



A comparative analysis of algorithms for fast computation of Zernike moments

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Abstract

This paper details a comparative analysis on time taken by the present and proposed methods to compute the Zernike moments, Z_{pq} . The present method comprises of Direct, Belkasim's, Prata's, Kintner's and Coefficient methods. We propose a new technique, denoted as q -recursive method, specifically for fast computation of Zernike moments. It uses radial polynomials of fixed order p with a varying index q to compute Zernike moments. Fast computation is achieved because it uses polynomials of higher index q to derive the polynomials of lower index q and it does not use any factorial terms. Individual order of moments can be calculated independently without employing lower- or higher-order moments. This is especially useful in cases where only selected orders of Zernike moments are needed as pattern features. The performance of the present and proposed methods are experimentally analyzed by calculating Zernike moments of orders 0 to p and specific order p using binary and grayscale images. In both the cases, the q -recursive method takes the shortest time to compute Zernike moments. © 2002 Pattern Recognition Society. Published by Elsevier Science Ltd. All rights reserved.

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1. Introduction

Zernike moments are used in pattern recognition applications [1–3] as invariant descriptors of the image shape. They have been proven to be superior to moment functions such as geometric moments in terms of their feature representation capabilities and robustness in the presence of image quantization error and noise [4–7]. Their orthogonality property helps in achieving a near zero value of redundancy measure in a set of moment functions. Thus, moments of different orders correspond to independent characteristics of the image.

Zernike moments are however computationally more complex and lengthy compared to that of geometric moments and Legendre moments. This is because the

definition of Zernike radial polynomials is heavily dependent on factorial functions and only a single Zernike moment can be obtained at one time. The computation has to be repeated to obtain the entire set of Zernike moments of order p with $q = p - 2, p - 4, p - 6$, etc. The computation time increases substantially as the order p increases. This limitation has prompted considerable study on algorithms for fast evaluation of Zernike moments [8–11].

Mukundan et al. have proposed two fast algorithms namely contour integration method and square-to-circular transformation method to compute Zernike moments [8]. However, these proposed algorithms have limitations. In the case of the contour integration method, it is applicable only for binary image and it requires off-line analysis to extract the image boundary points. The accuracy of the Zernike moments has to be compromised when the square-to-circular transformation method is used.

Belkasim too has introduced an algorithm for fast computation of Zernike moments based on the series angular and

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radial expansion of the radial polynomials [9]. This method can still be further improved if the factorial functions in Zernike radial polynomial coefficients are avoided. It is only possible to obtain a single Zernike moment, Z_{pq} , each time through a loop. Zernike moments can also be expressed using a combination of geometric moments. Here too, the computation of polynomial coefficients is heavily dependent on factorial terms. This will substantially increase the computation time. Besides that, only a single Zernike moment can be obtained at a single loop.

Prata and Kintner have, respectively, proposed a recurrence relation for radial polynomials of Zernike moments [10,11]. However, their techniques are not applicable in cases where $(p = q)$ and $(q = 0)$, and $(p - q < 4)$, respectively. Although these values can be directly computed from the definition of Zernike radial polynomials, the overall computation again will be delayed by the factorial functions in the polynomial coefficients. Moreover, it is difficult to obtain any specific order of moments independently using Prata's or Kintner's method due to the interdependency of various orders of the radial polynomials.

Recently, we introduced a method for recursive calculation of Zernike polynomial coefficients, denoted as Coefficient method, specifically for fast computation of Zernike moments [12]. This method employs a recurrence relation to derive the polynomial coefficients without using any factorial functions. The entire set of p order of Zernike moments can be obtained independently and concurrently without using either lower- or higher-order moments. Nevertheless, this method still needs to compute every coefficient and power series of radius for each order of radial polynomial. The computation time will increase considerably when the order increases.

In this paper, we introduce a new algorithm, denoted as q -recursive method, specifically for fast computation of the Zernike moments. This algorithm aims to reduce the amount of complex computation involved in calculating the Zernike radial polynomials. The q -recursive method uses Zernike radial polynomials of fixed order p with higher index q to derive the polynomial of the lower index q without computing the polynomial coefficients and the power series of radius. It does not use any factorial functions, which in turn reduces substantially the computation time to derive the Zernike moments. Moreover, it also allows the computation of any specific order of Zernike moments to be carried out without using moments of different orders. This is especially useful in cases where only selected Zernike moments are needed as pattern features. The entire set of Zernike moments of order p can be obtained concurrently in a single iteration of the loop in the algorithm. Since each order of Zernike moments can be computed independently, it is ideally suited for parallel processing environment. Together with the q -recursive method, we also propose to use a couple of relations to improve the speed of the existing Kintner's method.

The performance of the present and proposed methods is experimentally verified by comparing their respective CPU elapsed time to calculate Zernike moments using binary and grayscale images. The present techniques comprise of Direct, Belkasim's, Prata's, Kintner's and Coefficient methods. The results show that the q -recursive method takes the shortest time in evaluating the entire set of Zernike moments of order 0 to p as well as in computing a specific order of Zernike moments.

In Section 2, we show the theory of Zernike moments, followed by a detailed analysis of Direct, Belkasim's, Prata's, Kintner's and Coefficient methods in Section 3. The q -recursive method is discussed in Section 4. Section 5 shows the experimental results and Section 6 concludes the study.

2. Zernike moments

The kernel of Zernike moments is the set of orthogonal Zernike polynomials defined over the polar coordinate space inside a unit circle. The two-dimensional Zernike moments of order p with repetition q of an image intensity function $f(r, \theta)$ are defined as [13]

$$Z_{pq} = \frac{p+1}{\pi} \int_0^1 \int_{-\pi}^{\pi} V_{pq}(r, \theta) r \, dr \, d\theta, \quad |r| \leq 1, \quad (2.1)$$

where the Zernike polynomials $V_{pq}(r, \theta)$ of order p are defined as

$$V_{pq}(r, \theta) = R_{pq}(r) e^{-j\hat{q}\theta}, \quad \hat{q} = \sqrt{-1} \quad (2.2)$$

and the real-valued radial polynomials, $R_{pq}(r)$ is defined as follows:

$$\begin{aligned} R_{pq}(r) &= \sum_{k=0}^{\frac{p-|q|}{2}} (-1)^k \frac{(p-k)!}{k! \left(\frac{p+|q|}{2} - k\right)! \left(\frac{p-|q|}{2} - k\right)!} r^{p-2k}, \end{aligned} \quad (2.3)$$

where $0 \leq |q| \leq p$, $p - |q|$ is even, $p \geq 0$.

