Managing the expectations of buyers with reference-dependent preferences

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Abstract

I consider a model of monopoly pricing where a firm makes a price offer to a buyer with reference-dependent preferences without being able to commit to it. The reference point is the ex-ante probability of trade and the buyer exhibits an attachment effect: the higher his expectations to buy, the higher his willingness-to-pay. When the buyer's valuation is private information, a unique equilibrium exists where the firm plays a mixed strategy and its profits are the same as in the reference-independent benchmark. The equilibrium always entails inefficiencies: even as the firm's information converges to complete information, it mixes on a non-vanishing support and the probability of no trade is greater than zero. Finally, I show that when the firm can design a test about the buyer's valuation, it can do strictly better than in the reference-independent benchmark by leveraging the uncertainty generated by a noisy test.

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The purpose of this paper is to study a monopoly pricing model where the buyer exhibits a specific type of reference-dependent preferences. I consider a buyer that values a good more when he expects to own it. For instance, a buyer expecting to own a specific car or house can get emotionally attached to it and finds it then harder to walk away from an offer. A job applicant, expecting to be employed, can build expectations about the prospects of a different lifestyle or social status, which can reduce his willingness to refuse an offer. More generally, this attachment effect is an "expectation-based endowment effect" and is a prediction of expectation-based reference-dependent preferences (Kőszegi and Rabin, 2006).

When can a firm benefit from facing a buyer with an attachment effect and what is its consequence on a firm's pricing strategy? To study this question, I adapt the monopoly model of Heidhues and Kőszegi (2014). In their model, a firm sells an indivisible good to a buyer by making a take-it-or-leave-it offer. To capture the attachment effect, the buyer's willingness-to-pay (WTP) increases linearly in the ex-ante probability of trading. Following the literature on expectation-based reference-dependent preferences, the buyer plays a Preferred Personal Equilibrium: when setting expectations, he correctly anticipates the firm's strategy and his own action and selects the most favourable plan of action.[1]

Importantly, in Heidhues and Kőszegi (2014), the firm can commit to a possibly random price offer distribution. They show that the optimal strategy is to randomise over prices. The low prices in the support ensure that the buyer expects to buy with positive probability. This increases the WTP through the attachment effect. The higher prices in the support exploit this higher WTP to increase profits. However, this strategy is not consistent with equilibrium behaviour when the firm cannot commit to a random price strategy.

In this paper, I characterise the firm's pricing strategy when it cannot commit to it. This model has two main features. First, like in Heidhues and Kőszegi (2014), the demand is endogenous: the probability of buying and hence the buyer's WTP depend on the firm's strategy. Second, unlike Heidhues and Kőszegi (2014), the firm has a commitment problem. There is a tension between offering low prices to induce expectations to buy and high prices to take advantage of a higher WTP. I characterise the equilibrium strategy under three different information environments: (1) the valuation is the buyer's private information, (2) the buyer's valuation is common knowledge, and (3) the firm can learn about the valuation.

Proposition 2 characterises the unique equilibrium of the game when valuations are private information. In equilibrium, the firm chooses the mixed strategy such that the resulting de-

[1] See for example Heidhues and Kőszegi (2014), Kőszegi and Rabin (2009), Herweg and Mierendorff (2013), Rosato (2016) or Macera (2018) for papers using this selection.

mand is unit-elastic on the support. That is, it creates the demand that makes it indifferent between any price on the support. This preserves the incentives for the mixed strategy and solves the firm's commitment problem. However, to induce this endogenous unit-elastic demand, different types must trade with different probability. This means that the equilibrium entails inefficiencies: the probability of no trade is always strictly greater than zero.

Moreover, the firm does not benefit from the attachment effect and equilibrium profits are independent of its strength. Indeed, types below the support of the mixed strategy only face prices above their valuation in equilibrium. By the PPE requirement, they cannot expect to buy and behave as if they have no attachment effect. This implies that the profits from the lowest price on the support must be the same as in the reference-independent benchmark[2], pinning down equilibrium profits. The commitment problem created by the attachment effect has thus two effects here: the firm must use random prices to overcome it and it does not benefit from having buyers with reference-dependent preferences.

In section 2.1, I characterise the firm's pricing strategy when it knows the buyer's valuation in two different ways, and get contrasting results. First, I use the incomplete information characterisation to study convergence to complete information. As the distribution over types concentrates on a singleton, the equilibrium strategy converges to a mixed strategy on a non-vanishing support and the limit probability of no trade is bounded away from zero. This follows from the incomplete information characterisation. In order to create a unit-elastic demand, the firm induces a large variation in the trading probability of almost identical types. This results in a positive probability of no trade. This problem is more severe for a stronger attachment effect: the probability of trading converges to zero as the attachment effect grows large.

However, there is a discontinuity at the limit: when the valuation is common knowledge, no equilibrium exists. Indeed, it is impossible for the firm to overcome its commitment problem. Any pricing strategy where the buyer is willing to accept increases his WTP above the price played in equilibrium.[3]

The first two sets of results consider extreme information structures: either incomplete information, where the firm does not benefit from the attachment effect or complete information, where no equilibrium exists. In section 3, I look at the intermediate case where the firm is partially informed about the buyer's valuation. Specifically, I allow the firm to design a test

[2]Throughout the paper, the reference-independent model refers to an equivalent model with no attachment effect, i.e., the WTP is the same as the valuation.

[3]Existence issues with PPE in strategic settings were already pointed out by Dato et al. (2017).

to learn about the buyer's valuation. The idea behind this section is to explore whether the firm can use the buyer's expectations about his performance on the test to take advantage of the attachment effect. For example, suppose that a firm designs a screening process before making a wage offer to a candidate.[4] The candidate will use his performance during the screening process to assess what his chances are of getting a high wage. If the candidate believes he did well on the test designed by the firm, he will expect high wages and thus has a high probability of accepting the job. If the candidate exhibits an attachment effect, this would weaken his bargaining position: he would be willing to accept lower wage offers to avoid the disappointment of not being employed. This could then be used by the firm to offer lower wages.

In this new environment, the firm first designs a publicly observed test, privately observes a signal realisation and then makes an offer. In Proposition 6, I show that the firm can be better off with a noisy test. Intuitively, when the test is noisy, the firm can create random price offers like in Heidhues and Kőszegi (2014). At the same time, given the uncertainty generated by the test, the firm can credibly offer low prices after a low signal and high prices after a high signal. This last section shows that in the presence of an attachment effect, a monopolist has an incentive to design imperfect tests. This allows the buyer to credibly entertain the idea that he will get a low price. The firm can then use these expectations to make higher profits. Another the way to put this is that the firm is better off by not perfectly price discriminate. In Proposition 7, in the case of uniform distributions, I characterise the firm's optimal testing strategy.

Relation to the literature

This paper is part of the literature that studies the implication of rational expectations as the reference point in reference-dependent preferences, following Kőszegi and Rabin (2006). The two closest papers in this literature are Heidhues and Kőszegi (2014) and Eliaz and Spiegler (2015).

In Heidhues and Kőszegi (2014), the firm can commit to a price distribution and the valuation is common knowledge. In contrast, I consider a firm that cannot commit to a price distribution and the buyer's valuation is private information. I show that inefficiencies are a general feature of this model and that, in the pricing model, the firm does not benefit from facing a

[4]The model is set up as a buyer-seller interaction. It can be easily rewritten as a firm making a wage offer to a candidate with an unknown outside option.

buyer with an attachment effect. Moreover, an equilibrium does not exist when the valuation is common knowledge. This shows that the commitment assumption is key to both existence and benefiting from the attachment effect. Noisy testing allows the firm to take advantage of the attachment effect. It is worth noting that my model also predicts random prices like Heidhues and Kőszegi's (2014).

Eliaz and Spiegler (2015) look at a more abstract model that nests the complete information environment with commitment of this paper as a special case. They show that uniqueness of the PE can be guaranteed through a first-order stochastic dominance property that is useful in this paper. This paper differs from the existing literature by not allowing the firm to commit to a price distribution. I show how to characterise the equilibrium pricing strategy by adapting the result of Eliaz and Spiegler (2015). I also show that an imperfect learning strategy can provide a foundation for the stochastic pricing strategy without commitment. Rosato (2016) also studies a monopoly pricing model where the uncertainty is used to exploit expectation-based reference-dependent preferences. There, the monopolist commits to the limited availability of substitutes to induce the expectation of buying.

The last section is related to the literature on optimal disclosure with a behavioural audiences as it is concerned with the design of the information environment with non-standard preferences, see e.g., Lipnowski and Mathevet (2018); Lipnowski et al. (2020); Levy et al. (2020). In particular, Karle and Schumacher (2017) study a model where a monopolist posts a public price as well as discloses a signal of the valuation of an initially uninformed buyer with expectation-based reference-dependent preferences. The firm benefits from imperfect disclosure when a low valuation is pooled with a high valuation. The buyer then expects to buy at the price posted and thus develops an attachment towards to good. In contrast, I consider a perfectly informed buyer and it is the firm that learns about the valuation. This has two implications. First, the price offered depend on the signal observed, so there is variation in price. Second, inducing expectations to buy is not enough for the firm to benefit from the attachment effect as this happens when prices are relatively low. So the firm must induce both high and low prices to benefit from it.[5]

Finally, there are links to the literature on optimal learning and price discrimination. Bergemann et al. (2015) characterise all the combination of consumers' surplus and monopoly

[5]Karle and Schumacher (2017) also show that the monopolist does not benefit from committing to its pricing strategy, unlike this paper. The type of commitment is however different: they consider a setting where the firm can commit to not change the price after the buyer has set expectations but they do not allow commitment to a random price strategy.

profit after some learning of the firm. I depart from their framework by introducing reference-dependent preferences. Where in their model the optimal learning strategy is to perfectly learn the valuation, introducing reference-dependent preferences incentivises the firm to create a stochastic environment. Roesler and Szentes (2017) and Condorelli and Szentes (2020) look at environments where an agent designs an optimal learning strategy taking into account the effect of information acquisition on the other agent's strategy. Here, the firm designs its optimal learning strategy taking into account its effect on the buyer's preferences.

1 The model

There is one firm and one buyer. The firm makes a take-it-or-leave-it offer $p \in \mathbb{R}$ for an indivisible good that the buyer can either accept, $a = 1$, or reject, $a = 0$. The distribution over prices resulting from the (potentially mixed) strategy of the firm is represented by the cdf F. The buyer has a reference-point $r \in \{0, 1\}$ and an exogenous valuation v. His payoffs are

$$u(p, v, a|r) = a(v - p) - \lambda \cdot v \cdot r(1 - a),$$

where $r = 1$ stands for "expecting to accept", and $r = 0$ for "expecting to reject". Here, the buyer "pays" a penalty λv whenever he rejects an offer he was expecting to accept. Like in Kőszegi and Rabin (2006), I allow the reference point to be stochastic. The reference point is then $q \in [0, 1]$ which stands for the probability of accepting. The utility of buyer v is written as

$$u(p, v, a|q) = q \cdot \big(a(v - p) - \lambda v \mathbb{1}[a = 0]\big) + (1 - q) \cdot a(v - p).$$

The firm's payoff is

$$\pi(p, a) = a \cdot p.$$

The buyer knows v. The firm only knows that $v \sim G$, where G denotes a cdf. It admits a strictly positive density g on the support $V = [\underline{v}, \overline{v}]$, $\underline{v} \geq 0$. Let $\Delta v = \overline{v} - \underline{v}$. I use γ to denote the probability measure associated with G: for any measurable set A, $Pr[v \in A] = \gamma(A)$. I will often refer to a valuation v as the buyer's type. I assume that there is a positive surplus with any type and so the assumption that the firm has no cost is a normalisation. A notation that I will follow throughout the paper is that when a capital letter represents a cdf, the lowercase letter represents the density, if it exists.

Buyer's behaviour Given his valuation v and his reference point q, the buyer's payoffs from accepting and refusing at price p are

$$u(p,v,a=1|q) = v - p,$$
$$u(p,v,a=0|q) = 0 - \lambda v \cdot q.$$

Therefore, he optimally plays a cutoff strategy: he accepts an offer p if and only if $p \leq v + \lambda vq$.[6] I denote the buyer's optimal strategy by $a^*(p,v|q) = \mathbb{1}[p \leq v + \lambda vq]$.

Following Kőszegi and Rabin (2006), the buyer forms his reference point based on the correct expectations of trading. I assume that the buyer first learns his type, then forms his expectations based on the price distribution F. The reference point is thus formed after learning his own type but before the price realisation. Therefore, different types can have different expectations of trading. A Personal Equilibrium (PE) is a reference point q such that the probability of trading is consistent with the optimal strategy given the reference point.

Definition 1. *Given a price distribution F, $(Q_v)_v$ is a profile of Personal Equilibria if for each $v \in V$, Q_v satisfies*

$$Q_v = \int_{\mathbb{R}} a^*(p,v|Q_v)dF(p)$$

and $a^*(p,v|Q_v) = \mathbb{1}[p \leq v + \lambda v Q_v] \in \arg\max u(p,v,a|Q_v)$.

In a PE, the buyer with valuation v correctly anticipates how his expectations change his strategy and how his strategy changes his expectations. The PE Q_v depends on the type but also on the distribution over prices. Therefore, the buyer's behaviour will depend directly on the firm's strategy.

The expected utility of type v, for a given PE Q_v and price distribution $F(p)$ is

$$W(v|F,Q_v) = \int_{-\infty}^{v+\lambda v Q_v} (v-p)dF(p) + \int_{v+\lambda v Q_v}^{+\infty} -\lambda v Q_v \, dF(p).$$

For any prices in $(-\infty, v + \lambda v Q_v]$, the buyer accepts the offer and gets a utility $v - p$. For prices larger than $v + \lambda v Q_v$, the buyer rejects the offer and gets a loss of $-\lambda v Q_v$.

Because there can be multiple PEs, I assume the buyer plays his Preferred Personal Equilibrium (PPE). The PPE is the Personal Equilibrium that gives the highest expected utility (Kőszegi and Rabin, 2006, 2007).

[6] Here, I assume that, when indifferent, the buyer accepts the price offer. Allowing for different strategies when indifferent could change the PPE outcome. However, one can show that it would not change the equilibrium strategies in this paper. Therefore, to simplify the exposition, I omit this possibility.

Definition 2. *Given a price distribution F, $(Q_v^*)_v$ is a profile of Preferred Personal Equilibria if for each $v \in V$, $Q_v^* \in \arg\max_{Q_v \in PE} W(v|F, Q_v)$.*

This (Personal) equilibrium selection is common in the literature using Personal Equilibria.[7] Its motivation is based on an introspection interpretation of the PE. The buyer can entertain multiple expectations of trading but cannot fool himself: his reference point must be correct given his optimal behaviour. Then, if he can "choose" amongst multiple reference points, he would choose the one with the highest expected utility.

It will be useful to think of the PE or PPE as the cutoff price it generates.

Definition 3. *Given a distribution over prices F and PE Q_v, the PE cutoff price of type $v \in V$ is $\hat{p}(v) = v + \lambda v Q_v$. Given PPE Q_v^*, the PPE cutoff price is $p^*(v) = v + \lambda v Q_v^*$.*

The PPE cutoff price determines buyer v's willingness-to-pay (WTP). Note also that we have $Q_v^* = F(p^*(v))$, as buyer v accepts any price below $p^*(v)$. In the rest of the paper, the valuation refers to a buyer's type v and his willingness-to-pay to his PPE cutoff price, $p^*(v)$.

Given a profile of PPE $(Q_v^*)_v$, let $V^*(p) = \{v \in V : p \leq p^*(v)\}$ be the set of types accepting price p. Define $v^*(p) = \inf\{v : v \in V^*(p)\}$, the lowest type in $V^*(p)$.

Equilibrium The firm's expected profits given the profile of PPE $(Q_v^*)_v$ are $\mathbb{E}[\pi(p)|(Q_v^*)_v] = p\,\gamma(V^*(p))$. I can now define an equilibrium in this model.

