\|_1$, where $y \in \mathcal{T}_{n+m} \setminus \mathcal{T}_n$.

Generalisations

Theorem (Lambert)

The Faber–Marcinkiewicz–Berman theorem remains true if we replace $C_{2\pi}$ with $L_1(0, 2\pi)$.

This is a (relatively) simple observation.
Our main result also remains true for functions $f \in L_1(0, 2\pi)$.
We also have the algebraic polynomial variants for partial sums of Chebyshev series for $f \in L_1(-1, 1)$.
Question: What is a minimal projection in the other cases? How far the DLVP operators are from being minimal?

Theorem (NZs)

$\|V_{n,m}\| \leq \|T^*_{n,m}\| + O(1)$.

Multivariate extension

Fix $d > 1$, $d \in \mathbb{N}$, and denote the d-dimensional torus by $\mathbb{T}^d = \mathbb{R}^d \pmod{2\pi\mathbb{Z}^d}$ ($\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$).
The Fourier series of $g \in C(\mathbb{T}^d)$:

$$g(\boldsymbol{\vartheta}) \sim \sum_{\mathbf{k}} \hat{g}(\mathbf{k})\, \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\vartheta}}, \qquad \hat{g}(\mathbf{k}) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} g(\mathbf{t})\, \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{t}}\, d\mathbf{t},$$

where $\boldsymbol{\vartheta} = (\vartheta_1, \vartheta_2, \ldots, \vartheta_d) \in \mathbb{T}^d$, $\mathbf{k} = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d$ and $\mathbf{k} \cdot \boldsymbol{\vartheta} = \sum_{l=1}^{d} k_l \vartheta_l$ (scalar product).
The n-th *triangular* partial sum:

$$S_{n,d}(g, \boldsymbol{\vartheta}) = \sum_{|\mathbf{k}|_1 \le n} \hat{g}(\mathbf{k})\, \mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\vartheta}} \qquad (n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}),$$

where $|\mathbf{k}|_1 = \sum_{l=1}^{d} |k_l|$ (the l_1 norm of multiindex $\mathbf{k}$).

d-dimensional de la Vallée Poussin means ($n, m \in \mathbb{N}_0$):

$$V_{n,m,d}(g,\vartheta) = \frac{1}{m+1}\sum_{j=0}^{m} S_{n+j,d}(g,\vartheta).$$

Denote the set of trigonometric polynomials

$$\sum_{|\mathbf{k}|_1 \le n} a_{\mathbf{k}}\, \mathrm{e}^{i\mathbf{k}\cdot\vartheta},$$

by $\mathcal{T}_{n,d}$, where $\mathbf{k} = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d$.

Theorem

Fix $d \geq 1$, $n, m \in \mathbb{N}$, $n \geq 1$. Let $T_{n,m,d} : \; C(\mathbb{T}^d) \to \mathcal{T}_{n+m,d}$ denote a de la Vallée Poussin type linear projection, i.e. $T_{n,m,d}(g,\vartheta) = g(\vartheta)$, where $g \in \mathcal{T}_{n,d}$. Now

$$\|T_{n,m,d}\| \geq c\left(\log\frac{n+m}{m+1}\right)^d,$$

The End

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