Since Zernike moments are defined in terms of polar coordinates (r, θ) with $|r| \leq 1$, the computation of Zernike polynomials requires a linear transformation of the image coordinates (i, j) , $i, j = 0, 1, 2, \dots, N-1$ to a suitable domain $(x, y) \in R^2$ inside a unit circle. Two commonly used cases of the transformations are shown in Fig. 1(b) and (c). Based on these figures, we have the following discrete approximation of the continuous Zernike moments' integral in (2.1):

$$\begin{aligned} Z_{pq} &= \lambda(p, N) \sum_{i=0}^{N-1} \sum_{j=0}^{N-1} R_{pq}(r_{ij}) e^{-j\hat{q}\theta_{ij}} f(i, j), \\ 0 &\leq r_{ij} \leq 1, \end{aligned} \quad (2.4)$$

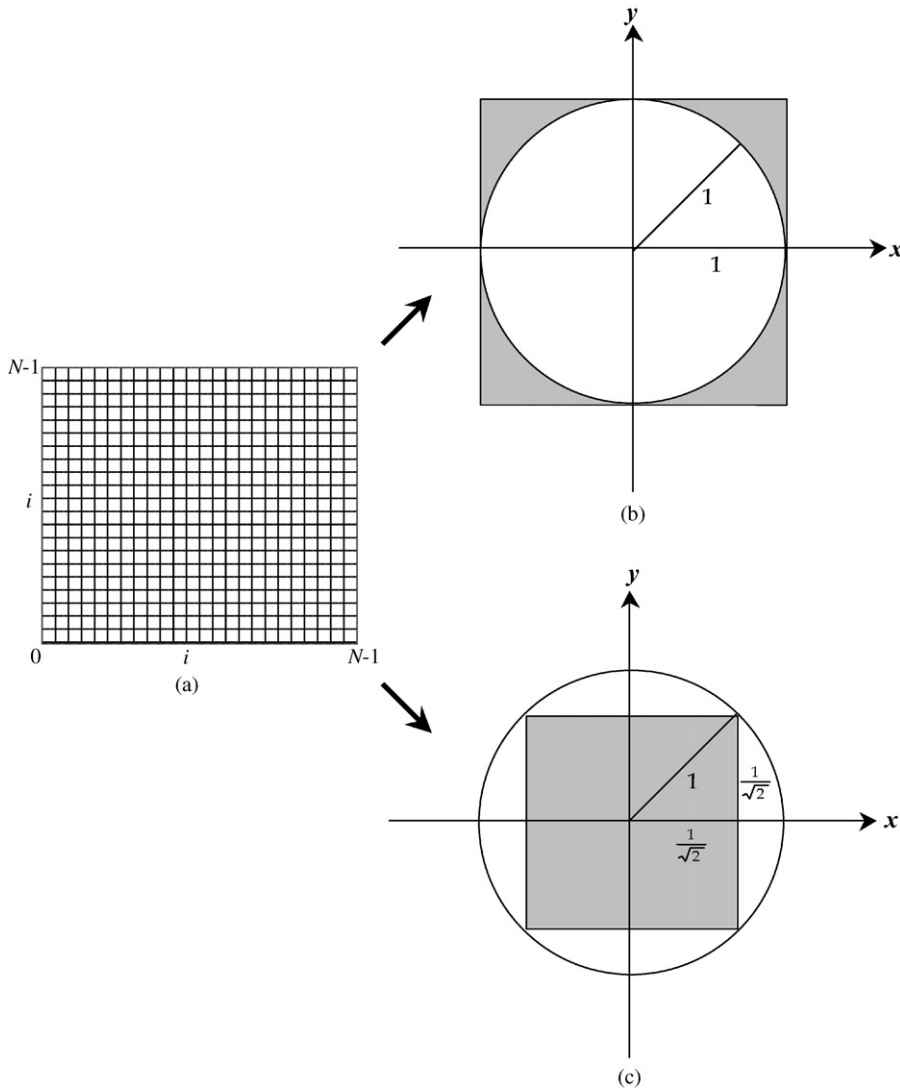


Fig. 1. (a) Image plane $N \times N$, (b) mapping of image plane $N \times N$ over a unit circle and (c) mapping of image plane $N \times N$ inside a unit circle.

where the most general image coordinate transformation to the interior of the unit circle is given by

$$r = \sqrt{(c_1 i + c_2)^2 + (c_1 j + c_2)^2},$$

$$\theta = \tan^{-1} \left(\frac{c_1 j + c_2}{c_1 i + c_2} \right). \quad (2.5)$$

In particular, for

Fig. 1(b): $\lambda(p, N) = \frac{p+1}{(N-1)^2}, \quad c_1 = \frac{2}{N-1},$

$$c_2 = -1, \quad (2.6)$$

Fig. 1(c): $\lambda(p, N) = \frac{2(p+1)}{\pi(N-1)^2}, \quad c_1 = \frac{\sqrt{2}}{N-1},$

$$c_2 = -\frac{1}{\sqrt{2}}. \quad (2.7)$$

3. Present methods to compute the Zernike moments

This section presents an analysis of some of the different techniques currently being used to compute the Zernike moments. The techniques include Direct, Belkasim, Prata's, Kintner's and Coefficient methods. All of the techniques except for Direct method focus on how to reduce the complexity of Zernike radial polynomials. The time taken to compute Zernike moments and the respective advantages and limitations of each method are discussed.