Definition 4. *A profile of strategy and reference points $\big(F(p), (Q_v^*)_v\big)$ is an equilibrium if for each $v \in V$, Q_v^* is type v's PPE given F and for each $p \in \operatorname{supp} F$, $p \in \arg\max_{\tilde{p}} \mathbb{E}[\pi(\tilde{p})|(Q_v^*)_v]$.*

In equilibrium, each buyer v forms his expectations based on the firm's equilibrium strategy and his type and the firm's strategy is a best response to the buyers' PPEs.

1.1 Comments

Utility function The utility function I use allows me to capture an attachment effect in the simplest possible way. With this utility function, the agent with valuation v pays a penalty λv weighted by the probability of accepting q when he does not accept the offer. The original

[7] See e.g., Heidhues and Kőszegi (2014), Herweg and Mierendorff (2013), Rosato (2016) or Macera (2018).

specification of Kőszegi and Rabin (2006) allows for a reference point that depends both on the distribution over consumption and price paid. Here, the utility function is similar to ones used in the literature with loss-aversion in one dimension only.[8] Having loss-aversion in one dimension only allows to cleanly isolate an effect of the reference-dependent preferences, for example aversion to price increases or in this case, the attachment effect.

The commitment assumption Whether commitment to a random pricing strategy is a reasonable assumption depends on the situation considered. When a patient firm posts publicly observed price, the commitment assumption can be justified by the incentives of a firm to develop a certain reputation for some price distribution. On the other hand, in many settings, prices are not directly observed. This is the case for example for goods or services that are the outcome of some bargaining or not often traded such as houses, cars or jobs. In this case, the take-it-or-leave-it bargaining structure captures a bargaining process where the firm has all the bargaining power.

1.2 Characterisation of the PPE

Proposition 1 establishes two properties of the PPE. First, the PPE cutoff is the smallest of the PE cutoffs. Second, it establishes that if $p^*(v)$ is the PPE cutoff then $F(p)$ must lie strictly above $\frac{p-v}{\lambda v}$ on $(-\infty, p^*(v))$. The proof of Proposition 1 also establishes existence of the PPE.

Proposition 1. *For a fixed type v and distribution F, these three statements are equivalent:*

- $p^*(v)$ *is the PPE cutoff price*
- $p^*(v) = \min\{p : F(p) = \frac{p-v}{\lambda v}\}$
- $v - p^*(v) = -\lambda v F\big(p^*(v)\big)$ *and for all* $p < p^*(v)$, $F(p) > \frac{p-v}{\lambda v}$

Moreover a PPE exists.

Proof. Fix a type v. First, I show that the PPE cutoff is the lowest PE cutoff. Fix two PE, Q_1,

[8] For example, section 4.1 in Heidhues and Kőszegi (2014), Herweg and Mierendorff (2013), Carbajal and Ely (2016), Rosato (2023) or Spiegler (2012).

Q_2 and their respective PE cutoffs, p_1, p_2. Then,

$$\begin{aligned} v - p_1 &= -\lambda v F(p_1) \\ \text{and } v - p_2 &= -\lambda v F(p_2). \end{aligned}$$

Note that we have $F(p_1) \neq F(p_2)$, for otherwise $p_1 = p_2$. The expected utility at PE Q_i is

$$W(v|Q_i) = \int_{-\infty}^{p_i} (v-p)dF(p) + \int_{p_i}^{+\infty} -\lambda v F(p_i) dF(p).$$

Using the equality defining the cutoff, p_1 is preferred to p_2 if and only if

$$\begin{aligned} \int_{-\infty}^{p_1} (v-p)dF(p) + (1-F(p_1))(v-p_1) &\geq \int_{-\infty}^{p_2} (v-p)dF(p) + (1-F(p_2))(v-p_2) \\ \Leftrightarrow \int_{p_1}^{p_2} p\, dF(p) &\geq (1-F(p_1))p_1 - (1-F(p_2))p_2 \\ \Leftrightarrow p_2 F(p_2) - p_1 F(p_1) - \int_{p_1}^{p_2} F(p)dp &\geq (1-F(p_1))p_1 - (1-F(p_2))p_2 \\ \Leftrightarrow p_2 - p_1 &\geq \int_{p_1}^{p_2} F(p)dp, \end{aligned}$$

where I obtain the third line by integrating by part. Because $F(p_1) \neq F(p_2)$, this is satisfied if and only if $p_1 < p_2$.

Now let $\tilde{p} = \inf\{p : \; F(p) \leq \frac{p-v}{\lambda v}\}$. If F is continuous at $\tilde{p}$, then $\inf\{p : \; F(p) \leq \frac{p-v}{\lambda v}\} = \min\{p : \; F(p) \leq \frac{p-v}{\lambda v}\}$. If F is not continuous at $\tilde{p}$, then because it is non-decreasing, $\lim_{p \nearrow \tilde{p}} F(p) < F(\tilde{p})$, which then contradicts that $\tilde{p} = \inf\{p : \; F(p) \leq \frac{p-v}{\lambda v}\}$.

Therefore, F is continuous at $\tilde{p}$ and $\min\{p : \; F(p) \leq \frac{p-v}{\lambda v}\}$ exists. Because F is continuous at $\tilde{p}$, it also implies that $\min\{p : \; F(p) \leq \frac{p-v}{\lambda v}\} = \min\{p : \; F(p) = \frac{p-v}{\lambda v}\}$ which establishes existence of the PPE and the first equivalence.

We can now show $p^*(v) = \min\{p : \; v - p = -\lambda v F(p)\} \Leftrightarrow F(p) > \frac{p-v}{\lambda v}$ for all $p < p^*(v)$ and $F(p^*(v)) = \frac{p^*(v)-v}{\lambda v}$.

$(\Rightarrow)$ Suppose $p^*(v)$ is a PPE and we have some $\hat{p} < p^*(v)$ with $F(\hat{p}) \leq \frac{\hat{p}-v}{\lambda}$. Because we have established that $\min\{p : \; F(p) \leq \frac{p-v}{\lambda v}\} = \min\{p : \; F(p) = \frac{p-v}{\lambda v}\}$, we get a contradiction.

$(\Leftarrow)$ If for all $p < p^*(v)$, $F(p) > \frac{p-v}{\lambda v}$, then there are no other PE smaller $p^*(v)$. $\square$

The PPE cutoff price is the smallest of the PE cutoff prices because for any distribution, this cutoff is weakly above the valuation v. Therefore, the lowest PE cutoff minimises trade

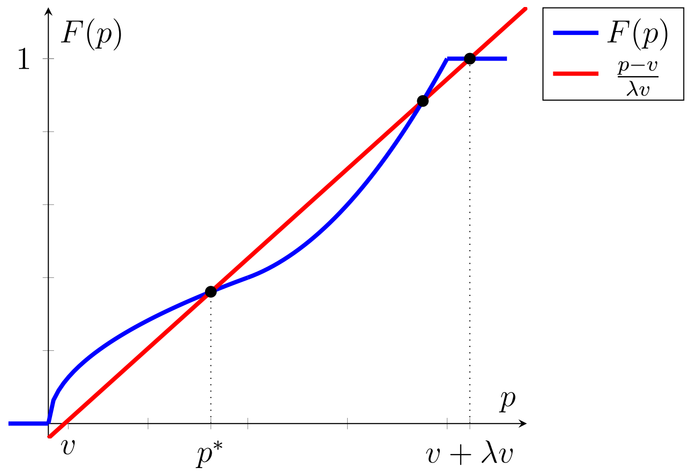

Figure 1: Each intersection of the blue and red curve is a PE. The lowest intersection, p^*, is the PPE.

when $p > v$, i.e., when the buyer has a negative utility. The condition that $F(p) > \frac{p-v}{\lambda v}$ when $p < p^*(v)$ was introduced by Eliaz and Spiegler (2015). This property is similar to the characterisation of first-order stochastic dominance albeit on only part of the support. In particular, it implies that for any F implementing PPE cutoff $p^*(v)$,

$$\int_{-\infty}^{p^*(v)} x\, dF(x) < \int_{v}^{p^*(v)} x \cdot \frac{1}{\lambda v} dx.$$

This observation will be useful in section 3 when we will design the firm's optimal testing strategy and I will refer to it as the FOSD interpretation of the PPE. Figure 1 illustrates graphically how to determine the PPE cutoff using Proposition 1. Intuitively, the prices below the PPE cutoff p^* are relatively low with high probability (compared to $\frac{p-v}{\lambda v}$) so the buyer can entertain expectations to buy at these prices. Whereas prices above p^* are relatively high with high probability (compared to $\frac{p-v}{\lambda v}$) so the buyer can expect to reject these prices.

2 Incomplete information

In this section, I first characterise the equilibrium when the valuation is private information. In section 2.1, I study the equilibrium when the set of types converges to a singleton and when the valuation is common knowledge. To simplify the analysis in this section, I restrict attention to strictly concave reference-independent profits.

Assumption 1. *The function $p(1 - G(p))$ is strictly concave.*

Assumption 1 is made to simplify the exposition of the paper. The results of this section extend qualitatively to more general distributions but the exact characterisation of the equilibrium differs. I comment after Proposition 2 on what would happen if we relax Assumption 1.

Proposition 2 characterises the unique equilibrium of the game. It shows that the firm plays a mixed strategy, the equilibrium demand is unit-elastic and the equilibrium profits are the same as in a reference-independent model.

Denote by $\pi^* = \max_p p(1-G(p))$ and $p_{ind} = \arg\max_p p(1-G(p))$, the equilibrium profits and price of the reference-independent benchmark.

Proposition 2. *There is a unique equilibrium* $\left(F, (Q_v^*)_v\right)$. *In equilibrium,*

- *The firm plays the mixed strategy*

$$F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)},$$

  with $\operatorname{supp} F = [p_{ind}, \overline{p}]$.

- *If* $p \in \operatorname{supp} F$, *then* $\gamma(V^*(p)) = \frac{\pi^*}{p}$, *i.e., the demand is unit-elastic on the support.*

- *The equilibrium profits are* $\pi^* = \max_p p(1-G(p))$.

The proof is in appendix A.

Proposition 2 illustrates how the firm's commitment problem constrains its behaviour and how it can solve it. First, any pure strategy is not credible. To see why, suppose the firm plays a pure strategy p. Then by the PPE requirement, all the types $v \geq p$ accept the offer and all types $v < p$ refuse it. This means that their WTPs are $p^*(v) = v + \lambda v$ for $v \geq p$ and $p^*(v) = v$ for $v < p$. The firm has then a profitable deviation to a higher price as all types that accept p are willing to pay strictly more than p.

The firm can play a mixed strategy only if it is indifferent between any price on the support, i.e., the demand is unit-elastic on the support. Therefore, the firm's mixed strategy induces expectations of trading such that the resulting distribution over WTP is unit-elastic.

Moreover, the firm does not benefit from facing buyers with reference-dependent preferences. The buyers whose valuation is below the support know they will only face prices higher than their valuation. Because they play a PPE, they cannot expect to trade and their WTP is equal to their valuation. Therefore, they behave like players with no attachment effect. When offering the lowest price on the support, only buyers with valuation above that price accept, exactly like in a reference-independent model. By the indifference condition, these must be the equilibrium profits. However, some types do end up buying the good at a price above their valuations with some probability. The exploitation of the buyers' attachment effect is compensated by higher probability of no trade when offering high prices.

If Assumption 1 is relaxed, the qualitative features of Proposition 2 remain unchanged: the unique equilibrium is in mixed strategy, the demand is unit-elastic on the support and equilibrium profits are equal to the reference-independent benchmark. What changes is that the cdf F as described in Proposition 2 might not be increasing and so we need to modify it to make sure the mixed strategy is well-defined.

2.1 (Almost) Complete information

In this subsection, I look at equilibrium strategies when the distribution over valuations converges to a singleton and when the valuation is common knowledge.

In what follows, I look at a sequence of games where the only varying primitive is the prior distribution G. Therefore, abusing notation, I will identify a sequence of games with a sequence of prior distributions. Note that I will maintain that each G along the sequence is continuous and strictly increasing. Denote by $\xrightarrow{\mathcal{D}}$ convergence in distribution and δ_v the Dirac measure on v.

Proposition 3. *Let $v > 0$. Take a sequence of games $\{G_i\}_{i=0}^{\infty}$ such that $G_i \xrightarrow{\mathcal{D}} \delta_v$. All other primitives of the model are fixed.*

Then, equilibrium profits converge to v and the firm's equilibrium strategy converges in distribution to $F_\infty(p) = \frac{p-v}{\lambda v}$ with $\operatorname{supp} F_\infty = [v, v + \lambda v]$.

Moreover, the limit probability of trade is

$$\frac{1}{\lambda} \log(1 + \lambda).$$

Proof. **Limit distribution and profits:**

Let F_i be the equilibrium strategy given G_i, $\underline{p}_i = \min \operatorname{supp} F_i$, $\overline{p}_i = \max \operatorname{supp} F_i$ and $\pi_i^* = \underline{p}_i(1 - G_i(\underline{p}_i))$. From Proposition 2, $F_i(p) = \frac{p - G_i^{-1}\left(\frac{p-\pi_i^*}{p}\right)}{\lambda G_i^{-1}\left(\frac{p-\pi_i^*}{p}\right)}$ for all $p \in [\underline{p}_i, \overline{p}_i]$. Using that

$G_i^{-1}(x) \to v$ for each $x \in (0,1)$, for each $p \in \mathbb{R}$,

$$F_i(p) \to \begin{cases} 0 & if \quad p < v \\ \frac{p-v}{\lambda v} & if \quad p \in [v, v+\lambda v] \\ 1 & if \quad p > v + \lambda v \end{cases} ,$$

and thus $F_i \xrightarrow{\mathcal{D}} F_\infty$.

Profits converge to v as $\max_p\, p(1 - G_i(p)) \to v$.

Probability of trade: Denote the probability of trading at price p by $\phi(p)$. This probability is pinned down by the indifference condition:

$$\pi_i^* = p\phi_i(p).$$

The probability of trading is thus $\int_{\mathbb{R}} \phi_i(p) f_i(p)\, dp$ where f_i is the density of F_i. It is easy to verify that $f_i(p) \to \frac{1}{\lambda v}$ for all $p \in [v, v+\lambda v]$ and $\phi_i(p) f_i(p)$ is uniformly bounded. Using the dominated convergence theorem, we get that

$$\int_{\mathbb{R}} \phi_i(p) f_i(p)\, dp \to \int_v^{v+\lambda v} \frac{v}{p} \cdot \frac{1}{\lambda v} dp = \frac{1}{\lambda} \log(1+\lambda).$$

□

As the distribution of types converges to the singleton v, the firm's strategy converges to a uniform distribution on $[v, v+\lambda v]$. The profits, on the other hand, are always equal to the reference-independent benchmark, $\pi^* = v$ in the limit. As the mass of types accumulate on $\{v\}$, the support does not converge to a singleton. Even though the interval of valuations could become arbitrarily small, the interval of potential WTP stays large: any $p \in [v, v+\lambda v]$ can be a PPE cutoff. Even as we converge to complete information, there needs to be uncertainty over the price to create the unit-elastic demand and guarantee an equilibrium. In equilibrium, the firm uses the full range of possible PPE cutoff and the support of its strategy is, in the limit, $[v, v+\lambda v]$. This variation leads to a strictly positive probability of no trade in the limit.

Moreover, $\frac{1}{\lambda}\log(1+\lambda)$, the probability of trade, decreases in the attachment effect λ and converges to 0 as $\lambda \to \infty$. Intuitively, a higher λ makes the buyer more vulnerable to exploitation but also increases the firm's commitment problem. The firm must lower the probability of trading to compensate for the higher demand induced by a higher λ. This can also be seen from the indifference condition: as profits converge to v for any λ, the higher prices must be compensated for by a higher probability of rejection.

