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Equilibrium conditions for the Malmquist–Takenaka systems

MARGIT PAP and FERENC SCHIPP

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Dedicated to the 80th birthday of Professor László Leindler

Abstract. In this paper we give an overview of the discretization results connected to Malmquist–Takenaka systems for the unit disc and upper half plane. We prove that the discretization nodes on the real line have similar properties like the discretization nodes on the unit circle: for example they satisfy some equilibrium conditions and they are stationary points of some logarithmic potential. The problems whether they are the minimum of a logarithmic potential is formulated and solved in a special case.

1. Introduction

1.1. Malmqiust–Takenaka systems

Let us denote by $L^2(\mathbb{T})$ the space of square integrable functions on the unit circle $\mathbb{T}$. The inner product of this space is given by the following formula:

$$\langle f, g\rangle := \frac{1}{2\pi}\int_{\mathbb{I}} f(e^{it})\overline{g(e^{it})}\,dt \quad (f, g \in L^2(\mathbb{T}),\ \mathbb{I} = [0, 2\pi]).$$

The trigonometric system $\epsilon_n(t) = e^{int} (t \in \mathbb{I}, n \in \mathbb{Z})$ is orthonormal and complete with respect to this inner product.

Denote by $H^2(\mathbb{T})$ the Hardy space on the unit circle, which is the closure in $L^2(\mathbb{T})$-norm of the set $\operatorname{span}\{\epsilon_n, n \in \mathbb{N}\}$. Let us consider $H^2(\mathbb{D})$, the analytic extensions on the unit disc $\mathbb{D}$ of the functions from $H^2(\mathbb{T})$.

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The upper half plane is denoted by $\mathbb{C}_+ = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. The Hardy spaces of the upper half plane $H^2(\mathbb{C}_+)$ is the set of all functions h which are analytic on the upper half plane and

$$\sup\left\{\int_{\mathbb{R}} |h(x+iy)|^2\, dx : y > 0\right\} < \infty.$$

The Hardy space of the upper half plane and the Hardy space of the unit disc $H^2(\mathbb{D})$ may be connected through the Cayley transform. The conformal mapping from $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ to $\overline{\mathbb{C}}$ defined by

$$\tilde{z} := K(z) := \frac{i-z}{i+z} \quad (z \in \overline{\mathbb{C}})$$

is called the Cayley transform. The maps $K\colon \mathbb{C}_+ \to \mathbb{D}$ and $K\colon \mathbb{R} \to \mathbb{T} \setminus \{-1\}$ are bijections and the inverse of K is

$$K^{-1}(z) = i\frac{1-z}{1+z} \quad (z \in \overline{\mathbb{C}}).$$

The correspondence between the boundaries is given by the following relation

$$e^{is} = K(t) = \frac{i-t}{i+t} \qquad (t \in \mathbb{R},\ s \in (-\pi, \pi)).$$

This implies that $s = 2\arctan(t)$ $(t \in \mathbb{R})$.

The linear transformation

$$(Tf)(z) := \frac{1}{\sqrt{\pi}}\frac{1}{i+z}\tilde{f}(z),\ \tilde{f}(z) := f(K(z)) \quad (z \in \mathbb{C}_+)$$

defined for function $f\colon \overline{\mathbb{C}} \to \mathbb{C}$ is an isomorphism between $H^2(\mathbb{C}_+)$ and $H^2(\mathbb{D})$.

To represent periodic signals some special orthonormal basis of rational functions, for example the Malmquist–Takenaka systems are useful.

The Malmquist–Takenaka system $\Phi_n^{\mathfrak{a}} = \Phi_n$ $(n \in \mathbb{N}^* := \{1, 2, \dots\})$ on the closed disc $\overline{\mathbb{D}}$ is generated by a given sequence $\mathfrak{a} = (a_1, a_2, \dots)$ of complex numbers of the unit disc $\mathbb{D}$ and can be expressed by the Blaschke and rational functions

$$b_a(z) := \frac{z-a}{1-\overline{a}z}, \quad r_a(z) := \frac{\sqrt{1-|a|^2}}{1-\overline{a}z} \quad (a \in \mathbb{D}, z \in \overline{\mathbb{C}})$$

(see [7, 15]):

$$\Phi_1(z) := r_{a_1}(z), \quad \Phi_n(z) := r_{a_n}(z)\prod_{k=1}^{n-1} b_{a_k}(z) \quad (z \in \overline{\mathbb{D}}, n = 2, 3, \dots). \tag{1.1}$$