3.1. Direct method

The calculation of Zernike moments using the definition of radial polynomials in Eq. (2.3) is denoted hereafter

as Direct method. From Eq. (2.3), it can be seen that the radial polynomial is heavily dependent on factorial functions. There are four factorial functions to be computed for each $R_{pq}(r)$. Assuming that $n!$ requires n multiplications, then the total number of multiplications involved in computing the radial polynomials of order p , i.e. $R_{pp}(r)$, $R_{p(p-2)}(r)$, $R_{p(p-4)}(r)$, \dots and $R_{pq}(r)$ is defined as

$$T(p) \approx \sum_{l=0}^p \frac{(p-l+2)(2p+l)}{2}, \quad p-l = \text{even} \quad (3.1)$$

and within the $T(p)$, the total number of multiplications involved in computing the factorial functions in the polynomial coefficient is

$$\sum_{l=0}^p \frac{(p-l+2)(3p+l)}{4}. \quad (3.2)$$

It can be seen from Eq. (3.1) that the number of multiplications involved for Zernike moments of order p is $O(p^3)$. If we compute 48 orders of radial polynomials, the computation will take 462, 800 multiplications, and 71% of them is just used for computing the factorial functions! This is the reason for the long computation time involved for computing the Zernike moments. Although this method can easily obtain any $R_{pq}(r)$, it can only derive a single $R_{pq}(r)$ each time as it goes through the loop. In other words, it has to go through $(p+1)/2$ (if p is odd) or $(p+2)/2$ (if p is even) iterations to obtain the entire set of Zernike moments of order p . It is inappropriate to be used for fast computation of Zernike moments. Moreover, the Zernike polynomial often involves powers of p and q . Zernike moments computed using Eq. (2.3) will have large variation in the values for different orders of p . It may also be necessary to develop methods for avoiding numerical instabilities when the image size is large.

3.2. Belkasim's method

Belkasim has introduced an algorithm for fast computation of Zernike moments based on factorizing some of the redundant terms in the radial and angular expansion of Zernike polynomials [9]. The radial polynomial, $R_{pq}(r)$, can be expressed in the following form:

$$R_{pq}(r) = B_{pq,p}R_{pp}(r) + B_{pq,(p-2)}R_{(p-2)(p-2)}(r) + \dots + B_{pq,k}R_{kk}(r) + \dots + B_{pq,q}R_{qq}(r), \quad (3.3)$$

where the polynomial coefficient, $B_{pq,k}$ is

$$B_{pq,k} = \frac{(-1)^{(p-k)/2}((p+k)/2)!}{((p-k)/2)!((k+q)/2)!((k-q)/2)!}. \quad (3.4)$$

Using Eq. (3.3), the Zernike moments, Z_{pq} in Eq. (2.1) can be rewritten as follows:

$$Z_{pq} = \frac{p+1}{N^2} \left(B_{pq,p} \sum_{\theta} Z_{pp}(\theta) e^{-jq\theta} \right.$$

$$+ B_{pq,(p-2)} \sum_{\theta} Z_{(p-2)(p-2)}(\theta) e^{-jq\theta} + \dots + B_{pq,k} \sum_{\theta} Z_{kk}(\theta) e^{-jq\theta} + \dots + B_{pq,q} \sum_{\theta} Z_{qq}(\theta) e^{-jq\theta} \Big). \quad (3.5)$$

This equation will further reduce the computation time if the following conditions are adopted:

- (1) The factors $Z_{kk}(p)$ need to be computed only once, and then multiplied by $\cos(q\theta) + j \sin(q\theta)$.
- (2) No multiplication of $e^{-jq\theta}$ for Z_{p0} moments.
- (3) For large p and q , the summation $\sum Z_{pp}(\theta) e^{-jq\theta}$ can be evaluated using the fast Fourier transform.

The conditions (1)–(3) shorten the time taken to evaluate the Zernike moments as compared to that of Direct method. However, the total number of multiplications involved in computing the radial polynomials of order p is still the same as Eq. (3.1) and $O(p^3)$. This is mainly due to the factorial terms in the polynomial coefficients. Belkasim's method is limited to obtain only a single $R_{pq}(r)$ each time through the loop. Similar to Direct method, moments computed using Eq. (3.5) will also have large variation in the values for different orders of p . It may also be necessary to develop methods for avoiding numerical instabilities when the image size is large.

3.3. Prata's method

Prata has proposed a recurrence relation that uses radial polynomials of lower order p , i.e. $R_{(p-1)(q-1)}(r)$ and $R_{(p-2)q}(r)$ to derive $R_{pq}(r)$. The equation is shown below and can be utilized for fast computation of Zernike moments [10]

$$R_{pq}(r) = L_1 R_{(p-1)(q-1)}(r) + L_2 R_{(p-2)q}(r), \quad (3.6)$$

where the coefficients L_1 and L_2 are given by

$$L_1 = \frac{2rp}{p+q}, \quad L_2 = -\frac{p-q}{p+q}.$$

It is quite evident from the equation that not all cases of p and q can be used to compute the radial polynomial. It is not possible to use Prata's equation in cases where $q=0$ and $p=q$. As a result, terms such as R_{00} , R_{11} , R_{22} , R_{20} , etc. must be obtained using other methods. To illustrate the recursive algorithm, an example to compute the radial polynomials of order 8 is diagrammatically shown in Fig. 2. As shown in Fig. 2, the Direct method is used in cases where $q=0$, whereas the equation $R_{pp}(r) = r^p$ is used for $p=q$. For other than these two cases, the Prata's equation in (3.6) is used. The linear relationship of Prata's recurrence relation in Eq. (3.6) enables the higher-order polynomials to be derived from the lower-order polynomials. However, the