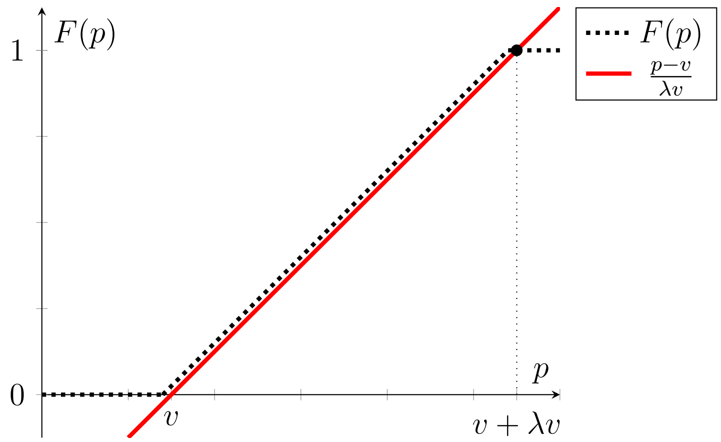

Figure 2: Almost optimal distribution over prices.

Finally, I note that the limit strategy is arbitrarily close to the one the firm would use if it could commit to a price distribution when knowing the valuation v. I call this problem the pricing with commitment problem.

Proposition 4 (Heidhues and Kőszegi, 2014)**.** *Suppose $v > 0$ is commonly known. The solution to the pricing with commitment problem:*

$$\sup_{F \in \Delta \mathbb{R}} \int_{-\infty}^{p^*} p \, dF(p) \text{ s.t. } p^* = \min\{p : v - p = -\lambda v F(p)\}$$

is

$$\frac{\lambda + 2}{2} \cdot v.$$

The distribution that attains the supremum profit is $F(p) = \frac{p-v}{\lambda v}$ on $[v, v(1+\lambda)]$.

The proof of this result is available in Heidhues and Kőszegi (2014) or Eliaz and Spiegler (2015). I provide it here for completeness.

Proof. Take some distribution F inducing WTP $p^* \in [v, v(1+\lambda)]$. We have $F(p) > \frac{p-v}{\lambda v}$ for all $p < p^*$ and $F(p^*) = \frac{p^*-v}{\lambda v}$. The firm is better off by setting another distribution F' such that for any price p on the support of F' with $p < p^*$, $F'(p) \in (\frac{p-v}{\lambda v}, F(p))$ and $F'(p^*) = \frac{p^*-v}{\lambda v}$ as this would give strictly larger profits and the same induced WTP. This operation is possible for any distribution F such that $F(p) > \frac{p-v}{\lambda v}$ for all $p < p^*$ and $F(p^*) = \frac{p^*-v}{\lambda v}$. Therefore, for a given WTP p^*, the supremum profits are $\int_v^{p^*} p \, d(\frac{p-v}{\lambda v})$. In turn, the firm is better off setting $p^* = v(1+\lambda)$. The resulting supremum profits are $\frac{\lambda+2}{2} \cdot v$. $\square$

The firm chooses a price distribution that maximises its profits amongst all the distribution that implement trade with probability one. By the FOSD interpretation of the PPE (Proposition 1), this is done by choosing a price distribution as close as possible to $\frac{p-v}{\lambda v}$, see figure 2 for an illustration. We need to solve for supremum profits because the constraint that p^* is PPE is not closed.[9] The supremum profits can be approached by taking distributions $F_\epsilon(p) = \frac{p-v+\epsilon}{\lambda v+\epsilon}$ on $[v-\epsilon, v(1+\lambda)]$ and let $\epsilon \to 0$.

In contrast, if the value of v is common knowledge and there is no commitment, no equilibrium exists.

Proposition 5. *When v is common knowledge and $v > 0$, no equilibrium exists.*

Proof. For any $p^*(v)$, there is a unique best-response of the firm, which is to offer $p^*(v)$. Let p be the equilibrium price, i.e., the equilibrium strategy is $F(\tilde{p}) = \mathbb{1}[\tilde{p} \geq p]$.

If $p \leq v$, then there is a unique PE cutoff $p^*(v) = v + \lambda v F(v+\lambda) = v + \lambda v$. There is a profitable deviation to $p' = v + \lambda v$.

If $p > v$, then $p^*(v) = v + \lambda v F(v) = v$ is a PE cutoff and also the smallest PE cutoff. By Proposition 1, it is the PPE cutoff and thus there is no trade in equilibrium. Then, there is a profitable deviation to any $p' \in (0, v]$. □

The key tension is that the firm wants to take advantage of the attachment effect. However, given the PPE requirement and a deterministic price, the buyer's WTP is only higher than his valuation if the price offered is below his valuation. An equilibrium with no trade is also impossible because any price below the valuation will be accepted for any PPE.

We obtain the non-existence result for the same reason there cannot be a pure strategy equilibrium in the incomplete information environment. Playing a pure strategy shifts the demand if the offer is accepted. Unlike the incomplete information environment, the firm cannot play a mixed strategy because it is facing only one type. Thus, it cannot solve its commitment problem. Note that this argument does not depend on "forcing" the buyer to buy when indifferent as, as noted by Dato et al. (2017), a mixed strategy cannot be part of a PPE.

Relation to other non-existence results: Dato et al. (2017) have already observed that in

[9] A sequence of distributions $F_i(\cdot) \to F(\cdot)$ that induce $p^* = p'$ for each i can have $p' \neq \min\{p : v - p = -\lambda v\, F(p)\}$. For example, the distribution $F(p) = \frac{1}{2} + \frac{p-v-\frac{\lambda v}{2}}{\lambda v+\epsilon}$ induces a PPE cutoff $p^* = v + \lambda v/2$ for all $\epsilon > 0$ but a PPE cutoff $p^* = v$ for $\epsilon = 0$.

games where players are constrained to play a PPE, an equilibrium does not always exist. They note that with binary actions, a PPE strategy never entails mixing. Therefore, if the equilibrium requires mixed strategies, these strategies can never be the players' PPE, even though they could be PEs. Here, the mechanism for non-existence is different as the equilibrium relaxing the PPE constraint would not be in mixed strategies. Instead, it occurs because the buyer's PPE price cutoff is always bounded away from the price offered.

Azevedo and Gottlieb (2012) show that games with prospect theory preferences can suffer from equilibrium existence issues in a game where a risk-neutral firm offers a gamble to an agent. In their case, they observe that for an exogenous reference point and some conditions on the value and probability weighting functions, there exists a bet with arbitrarily low expected value that an agent with prospect theory preferences is willing to accept. Our analyses differ both in the choice of reference-point as well as the choice of payoff function – they allow for probability weighting as well as restrict attention to gain-loss value function. In the model considered here, the payoffs are always bounded so the mechanism for non-existence is also different.

3 Intermediate case: Testing the valuation

We have so far looked at "extreme" information structures, either complete information or complete lack of information. In this section, I allow the firm to collect additional information on the buyer's valuation before setting a price. In Proposition 6, I first establish that having only partial information about the buyer's valuation can be beneficial for the firm. In Proposition 7, I characterise the firm's preferred testing strategy and its profits when valuations are uniformly distributed.

In many bilateral trade settings, one party can gather information about the other before making an offer. A leading example is job applications where an employer designs a screening process before making an offer to a candidate.[10] In the process, he collects information about the candidate's productivity and outside option. These can be explicit tests through an assessment centre or asking to submit certain documents like a CV or recommendation letters. The question being asked in this section is whether a firm can use a combination of the screening process and a candidate's attachment effect to offer lower wages.

[10]The model is cast as a buyer-seller interaction but it can be rewritten as a firm-candidate interaction where the candidate has private information regarding his outside option and the firm makes a take-it-or-leave-it wage offer.

This section also applies to settings where a seller can gather information about a buyer's valuation. For example, a car dealer can learn about a consumer's valuation by asking him the right questions during the sales pitch.

All functions and sets are assumed to be measurable. For this section, assume also that the density g is Lipschitz continuous and $\underline{v} > 0$. I discuss how to relax these assumptions in appendix C.4.

Test: Let S be a set of signals with $S \supset \mathbb{R}$. A test is a mapping from types to distributions over signals $F : V \to \Delta(S)$. Denote by $F(s|v)$ the distribution of s conditional on v. Abusing notation, $F(v, s)$ is the joint distribution of (v, s) induced by $F(s|v)$ and $G(v)$. The set of signal S is a large set that must at least contain the real numbers.

Players' information: The test is common knowledge but only the firm observes the signal realisation. The valuation v is still privately known.

Players' strategy: A strategy for the firm is $P : S \to \Delta(\mathbb{R})$, a mapping from signals to distributions over prices.

The assumption regarding the players' information and strategies corresponds to a setting where the seller can commit to a test or where the test is publicly observable. In the job application setting, this would be a case where the candidate can observe the selection process he goes through. An important assumption is that the reference point is set without knowing the signal realisation. This would hold if the candidate does not observe the signal realisation, e.g., he does not know the result of the test, does not observe his recommendation letters or cannot fully predict the interviewer's assessment of the interview. Alternatively, this information could be revealed to him as long as his reference point is set before learning the outcome of the test. The case of public signals is discussed at the end of the section.

The PE and PPE are defined in the same way as before. I assume that the buyer forms expectations after having observed his type and the test. Given the test F and strategy P, a reference point Q_v is type v's PE if

$$Q_v = \int_S \int_{\mathbb{R}} \mathbb{1}[p \leq v + \lambda v Q_v] \, dP(p|s) dF(s|v),$$

and the expected utility of type v given F, P and Q_v is

$$W(v|Q_v) = \int_S \int_{\mathbb{R}} \left(\mathbb{1}[p \leq v + \lambda v Q_v](v - p) + \mathbb{1}[p > v + \lambda v Q_v](-\lambda v Q_v) \right) dP(p|s) dF(s|v).$$

Then, Q_v^* is a PPE if $Q_v^* \in \arg\max_{Q_v \in PE} W(v|Q_v)$.

The firm's ex-ante payoffs are

$$\mathbb{E}[\pi(P)|(Q_v^*)_v] = \int_V \int_S \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, dP(p|s) dF(s|v) dG(v).$$

We can now define the relevant equilibrium definition for this setting.

Definition 5. *Fix a test F. An equilibrium is a profile $(P, (Q_v^*)_v)$ such that*

- $P \in \arg\max \mathbb{E}[\pi(\cdot)|(Q_v^*)_v]$,
- *and for all $v \in [\underline{v}, \overline{v}]$, Q_v^* is type v's PPE.*

In equilibrium, the buyer plays according to his PPE based on the test and the firm's strategy. The firm best replies to the buyers' PPE based on its information.

A first observation is that we can focus on tests that generate price recommendations that the firm follows in equilibrium. This simplifies the characterisation of tests and follows from standard revelation principle arguments. For any $s \in \mathbb{R}$, let δ_s, the probability distribution setting price equal to s with probability one.

Lemma 1. *Consider a test F and an equilibrium $\big(P, (Q_v^*)_v\big)$ in F. Then, there exists a test $\tilde{F}$ with support $\cup_{s \in S} \operatorname{supp} P(\cdot|s)$ and an equilibrium such that the PPE cutoffs are Q_v^* for all v, the firm's strategy is δ_s for any $s \in \operatorname{supp} \tilde{F}$ and each player gets the same payoffs as in the original equilibrium.*

The proof is in appendix B.

Lemma 1 holds because the only thing that matters for the buyer's PPE is the distribution over prices given his type. Therefore, a standard revelation principle argument holds. If after two different signals, the firm offers the same price, we can modify the test to "merge" these two signals. This will not change the distribution over actions and thus all PPEs are preserved. In addition, any randomness generated by a mixed pricing strategy can be created through the test.

From now on, we will only consider tests that generate action recommendations that the firm finds optimal to follow. In this way, we can think of a test as a distribution over prices for each type. We can rewrite the firm's profits as $\int_{V\times S} \mathbb{1}[v \in V^*(s)]\, s\, dF(v,s)$. The equilibrium conditions are the PPE requirement (PPE) and obedience constraints (OB):

$$\text{for all } v,\ p^*(v) = \min\{p:\ v - p = -\lambda v\, F(p|v)\}, \tag{PPE}$$

$$\text{for all } S' \subseteq S,$$
$$\int_{V\times S'} \mathbb{1}[v \in V^*(x)]\, x\, dF(v,x) \geq \int_{V\times S'} \mathbb{1}[v \in V^*(\tilde{P}(x))]\, \tilde{P}(x) dF(v,x) \tag{OB}$$
$$\text{for all } \tilde{P} : S \to \mathbb{R}.$$

The constraint (PPE) pins down the WTP of each buyer. The obedience constraint (OB) ensures that the firm is willing to follow the price recommendations. Obedience constraints are required to hold for any subset S' to take into account that some signal realisations might be zero probability events and have no well-defined density.

In the reference-independent model, the firm would learn the buyer's valuation perfectly and offer the valuation, i.e., it would use perfect price discrimination. "Almost" perfect price discrimination is also a feasible strategy if the test is arbitrarily close to fully revealing the valuation (Proposition 3).[11] The profits in this case would be $\int_V v\, dG$. However the firm can always do strictly better than that. Call a test completely noisy if for all s and v, $Pr_F[s|v] < 1$ and there is (almost) no unique v such that $s \in \operatorname{supp} F(\cdot|v)$. That is, a test is completely noisy if no type sends a signal deterministically and no signal realisation reveals any type. Therefore, in a completely noisy test, the buyer is uncertain about which signal he generated and the firm is uncertain about which type it is facing.

Proposition 6. *There is a completely noisy test F^* respecting (PPE) and (OB) such that the firm's profits are strictly greater than $\int_V v\, dG$.*

The proof is in appendix C. The discussion below provides the intuition for how to construct F^*.

Proposition 6 shows that the firm benefits from not fully learning the buyer's type and perfect price discrimination is therefore suboptimal. Considering the case of a screening process for job candidates, Proposition 6 suggests that firms could benefit from designing noisy or opaque screening procedures. The reason is that this would let candidates entertain the idea that they might get a good wage offer and thus weakens their bargaining position.

[11] Remember that if the firm perfectly learns the valuation, there is no equilibrium in the resulting game.

A completely noisy test uses the two types of uncertainty it generates to credibly exploit the buyer's attachment effect. First, the buyer is uncertain about which signal he generated and therefore which types he is pooled with. At low signals, the firm offers low prices, inducing expectations to buy. At higher signals, the firm offers higher prices, taking advantage of the higher WTP. From the buyer's perspective, he is facing random prices like in Heidhues and Kőszegi (2014). Second, the firm uses the uncertainty it has about the buyer's type to credibly offer low prices after a low signal, despite facing some buyers willing to accept higher prices. In the following, I characterise the test F^* which satisfies Proposition 6 and provide conditions under which it is also optimal among all tests.

Construction of F^* from Proposition 6 To understand how to construct F^*, it is useful to first look at the test the firm would design if it would not have to follow obedience constraints and thus could commit to follow the signal's action recommendation. The test F^* will be obtained by modifying this optimal test under commitment and apply it on subintervals of V.

When the firm is not constrained by the obedience constraints, solving for the profit-maximising test is equivalent to choosing the optimal distribution over prices for each valuation as in the setting studied by Heidhues and Kőszegi (2014). Proposition 4 in the previous section has already characterised this solution and the findings are summarised below.

Claim 1. *Let the commitment solution be the solution to*

$$\sup_F \int_{V\times S} \mathbb{1}[v \in V^*(s)]\, s\, dF(v,s)$$
$$\textit{s.t. for all } v, p^*(v) = \min\{p : \ v - p = -\lambda v\, F(p|v)\}.$$

The supremum can be attained by taking a sequence $\{F_i\}$ such that

- $F_i(s|v)$ *converges pointwise to* $\frac{s-v}{\lambda v}$ *on* $[v, v+\lambda v]$ *for all* v.
- *For each* i *and* v*, the WTP is* $p^*(v) = v + \lambda v$ *and there is trade with probability one.*
- *The supremum profits are* $\int_V \int_v^{v(1+\lambda)} s \frac{1}{\lambda v} ds dG$.

As in Proposition 4, we can only solve for the supremum because the set of constraints is not closed (see footnote 9 for an example). Under this test, each type v generates a uniform distribution over signals on $[v, v(1+\lambda)]$ and the firm commits to following the price recommendation. This induces a WTP for each type $p^*(v) = v(1+\lambda)$. I call the distribution $\frac{s-v}{\lambda v}$

on $[v, v + \lambda v]$ the commitment distribution. This is the best possible way the firm could use the randomness of prices to take advantage of the attachment effect.