This system is orthonormal on the unit circle $\mathbb{T}$, for example

$$\langle \Phi_n, \Phi_m \rangle = \frac{1}{2\pi} \int_0^{2\pi} \Phi_n(e^{it}) \overline{\Phi_m(e^{it})} dt = \delta_{mn} \quad (m, n \in \mathbb{N}^*),$$

where δ_{mn} is the Kronecker symbol. In the special case when $a_1 = a_2 = \cdots = 0$ we reobtain the trigonometric system. If $a_1 = a_2 = \cdots = a$, then $\Phi_n = L_n^a$ $(n \in \mathbb{N}^*)$ is the discrete Laguerre-system, and if $a_{2k-1} = a, a_{2k} = \overline{a}$ $(k \in \mathbb{N}^*)$, then $\Phi_n (n \in \mathbb{N}^*)$ is the Kautz-system investigated in [2, 10].

If the Blaschke condition for the unit disc is satisfied, i.e.,: $\sum_{n=1}^{\infty}(1-|a_n|) = \infty$, then this system is complete in $H^2(\mathbb{T})$.

If $a \in \mathbb{D}$, then b_a is an 1-1 map on $\mathbb{D}$ and on $\mathbb{T}$, respectively. Moreover (see [2]), b_a can be written in the form

$$b_a(e^{it}) = e^{i\beta_a(t)} \quad (t \in \mathbb{R}, a = re^{i\tau} \in \mathbb{D}),$$

where

$$\beta_a(t) := \tau + \gamma_s(t - \tau), \quad \gamma_s(t) := 2 \arctan \left(s \tan \frac{t}{2} \right) \quad \left(t \in [-\pi, \pi),\ s := \frac{1+r}{1-r} \right)$$

and γ_s is extended to $\mathbb{R}$ by $\gamma_s(t + 2\pi) = 2\pi + \gamma_s(t)$ $(t \in \mathbb{R})$. For the detailed description of the β functions see [4].

The system

$$\Psi_n(z) := (T\Phi_n)(z) = (Tf)(z) := \frac{1}{\sqrt{\pi}} \frac{1}{i+z} \Phi_n(K(z)) \quad (\operatorname{Im} z \geq 0,\ n \in \mathbb{N}^*),$$

which is an analogue of the Malmquist Takeneka system for the upper half plane, is ortonormal in $L^2(\mathbb{R})$. It is easy to check that for $a \in \mathbb{D}$ with $a^* := 1/\overline{a}$

$$\lambda_a := K^{-1}(a) = i\frac{1-a}{1+a} \in \mathbb{C}_+,\ \lambda_{a^*} = \overline{\lambda}_a, \quad \frac{\sqrt{1-|a|^2}}{|1+\overline{a}|} = \sqrt{\operatorname{Im} \lambda_a}, \tag{1.2}$$

and

$$\tilde{b}_a(z) = b_a(-1) \frac{z - \lambda_a}{z - \overline{\lambda}_a}, \quad \tilde{r}_a(z) = r_a(-1) \frac{z+i}{z - \overline{\lambda}_a} \quad (z \in \overline{\mathbb{C}}_+). \tag{1.3}$$

This implies that the functions $\Psi_n = T\Phi_n$ $(n \in \mathbb{N}^*)$ are of the form

$$\Psi_1(z) = \frac{1}{\sqrt{\pi}} \frac{\Phi_1(-1)}{z - \overline{\lambda}_{a_1}}, \quad \Psi_n(z) = \frac{1}{\sqrt{\pi}} \frac{\Phi_n(-1)}{z - \overline{\lambda}_{a_n}} \prod_{k=1}^{n-1} \frac{z - \lambda_{a_k}}{z - \overline{\lambda}_{a_k}}. \tag{1.4}$$

Moreover, if the following Blaschke condition for the upper half plane is satisfied

$$\sum_{k=1}^{\infty} \frac{\operatorname{Im} \lambda_k}{1+|\lambda_k|^2} = \infty,$$

then $(\Psi_n, n \in \mathbb{N}^*)$ is a complete orthonormal system for $H^2(\mathbb{C}_+)$.