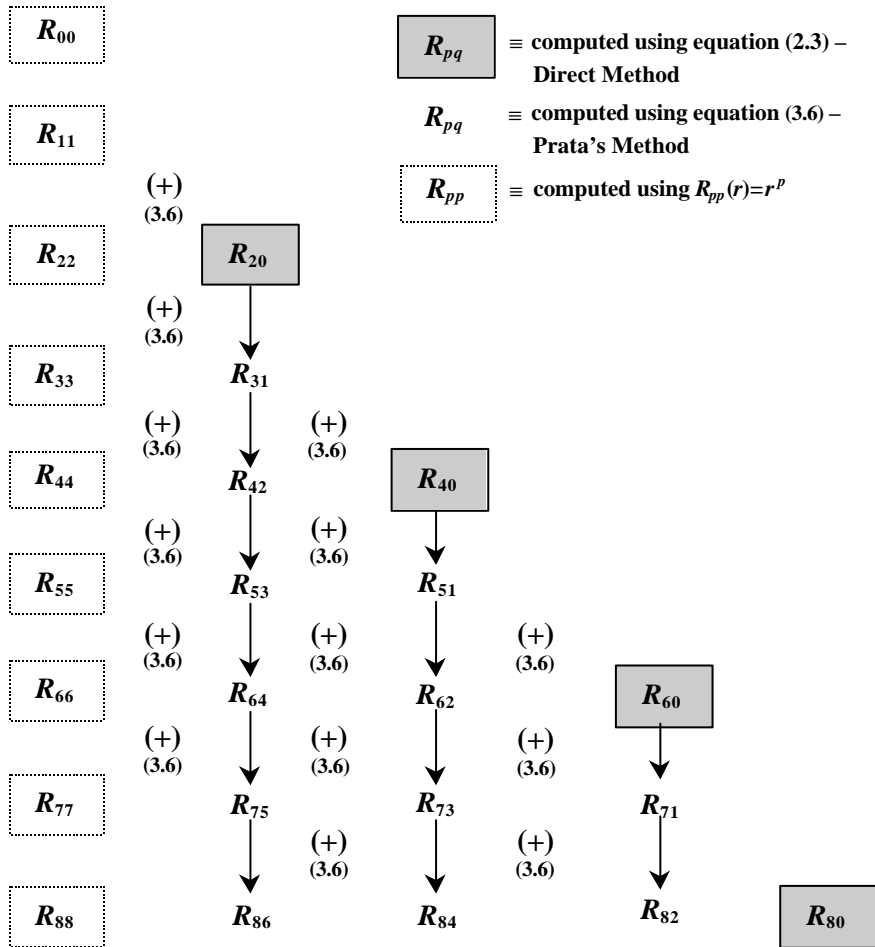


Fig. 2. Computational flow of Prata's method.

usage of Direct method for radial polynomials with $q = 0$ will considerably increase the computation time especially when p is large. This can be seen from the total number of multiplications used by Prata's method, Direct method and $R_{pp}(r)=r^p$ to compute the radial polynomials of orders $\leq p$

$$T(\leq p) \approx \begin{cases} \frac{3p^2}{4} + p(p+2) & \text{for } p = \text{even}, \\ \frac{3p^2+1}{4} + (p-1)(p+1) & \text{for } p = \text{odd}. \end{cases} \quad (3.7)$$

The total number of multiplications involved for orders $\leq p$ is $O(p^2)$. The first term of the RHS of Eq. (3.7) represents the total number of multiplications involved for Prata's method and $R_{pp}(r)$. The second term of RHS of Eq. (3.7) represents the multiplications used in Direct method. If we compute 48 orders of radial polynomials, the computation

will take 4128 multiplications, and 60% of them is just used for Direct method to compute the factorial functions! The involvement of the factorial terms significantly delays the moment computation especially when p is large. Apart from this, Prata's method is not able to derive the entire set of any order of Zernike moments independently due to the interdependency of various orders of the radial polynomials. The computational flow of Zernike radial polynomials must be performed in ascending order but not vice versa. This limitation causes the total number of multiplications needed to compute radial polynomials of order p is equal to total number of multiplications needed for radial polynomials of orders $\leq p$, i.e. $T(p) = T(\leq p)$!

3.4. Kintner's method

Kintner has proposed the following recurrence relation that uses polynomials of a varying low-order p with a fixed

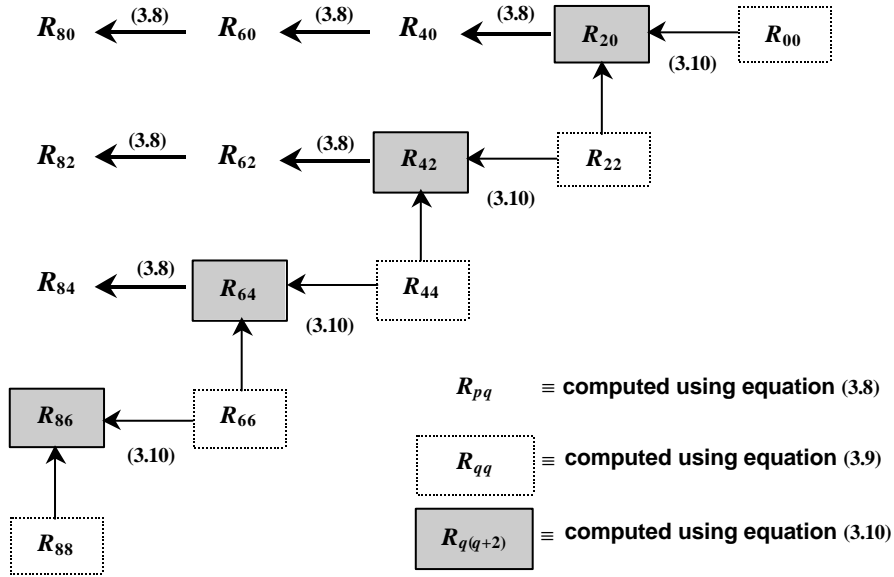


Fig. 3. Computational flow of modified Kintner's method.

index q to compute the radial polynomials $R_{pq}(r)$ [11]:

$$R_{pq}(r) = \frac{(K_2 r^2 + K_3)R_{(p-2)q}(r) + K_4 R_{(p-4)q}(r)}{K_1}, \quad (3.8)$$

where the coefficients K_1, K_2, K_3 and K_4 are given by

$$K_1 = \frac{(p+q)(p-q)(p-2)}{2},$$

$$K_2 = 2p(p-1)(p-2),$$

$$K_3 = -q^2(p-1) - p(p-1)(p-2),$$

$$K_4 = \frac{-p(p+q-2)(p-q-2)}{2}.$$

As can be seen from the equation, Kintner's method cannot be applied in cases where $p = q$ and $p - q = 2$. For these two cases, the normal approach is to use the Direct method, which takes too long to compute. We propose to use the following two relations to avoid the involvement of Direct method:

$$R_{qq}(r) = r^q \quad \text{for } p = q, \quad (3.9)$$

$$R_{(q+2)q}(r) = (q+2)R_{(q+2)(q+2)}(r) - (q+1)R_{qq}(r) \quad \text{for } p - q = 2. \quad (3.10)$$