This test however would not respect the obedience constraints: for any $s < \underline{v}(1+\lambda)$, there is a profitable deviation to $\underline{v}(1+\lambda)$ as this price would be accepted for sure by any buyer type. To prevent these upward deviations, the firm needs to make sure that the WTPs do not grow too fast. Or put differently, it needs to make sure that the lowest type willing to accept at each price, $v^*(s)$, increases fast enough with s. (Recall that if p^* is strictly increasing, $v^* = p^{*-1}$.)

One possibility to characterise F^* would be to maintain a cdf equal to the commitment distribution on the support of accepted prices and to engineer v^* to make the test obedient. I call these distributions censored commitment distributions and define them as follows:

Definition 6 (Censored commitment distribution)**.** *$F(\cdot|v)$ is a censored commitment distribution if it is a cdf and there exist a $\tilde{p} \in [v, v + \lambda v]$ and $\tilde{F}(s|v) < \frac{s-v}{\lambda v}$ such that*

$$F(s|v) = \begin{cases} 0 & if \quad s < v, \\ \frac{s-v}{\lambda v} & if \quad s \in [v, \tilde{p}], \\ \tilde{F}(s|v) & if \quad s > \tilde{p}. \end{cases}$$

A censored commitment distribution behaves like the commitment distribution for prices below the price $\tilde{p}$ but stays below $\frac{s-v}{\lambda v}$ for higher prices. Therefore this distribution is arbitrarily close to a distribution that implements WTP $\tilde{p}$.[12] The commitment distribution is a censored commitment distribution with $\tilde{p} = v + \lambda v$. By the FOSD interpretation of the PPE (Proposition 1), the censored commitment distribution is the best the firm can use given a target WTP $\tilde{p}$. Thus it is a good candidate to achieve higher profits using a noisy test.

To construct F^*, we will start by assuming that the firm uses censored commitment distributions and use the local (upward) obedience constraints to characterise v^*. The first-order conditions of the firm's obedient constraints deliver the following differential equation:

$$\frac{\partial^+}{\partial s'} \int_{v^*(s')}^{\overline{v}} s' f(s|v) g(v) dv \Bigg|_{s'=s} = 0, \text{ with } v^*(\underline{v}) = \underline{v}. \tag{L-OB}$$

This is the derivative of the profit function when observing signal s and offering price s', assuming that $V^*(s') = [v^*(s'), \overline{v}]$. We focus on upward deviations because this is where the censored commitment distribution is identical to the commitment distribution with $f(s|v) =$

[12] For example, $F(s|v) = \frac{\tilde{p}-v}{\lambda v(\tilde{p}-v+\epsilon)}(s - v - \epsilon)$ for $s \leq \tilde{p}$ and $F(s|v) = \tilde{F}(s|v)$ otherwise.

$\frac{1}{\lambda v}$ if $s \geq v$ and 0 otherwise. The initial condition $v^*(\underline{v}) = \underline{v}$ is obtained by assuming that the lowest signal used is $\underline{v}$ and therefore accepted by all types.

Note however that $v^*(s)$ is not allowed to be arbitrarily low: because $p^*(v) \leq v(1+\lambda)$, it must always be the case that $v^*(s) \geq \frac{s}{1+\lambda}$. If $\tilde{v}(s)$ is the solution to (L-OB), we will impose $v^*(s) = \max\{\tilde{v}(s), \frac{s}{1+\lambda}\}$. This gives the following candidate for F^* to approximate:

Definition 7 (Candidate test)**.** *A candidate test F is a test such that for each v, $F(\cdot|v)$ is a censored commitment distribution and the inverse of the induced WTPs is $v^*(s) = \max\{\tilde{v}(s), \frac{s}{1+\lambda}\}$ where $\tilde{v}(s)$ is the solution to (L-OB).*

There are three challenges remaining to characterise F^*. First, we must find a test that is arbitrarily close to a censored commitment distribution, but actually not equal to it, and we must also define its values for $s > p^*(v)$. Second, we need to verify that the candidate test indeed gives higher profits than the full information benchmark. Finally, we need to check that global obedience constraints are satisfied. This is addressed in the proof of Proposition 6.

It is worth pointing out that the candidate test identified does not generally respect global obedience constraints. However, it does when G is the posterior distribution of a sufficiently fine partition of the type space. This is enough to show Proposition 6. The relevant F^* would be a test that first partitions V and then within each element of the partition, $[v_{i-1}, v_i]$, the firm would use an approximation of a candidate test. Because each buyer type knows his own type, the information that $v \in [v_{i-1}, v_i]$ is common knowledge, but not the exact realisation of the signal. Within each element of the partition, the firm would do better than fully learning the valuations, therefore, it would be strictly better off overall.

Design of the optimal test In Proposition 6 and through the discussion above, we have shown that the firm can always do better than the full information benchmark by using censored commitment distributions. I show here that whenever the candidate test (Definition 7) satisfies global obedience constraints then this is the best the firm can do. That is, it solves the following problem:

$$\begin{aligned}
\pi^* = \sup_F & \int_{V\times S} \mathbb{1}[v \in V^*(s)]\, s\, dF(v,s) \\
\text{s.t. } & p^*(v) = \min\{p:\ v - p = -\lambda v\, F(p|v)\}, && \text{(PPE)} \\
& \text{for all } S' \subseteq S, \\
& \quad \int_{V\times S'} \mathbb{1}[v \in V^*(x)]\, x\, dF(v,x) \geq \int_{V\times S'} \mathbb{1}[v \in V^*(\tilde{P}(x))]\, \tilde{P}(x) dF(v,x) && \text{(OB)} \\
& \text{for all } \tilde{P} : S \to \mathbb{R}.
\end{aligned}$$

This is the optimal test design problem of the firm. The firm chooses distributions over prices that will determine its profits and the WTP of each type subject to the obedience constraints. Again, we need to solve for the supremum because the set of constraints is not closed.

The following lemma shows that the profits from a candidate test are an upper bound to the firm's supremum profits.

Lemma 2. *Denote by π^c the profits from a candidate test and recall that π^* are the profits from the optimal test. Then,*

$$\pi^c \geq \pi^*.$$

The proof is in appendix C.

Lemma 2 confirms the intuition that candidate tests can solve the firm's optimal test design problem. This comes from the fact that they are derived from a subset of obedience constraints and that given the WTPs, a censored commitment distribution is optimal for the firm. Therefore, we know that whenever a sequence of tests satisfying global obedience constraints converges to a candidate test, it solves the optimal test design problem of the firm. In particular, we know from Lemma 8 (proven in the appendix) that this is the case for $v \sim U[\underline{v}, \overline{v}]$.

Proposition 7. *Assume $v \sim U[\underline{v}, \overline{v}]$ and let $\hat{v}(s) = \min\{\overline{v}, s\}$. The firm's supremum profits are*

$$\pi^* = \int_{\underline{v}}^{\overline{v}(1+\lambda)} \frac{\min\left\{\hat{v}(s) - \underline{v},\ s \log \frac{(1+\lambda)\hat{v}(s)}{s}\right\}}{\lambda \Delta v} ds,$$

and there exists a sequence of tests $\{F_i\}$ that approximates the firm's supremum profits such that

- *The sequence $\{F_i\}$ converges pointwise to a candidate test.*

- *The sequence $\{F_i\}$ converges pointwise to a completely noisy test.*
- *There is downward distortion: the probability of trading is increasing v.*

In the limit, we have $v^*(s) = \max\{\hat{v}(s)\exp\left(\frac{-\hat{v}(s)+v}{s}\right), \dfrac{s}{1+\lambda}\}$ and $p^*(\cdot)$ is the inverse of $v^*(\cdot)$.

The proof is in appendix C. We look at the case of uniformly distributed valuations but the same result would hold whenever a sequence of tests approaching a candidate test are obedient. So we could say that the result holds "whenever local obedience constraints are sufficient".

Proposition 7 characterises the firm's preferred test and supremum profits. As in Proposition 6, the optimal test is a completely noisy test. Moreover, higher types are more likely to trade and also face, and accept, higher prices compared to their valuation. In our job screening example, it means that less productive candidates (the equivalent of high valuation buyers) are more likely to suffer from their attachment towards the job as they are more likely to receive offers that are better than their outside option.

The firm can credibly follow the equilibrium strategy because it is uncertain about which type it is facing. However, there are limits to the uncertainty the firm can generate. For example, the firm cannot pool the lowest type in the support with even lower types. Therefore, this type always expects prices higher than his WTP. By the PPE requirement, he must have a WTP equal to his valuation. This, in turn, means that he must trade with probability 0. This logic can be extended to more types: lower valuations can be pooled with fewer lower types. The firm can take advantage of their attachment effect but not fully. Their probability of trade is then smaller than one. Figure 3 shows the WTPs $p^*(v)$ for the case of $v \sim U[0.5, 1.5]$ and $\lambda = 1.5$. For low types, the WTP is lower than $v(1+\lambda)$ and their probability of trade is lower than one. For high enough types, the WTP is equal to $v(1+\lambda)$ and they face a commitment distribution.

The two regimes in the profit functions correspond to when $v^*(s) = \frac{s}{1+\lambda}$ and when $v^*(s) = \hat{v}(s)\exp\left(\frac{-\hat{v}(s)+v}{s}\right)$. In the first case, the WTP of the lowest type willing to accept s, $v^*(s)$, is $v^*(s)(1+\lambda)$ and that type faces a commitment distribution over prices. In the second, the WTP is lower and $v^*(s)$ faces a censored commitment distribution with a probability of trade lower than 1. In the first case, the obedience constraints don't bind and in the second case they do.

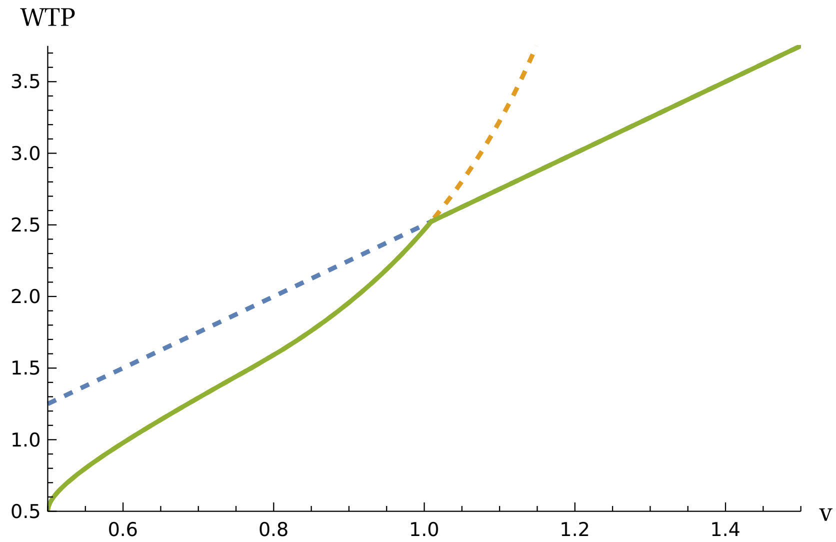

Figure 3: The WTPs at the optimal test for $v \sim U[0.5, 1.5]$ and $\lambda = 1.5$. The green line shows $p^*(v)$, the blue and red dashed lines show $v(1+\lambda)$ and the inverse of $\hat{v}(s)\exp\left(\frac{-\hat{v}(s)+v}{s}\right)$.

This observation allows us to understand better how a change in λ affects the profits. On the one hand, a larger λ allows the firm to charge higher prices because of a larger attachment effect. On the other hand, a larger λ makes a deviation more tempting as the firm can benefit more from it. In a sense, the commitment problem of the firm is bigger. This can be seen from the fact that as λ grows larger, the set of signals for which $\frac{s}{1+\lambda} \leq \hat{v}(s)\exp\left(\frac{-\hat{v}(s)+v}{s}\right)$ is larger, i.e., the set of signals for which the obedience constraints bind is larger. This is illustrated numerically in figure 4 for $v \sim U[0.5, 1.5]$. For low values of λ, the profits under commitment and under the optimal test are very close to each other but at some point the profits from the optimal test and the commitment solution diverge.

Finally, note that as in section 2, the firm generates inefficiencies to maintain the credibility of its own strategy. In both cases, the firm must manage the buyers' expectations to ensure that it is willing to follow its strategy. Unlike the environment of section 2 with no test, the firm can now make more profits than in the reference-independent benchmark. In Proposition 2, the firm can only make the reference-independent profits because types below the support are always expecting prices higher than their valuation. Here, the support of types conditional on the signal is not common knowledge anymore. Therefore, only the lowest type in the prior distribution expects to face prices higher than his valuation.

Public signals Consider a model where the signal realisation is public and the buyer's reference point is set after having observed the signal realisation. In this environment, the firm's

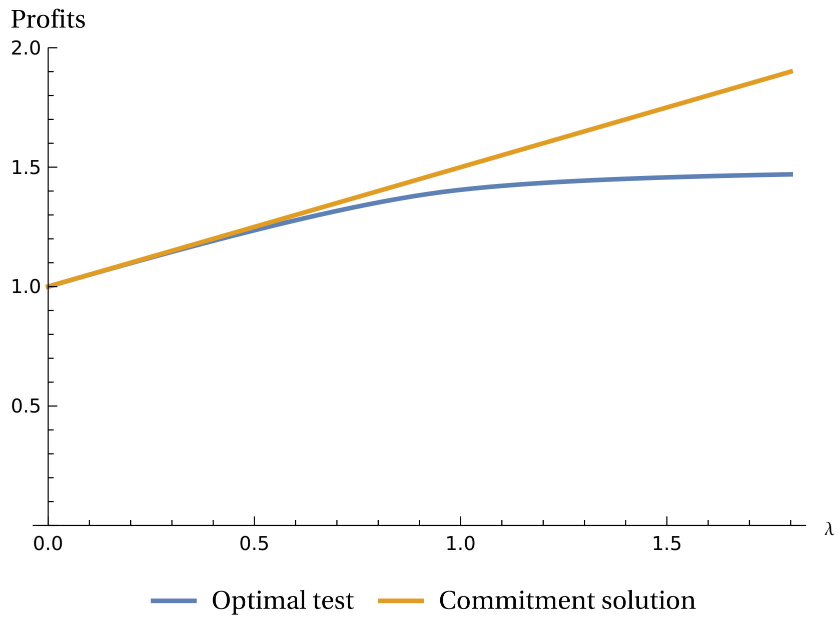

Figure 4: Profits from the commitment solution and the optimal test as a function of λ for $v \sim U[0.5, 1.5]$.

information is common knowledge. Thus, after each signal realisation, we are back to the environment of section 2. The optimal test for the firm is then to take an arbitrarily fine partition of the type space and play the equilibrium of Proposition 2 in each element of the partition.

The profits are the same as in the reference-independent benchmark but like in section 2, there is a positive probability of no-trade despite the near-complete information. Another difference is that the WTP is no longer monotonic in the valuation: in each element of the partition $[v, v + \epsilon)$, the support of the mixed strategy is $\approx [v, v + \lambda v]$ and the WTPs vary on an interval $\approx [v, v + \lambda v)$. Therefore, some types might be on higher elements of the partition, but have a lower WTP.

4 Conclusion

In this paper, I study a model of monopoly pricing where the buyer has expectation-based reference-dependent preferences, focusing on an attachment effect. The model has two main features. The expectation-based reference point renders the demand an endogenous object. The PPE requirement creates a commitment problem for the firm.

On a theoretical level, this model offers two main lessons. First, uncertainty can help overcome the firm's commitment problem. In all the environments studied, the firm must manage the buyers' expectations, and thus the demand, to maintain a credible strategy. In the incomplete information environment, the firm needs the uncertainty to induce a unit-elastic demand. For its testing strategy, the firm uses the uncertainty to create obedient distributions over prices. While it can deliver equilibrium existence or credible price distributions, using uncertainty necessarily entails inefficiencies. Furthermore, a higher λ, associated with a stronger commitment problem, implies a higher probability of no trade.