1.2. Discrete orthogonality

First let us recall the discrete orthogonality of the Malmquist–Takenaka system for the unit disc. The Blaschke product $B_N = \prod_{j=1}^{N} b_{a_j}$ on the unit circle can be written as

$$B_N(e^{it}) = \prod_{j=1}^{N} b_{a_j}(e^{it}) = e^{i(\beta_{a_1}(t)+\cdots+\beta_{a_N}(t))} \quad (t \in \mathbb{R}, N = 1, 2 \ldots).$$

This implies that the solution of the equation

$$\frac{w-a_1}{1-\overline{a}_1 w} \cdot \frac{w-a_2}{1-\overline{a}_2 w} \cdots \frac{w-a_N}{1-\overline{a}_N w} = e^{2\pi i \delta} \quad (\delta \in \mathbb{R}) \tag{1.5}$$

are given by

$$w_k := e^{i\tau_k}, \quad \tau_k := \theta_N^{-1}(2\pi((k-1)+\delta)/N) \quad (k = 1, 2, \ldots, N), \tag{1.6}$$

where θ_N^{-1} is the inverse of the function

$$\theta_N(t) := \frac{1}{N}(\beta_{a_1}(t) + \cdots + \beta_{a_N}(t)) \quad (t \in \mathbb{R}).$$

Let us consider

$$\mathbb{T}_N := \mathbb{T}_N^{a,\delta} := \{w_k = e^{i\tau_k} : k = 1, 2, \ldots, N\} \quad (N = 1, 2, \ldots), \tag{1.7}$$

the set of solutions of the equation (1.5). We name $\mathbb{T}_N$ the set of discretization nodes on the unit circle. Let us consider the weight function ρ_N given for $w \in \mathbb{T}$ by

$$\frac{1}{\rho_N(w)} := \sum_{k=1}^{N} \frac{1-|a_k|^2}{|1-\overline{a}_k w|^2} \quad (w \in \mathbb{T}, N = 1, 2, \ldots).$$

The discrete orthogonality of the Laguerre, Kautz and Malmquist–Takenaka systems was investigated in [2, 9, 10]. It was proved.

The following theorem

Theorem 1. ([9]) *The finite collection of Φ_n $(1 \le n \le N)$ forms a discrete orthonormal system with respect to the scalar product*

$$[F,G]_N := \sum_{w \in \mathbb{T}_N^{\mathfrak{a},\delta}} F(w)\overline{G(w)}\rho_N(w),$$

namely $[\Phi_n, \Phi_m]_N = \delta_{mn} \quad (1 \le m, n \le N)$.

The analogue of the previous theorem for the Malmquist–Takenaka system of the upper half plane was proved recently in [6]. The transformation formulas (1.3) imply

$$\tilde{B}_N(z) := B_N(K(z)) = B_N(-1) \prod_{k=1}^{N} \frac{z - \lambda_{a_k}}{z - \overline{\lambda}_{a_k}}.$$

Let $\mathbb{R}_N^{\mathfrak{a},\delta} := \{K^{-1}(w) : w \in \mathbb{T}_N^{\mathfrak{a},\delta}\} = K^{-1}(\mathbb{T}_n^{\mathfrak{a},\delta})$, $\tilde{\rho}_N(t) := \pi(1+t^2)\rho_N(K(t))$ $(t \in \mathbb{R})$. Then by Theorem 1 we have

$$\begin{aligned}\delta_{mn} &= \sum_{z \in K^{-1}(\mathbb{T}_N^{\mathfrak{a},\delta})} \Phi_n(K(z))\overline{\Phi}_m(K(z))\rho_N(K(z)) \\ &= \pi \sum_{z \in \mathbb{R}_N^{\mathfrak{a},\delta}} \Psi(z)\overline{\Psi}_m(z)|i + z|^2 \rho_N(K(z)).\end{aligned}$$

Thus we get

Theorem 2. ([6]) *The finite collection of Ψ_n $(1 \le n \le N)$ forms a discrete orthonormal system with respect to the weight function $\tilde{\rho}_N$:*

$$\sum_{z \in \mathbb{R}_N^{\mathfrak{a},\delta}} \Psi(z)\overline{\Psi}_m(z)\tilde{\rho}_N(z) = \delta_{mn} \quad (1 \le m, n \le N).$$

We mention that, based on the discrete orthogonality of the Malmquist–Takenaka systems, in [13] new rational interpolation operators were introduced. In the case of the upper half plane the introduced interpolation operator gives an exact interpolation for the Runge test function.