This improved version of Kintner's method is denoted as modified Kintner's method. The total number of multiplications used in modified Kintner's method to compute the radial polynomials of orders $\leq p$ is given as follows:

$$T(\leq p) \approx \begin{cases} \frac{3p^2 + 2p}{4} & \text{for } p = \text{even}, \\ \frac{3p^2 + 2p - 1}{4} & \text{for } p = \text{odd}. \end{cases} \quad (3.11)$$

As can be seen from Eq. (3.11), the total number of multiplications involved to compute the radial polynomials of orders $\leq p$ is $O(p^2)$. Using an example of order 8, the effectiveness of the modified Kintner's method is illustrated in Fig. 3. This method provides a recursive scheme over the index p except for cases when $(p - q < 4)$. As a result, $R_{00}, R_{22}, R_{20}, R_{42}$, etc. are obtained using Eqs. (3.9) and (3.10) and these are indicated using shaded and dotted textboxes in Fig. 3. These initial values are then used to derive the remaining polynomials of orders $(p - q \geq 4)$ using Eq. (3.8). It can be seen that there is no factorial function involved in the modified Kintner's method. This method is useful in applications when q is fixed. For instance in optics, modified Kintner's method can be used to analyze the radial function with fixed azimuthal frequency q [10]. The evaluation for power series of radius, r^p , is not required. Moments computed using Eq. (3.8) will therefore avoid large variation in the dynamic range of values for different orders of p . Similar to Prata's algorithm, this method cannot be used to compute any single order radial polynomial, $R_{pq}(r)$ without using the lower order radial polynomials.

For example to compute R_{82} , we need to use the values of R_{22}, R_{42}, R_{44} and R_{62} and its computational flow must be carried out from the lowest order until the highest, but not vice versa. As a result, the total number of multiplications used to derive a single order of radial polynomials of order p is still $O(p^2)$, which is defined as follows:

$$T(p) \approx \begin{cases} \frac{3p(p+2)}{8} & \text{for } p = \text{even}, \\ \frac{(3p+1)(p+1)}{8} & \text{for } p = \text{odd}. \end{cases} \quad (3.12)$$

Table 1
Computational flow of coefficients of radial polynomials

R_{pp}	$R_{p,p-2}$	$R_{p,p-4}$
$B_{ppp}=1$ \downarrow (3.15) $B_{p,p-2,p}$ \downarrow (3.16) $B_{p,p-2,p-2}$ \downarrow (3.15) $B_{p,p-4,p}$	$B_{p,p-2,p-2}$ \downarrow (3.16) $B_{p,p-4,p-2}$ \downarrow (3.16) $B_{p,p-4,p-4}$	$B_{p,p-4,p-4}$ \downarrow (3.16) $B_{p,p-4,p-2}$ \downarrow (3.16) $B_{p,p-4,p}$
B_{ppp}	$B_{p,p-2,p}$ $B_{p,p-2,p-2}$	$B_{p,p-4,p}$ $B_{p,p-4,p-2}$ $B_{p,p-4,p-4}$

3.5. Coefficient method

In this section, we present a method for recursive calculation of Zernike polynomial coefficients, denoted as Coefficient method, specifically for fast computation of Zernike moments [12]. This method removes the factorial functions contained in the polynomial coefficients and shortens the computation time to obtain the Zernike moments. If we apply the transformation of the index $k \leftarrow (p - 2k)$ in Eq. (2.3), then it can be rewritten in the following form:

$$R_{pq}(r) = \sum_{k=q}^p B_{pqk} r^k \quad (3.13)$$

and the recurrence relations for the coefficients B_{pqk} are derived

$$B_{ppp} = 1, \quad (3.14)$$

$$B_{p(q-2)p} = B_{pqp} \frac{p+q}{p-q+2}, \quad (3.15)$$

$$B_{pq(k-2)} = -B_{pqk} \frac{(k+q)(k-q)}{(k+p)(p-k+2)}. \quad (3.16)$$

The computational flow of p th-order Zernike radial polynomials using the above equations are schematically tabulated in Table 1. According to Table 1, the computational flow starts from $B_{ppp}(=1)$ and proceeds to obtain $B_{p(p-2)p}$ and $B_{p(p-2)(p-2)}$. Using this relationship, another set of the coefficients $B_{p(p-4)p}$, $B_{p(p-4)(p-2)}$ and $B_{p(p-4)(p-4)}$ is obtained. It continues to generate the coefficients of p th-order radial polynomials until the conditions of Eq. (3.13) are met.

Fig. 4 shows how to derive the entire set of polynomial coefficients of order 8 using Coefficient method. Starting from the initial value of $B_{888} = 1$ using Eq. (3.14), the coefficients B_{868} , B_{848} , B_{828} and B_{808} can be recursively computed using Eq. (3.15). Eq. (3.16) is then used to generate the remaining coefficients B_{866} , B_{846} , B_{844} , B_{826} , B_{824} , B_{822} , B_{806} , B_{804} , B_{802} and B_{800} . The figure shows that the computation of the 8th-order polynomials requires fifteen coefficients B_{pqk} , and five polynomials $R_{pq}(r)$. In general, the size of B_{pqk} , and

$R_{pq}(r)$ can be determined from the equations shown below:

$$|B_{pqk}| = \begin{cases} \frac{(p+2)(p+4)}{8} & \text{for } p = \text{even}, \\ \frac{(p+1)(p+3)}{8} & \text{for } p = \text{odd}, \end{cases} \quad (3.17)$$

$$|R_{pq}| = \begin{cases} \frac{p+2}{2} & \text{for } p = \text{even}, \\ \frac{p+1}{2} & \text{for } p = \text{odd}. \end{cases} \quad (3.18)$$

This recurrence relation is free from any factorial terms. The whole set of Zernike radial polynomials of order p namely $R_{pp}(r)$, $R_{p(p-2)}(r)$, $R_{p(p-4)}(r)$, \dots and $R_{p1}(r)$ (p is odd) or $R_{p0}(r)$ (p is even) can be computed without relying on lower- or higher-order polynomials. This feature enables the entire set of any p th order of Zernike moments to be obtained independently. Each order of Zernike moments can be computed in parallel. The overall speed will be considerably improved as the number of parallel computing elements increases. Coefficient method can also be efficiently used to obtain a single $R_{pq}(r)$ without depending on lower or higher order of radial polynomials.