The other recurring theme is the impossibility for the firm to exploit the low types. This follows from the buyers' rational expectations as a low type always anticipate prices above his valuation and therefore cannot expect to buy in a PPE. The consequence was particularly stark in the incomplete information model where it made the profits the same as in the reference-independent model. In the optimal testing environment, it generated downward distortions.

One issue put aside in this paper is the possibility for the buyer to experience gain-loss utility in the money dimension as well. This modification would change the characterisation of the PPE. For example, a PE price cutoff would not be determined by the probability of reaching the cutoff but also on the expected loss in price and thus a result like Proposition 1 would not hold.

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A Proof of Proposition 2

Proof. **Preliminary lemmas:**

The first lemmas guarantee the good behaviour of two equilibrium objects, F, the firm's strategy, and $p^*(v)$ the WTP of each buyer as a function of his type.

Lemma 3. *In any equilibrium, $p^*(v)$ is strictly increasing.*

Proof. Take $v_1 < v_2$. Let $p^*(v_1) = p_1$ and $p^*(v_2) = p_2$ be their PPE cutoffs. Assume that $p_1 \geq p_2$. Because p_1, p_2 are PE cutoffs,

$$v_1 - p_1 = -\lambda v_1 F(p_1), \tag{1}$$

$$v_2 - p_2 = -\lambda v_2 F(p_2). \tag{2}$$

Clearly, $p_1 = p_2$ cannot hold. We either have $v_1 - p_2 \geq -\lambda v_2 F(p_2)$ or $v_1 - p_2 < -\lambda v_2 F(p_2)$. In the first case, we have

$$v_2 - p_2 > v_1 - p_2 \geq -\lambda v_2 F(p_2),$$

contradicting equation (2). In the second case, $F(p_2) < \frac{p_2 - v_1}{\lambda v_2}$ and $p_1 > p_2$ contradict Proposition 1. □

Lemma 4. *Let F be an equilibrium strategy. If $p \in \operatorname{supp} F$, then there exists $v \in V$ such that $p^*(v) = p$.*

Proof. Assume not: there is a $p \in \operatorname{supp} F$ and no v such that $p^*(v) = p$. First, if $p \in \operatorname{supp} F$, then $p^*(v) \geq p$ for some v, for otherwise the firm makes zero profits. The firm can always make strictly positive profits by offering a price $p < \overline{v}$. This would be accepted by all types $v \in [p, \overline{v}]$ because $p^*(v) \geq v$ in any PPE. Because $p^*(\cdot)$ is strictly increasing, this implies that there is a v such that $p^*(\cdot)$ is not continuous at v and $p \in [\lim_{x \searrow v} p^*(x), \lim_{x \nearrow v} p^*(x)]$. By continuity of G, $\gamma(V^*(p)) = \gamma(V^*(p^*(v)))$. But then both $p^*(v), p \in \operatorname{supp} F$ but they give different profits, a contradiction. □

Lemma 5. *Any equilibrium strategy F is continuous.*

Proof. Assume not. Let $\tilde{p}$ be a point of discontinuity of F. If $\tilde{p}$ is a PE cutoff for some v, then $F(\tilde{p}) = \frac{\tilde{p} - v}{\lambda v}$. Using the upper semicontinuity of F and continuity of $\frac{p - v}{\lambda v}$, there exists $p' < \tilde{p}$ such that $F(p') < \frac{p' - v}{\lambda v}$. By Proposition 1, $\tilde{p}$ cannot be a PPE cutoff of v. This contradicts Lemma 4. □

Lemma 6. *In any equilibrium, $p^*(v)$ is continuous.*

Proof. Assume there exists a point of discontinuity $\tilde{v}$, i.e., $p_1 \equiv \lim_{v \nearrow \tilde{v}} p^*(v) < \lim_{v \searrow \tilde{v}} p^*(v) \equiv p_2$. We have that $F(p^*(v)) = \frac{p^*(v)-v}{\lambda v}$ and F is continuous, therefore,

$$F(p_1) = \lim_{v \nearrow \tilde{v}} \frac{p^*(v) - v}{\lambda v} < \lim_{v \searrow \tilde{v}} \frac{p^*(v) - v}{\lambda v} = F(p_2).$$

We can then find $\tilde{p} \in (p_1, p_2)$ such that $\tilde{p} \in \operatorname{supp} F$ and there exist no v such that $p^*(v) = \tilde{p}$. This contradicts Lemma 4. $\square$

Lemma 5 rules out pure strategies for the firm. It shows that if the firm puts strictly positive mass at one point of the support, it creates a discontinuity in the demand exactly at that point. Then, it wants to take advantage of it.

Lemma 3 and Lemma 6 also imply that we can think of $v^*(p) = \inf\{v : p^*(v) \geq p\}$ as the inverse of $p^*(v)$: for any p in the support, $p^*(v^*(p)) = p$. Furthermore, $V^*(p) = \{v : p^*(v) \geq p\} = [v^*(p), \overline{v}]$ and the demand at any price $\gamma(V^*(p)) = 1 - G(v^*(p))$.

Let $\underline{p} = \min \operatorname{supp} F$ and $\overline{p} = \max \operatorname{supp} F$. Note that $\underline{p} > 0$, as otherwise the firm would make zero profits and there is always a profitable deviation to some price greater than zero.

Profits from $\underline{p}$

The profits from $\underline{p}$ are $\underline{p}(1 - G(v^*(\underline{p}))$. Indeed, for any $v \leq \underline{p}$, $F(v) = 0$. Therefore, $p^*(v) = v + \lambda v F(v) = v$ is a PE cutoff. This being the smallest PE cutoff possible, it is the PPE cutoff by Proposition 1. Moreover, for any v, $p^*(v) \geq v$. Therefore, all types above $\underline{p}$ accepts it and all types below reject it, i.e., $v^*(\underline{p}) = \underline{p}$. Profits when offering $\underline{p}$ are then $\underline{p}\,(1 - G(\underline{p}))$. These must be the equilibrium profits.

Finding the equilibrium strategy

For any $p \in \operatorname{supp} F$, by indifference on the support,

$$\pi^* \equiv \underline{p}(1 - G(\underline{p})) = p(1 - G(v^*(p))).$$

Therefore,

$$v^*(p) = G^{-1}\left(\frac{p - \pi^*}{p}\right),$$

for all $p \in \operatorname{supp} F$. Since $\frac{p-\pi^*}{p} \in [0, 1)$ for all $p \geq \underline{p}$, the expression above is well-defined.

The equilibrium strategy F must guarantee that a PE cutoff of $v^*(p)$ is $p^*(v^*(p)) = p$:

$$v^*(p) - p = -\lambda v^*(p)F(p) \Rightarrow F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)}, \tag{3}$$

using that $p^*(v^*(p)) = p$. Note that this discussion also implies that in equilibrium, $p^*(v) = \frac{\pi^*}{1-G(v)}$.

Pinning down $\underline{p}$. For any $p < \underline{p}$, $F(p) = 0$. Therefore, $v^*(p) = p - \lambda v^*(p)F(p) = p$. In equilibrium, we must have

$$\pi^* \geq p(1 - G(p)).$$

For any $p > \underline{p}$, we have:

$$F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)} \Rightarrow G^{-1}(\frac{p-\pi^*}{p}) = p - \lambda G^{-1}\big(\frac{p-\pi^*}{p}\big)F(p) < p$$
$$\Leftrightarrow p(1 - G(p)) < \pi^*,$$

using that $F(p) > 0$ for $p > \underline{p}$. Therefore, we have $\underline{p} = \arg\max_p p(1 - G(p))$.[13]

F is well-defined on the support

I check here that $\frac{p-G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)}$ is a strictly increasing and positive function. For all $p \geq \underline{p}$,

$$p - G^{-1}(\frac{p-\pi^*}{p}) \geq 0 \Leftrightarrow G(p) \geq \frac{p-\pi^*}{p} \Leftrightarrow \pi^* \geq p(1 - G(p))$$

This is satisfied because $\pi^* = \max p(1 - G(p))$.

I now show that for all $p > \underline{p}$, the derivative of F is strictly positive. This follows from the following fact

1. For $p > \underline{p}$, $\frac{1-G(p)}{p} < g(p)$: by strict concavity of the profit function, the derivative is negative after the maximum.

Taking the derivative of F,

$$F'(p) \propto \frac{G^{-1}\left(\frac{p-\pi^*}{p}\right) - p\frac{\pi^*}{p^2}\frac{1}{g\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)}}{\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)^2} > \frac{G^{-1}\left(\frac{p-\pi^*}{p}\right) - \frac{\pi^*}{p}\frac{G^{-1}\left(\frac{p-\pi^*}{p}\right)}{1-G\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)}}{\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)^2} = 0,$$

[13] $\underline{p}$ is well-defined by the strict concavity of $p(1 - G(p))$.

using fact 1 to get the inequality and rearranging to get the equality.

Pinning down $\overline{p}$. We have to check that there exists a $\overline{p}$, such that $F(\overline{p}) = 1$. To do that, we will check that there exists p such that $F(p) = 1$. Note that $F(\underline{p}) = 0$ and $F(\overline{v} + \lambda\overline{v}) = \frac{\overline{v}+\lambda\overline{v}-G^{-1}\left(\frac{\overline{v}+\lambda-\pi^*}{\overline{v}+\lambda\overline{v}}\right)}{\lambda G^{-1}\left(\frac{\overline{v}+\lambda\overline{v}-\pi^*}{\overline{v}+\lambda\overline{v}}\right)} > 1$ (rearranging and using that $G(\overline{v}) = 1$). Therefore, by continuity of F, there exists, $\underline{p} < p < \overline{v} + \lambda\overline{v}$ such that $F(p) = 1$.

Preferred Personal Equilibrium The last step is to check that the PE cutoffs pinned down by equation (3) are PPE cutoffs. This follows from the fact that the PE pinned down by equation (3) is unique:

$$\begin{aligned}
\frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)} &\geq \frac{p-v}{\lambda v} \\
\Leftrightarrow G(v) &\geq \frac{p-\pi^*}{p} \\
\Leftrightarrow p &\leq \frac{\pi^*}{1-G(v)} = p^*(v).
\end{aligned}$$

Hence, it is also a PPE cutoff. □

B Proof of Lemma 1

The firm's strategy and test as defined are Markov kernels. For simplicity, for any measurable space $(X, \mathcal{X})$, I simply write X. A mapping $Q : X \times Y \to [0, 1]$ is a Markov kernel if (i) for any measurable $A \subseteq Y$, $Q(A|\cdot)$ is measurable and (ii) for any $x \in X$, $Q(\cdot|x)$ is a probability measure. I will make repeated use of composition of Markov kernels. Let $Q : X \times Y \to [0, 1]$ and $P : Y \times Z \to [0, 1]$ be two Markov kernels. Then the composition of $P \circ Q : X \times Z \to [0, 1]$ defined as

$$(P \circ Q)(A|x) = \int_Y P(A|y)dQ(y|x) \text{ for all measurable } A \subseteq Z \text{ and } x \in X$$

is a Markov kernel. Furthermore, for all bounded measurable $f : Z \to \mathbb{R}$,

$$\int f(z)d(P \circ Q)(z|x) = \int \int f(z)dP(z|y)dQ(y|x)$$

See e.g., Bauer (1996), chapter VIII, §36.

Proof. Start with a test (F, S) and an equilibrium $\big(P, (Q_v^*)_v\big)$. We are going to construct a new test $(\tilde{F}, \tilde{S})$ and an equilibrium $\big(\delta_s, (Q_v^*)_v\big)$ such that all players get the same payoffs. We can construct the Markov kernel $\tilde{F} : V \times \tilde{S} \to [0, 1]$, where $\tilde{S} = \cup_{s \in S} \operatorname{supp} P(s) \subseteq \mathbb{R}$ as $\tilde{F} = P \circ F$.

Let's first verify that the PPE do not change. Fix a v.

$$
\begin{aligned}
Q_v &= \int_S \int_{\mathbb{R}} \mathbb{1}[p \leq v + \lambda v Q_v] dP(p|s) dF(s|v) \\
&= \int_{\tilde{S}} \mathbb{1}[p \leq v + \lambda v Q_v] d\tilde{F}(p|v) \\
&= \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[p \leq v + \lambda v Q_v] d\delta_{\tilde{s}}(p) d\tilde{F}(\tilde{s}|v),
\end{aligned}
$$

and
$$
\begin{aligned}
W_{(F,S)}(v|Q_v) &= \int_S \int_{\mathbb{R}} \Bigg(\mathbb{1}[p \leq v + \lambda v Q_v](v - p) + \mathbb{1}[p > v + \lambda v Q_v](-\lambda v Q_v) \Bigg) dP(p|s) dF(s|v) \\
&= \int_{\tilde{S}} \Bigg(\mathbb{1}[p \leq v + \lambda v Q_v](v - p) + \mathbb{1}[p > v + \lambda v Q_v](-\lambda v Q_v) \Bigg) d\tilde{F}(p|v) \\
&= \int_{\tilde{S}} \int_{\mathbb{R}} \Bigg(\mathbb{1}[p \leq v + \lambda v Q_v](v - p) + \mathbb{1}[p > v + \lambda Q_v](-\lambda v Q_v) \Bigg) d\delta_{\tilde{s}}(p) d\tilde{F}(\tilde{s}|v) \\
&= W_{(\tilde{F},\tilde{S})}(v|Q_v).
\end{aligned}
$$

This shows that the set of PE and the payoff from each of them does not change under the new test and equilibrium. Moving to the firm to the firm's payoffs, we get similarly

$$
\begin{aligned}
\mathbb{E}_{(F,S)}[\pi(P)|(Q_v^*)_v] &= \int_V \int_S \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)] \, p \, dP(p|s) dF(s|v) dG \\
&= \int_V \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)] \, p \, d\delta_{\tilde{s}}(p) d\tilde{F}(\tilde{s}|v) dG \\
&= \mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\delta_{\tilde{s}})|(Q_v^*)_v].
\end{aligned}
$$

Note that the integrand is bounded because $\mathbb{1}[v \in V^*(p)] = 0$ for $p > \overline{v}(1 + \lambda)$ and offering a negative is a strictly dominated action. The last step is to check that any deviation from $\delta_{\tilde{s}}(p)$ is suboptimal in the new test. I show that from any strategy in $(\tilde{F}, \tilde{S})$, we can construct a strategy in (F,S) that yields the same payoff. Let $\tilde{P}:\tilde{S}\times\mathbb{R}\to[0,1]$ a strategy in $(\tilde{F},\tilde{S})$. Define the Markov kernel $P':S\times\tilde{S}\to[0,1]$ as $P'=\tilde{P}\circ P$. Then,

$$\begin{aligned}\mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\tilde{P})|(Q_v^*)_v] &= \int_V\int_{\tilde{S}}\int_{\mathbb{R}} \mathbb{1}[v\in V^*(p)]\,p\,d\tilde{P}(p|\tilde{s})d\tilde{F}(\tilde{s}|v)dG(v)\\ &= \int_V\int_S\int_{\tilde{S}}\int_{\mathbb{R}} \mathbb{1}[v\in V^*(p)]\,p\,d\tilde{P}(p|\tilde{s})dP(\tilde{s}|s)dF(s|v)dG(v)\\ &= \int_V\int_S\int_{\mathbb{R}} \mathbb{1}[v\in V^*(p)]\,p\,dP'(p|s)dF(s|v)dG(v)\\ &\le \mathbb{E}_{(F,S)}[\pi(P)|(Q_v^*)_v] = \mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\delta_{\tilde{s}})|(Q_v^*)_v].\end{aligned}$$

□

C Proofs of Proposition 6, Lemma 2 and Proposition 7

This section is organised as follows. I first introduce some notation, definition and results that will be useful for the proofs of Proposition 6, Lemma 2 and Proposition 7. I show how to approximate a candidate test (Definition 8 and Lemma 7) and give sufficient condition for this approximation to satisfy all the obedience constraints (Lemma 8). I then show Proposition 6 in appendix C.1, Lemma 2 in appendix C.2 and Proposition 7 in appendix C.3.