1.3. Equilibrium condition on the unit circle

The set of the discretization nodes on the unit circle satisfies an equilibrium condition and is a stationary point for a logarithmic potential function. These results were published in [8–10]. We present a short overview of them.

Let us consider the following notations: for any complex number $z \in \mathbb{C} \setminus \{0\}$ set $z^* := 1/\overline{z}$ and introduce the polynomials

$$\omega_1(w) := \prod_{k=1}^{N}(w - a_k), \quad \omega_2(w) := \prod_{k=1}^{N}(1 - \overline{a}_k w),$$

$$\omega(w) := \omega_1'(w)\omega_2(w) - \omega_2'(w)\omega_1(w) \quad (z \in \mathbb{C}).$$

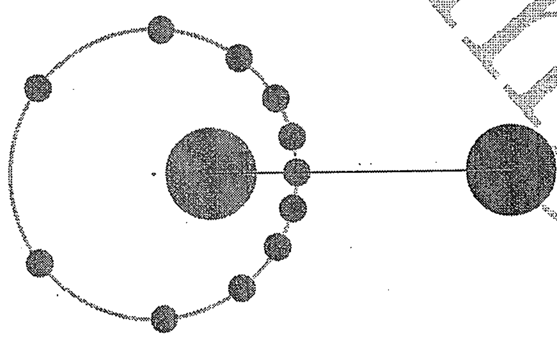
It is clear that ω is a polynomial of degree $2N-2$. It can be proved (see [9]) that if c is a root of ω with multiplicity m then c^* is also a root of ω with the same multiplicity. Denote by $c_1, c_1^*, \dots, c_{N-1}, c_{N-1}^*$ the roots of ω.

Theorem 3. ([9]) *For every $\delta \in \mathbb{R}$ the numbers $w_n := e^{i\tau_n} \in \mathbb{T}_N^{a,\delta}$, $\tau_n := \theta_N^{-1}(2\pi((n-1)+\delta)/N)$ $(n = 1, 2, \dots, N)$ are the solutions of the equilibrium equations*

$$\sum_{k=1, k\neq n}^{N} \frac{1}{w_n - w_k} = \frac{1}{2}\sum_{j=1}^{N-1}\left(\frac{1}{w_n - c_j} + \frac{1}{w_n - c_j^*}\right) \quad (n = 1, \dots, N). \tag{1.8}$$

The electrostatic interpretation of (1.8) is the following: if negative unit charges are placed to the points c_k and c_k^*, then n positive unit charges places to the points w_j will be in equilibrium in the external field generated by the negative charges. d

In the case of discrete Laguerre functions $a_1 = \dots = a_N = a$ and $\omega(w) = N(1-|a|^2)(w-a)^{N-1}(1-\overline{a}w)^{N-1}$. Thus the roots of ω are a and a^* with multiplicity $N-1$, i.e., $c_1 = \dots = c_{N-1} = a$.



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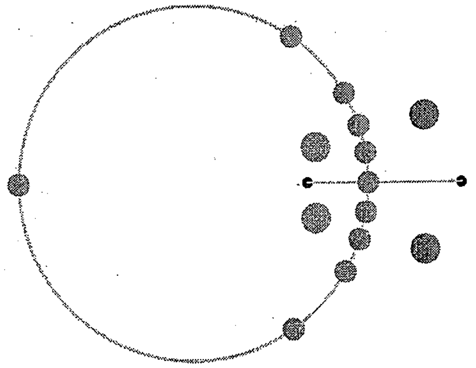
Equilibrium position on the circle in Laguerre case

In the case of Kautz system $N = 2M$, $a_1 = a_3 = \dots = a_{2M-1} = a$, $a_2 = a_4 = \dots = a_{2M} = b := \overline{a}$ and consequently

$$\omega(w) = \Omega(w)[(w-a)(w-b)(1-\overline{a}w)(1-\overline{b}w)]^{M-1},$$

where $\Omega(w) = M[(1-|a|^2)(w-b)(1-\overline{b}w) + (1-|b|^2)(w-a)(1-\overline{a}w)]$. Thus in

s this case c_1, c_1^* are the root of Ω, $c_2 = \dots = c_M = a$, $c_{M+1} = \dots = c_{2M-1} = b$.