However, the Coefficient method still needs to evaluate each and every coefficient and power of radius for each radial polynomial. The size of B_{pqk} , $|B_{pqk}|$, increases with a rate of p^2 as the order p increases. Thus, the overall computation time for Coefficient method is still long especially when the order p is large. This can be seen from its total number of multiplications needed to compute the entire set of radial polynomials of order p defined as below:

$$T(p) = \sum_{l=0}^p \frac{(p-l+2)(p+l)}{4}, \quad p-l = \text{even}. \quad (3.19)$$

The number of multiplications is still $O(p^3)$ even though all the factorial functions have been completely removed from the polynomial coefficients. It is also difficult for the Coefficient method to derive a set of Zernike moments of fixed index q with varying order p as in Kintner's method.

4. Proposed method to compute Zernike moments

In this section, we introduce a new recurrence relation, denoted as q -recursive method, for fast computation of Zernike moments. The q -recursive method uses Zernike radial polynomials of fixed order p with higher index q to derive the polynomial of the lower index q without computing the polynomial coefficients and the power series of radius. The recurrence relation and its coefficients are given as follows:

$$R_{p(q-4)}(r) = H_1 R_{pq}(r) + \left(H_2 + \frac{H_3}{r^2} \right) R_{p(q-2)}(r), \quad (4.1)$$

where the coefficients H_1 , H_2 and H_3 are given by

$$H_1 = \frac{q(q-1)}{2} - qH_2 + \frac{H_3(p+q+2)(p-q)}{8},$$

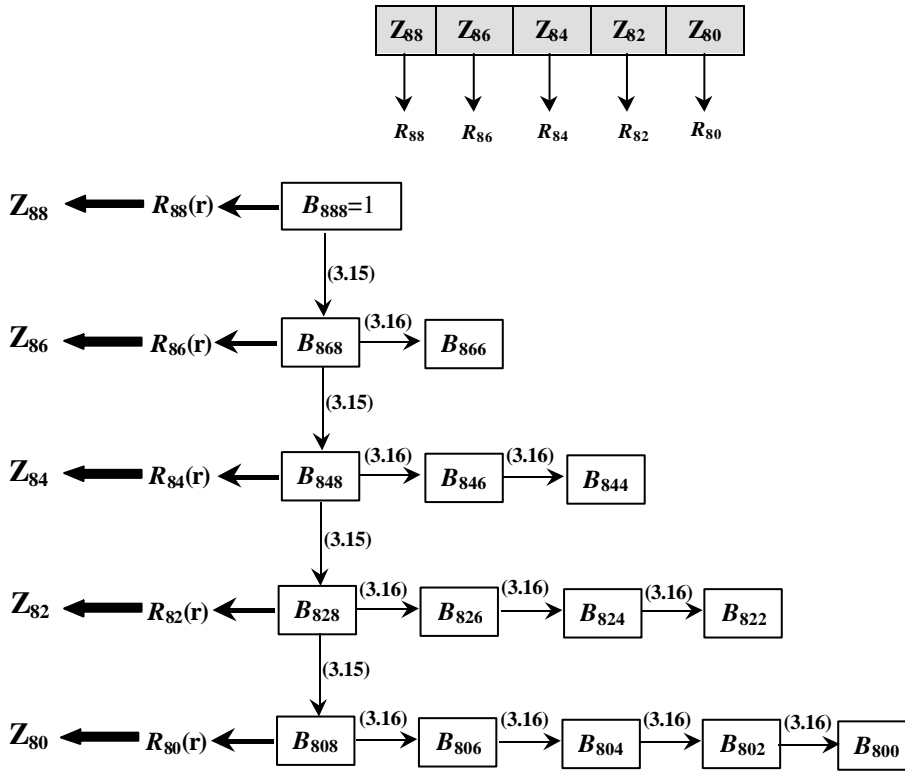


Fig. 4. Computational flow of radial polynomials using Coefficient method.

$$H_2 = \frac{H_3(p+q)(p-q+2)}{4(q-1)} + (q-2),$$

$$H_3 = \frac{-4(q-2)(q-3)}{(p+q-2)(p-q+4)}.$$

This equation is not applicable in cases where $p = q$ and $p - q = 2$. For these cases, the following equations are used:

$$R_{pp}(r) = r^p \quad \text{for } p = q, \quad (4.2)$$

$$R_{p(p-2)}(r) = pR_{pp}(r) - (p-1)R_{(p-2)(p-2)}(r) \quad \text{for } p - q = 2. \quad (4.3)$$

The total number of multiplications used to compute the entire set of radial polynomials of order p is given as follows:

$$T(p) \approx \begin{cases} \frac{5p-6}{2} & \text{for } p = \text{even}, \\ \frac{5p-7}{2} & \text{for } p = \text{odd}. \end{cases} \quad (4.4)$$

The number of multiplications for a single order of radial polynomials is merely $O(p)$. It implies that the time taken to compute the Zernike moments using the q -recursive method is significantly reduced as compared to its corresponding $T(p)$ of the present methods. Using an example of order 8, the computational flow of q -recursive method is illustrated

in Fig. 5. The recurrence relation in Eq. (4.1) provides a recursive scheme over index q with fixed order p except for cases where $(p - q \leq 2)$. Therefore, the initial values $R_{pp}(r)$ and $R_{p(p-2)}(r)$ such as $R_{88}(r)$, $R_{86}(r)$, $R_{66}(r)$, $R_{64}(r)$, etc. are obtained from relations (4.2) and (4.3). These initial values enable the remaining polynomials of $(p - q > 2)$ to be recursively computed using Eq. (4.1). The q -recursive method uses the radial polynomials of higher q index to derive the polynomials of lower q index. Because of this, the computation of polynomial coefficients, which are dependent on factorial functions, and the power of radius, r^k for each radial polynomial are not required. Using the q -recursive method, individual p th-order moments can be calculated without employing higher or lower order moments. This allows the entire set of any order of Zernike moments to be derived independently and concurrently in a single iteration through the loop, which is not possible with the existing methods. This feature is useful for applications where only selected orders of Zernike moments are needed as pattern features, and fast computation is required. The moment computation can be carried out in ascending order or descending order. The q -recursive is ideal for parallel computation. This is because each order of Zernike moments can be computed independently.