Fix some $\epsilon>0$ and let $\eta=\epsilon\cdot\frac{(\overline{v}(1+\lambda)-\underline{v})^2}{1+\epsilon(\overline{v}(1+\lambda)-\underline{v})}$, $A=\frac{\overline{v}\max g(v)}{\underline{v}^2\min g(v)}$, $B=\frac{\underline{v}\min g(v)}{\overline{v}^2\max g(v)(1+\lambda)}$. Let

$$\nu(t)=\begin{cases}\min\left\{\frac{1}{1+\lambda},B(\frac{1}{A}-\eta)(1-e^{-A(t-\underline{v})})\right\}, & t\in[\underline{v},\overline{v}-\eta],\\ \min\left\{\frac{1}{1+\lambda},B\Big(t-\overline{v}-e^{-At}\big((\frac{1}{A}-\eta)e^{A\underline{v}}-\frac{e^{A(\overline{v}-\eta)}}{A}\big)\Big)\right\}, & t\in[\overline{v}-\eta,\overline{v}],\\ \min\left\{\frac{1}{1+\lambda},-Be^{-At}\big((\frac{1}{A}-\eta)e^{A\underline{v}}-\frac{e^{A(\overline{v}-\eta)}}{A}\big)\right\}, & t\in[\overline{v},\overline{v}(1+\lambda)].\end{cases}$$

Note that for η small enough, $\nu(t)$ is strictly positive everywhere but at $t=\underline{v}$.

Let $\hat{v}(s):=\min\{\overline{v},s\}$.We can rewrite equation (L-OB) as

$$-v'(s)\cdot\frac{sg\big(v(s)\big)}{\lambda v(s)}+\int_{v(s)}^{\hat{v}(s)}\frac{1}{\lambda v}g(v)dv=0,\ \text{ with } v(\underline{v})=\underline{v}, \tag{L-OB}$$

by using the fact that for a censored commitment distribution, $f(s|v)=\frac{1}{\lambda v}$ for $v\in[v^*(s),\hat{v}(s)]$ and $f(s|v)=0$ for $v>\hat{v}(s)$.

The following test will be our way of approximating the candidate test (Definition 7).

Definition 8 (ϵ-candidate test)**.** *Let $\epsilon > 0$. An ϵ-candidate test F_ϵ is a test such that there exist functions $p_\epsilon : V \to [\underline{v}, \overline{v}(1+\lambda)]$, $v_\epsilon : [\underline{v}, \overline{v}(1+\lambda)] \to V$, $\underline{p}_\epsilon : V \to V$ and for each v, a continuous cdf*

$$F_\epsilon(s|v) = \begin{cases} 0 & s < \underline{p}_\epsilon(v) \\ \int_{\underline{p}_\epsilon(v)}^{s} \frac{1}{\lambda v(1+\epsilon(t-\underline{v}))} dt & s \in [\underline{p}_\epsilon(v), p_\epsilon(v)] \\ \tilde{F}_\epsilon(s|v) & s > p_\epsilon(v), \end{cases}$$

such that $v_\epsilon(s) = \max\{\tilde{v}(s), \frac{s}{1+\lambda}\}$, where $\tilde{v}(s)$ is the solution to

$$\tilde{v}'(s) = \frac{\tilde{v}(s)}{sg(\tilde{v}(s))} \int_{\{v: v \geq \tilde{v}(s),\, s \geq \underline{p}_\epsilon(v)\}} \frac{g(v)}{v} dv, \text{ with } \tilde{v}(\underline{v}) = \underline{v}, \tag{4}$$

and p_ϵ are the WTPs induced by the test F_ϵ, $p_\epsilon = v_\epsilon^{-1}$ and $\tilde{F}_\epsilon(\cdot|v)$ is either flat or $= \frac{s - v_\epsilon(s)}{\lambda v_\epsilon(s)}$.

Note that existence of an ϵ-candidate test is not guaranteed because $\underline{p}_\epsilon$ depends implicitly on v_ϵ and thus equation (4) is a non-standard differential equation.[14] The following lemma guarantees that ϵ-candidate tests do exist and that they converge to the desired test. Lemma 7 is the only result for which I need the assumption of Lipschitz continuity on g.

Lemma 7. *If ϵ is small enough, there exists an ϵ-candidate test F_ϵ.*

Furthermore, the distributions $(F_\epsilon(\cdot|v))_v$, $\underline{p}_\epsilon$ and the solution of (4) converge pointwise to censored commitment distributions, $\underline{p}(v) = v$ and to the solution of (L-OB) respectively as $\epsilon \to 0$.

Proof. I first show existence of an ϵ-candidate test. To construct such test, I first show the existence of a function v_ϵ as described in the definition of ϵ-candidate tests. To do so, I define a map $\Phi : \Psi \to \Psi$, where Ψ is a set of functions defined below. The goal is to show that a fixed-point of Φ defines the function v_ϵ. We will use Schauder fixed point theorem: if Φ is a continuous self-map from a non-empty compact subset of a normed linear space, a fixed point of Φ exists (see e.g., Ok, 2007, p. 469). The proof defines Φ then verifies that all the condition to apply the theorem are satisfied.

Define Ψ as the set of functions $\psi : [\underline{v}, \overline{v}(1+\lambda)] \to [\underline{v}, \overline{v}]$ such that

(i) $\psi(s) \leq s$ and $\psi(\underline{v}) = \underline{v}$

[14] To see this note that to have $p_\epsilon(v)$ respect the PPE constraint, we must have $\int_{\underline{p}_\epsilon(v)}^{p_\epsilon(v)} \frac{1}{\lambda v(1+\epsilon(t-\underline{v}))} dt = \frac{p_\epsilon(v) - v}{\lambda v}$ and $v_\epsilon = p_\epsilon^{-1}$.

(ii) For any x, y, $x > y$, $\psi(x) - \psi(y) \geq (x-y) \cdot \min_{t \in [y,x]} \nu(t)$

(iii) ψ is Lipschitz continuous with Lipschitz constant $\max\{A(\overline{v} - \underline{v}), \frac{1}{1+\lambda}\}$

Note that property (ii) implies that any $\psi \in \Psi$ is strictly increasing because $\nu(t) > 0$ for all $t > \underline{v}$.

Let's define the map $\Phi : \Psi \to \Psi$ as follows.

For any $\psi \in \Psi$, define $p := \psi^{-1}$. The function p is well-defined because ψ is continuous and strictly increasing. Next define $\underline{p} : V \to [\underline{v}, \overline{v}(1+\lambda)]$ as the solution to

$$\int_{\underline{p}(v)}^{p(v)} \frac{1}{1+\epsilon(s-\underline{v})} ds = p(v) - v.$$

Note that it is continuous, $\underline{p}(v) \leq v$ and $\underline{p}(\underline{v}) = \underline{v}$. Finally, I show below that if ϵ is small enough, it is strictly increasing. Take v, v' with $v > v' > \underline{v}$. Then,

$$\begin{aligned}
\int_{\underline{p}(v)}^{\underline{p}(v')} \frac{1}{1+\epsilon(s-\underline{v})} ds &= \int_{p(v')}^{p(v)} (1 - \frac{1}{1+\epsilon(s-\underline{v})}) - (v - v') \\
&< \frac{\epsilon(p(v) - \underline{v})}{1+\epsilon(p(v)-\underline{v})}(p(v) - p(v')) - (v - v') \\
&\leq (\frac{\epsilon(p(v) - \underline{v})}{1+\epsilon(p(v)-\underline{v})} \cdot \frac{1}{\nu(p(v'))} - 1)(v - v'),
\end{aligned}$$

where the inequality is strict on the second line because p is strictly increasing and I have used property (ii) and the fact that $\psi = p^{-1}$ for the last inequality. Now note that it is enough to establish that $\frac{\epsilon(p(v') - \underline{v})}{1+\epsilon(p(v')-\underline{v})} \cdot \frac{1}{\nu(p(v'))} - 1 < 0$ for all $v' > \underline{v}$ to get that $\underline{p}(v) > \underline{p}(v')$ for v close enough to v'. The expression $\frac{p(v') - \underline{v}}{\nu(p(v'))}$ is continuous and bounded except possibly as $v' \to \underline{v}$. In a neighbourhood of $p(\underline{v})$, $\nu(t) = B(\frac{1}{A} - \eta)(1 - e^{-A(t-\underline{v})})$, therefore we get

$$\lim_{v' \to \underline{v}} \frac{(p(v') - \underline{v'})}{\nu(p(v'))} = \lim_{x \to \underline{v}} \frac{(x - \underline{v'})}{B(\frac{1}{A} - \eta)(1 - e^{-A(x-\underline{v})})} = \frac{1}{B(\frac{1}{A} - \eta)A}.$$

Therefore, if ϵ is small enough, $\underline{p}(v) > \underline{p}(v')$ for v close enough to v'. This in turn implies that $\underline{p}$ is everywhere strictly increasing.

Abusing notation, extend $\underline{p}^{-1}(s)$ so that $\underline{p}^{-1}(s) = \overline{v}$ for $s \in (\underline{p}(\overline{v}), \overline{v}(1+\lambda)]$. Let $\tilde{v} : [\underline{v}, \overline{v}(1+\lambda)] \to V$ be the solution to

$$\tilde{v}'(s) = \frac{\tilde{v}(s)}{sg(\tilde{v}(s))} \int_{\tilde{v}(s)}^{\underline{p}^{-1}(s)} \frac{g(v)}{v} dv,$$

with $\tilde{v}(\underline{v}) = \underline{v}$. This equation satisfies the conditions of Picard's existence theorem (recall that we assume that $\underline{v} > 0$, that g is Lipschitz continuous and that we showed that $\underline{p}^{-1}$ is continuous), therefore a unique solution exists.

Finally, let $\Phi(\psi)(s) = \max\{\tilde{v}(s), \frac{s}{1+\lambda}\}$.

Now let's verify that $\Phi(\psi)$ belongs to Ψ.

Property (iii) is easily verified by using equation (4).

To verify (i) and (ii), observe that $\max_{v\in V} v - \underline{p}(v) \leq \eta$. Therefore,

$$\tilde{v}'(s) \leq A(\min\{s+\eta, \overline{v}\} - \tilde{v}(s))$$

where $\eta = \epsilon \cdot \frac{(\overline{v}(1+\lambda)-\underline{v})^2}{1+\epsilon(\overline{v}(1+\lambda)-\underline{v})}$ and $A = \frac{\overline{v}\max g(v)}{\underline{v}^2 \min g(v)}$. This implies that

$$\tilde{v}(s) \leq b(s) \equiv \begin{cases} -(\frac{1}{A} - \eta)(1 - e^{-A(s-\underline{v})}) + s, & s \in [\underline{v}, \overline{v} - \eta], \\ \overline{v} + e^{-As}\big((\frac{1}{A} - \eta)e^{A\underline{v}} - \frac{e^{A(\overline{v}-\eta)}}{A}\big), & s \in [\overline{v} - \eta, \overline{v}(1+\lambda)], \end{cases}$$

where $b(s)$ is the solution to $b'(s) = A(\min\{s+\eta, \overline{v}\} - b(s))$ with $b(\underline{v}) = \underline{v}$ (see e.g., Chapter III, Theorem 4.1 in Hartman, 2002). In fact, $\nu(s) = \min\{\frac{1}{1+\lambda}, B(\min\{s, \overline{v}\} - b(s))\}$. Because $s \geq \min\{s, \overline{v}\} \geq b(s) \geq \tilde{v}(s)$ and $\tilde{v}(\underline{v}) = \underline{v}$, property (i) is verified.

Finally, to verify property (ii), we can bound $\tilde{v}'(s)$ from below by observing that

$$\tilde{v}'(s) \geq B(\min\{s, \overline{v}\} - \tilde{v}(s)) \geq B(\min\{s, \overline{v}\} - b(s)) \geq \nu(s).$$

Take two x, y with $x > y$. There are four possibilities to consider. First, if $\Phi(\psi)(s) = \tilde{v}(s)$ for $s = x, y$. In this case, by the mean value theorem, there is $t \in [y, x]$ such that

$$\frac{\tilde{v}(x) - \tilde{v}(y)}{x - y} = \tilde{v}'(t) \geq \nu(t) \geq \min_{s\in[y,x]} \nu(s).$$

If $\Phi(\psi)(s) = \frac{s}{1+\lambda}$ for $s = x, y$, then

$$\frac{\frac{x}{1+\lambda} - \frac{y}{1+\lambda}}{x - y} = \frac{1}{1+\lambda} \geq \min_{s\in[y,x]} \nu(s).$$

If $\Phi(\psi)(x) = \tilde{v}(x)$ and $\Phi(\psi)(y) = \frac{y}{1+\lambda}$, then $\tilde{v}(x) \geq \frac{x}{1+\lambda}$ and therefore,

$$\frac{\tilde{v}(x) - \frac{y}{1+\lambda}}{x - y} \geq \frac{\frac{x}{1+\lambda} - \frac{y}{1+\lambda}}{x - y} \geq \min_{s\in[y,x]} \nu(s).$$

And finally, if $\Phi(\psi)(x) = \frac{x}{1+\lambda}$ and $\Phi(\psi)(y) = \tilde{v}(y)$, we can use a similar argument using $\frac{x}{1+\lambda} \geq \tilde{v}(x)$.

Endow Ψ with the sup-norm. The set Ψ is convex, non-empty and bounded. The set of Lipschitz continuous function under the sup-norm is closed and property (ii) is also preserved under limits using similar arguments. Property (i) is also preserved under limits, therefore Ψ is closed.

I now argue that Φ is continuous in the sup-norm. The function Φ is the composition of different functions and I will show the continuity of each one. First, the inverse function of strictly increasing and continuous function on a compact space is continuous in the sup-norm (Barvinek et al., 1991). The function $\phi \to \max\{\frac{s}{1+\lambda}, \phi(s)\}$ is also continuous. Let's show continuity of the function that takes continuous $\phi : [\underline{v}, \overline{v}(1+\lambda)] \to V$ to the solution of

$$\tilde{v}'(s) = \frac{\tilde{v}(s)}{sg(\tilde{v}(s))} \int_{\tilde{v}(s)}^{\phi(s)} \frac{g(v)}{v} dv, \quad \text{with } \tilde{v}(\underline{v}) = \underline{v}.$$

Take ϕ_n converging uniformly to ϕ. For any $\delta > 0$, there is N such that $|\phi_n(s) - \phi(s)| < \delta$ for all s and $n > N$. Therefore,

$$\Big|\frac{\tilde{v}(s)}{sg(\tilde{v}(s))} \int_{\tilde{v}(s)}^{\phi_n(s)} \frac{g(v)}{v} dv - \frac{\tilde{v}(s)}{sg(\tilde{v}(s))} \int_{\tilde{v}(s)}^{\phi(s)} \frac{g(v)}{v} dv\Big| \leq A|\phi_n(s) - \phi(s)| < A\delta,$$

for all s and $n > N$. Therefore, $\frac{\tilde{v}(s)}{sg(\tilde{v}(s))} \int_{\tilde{v}(s)}^{\phi_n(s)} \frac{g(v)}{v} dv$ converges uniformly to $\frac{\tilde{v}(s)}{sg(\tilde{v}(s))} \int_{\tilde{v}(s)}^{\phi(s)} \frac{g(v)}{v} dv$. By Theorem I 2.4 in Hartman (2002), $\tilde{v}_n \to \tilde{v}$ uniformly.

The last function we need to care of is the function that takes $\phi : V \to [\underline{v}, \overline{v}(1+\lambda)]$ and outputs the solution to $\int_{\underline{p}(v)}^{\phi(v)} \frac{1}{1+\epsilon(s-\underline{v})} ds - \phi(v) + v = 0$. Take ϕ_n converging uniformly to ϕ. For any $\delta > 0$, there is N such that $|\phi_n(v) - \phi(v)| < \delta$ for all v and $n > N$. We have for any v and $n > N$, by definition of $\underline{p}$,

$$\begin{aligned}
\Big|\frac{1}{1+\epsilon(\overline{v}(1+\lambda) - \underline{v})}\Big| \cdot |\underline{p}_n(v) - \underline{p}(v)| &\leq \Big| \int_{\underline{p}_n(v)}^{\underline{p}(v)} \frac{1}{1+\epsilon(s-\underline{v})} ds\Big| \\
&= \Big| \int_{\phi(v)}^{\phi_n(v)} \Big(\frac{1}{1+\epsilon(s-\underline{v})} - 1\Big) ds\Big| \\
&\leq \max_s \Big|\frac{1}{1+\epsilon(s-\underline{v})} - 1\Big| \cdot |\phi_n(v) - \phi(v)| \\
&< \max_s \Big|\frac{1}{1+\epsilon(s-\underline{v})} - 1\Big|\delta
\end{aligned}$$

Therefore $\underline{p}_n \to \underline{p}$ in the sup-norm.