Equilibrium position on the circle in Kautz case the

In the special case $a_1 = a_2 = \cdots = 0$ from the Malmquist–Takenaka system we reobtain the trigonometric system, the corresponding sets of discretization are

$$\mathbb{T}_N^\delta := \{e^{2\pi i(n-1+\delta)/N} : n = 1, 2, \ldots, N\} \ (0 \le \delta < 1).$$

In this case the equilibrium condition becomes:

$$\sum_{k=1, k\neq n}^{N} \frac{1}{w_n - w_k} = \frac{N-1}{2} \cdot \frac{1}{w_n} \quad (n = 1, \ldots, N).$$

[1, p. 425]

This special case can be found for example in $\boxed{[1] \text{ p. } 425}$, moreover for this case the following minimum property is true: the potential energy

$$W(v_1, \cdots, v_N) = -\log \prod_{1\le j<k\le N} |v_k - v_j| \quad (v_1, \cdots, v_N \in \mathbb{T})$$

attains its minimum when $v_k = w_k \in \mathbb{T}_n^\delta$ ($k = 1, \ldots, N$).

This special case is the so called *Stieltjes problem on the unit circle* and has the following interpretation: if the N freely moving unit charges lie on thin circular conductor of unit radius, then the potential energy of the system is $W(v_1, \boxed{\cdots}, v_N)$ and this is minimal if the charges are located in $\mathbb{T}_N^\delta$, and this minimum is equal to $-\frac{N}{2} \log N$.

This motivate the interest in the examination of similar minimum property of s

$$w_n = e^{i\tau_n} \in \mathbb{T}_N^{a,\delta}, \tau_n := \theta_N^{-1}(2\pi((n-1)+\delta)/N) \quad (n = 1, 2, \ldots, N).$$

As a first step to answer this question in a special case was made in [10]. In [8] the following theorem was proved:

Theorem 4.([8]) *The point* $(w_1, w_2, \dots, w_N) \in \mathbb{T}_N^{a,\delta}$ *is a stationary point of the logarithmic potential*

$$W(v_1, \dots, v_N) = -\log \Big(\prod_{1 \le j < k \le N} |v_j - v_k| \prod_{k=1}^{N-1} \prod_{j=1}^{N} (|v_j - c_k||v_j - c_k^*|)^{-1/2} \Big) \quad (1.9)$$

$$(v_1 = e^{it_1}, \dots, v_N = e^{it_N} \in \mathbb{T}),$$

i.e.,

$$\frac{\partial W(e^{i\tau_1}, \dots, e^{i\tau_N})}{\partial t_n} = 0 \quad (n = 1, \dots, N). \quad (1.10)$$

For $v_j \in \mathbb{T}$ we have

$$|v_j - c_k^*| = |\overline{v}_j - 1/c_k| = |1/v_j - 1/c_k| = |v_j - c_k"/|c_k|,$$

and consequently replacing in (1.9) c_k by c_k^* we get a function which differ from W by an additive constant. Thus if negative unit charges are placed to the points c_k, the n positive unit charges places to the points w_j, will be in equilibrium in the external field generated by the negative charges.

In natural way arises the following question for the general case: is the stationary point $(w_1, w_2, \dots, w_N)$ a minimum point for the potential function W? In the next section we investigate this problem for both types of Malmquist–Takenaka systems.

2. New results

2.1. Equilibrium condition on the real line

In what follows we prove that the discretization nodes on the real line satisfy an analogue equilibrium property. For $\lambda_1, \boxed{\dots}, \lambda_N \in \mathbb{C}_+$ let us consider the polynomials \ldots

$$\phi_1(z) := \prod_{k=1}^{N} (z - \lambda_k), \qquad \phi_2(z) := \prod_{k=1}^{N} (z - \overline{\lambda}_k),$$

$$\phi(z) := \phi_1'(z)\phi_2(z) - \phi_2'(z)\phi_1(z) \quad (z \in \mathbb{C}).$$

It is clear that ϕ is a polynomial of degree $2N-2$. It is easy to prove if d is a root of ϕ with multiplicity m then $\overline{d}$ is also a root of ϕ with the same multiplicity. Let us denote by $d_1, \overline{d}_1, \dots, d_{N-1}, \overline{d}_{N-1}$ the roots of ϕ, i.e., that

$$\phi(z) = \prod_{j=1}^{N-1} (z - d_j)(z - \overline{d}_j) \quad (z \in \mathbb{C}). \quad (2.1)$$