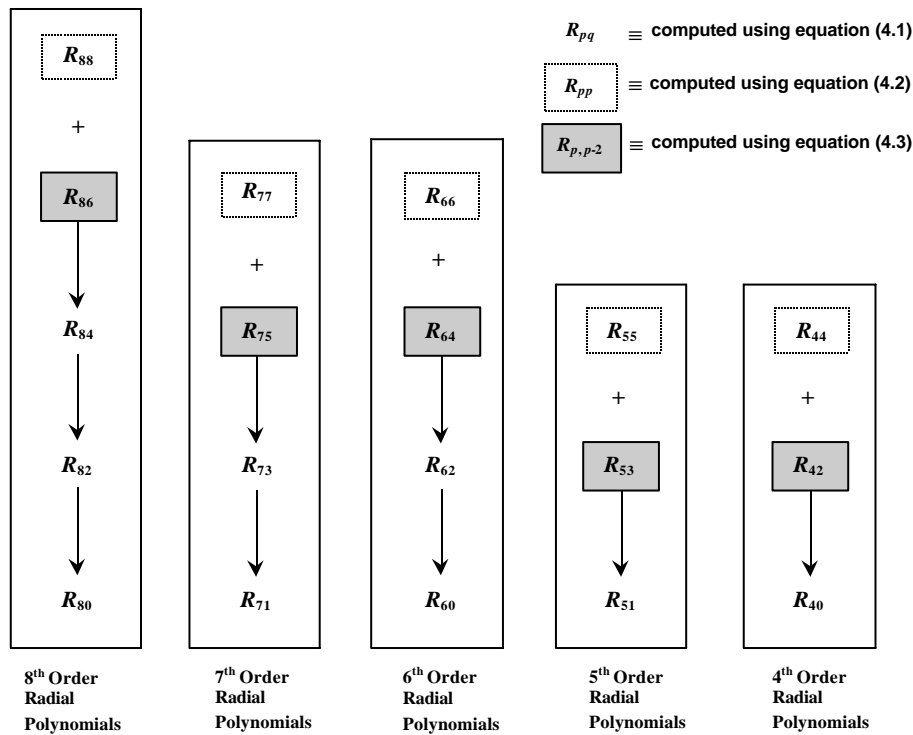


Fig. 5. Computational flow of q -recursive method.

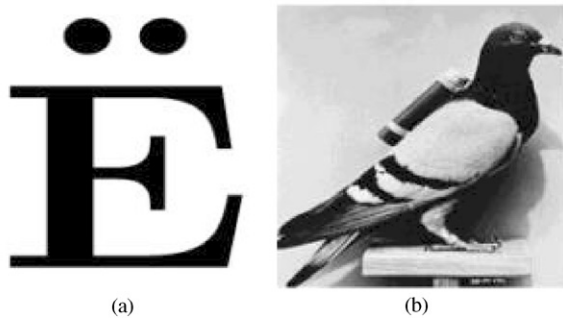


Fig. 6. (a) 50×50 binary latin character, (b) 50×50 gray level real image.

5. Experimental study

In this experimental study, a binary Latin character and a grayscale real image with 50×50 resolutions as shown in Fig. 6(a) and (b) are used in the computation of Zernike moments using the present and the proposed methods. They comprise of Direct, Belkasim's, Prata's, Kintner's, Coefficient methods, modified Kintner's method and q -recursive method. Two experiments are carried out to determine the time taken to compute the Zernike moments. In the first experiment, the entire set of Zernike moments from 0 to 24 orders, 0 to 36 orders and 0 to 48 orders are computed.

Table 2
The CPU elapsed time (s) for computation of Zernike moments

Maximum order	Direct method	Belkasim's method	Prata's method	Kintner's method	Coefficient method	q -Recursive method & modified Kintner's method
(a) Using binary image						
$N_{\text{MAX}} = 24$	1.10	0.55	0.16	0.12	0.27	0.05
$N_{\text{MAX}} = 36$	3.74	1.60	0.33	0.20	0.82	0.11
$N_{\text{MAX}} = 48$	9.39	4.07	0.55	0.32	1.87	0.16
(b) Using grayscale real image						
$N_{\text{MAX}} = 24$	8.19	3.29	0.99	0.66	2.09	0.50
$N_{\text{MAX}} = 36$	28.07	11.04	2.14	1.37	6.26	0.88
$N_{\text{MAX}} = 48$	70.36	28.34	3.90	2.41	13.84	1.38

Table 3

The CPU elapsed time (s) for individual order of Zernike moments using grayscale image

Order, p	Prata's method	Kintner's method	Coefficient method	Modified Kintner's method	q -Recursive method
24	0.99	0.33	0.22	0.25	0.05
36	2.14	0.70	0.50	0.44	0.06
48	3.90	1.21	0.77	0.70	0.07

Table 4

The summary of features for various methods to compute Zernike radial polynomials

Features	Direct method	Belkasim's method	Prata's method	Kintner's method	Coefficient method	Modified Kintner's method	q -Recursive method
To obtain a single $R_{pq}(r)$	Very slow	Slow	Moderate	Quite fast	Quite moderate	Very fast	Very fast
To obtain the entire set of radial polynomials of order p with varying q without using lower or higher order p	Very slow	Slow	Not possible	Not possible	Fast	Not possible	Very fast
To obtain the entire set of radial polynomials of fixed index q with varying order p , i.e. $R_{q+4,q}(r)$, $R_{q+2,q}(r)$, $R_{qq}(r)$, etc.	Very slow	Slow	Moderate	Fast	Quite moderate	Very fast	Quite fast
Factorial functions involved in overall computation	Many	Many	Few	Free	Free	Free	Free
To obtain the entire set of radial polynomials of orders $\leq p$, i.e. p th, $(p-1)$ th, $(p-2)$ th, ..., 0th	Very slow	slow	Moderate	Quite fast	Quite moderate	Very fast	Very fast
Flow of computation	Any order	Any order	Ascending order	Ascending order	Any order	Ascending order	Any order
Time complexity for specific order p	$O(p^3)$	$O(p^3)$	$O(p^2)$	$O(p^2)$	$O(p^3)$	$O(p^2)$	$O(p)$

Evaluation of relative speeds: Very slow—the computation is very slow; Slow—the computation is slow; Quite moderate—the computation is quite moderate; Moderate—the computation is moderate; Quite fast—the computation is quite fast; Fast—the computation is fast; and Very fast—the computation is very fast.