Finally, by the Arzela-Ascoli theorem, a sequence of bounded Lipschitz continuous functions (with same Lipschitz constant) admits a convergent subsequence. Because Ψ is closed, this also means that Ψ is compact.

Therefore, because Φ is a continuous self-map from a non-empty compact subset of a normed linear space, a fixed point of Φ exists (Schauder fixed point theorem, see e.g., Ok, 2007, p. 469). Call this fixed-point v_ϵ. Let p_ϵ be the inverse of v_ϵ and define $\underline{p}_\epsilon$ as in the second step of Φ. Define the test as follows:

$$F_\epsilon(s|v) = \begin{cases} 0 & s < \underline{p}_\epsilon(v) \\ \int_{\underline{p}_\epsilon(v)}^{s} \frac{1}{\lambda v(1+\epsilon(t-\underline{v}))} dt & s \in [\underline{p}_\epsilon(v), p_\epsilon(v)] \\ \tilde{F}_\epsilon(s|v) & s > p_\epsilon(v), \end{cases}$$

where $\tilde{F}_\epsilon(s|v)$ is the lowest function that is everywhere above $\frac{s-v_\epsilon(s)}{\lambda v_\epsilon(s)}$, weakly increasing and with $\tilde{F}_\epsilon(p_\epsilon(v)|v) = \int_{\underline{p}_\epsilon(v)}^{p_\epsilon(v)} \frac{1}{\lambda v(1+\epsilon(t-\underline{v}))} dt$. This function is therefore either flat or increasing on a section where $\frac{s-v_\epsilon(s)}{\lambda v_\epsilon(s)}$ is as well. This means that the density is always smaller than $\max\{0, \big(\frac{s-v_\epsilon(s)}{\lambda v_\epsilon(s)}\big)'\}$. Note also that if $v_\epsilon(s) = \frac{s}{1+\lambda}$, then $f_\epsilon(s|v) = 0$ for $s > p_\epsilon(v)$. Furthermore, $\underline{p}_\epsilon(v) \in [v-\eta, v]$.

Because $\underline{p}_\epsilon(v) \in [v-\eta, v]$ for each v, $\underline{p}_\epsilon$ converges uniformly to v. It also implies that $\tilde{v}$ converges pointwise to the solution of (L-OB) by the same reasoning that showed uniform continuity with respect to $\underline{p}$. Given, that $\underline{p}_\epsilon(v) \to v$ as $\epsilon \to 0$, $F_\epsilon(\cdot|v)$ converges pointwise to censored commitment distributionsfor each v. □

In the following lemma, I give two sufficient conditions for ϵ-candidate tests to be obedient.

Lemma 8. *Let $\tilde{G} \in \Delta[\underline{v}, \overline{v}]$ such that $\tilde{G}$ admits a strictly positive and Lipschitz continuous density $\tilde{g}$. Let F_ϵ be an ϵ-candidate test.*

- *If $v \sim U[\underline{v}, \overline{v}]$, then F_ϵ satisfies all obedience constraints if ϵ is small enough.*
- *If $v \sim \tilde{G}(\cdot \,|\, [a, a+\delta])$ for $a < \overline{v}$ and $\delta > 0$, then, for any $a \in [\underline{v}, \overline{v})$, F_ϵ satisfies all obedience constraints if δ and ϵ are small enough.*

Proof. Take an ϵ-candidate test and associated WTP and its inverse, p_ϵ and v_ϵ. Define $\underline{p}_\epsilon$ as $F_\epsilon(\underline{p}_\epsilon(v)|v) = 0$. Let $\hat{v}_\epsilon(s)$ be the inverse of $\underline{p}_\epsilon$ extended such that $\hat{v}_\epsilon(s) = \overline{v}$ for $s \in (\underline{p}_\epsilon(\overline{v}), \overline{v}(1+\lambda)]$. This implies $\hat{v}_\epsilon(s) \in [\min\{\overline{v}, s\}, \min\{\overline{v}, s+\eta\}]$. Let $\pi(s, s') = \int_{v_\epsilon(s')}^{\hat{v}_\epsilon(s)} s' f_\epsilon(s|v) g(v) dv$. To satisfy the obedience constraints, we must have

$$s \in \arg\max_{s'} \pi(s, s').$$

Let's examine upward and downward deviations separately. First, upward deviation. We can write the profits from offering price s' at signal s as $\pi(s,s') = \int_{v_\epsilon(s')}^{\hat{v}_\epsilon(s)} s' f_\epsilon(s|v)g(v)dv$. The derivative with respect s' is proportional to

$$-v_\epsilon'(s')\frac{s'g(v_\epsilon(s'))}{v_\epsilon(s')} + \int_{v_\epsilon(s')}^{\hat{v}_\epsilon(s)} \frac{g(v)}{v}dv.$$

Setting $s' = s$, we get $-v_\epsilon'(s)\frac{sg(v_\epsilon(s))}{v_\epsilon(s)} + \int_{v_\epsilon(s)}^{\hat{v}_\epsilon(s)} \frac{g(v)}{v}dv = 0$ when $v_\epsilon(s) = \tilde{v}(s)$ (where $\tilde{v}$ solves equation (4)). When $v_\epsilon(s) = \frac{s}{1+\lambda}$, we need

$$-s\frac{1/(1+\lambda)}{s/(1+\lambda)}g(v_\epsilon(s)) + \int_{\frac{s}{1+\lambda}}^{\hat{v}_\epsilon(s)} \frac{g(v)}{v}dv \le 0.$$

Case where $v \sim U[\underline{v}, \overline{v}]$: In this case, we can solve for $\tilde{v}$ in closed form and we obtain $\tilde{v}(s) = \hat{v}_\epsilon(s)\exp\frac{-\hat{v}_\epsilon(s)+\underline{v}}{s}$.

Note that $v^*(s) = \frac{s}{1+\lambda}$ when $\frac{s}{1+\lambda} \ge \hat{v}_\epsilon(s)\exp\frac{-\hat{v}_\epsilon(s)+\underline{v}}{s}$. Using that $\hat{v}_\epsilon(s) \le s+\eta$, we get

$$\log\frac{\hat{v}_\epsilon(s)(1+\lambda)}{s} \le \frac{\hat{v}_\epsilon(s)-\underline{v}}{s} \le \frac{s+\eta-\underline{v}}{s} \le 1, \text{ for } \eta \le \underline{v}.$$

Case where $v \sim \tilde{G}(\cdot|v \in [a, a+\delta])$: In this case, we obtain, $g(v) = \frac{\tilde{g}(v)}{\tilde{G}(a+\delta)-\tilde{G}(a)}$. The derivative is negative if

$$-\frac{\tilde{g}(\frac{s}{1+\lambda})}{\tilde{G}(a+\delta)-\tilde{G}(a)} + \int_{\frac{s}{1+\lambda}}^{\hat{v}_\epsilon(s)} \frac{\tilde{g}(v)}{(\tilde{G}(a+\delta)-\tilde{G}(a))v}dv \le 0$$
$$\Leftrightarrow -\tilde{g}(\frac{s}{1+\lambda}) + \int_{\frac{s}{1+\lambda}}^{\hat{v}_\epsilon(s)} \frac{\tilde{g}(v)}{v}dv \le 0$$

And observe that because $[\frac{s}{1+\lambda}, \hat{v}_\epsilon(s)] \subseteq [a, a+\delta]$,

$$-\tilde{g}(\frac{s}{1+\lambda}) + \int_{\frac{s}{1+\lambda}}^{\hat{v}_\epsilon(s)} \frac{\tilde{g}(v)}{v}dv \le -\min_{v\in V}\tilde{g}(v) + \frac{\max_{v\in V}\tilde{g}(v)}{\underline{v}}\delta < 0 \text{ for } \delta \text{ small enough.}$$

Then observe that for $s' > s$,

$$-v_\epsilon'(s')\frac{s'g(v_\epsilon(s'))}{v_\epsilon(s')} + \int_{v_\epsilon(s')}^{\hat{v}_\epsilon(s)} \frac{g(v)}{v}dv \le -v_\epsilon'(s')\frac{s'g(v_\epsilon(s'))}{v_\epsilon(s')} + \int_{v_\epsilon(s')}^{\hat{v}_\epsilon(s')} \frac{g(v)}{v}dv \le 0$$

because $\hat{v}_\epsilon(\cdot)$ is weakly increasing. Therefore, there is no profitable upward deviation.

Now, for downward deviations, the payoffs are

$$\int_{v_\epsilon(s)}^{\hat{v}_\epsilon(s)} s' \frac{g(v)}{(\lambda(1+\epsilon(s-\underline{v}))v} dv + \int_{v_\epsilon(s')}^{v_\epsilon(s)} s' \tilde{f}_\epsilon(s|v) g(v) dv$$

Note that $\left(\frac{s-v_\epsilon(s)}{\lambda v_\epsilon(s)}\right)' = \frac{v_\epsilon(s)-(v_\epsilon(s))'s}{\lambda v_\epsilon(s)^2}$. Note also that $(1+\epsilon(s-\underline{v}))\cdot\frac{v_\epsilon(s)-(v_\epsilon(s))'s}{v_\epsilon(s)} \leq 1$ for ϵ small enough for all s. To see this first let's rearrange this inequality as $\epsilon(s-\underline{v}) \leq (1+\epsilon(s-\underline{v}))\frac{v'_\epsilon(s)s}{v_\epsilon(s)}$. All the functions are bounded away from zero except $v'_\epsilon(s)$ and $(s-\underline{v})$ at $s=\underline{v}$. So it is enough to check that this inequality can be satisfied as $s \to \underline{v}$. From the proof of Lemma 7, $v'_\epsilon(s) \geq \nu(s)$ therefore it is enough to check that $\lim_{s\to\underline{v}} \frac{\nu(s)}{s-\underline{v}} > 0$:

$$\begin{aligned}
\lim_{s\to\underline{v}} \frac{\nu(s)}{s-\underline{v}} &= \lim_{s\to\underline{v}} \frac{B(\frac{1}{A}-\eta)(1-e^{-A(s-\underline{v})})}{s-\underline{v}} \\
&= \lim_{s\to\underline{v}} \frac{B(\frac{1}{A}-\eta)Ae^{-A(s-\underline{v})}}{1} = B(\frac{1}{A}-\eta)A > 0, \text{ for } \eta \text{ small enough.}
\end{aligned}$$

Therefore for ϵ small enough, $(1+\epsilon(s-\underline{v}))\cdot\frac{v_\epsilon(s)-(v_\epsilon(s))'s}{v_\epsilon(s)} \leq 1$. Taking the derivative and evaluating it at $s'=s$, we have, using the notation $z_+ = \max\{0,z\}$,

$$\begin{aligned}
&\int_{v_\epsilon(s)}^{\hat{v}_\epsilon(s)} \frac{g(v)}{(\lambda(1+\epsilon(s-\underline{v}))v} dv - v'_\epsilon(s) s g(v_\epsilon(s)) \tilde{f}_\epsilon(s|v_\epsilon(s)) \\
&\qquad \geq \int_{v_\epsilon(s)}^{\hat{v}_\epsilon(s)} \frac{g(v)}{(\lambda(1+\epsilon(s-\underline{v}))v} dv - v'_\epsilon(s) s g(v_\epsilon(s)) \Big(\frac{v_\epsilon(s)-(v_\epsilon(s))'s}{\lambda v_\epsilon(s)}\Big)_+.
\end{aligned}$$

Furthermore, we have $(1+\epsilon(s-\underline{v}))\cdot\frac{v_\epsilon(s)-(v_\epsilon(s))'s}{v_\epsilon(s)} \leq 1$ for all s, thus

$$\begin{aligned}
&\int_{v_\epsilon(s)}^{\hat{v}_\epsilon(s)} \frac{g(v)}{v} dv - \frac{v'_\epsilon(s) s g(v_\epsilon(s))}{v_\epsilon(s)} \cdot (1+\epsilon(s-\underline{v}))\cdot \Big(\frac{v_\epsilon(s)-(v_\epsilon(s))'s}{v_\epsilon(s)}\Big)_+ \\
&\qquad \geq \int_{v_\epsilon(s)}^{\hat{v}_\epsilon(s)} \frac{g(v)}{v} dv - \frac{v'_\epsilon(s) s g(v_\epsilon(s))}{v_\epsilon(s)} = 0.
\end{aligned}$$

Thus we can conclude that $\int_{v_\epsilon(s)}^{\hat{v}_\epsilon(s)} \frac{g(v)}{(\lambda(1+\epsilon(s-\underline{v}))v} dv - v'_\epsilon(s) s g(v_\epsilon(s)) \tilde{f}_\epsilon(s|v_\epsilon(s)) \geq 0$.

We are then left to check that the profit function is concave when $s' < s$. If $f(s|v_\epsilon(s'))=0$, the second derivative is zero. If it is not then we need to show that

$$-2v'_\epsilon(s')g(v_\epsilon(s')) - v''_\epsilon(s')s'g(v_\epsilon(s')) - g'(v_\epsilon(s'))(v'_\epsilon(s'))^2 s' \leq 0$$

To see this is verified, note that $v_\epsilon'(s)sg(v_\epsilon(s)) = v_\epsilon(s)\int_{v_\epsilon(s)}^{\hat{v}_\epsilon(s)} \frac{g(v)}{v}dv$. Taking the derivative on both sides and noting that $\hat{v}_\epsilon(s)$ is differentiable everywhere but at $\underline{p}^{-1}(\overline{v})$, we get,

$$
\begin{aligned}
&2v_\epsilon'(s)g(v_\epsilon(s)) + v_\epsilon''(s)sg(v_\epsilon(s)) + g'(v_\epsilon(s))(v_\epsilon'(s))^2 s \\
&\qquad = v_\epsilon'(s)\int_{v_\epsilon(s)}^{\hat{v}_\epsilon(s)} \frac{g(v)}{v}dv + v_\epsilon(s)\hat{v}_\epsilon'(s)\frac{g(\hat{v}_\epsilon(s))}{\hat{v}_\epsilon(s)} \geq 0.
\end{aligned}
$$

□

C.1 Proof of Proposition 6

The plan of the proof is the following. In Lemma 9 below, I show that the limit profits of ϵ-candidate tests are greater than the profits from perfect price discrimination. I then use the fact that ϵ-candidate tests satisfy all the obedience constraints when G is the posterior distribution of a sufficiently fine partition of the type space (Lemma 8) to construct a test F^* such that the profits from this test are higher than in the perfect price discrimination benchmark.