The numbers $a_k := K(\lambda_k)$ $(k = 1, \cdots, N)$ are in $\mathbb{D}$ and by (1.3)

$$B_N(K(z)) = B_N(-1) \prod_{k=1}^{N} \frac{z - \lambda_k}{z - \overline{\lambda}_k} = B_N(-1) \frac{\phi_1(z)}{\phi_2(z)}.$$

The functions $\omega_1, \phi_1, \omega_2, \phi_2$ and ω, ϕ can be expressed by each others:

$$(i + z)^N \omega_1(K(z)) = \omega_1(-1)\phi_1(z), \qquad (i + z)^N \omega_2(K(z)) = \omega_2(-1)\phi_2(z),$$ b/

$$(i + z)^{2N-2} \omega(K(z)) = -\omega_1(-1)\omega_2(-1)\phi(z) \tag{2.2}$$

and consequently $\omega(K(d_j)) = 0$, if $d_j \neq -i$.

\ldots (2x) Denote $w_1^\delta, \dots, w_N^\delta \in \mathbb{T}$ the N (pairwise distinct) solution of the equation $B_N(w) = e^{2\pi i\delta}$. Then the numbers $t_k := t_k^\delta := K^{-1}(w_k^\delta) \in \mathbb{R}$ $(k = 1, \dots, N)$ are the solutions of φ by λs

$$\frac{\phi_1(z)}{\phi_2(z)} = q := e^{2\pi i\delta}/B_N(-1) \in \mathbb{T} \tag{2.3}$$

and we have

Theorem 5. *Let $q \in \mathbb{T}$ and denote $t_n = t_n^\delta \in \mathbb{R}$ $(n = 1, 2, \ldots, N)$ the solutions of (2.3), where $\delta \in [0, 1)$. Then the following equilibrium conditions are satisfied:* φ by

$$\sum_{k=1, k\neq n}^{N} \frac{1}{t_n - t_k} = \frac{1}{2} \sum_{j=1}^{N-1} \left(\frac{1}{t_n - d_j} + \frac{1}{t_n - \overline{d}_j} \right) \quad (n = 1, \dots, N). \tag{2.4}$$ \ldots

Proof. By the definition of t_k it follows that $g(z) := \phi_1(z) - q\phi_2(z) = 0$ if and only if $z = t_k$ $(k = 2, \ldots, N)$. Set

$$f(z) = \prod_{k=1}^{N} (z - t_k).$$

The polynomials f and g have the same degree and roots, therefore $f = \lambda g$ with $\lambda \in \mathbb{C}$, a constant.

It is easy to see that

$$\frac{g''(t_n)}{2g'(t_n)} = \frac{f''(t_n)}{2f'(t_n)} = \sum_{k=1, k\neq n}^{N} \frac{1}{t_n - t_k} \quad (n = 1, 2, \ldots, N).$$

By the definition of t_n

$$\frac{\phi_1(t_n)}{\phi_2(t_n)} = q \quad (n = 1, 2, \ldots, N).$$

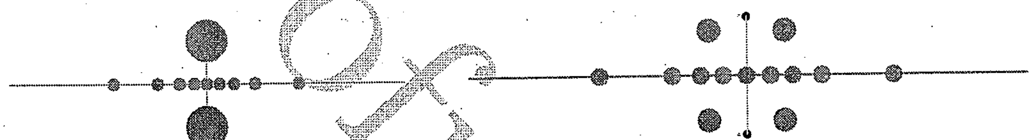
On the other hand we get that:

$$\frac{g''(t_n)}{g'(t_n)} = \frac{\phi_1''(t_n) - q\phi_2''(t_n)}{\phi_1'(t_n) - q\phi_2'(t_n)} = \frac{\phi_2(t_n)\phi_1''(t_n) - \phi_1(t_n)\phi_2''(t_n)}{\phi_2(t_n)\phi_1'(t_n) - \phi_1(t_n)\phi_2'(t_n)} = \frac{\phi'(t_n)}{\phi(t_n)}.$$

From (2.1) we have

$$\frac{\phi'(t)}{\phi(t)} = \sum_{k=1}^{N-1}\Big(\frac{1}{t-d_k} + \frac{1}{t-\overline{d}_k}\Big)$$

and our claim is proved. ∎



Equilibrium position on the real line in Laguerre and Kautz case

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In the special case when $a_1 = \cdots = a_N = a = r \in [0,1)$, then by (1.2)

$$\lambda_j := K^{-1}(a) = i\frac{1-r}{1+r} = ip,\ p := \frac{1-r}{1+r} > 0\ (j = 1,2,\ldots,N).$$