In the second experiment, the computation is only done for the selected orders of 24, 36 and 48.

In the first experiment, the present and proposed methods are used to computing the Zernike moments of the binary and grayscale images, and the results are tabulated in Table 2(a) and (b), respectively. In all the three cases, the q -recursive method and modified Kintner's method perform equally well. They take the shortest time to calculate the Zernike moments followed by Kintner's method for both binary and grayscale images. As can be seen in Table 2(a) and (b), they take only half the time of Kintner's method,

for all the three cases. For a maximum p order of 48, the q -recursive method and modified Kintner's method take only 0.16 and 1.38 s followed by Kintner's method which takes 0.32 and 2.41 s for both binary and grayscale images, respectively. The computation speed of modified Kintner's method has improved 50% over that of Kintner's method. For the same case, Direct method takes 9.39 and 70.36 s, respectively, which is the longest time.

In the second experiment, only three present methods and two proposed methods are selected. They are Prata's, Kintner's, Coefficient methods, modified Kintner's method and

q -recursive method. The time taken to compute the Zernike moments of individual orders of 24, 36 and 48 using the grayscale image is tabulated in Table 3. The q -recursive method takes the shortest time for all the three cases. It can be seen from Table 3 that the time taken by q -recursive method to compute the 24th, 36th and 48th order of Zernike moments is 0.05, 0.06 and 0.07 s, respectively. For the same orders, the next best performer, the modified Kintner's method takes 0.25, 0.44 and 0.70 s and Prata's method takes the longest time for all the three cases. The rate of increase in computation time as the order increases is minimal in the case of the q -recursive method. It takes between five to ten times less time than the next best performer to compute 24th, 36th and 48th order of Zernike moments.

Table 4 shows the advantages and limitations of the present and proposed methods to compute Zernike moments. As shown in the Table 4, seven different speeds are categorized to describe the relative computation time of the present and proposed methods. The seven mentioned categories are "Very Fast", "Fast", "Quite Fast", "Moderate", "Quite Moderate", "Slow" and "Very Slow". The category "Very Fast" is the fastest and the "Very Slow" is the slowest.

6. Concluding remarks

In this paper, we have presented a detailed analysis on the present methods to compute Zernike moments based on Zernike radial polynomials. We have discussed the advantages as well as the limitations of Direct, Belkasim's, Prata's, Kintner's and Coefficient methods. We have proposed to use a couple of relations to improve the computation speed of Kintner's method. We have also proposed a new method denoted as q -recursive method for fast computation of Zernike moments. The q -recursive method uses Zernike radial polynomials of fixed order p with higher index q to derive the polynomial of the lower index q . It does not use any factorial terms. It enables the entire set of any fixed order p with varying q of Zernike moments to be obtained without using lower or higher orders of p . This feature is useful for applications that require only selected orders as pattern features. The proposed algorithm is ideal for parallel processing environment. The performance of each method has been experimentally analyzed by taking their respective CPU elapsed time to calculate the Zernike

moments using binary and grayscale images. Two experiments are carried out to determine the time taken to compute a single order and a set of orders of Zernike moments. In both experiments, the q -recursive method takes the shortest time to compute the Zernike moments.

References

- [1] R.J. Prokop, A.P. Reeves, A survey of moment based techniques for unoccluded object representation, *Graphical Models Image Process.* 54 (5) (1992) 438–460.
- [2] Y.S. Abu Mostafa, Recognitive aspects of moment invariants, *IEEE Trans. Pattern Anal. Mach. Intell.* 6 (6) (1984) 698–706.
- [3] M.R. Teague, Image analysis via the general theory of moments, *J. Opt. Soc. Am.* 70 (8) (1980) 920–930.
- [4] C.H. Teh, R.T. Chin, On image analysis by the method of moments, *IEEE Trans. Pattern Anal. Mach. Intell.* 10 (4) (1988) 496–513.
- [5] S.O. Belkasim, Pattern recognition with moment invariants—a comparative study and new results, *Pattern Recognition* 24 (12) (1991) 1117–1138.
- [6] A. Khotanzad, Invariant image recognition by Zernike moments, *IEEE Trans. Pattern Anal. Mach. Intell.* 12 (5) (1990) 489–497.
- [7] A. Khotanzad, Y.H. Hong, Rotation invariant image recognition using features selected via a systematic method, *Pattern Recognition* 23 (10) (1990) 1089–1101.
- [8] R. Mukundan, K.R. Ramakrishnan, Fast computation of Legendre and Zernike moments, *Pattern Recognition* 28 (9) (1995) 1433–1442.
- [9] S.O. Belkasim, M. Ahmadi, M. Shridhar, An efficient algorithm for fast computation of Zernike moments, *Circuits and Systems, IEEE 39th Midwest Symposium*, Vol. 3, 1996, pp. 1401–1404.
- [10] A. Prata, W.V.T. Rusch, Algorithm for computation of Zernike polynomials expansion coefficients, *Appl. Opt.* 28 (1989) 749–754.
- [11] E.C. Kintner, On the mathematical properties of the Zernike polynomials, *Opt. Acta* 23 (8) (1976) 679–680.
- [12] C.W. Chong, R. Mukundan, P. Raveendran, An efficient algorithm for fast computation of Zernike moments, *International Conference on Computer Vision, Pattern Recognition and Image Processing*, Durham, NC, 2002, accepted.
- [13] R. Mukundan, K.R. Ramakrishnan, *Moment Functions in Image Analysis—Theory and Applications*, World Scientific, Singapore, 1998.

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