By the definition of β_a and θ_N in this case we have

$$\theta_N(t) = \beta_r(t) = \gamma_s(t) = 2\arctan(s\tan(t/2)),\ \theta_N^{-1}(t) = \gamma_{1/s}(t),\ s = \frac{1+r}{1-r}.$$

By (1.6) the solution of $B_N(w) = e^{2\pi i\delta}$ are $w_k = e^{i\tau_k}$, where

$$\tau_k = \gamma_{1/s}(2\pi(k-1+\delta)/N)\ (k = 1,2,\ldots,N),$$

consequently for $t_k = K^{-1}(e^{i\tau_k})$ we get

$$t_k = \tan(\tau_k/2) = \frac{1}{s}\tan\Big(\frac{\pi}{N}(k-1+\delta)\Big) = p\tan\Big(\frac{\pi}{N}(k-1+\delta)\Big).$$

\ldots =

Thus in the case $\lambda_1 = \lambda_2 = \cdots \lambda_N = pi$ $(p > 0)$ the corresponding nodal points are $t_n = p\tan((n-1+\delta)\pi/N)$ $(0 \le \delta < 1)$. In this case the equilibrium condition becomes:

$$\sum_{k=1,k\neq n}^{N}\frac{1}{t_n - t_k} = \frac{N-1}{2}\Big(\frac{1}{t_n - pi} + \frac{1}{t_n + pi}\Big)\ \ (n = 1,\ldots,N).$$

Let us introduce the analogue of the potential function W in (1.9):

$$V(x_1,\dots,x_N) = -\log\Big(\prod_{1\le j<k\le N}|x_j-x_k|\prod_{k=1}^{N-1}\prod_{j=1}^{N}(|x_j-d_k||x_j-\overline{d}_k|)^{-1/2}\Big) \tag{2.5}$$

6,

$$(x_1,\dots,x_N)\in\mathbb{R}^N,\quad (d_1,\dots,d_{N-1})\in\mathbb{C}_+^{N-1}.$$

Obviously (2.4) is equivalent to

$$\frac{\partial V(t_1,\dots,t_N)}{\partial x_n}=0\quad (n=1,\dots,N), \tag{2.6}$$

i.e., $(t_1,t_2,\dots,t_N)$ is a stationary point of the potential function V. The function V can be expressed by

$$V_d(s,t):=\frac{|s-t|}{|(s-d)(t-d)|}\quad (s,t\in\mathbb{R}, d\in\mathbb{C}_+).$$

Namely using the identity $(|x-d||x-\overline{d}|)^{-1/2}=|x-d|^{-1}$ $(x\in\mathbb{R}, d\in\mathbb{C}_+)$ we get

$$V(x_1,x_2,\cdots,x_N)=-\frac{1}{N-1}\log\Big(\prod_{1\le j<k\le N}\prod_{\ell=1}^{N-1}V_{d_\ell}(x_j,x_k)\Big). \tag{2.7}$$

In a similar way the potential W in (1.9) can by expressed by the functions

$$W_c(v,w):=\frac{|v-w|}{|(v-c)(w-c)|}\quad (c\in\mathbb{D}, v,w\in\mathbb{T}).$$

Namely

$$W(v_1,\square,v_N)=-\frac{1}{N-1}\log\Big(\prod_{1\le j<k\le N}\prod_{\ell=1}^{N-1}W_{c_\ell}(v_j,v_k)\Big).$$

\ledots (3x)

The functions V and W are closely connected. It is easy to see that

$$W_c(K(x_1),K(x_2))=\frac{2}{|1+c|^2}V_{K^{-1}(c)}(x_1,x_2)\quad (x_1,x_2\in\mathbb{R}, c\in\mathbb{D}).$$

This implies

Theorem 6. *If $d_k\in\mathbb{C}_+$, $c_k=K(d_k)$ $(k=1,\dots,N-1)$ then*

$$W(K(x_1),\dots,K(x_N))=V(x_1,\dots,x_N)+const\quad ((x_1,\square,x_N)\in\mathbb{R}^N),$$

consequently $(t_1,\square,t_N)\in\mathbb{R}^N$ is a minimum position for V, if and only if $w_1=K(t_1),\dots,w_N=K(t_N)\in\mathbb{T}$ is a minimum position for W.

For the function V we can give a geometrical interpretation. Let us consider the triangle in the complex plan with vertices $s, t \in \mathbb{R}, d \in \mathbb{C}_+$ and denote α the angle at the vertex d. Then by the area formula $|s-t| \operatorname{Im} d = |d-s||d-t| \sin\alpha$, consequently e by

$$V_d(s,t) = \frac{\sin\alpha}{\operatorname{Im} d}.$$

Instead of V we investigate the maximum of the function

$$T(x) := T(x_1, x_2, \square, x_N) := \prod_{1 \le j < k \le N} \prod_{\ell=1}^{N-1} V_{d_\ell}(x_j, x_k) \ (x \in \mathbb{R}^N).$$ \ldots

Theorem 7. *In the special case of discrete Laguerre-system i.e., if* the

$$\lambda_1 = \cdots = \lambda_N = ip \ (p > 0) \tag{2.8}$$,

the function T attains its maximum at $t_n^\delta = p\tan((n-1+\delta)\pi/N)$, where $(0 \le \delta < 1, n = 1, \ldots, N)$.

Proof. Let $x_1 < x_2 < \cdots < x_N$, $P = ip$ and denote the angle $x_j P x_{j+1}\sphericalangle$ by α_j $(1 \le j < N)$. Then the function T is constant ~~times of~~ a multiple of

$$S(\alpha) := \prod_{1 \le j < k \le N} \sin\alpha_{jk}, \qquad \alpha_{jk} := \alpha_j + \cdots + \alpha_{k-1},$$

$$\alpha = (\alpha_1, \ldots, \alpha_{N-1}) \in A := \{\alpha \in \mathbb{R}^{N-1} : 0 < \alpha_j, \alpha_1 + \cdots + \alpha_{N-1} < \pi\}.$$

The function S does not depend on the position of x_1. In addition S is continuous and non negativ and vanishes at the boundary of A. Thus S has a maximum position in A. In addition, this position is uniquely determined. To show this, let us suppose that $\alpha'_1, \ldots, \alpha'_{N-1}$ and $\alpha''_1, \ldots, \alpha''_{N-1}$ are two position of this kind. Denote $\alpha_j := (\alpha'_j + \alpha''_j)/2$ $(j = 1, \ldots, N-1)$. Since

- e

$$\sin s \sin t \le \sin^2\left(\frac{s+t}{2}\right) \ (0 \le s, t \le \pi),$$

therefore

$$S(\alpha')S(\alpha^{\square}) \le S^2(\alpha)$$ $\square''$

and the equality sign being taken if and only if $\alpha'_j = \alpha''_j = \alpha_j$ $(j = 1, \ldots, N-1)$. This establishes the uniqueness. The maximum position $\tilde\alpha$ is the solution of

$$\frac{\partial S(\tilde\alpha)}{\partial \alpha_j} = 0 \ \ (j = 1, \ldots, N-1).$$

It is easy to see that $\tilde{\alpha}_j = \pi/N$ $(j = 1, \ldots, N-1)$. Fix x_1 and denote δ_j the angle at vertices x_j of the triangle Px_j0. Then in the maximum position of S we have $\delta_j = \delta_{j-1} + \pi/N$ $(j = 2, \ldots, N)$, i.e., $\delta_j = \delta_1 + (j-1)\pi/N$ $(j = 1, 2, \ldots, N)$. Consequently

$$\begin{aligned} x_{N-j+1} &= p\tan(\pi/2 - \delta_{N-j+1}) = p\tan(\pi/2 - \delta_1 - (N-j)\pi/N) = \\ &= p\tan(-\pi/2 - \delta_1 + \pi/N + (j-1)\pi/N) = p\tan(\delta\pi/N + (j-1)\pi/N) \\ &\qquad (j = 1, 2, \ldots, N), \end{aligned}$$

where $\delta\pi/N := -\pi/2 - \delta_1 + \pi/N$. Thus $x_{N-j+1} = t_j^{\delta}$ $(j = 1, 2, \ldots N)$ and we get the equilibrium positions in Theorem 7. ∎

The following general case seems to be an

Open problem. Does the function V defined before in the position $t_1, t_2, \ldots, t_N \in \mathbb{R}$ attain its minimum?

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M. PAP, University of Pécs

F. SCHIPP, Eötvös Loránd University, e-mail: schipp@numanal.inf.elte.hu