Lemma 9. *Let $\epsilon > 0$ and let F_ϵ be an ϵ-candidate test. Then if ϵ is small enough, we have*

$$
\int_V \int_{\{v \geq v_\epsilon(s), s \geq \underline{p}_\epsilon(v)\}} s dF_\epsilon(s|v)dG(v) > \int_V vdG(v).
$$

Proof. Let $\hat{v}(s) = \min\{\overline{v}, s\}$. The limit profits from almost fully learning the valuations is

$$
\begin{aligned}
\int_V vg(v)dv &= \int_V \int_v^{v(1+\lambda)} v\frac{g(v)}{\lambda v}dsdv \\
&= \int_{\underline{v}}^{\overline{v}(1+\lambda)} \int_{\max\{\underline{v}, \frac{s}{1+\lambda}\}}^{\hat{v}(s)} \frac{g(v)}{\lambda}dvds = \int_{\underline{v}}^{\overline{v}(1+\lambda)} \frac{G(\hat{v}(s)) - G(\max\{\underline{v}, \frac{s}{1+\lambda}\})}{\lambda}ds.
\end{aligned}
$$

Take a sequence of ϵ-candidate tests. Because F_ϵ, $\underline{p}_\epsilon$ and v_ϵ converge pointwise to a candidate test, $\underline{p}(v) = v$ and to $v^*(s) = \max\{\tilde{v}(s), \frac{s}{1+\lambda}\}$ with $\tilde{v}$ the solution to (L-OB) and the profits are bounded, by the dominated convergence theorem, the limit profits are $\int_{\underline{v}}^{\overline{v}(1+\lambda)} \int_{v^*(s)}^{\hat{v}(s)} s\frac{g(v)}{\lambda v}dvds$. First I will show that if $v^*(s) = \tilde{v}(s)$, then

$$
\int_{\tilde{v}(s)}^{\hat{v}(s)} s\frac{g(v)}{\lambda v}dv = \frac{G(\hat{v}(s)) - G(\underline{v})}{\lambda}.
$$

To see this note, that, except at $s = \overline{v}$,

$$
\hat{v}'(s)\frac{s}{\hat{v}(s)}g(\hat{v}(s)) - \tilde{v}'(s)\frac{s}{\tilde{v}(s)}g(\tilde{v}(s)) + \int_{\tilde{v}(s)}^{\hat{v}(s)} \frac{g(v)}{v}dv = \hat{v}'(s)g(\hat{v}(s)),
$$

as the two term in the middle are zero by Equation (L-OB) and that $\hat{v}'(s)\frac{s}{\hat{v}(s)}g(\hat{v}(s)) = \hat{v}'(s)g(\hat{v}(s))$. Integrating from $\underline{v}$ to s and using that $\tilde{v}(\underline{v}) = \underline{v}$, we get the desired result. Therefore we get,

$$\int_{\tilde{v}(s)}^{\hat{v}(s)} s\frac{g(v)}{\lambda v}dv = \frac{G(\hat{v}(s)) - G(\underline{v})}{\lambda} \geq \frac{G(\hat{v}(s)) - G(\max\{\underline{v}, \frac{s}{1+\lambda}\})}{\lambda}.$$

When $v^*(s) = \frac{s}{1+\lambda}$,

$$\int_{v^*(s)}^{\hat{v}(s)} \frac{s}{\lambda v}g(v)dv > \int_{v^*(s)}^{\hat{v}(s)} \frac{g(v)}{\lambda}dv \geq \frac{G(\hat{v}(s)) - G(\max\{\underline{v}, \frac{s}{1+\lambda}\})}{\lambda},$$

using that if $v \in [v^*(s), \hat{v}(s))$, $s/v > 1$.

When $s > \underline{v}(1+\lambda)$ s, both inequalities holds strictly.

Because this inequality is strict, we have that for ϵ small enough the profits from an ϵ-candidate test is strictly higher than $\int_V vg(v)dv$. □

We can conclude the proof of Proposition 6.

Let $\delta > 0$ and $\epsilon > 0$ such that there exists an ϵ-candidate test that satisfies global obedience constraints whenever $v \sim G(\cdot|[a, a+\delta])$ for any $a < \overline{v}$ and profits from the test are higher than $\int_a^{a+\delta} \frac{v}{G(a+\delta)-G(a)}dG$. By Lemma 7, Lemma 8 and Lemma 9, these exist.

Take a partition of V, $\mathcal{V} = \{[v_0 = \underline{v}, v_1), ..., [v_{n-1}, \overline{v} = v_n]\}$ such that $v_i - v_{i-1} < \delta$. For any $i = 1, ..., n$, define an ϵ-candidate test on $[v_{i-1}, v_i]$, denoted $F_{\epsilon,i}$. The test $F_{\epsilon,i}$ on $[v_{i-1}, v_i]$ gives higher profits than $\int_{v_{i-1}}^{v_i} \frac{v}{G(v_i)-G(v_{i-1})}dG$ and satisfies all obedience constraints, therefore we have:

$$\sum_i (G(v_i) - G(v_{i-1})) \int_{v_{i-1}}^{v_i(1+\lambda)} \int_{v_{i-1}}^{v_i} s\mathbb{1}[v \in V^*(s)]\frac{g(v)}{G(v_i) - G(v_{i-1})}dF_{\epsilon,i}(s|v)dv$$
$$> \sum_i (G(v_i) - G(v_{i-1})) \int_{v_{i-1}}^{v_i} \frac{v}{G(v_i) - G(v_{i-1})}dG.$$

C.2 Proof of Lemma 2

When designing a test, the firm has a lot of freedom and the optimal test is not guaranteed to be well-behaved in any ways. In particular, the definition of a candidate test uses two properties that are not guaranteed to hold at the optimum: (1) that the WTPs $p^*(v)$ are increasing and that therefore we can express the set $V^*(s)$ as $[v^*(s), \overline{v}]$ and (2) that the profit function is right-differentiable at each s.

To show the result without assuming any of these properties, the proof follows the following plan:

1. Relax the problem by requiring that obedience constraints only hold on intervals of signals $[\underline{s}, s]$ for all s and only for upward deviations, $\tilde{P}(x) = x + \epsilon$ for all $\epsilon > 0$.
2. Use that $F(s|v) > \frac{s-v}{\lambda v}$ for all $s < p^*(v)$ to relax the obedience constraints and make them only depend on a new object $d(x) := \int_{V^*(x)} \frac{1}{v} dG$. If the obedience constraints depend only d, then it is optimal to choose a censored commitment distribution, using the FOSD interpretation of the PPE (Proposition 1). We are left with optimising over d.
3. Look only at local deviations, i.e., $\epsilon \to 0$, to get an integral inequality that pins down d.
   (a) Because d is not necessarily Lipschitz continuous, which is needed for the operation described above, I construct a sequence of relaxed problems with a smaller set of relaxed obedience constraints where deviations are bounded away from 0. For each problem, I show that it is without loss to focus on Lipschitz continuous d.
   (b) Then, I look at the limit of these problems with Lipschitz continuous d and focusing on the smallest possible deviation in each element of the sequence to derive a condition on d.

Proof. For any test F, let $\underline{s} = \inf\{\cup_v \operatorname{supp} F(\cdot|v)\}$ and $\overline{s} = \sup\{\cup_v \operatorname{supp} F(\cdot|v)\}$ be the lowest and highest signal used in that test.

Lemma 10. *For any test F respecting obedience constraints and the PPE requirement, $\underline{s} \geq \underline{v}$, $\overline{s} \leq \overline{v}(1+\lambda)$ and $V^*(\underline{s}) = [\underline{s}, \overline{v}]$.*

Proof. If $\underline{s} < \underline{v}$, then at any signal $s \in [\underline{s}, \underline{v})$, the firm has a profitable deviation to $\underline{v}$ as $p^*(v) \geq \underline{v}$ for all v.

If $\overline{s} > \overline{v}(1+\lambda)$, then for any signal in $(\overline{v}(1+\lambda), \overline{s}]$, there is a profitable deviation to $\overline{v}(1+\lambda)$ as $p^*(v) \leq \overline{v}(1+\lambda)$ for all v.

By definition of PPE and $\underline{s}$, for all $v < \underline{s}$, $p^*(v) = v - \lambda v F(v|v) = v < \underline{s}$, using that $F(v|v) = 0$.

On the other hand, $p^*(v) \geq v$, therefore, $p^*(v) \geq \underline{s}$ for all $v \geq \underline{s}$. Thus, $V^*(\underline{s}) = [\underline{s}, \overline{v}]$. $\square$

Let Σ be the support of a test F. By Lemma 10, it is without loss to look at $\Sigma \subseteq [\underline{v}, \overline{v}(1+\lambda)]$. Let $\mu(s) := \int_s^{\max\{s,\overline{v}\}} \frac{1}{v} dG$.

The following lemma corresponds to step 1 and 2 in the plan of the proof. It shows that by focusing on upward deviations, we can obtain a relaxed problem that only depend on a function d. Implicitly, this function will depend on the set of types willing to accept s, $V^*(s)$, through the relation $d(s) = \int_{V^*(s)} \frac{1}{v} dG$. To get there, the proof uses the FOSD interpretation of the PPE.

Lemma 11. *The following problem is a relaxation of the firm's problem:*

$$\sup_{\Sigma, d \in L^1(\Sigma)} \int_\Sigma x \, \frac{d(x) - \mu(x)}{\lambda} dx$$

s.t. for all $s \in \Sigma$ *and* $\epsilon > 0$,

$$(d(s) - d(s+\epsilon)) s\, \epsilon + \int_{\underline{s}}^{s} x(d(x) - d(x+\epsilon)) dx \geq \int_{\underline{s}}^{s} \epsilon\, (d(x+\epsilon) - \mu(x)) dx \tag{5}$$

$$d(s) \in [\int_s^{\max\{s,\overline{v}\}} \frac{1}{v} dG, \int_{\max\{\underline{v}, \frac{s}{1+\lambda}\}}^{\overline{v}} \frac{1}{v} dG] \text{ for all } s \in \Sigma \tag{6}$$

$$d(\underline{s}) = \int_{\underline{s}}^{\max\{\underline{s},\overline{v}\}} \frac{1}{v} dG; \; d \text{ non-increasing.} \tag{7}$$

Proof. First, focus on the following subset of obedience constraints: for all $s \in \Sigma$,

$$\int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x)]\, x\, dF(v,x) \geq \int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x+\epsilon)]\, (x+\epsilon)\, dF(v,x) \text{ for all } \epsilon > 0$$

Noting that $V^*(x+\epsilon) \subseteq V^*(x)$, we can rearrange the relaxed obedience constraint as

$$\int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x) \setminus V^*(x+\epsilon)]\, x\, dF(v,x) \geq \int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x+\epsilon)]\, \epsilon\, dF(v,x)$$

This is equivalent to (see figure 5 for an illustration)

$$\int_{V^*(s) \setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} x\, dF(x|v) dG + \int_{V \setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} x\, dF(x|v) dG$$
$$\geq \int_{V^*(s+\epsilon)} \int_{\underline{s}}^{s} \epsilon\, dF(x|v) dG + \int_{V \setminus V^*(s+\epsilon)} \int_{\underline{s}}^{p^*(v)-\epsilon} \epsilon\, dF(x|v) dG$$

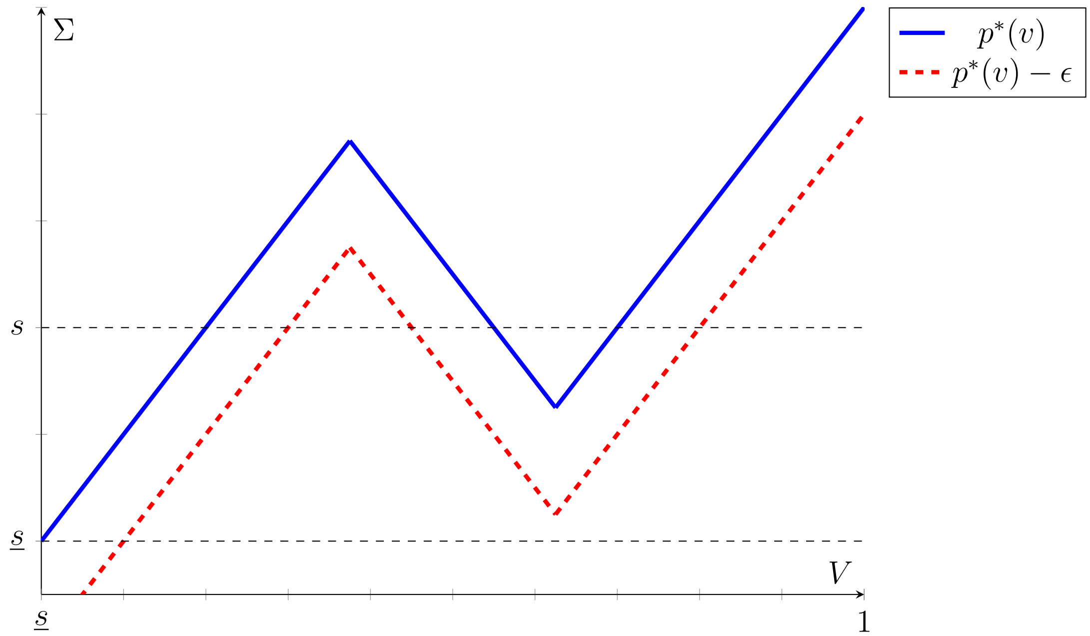

Figure 5: Area of integration with $V^*(x) = \{v : p^*(v) \geq x\}$ and $V^*(x+\epsilon) = \{v : p^*(v)-\epsilon \geq x\}$

I will now use repeatedly the FOSD interpretation of the PPE (Proposition 1): $F(x|v) > \frac{x-v}{\lambda v}$ for $x < p^*(v)$ and $F(p^*(v)|v) = \frac{p^*(v)-v}{\lambda v}$.

Take the RHS first.

$$
\begin{aligned}
&\int_{V^*(s+\epsilon)} \int_{\underline{s}}^{s} \epsilon\, dF(x|v) dG + \int_{V \setminus V^*(s+\epsilon)} \int_{\underline{s}}^{p^*(v)-\epsilon} \epsilon\, dF(x|v) dG \\
&\geq \int_{V^*(s+\epsilon)} \int_{\underline{s}}^{s} \epsilon\, \mathbb{1}[x \geq v] \frac{1}{\lambda v} dx\, dG + \int_{V \setminus V^*(s+\epsilon)} \int_{\underline{s}}^{p^*(v)-\epsilon} \mathbb{1}[x \geq v] \epsilon \frac{1}{\lambda v} dx\, dG \\
&= \int_{\underline{s}}^{s} \int_{V^*(x+\epsilon)} \mathbb{1}[x \geq v] \epsilon \frac{1}{\lambda v} dG\, dx \\
&\geq \int_{\underline{s}}^{s} \frac{\epsilon}{\lambda} \Big[\int_{V^*(x+\epsilon)} \frac{1}{v}\, dG - \int_{[x, \max\{x, \overline{v}\}]} \frac{1}{v}\, dG \Big] dx
\end{aligned}
$$

using the FOSD property on the second line, changing the order of integration in the third and using that $1 \geq \gamma(V^*(x+\epsilon)) + \gamma([0,x]) - \gamma\big(V^*(x+\epsilon) \cap [0,x]\big)$ on the last.

Now focusing on the LHS,

$$
\begin{aligned}
&\int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} x\, dF(x|v)dG + \int_{V\setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} x\, dF(x|v)dG \\
&\leq \underbrace{\int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} x\, dF(x|v)dG}_{I} \\
&+ \underbrace{\int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} \mathbb{1}[x \geq v] x \frac{1}{\lambda v}\, dx\, dG + \int_{V\setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} x\, dF(x|v)dG}_{II}
\end{aligned}
$$

where the inequality simply follows from adding a positive term on the second line.

$$
\begin{aligned}
I &\leq \int_{V^*(s)\setminus V^*(s+\epsilon)} s\big(F(s|v) - F(p^*(v)-\epsilon|v)\big)dG \\
&\leq \int_{V^*(s)\setminus V^*(s+\epsilon)} s\big(\frac{p^*(v)-v}{\lambda v} - \frac{p^*(v)-\epsilon-v}{\lambda v}\big)dG = \frac{s\epsilon}{\lambda} \int_{V^*(s)\setminus V^*(s+\epsilon)} \frac{1}{v} dG
\end{aligned}
$$

using that $s \geq x$ and the FOSD property.

$$
\begin{aligned}
II &\leq \int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} \mathbb{1}[x \geq v] x \frac{1}{\lambda v}\, dx\, dG + \int_{V\setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} \mathbb{1}[x \geq v] x \frac{1}{\lambda v}\, dx\, dG \\
&= \int_{\underline{s}}^{s} \int_{V^*(x)\setminus V^*(x+\epsilon)} \mathbb{1}[x \geq v] x \frac{1}{\lambda v}\, dG\, dx \\
&\leq \int_{\underline{s}}^{s} \int_{V^*(x)\setminus V^*(x+\epsilon)} x \frac{1}{\lambda v}\, dG\, dx
\end{aligned}
$$

where I use the FOSD property on the first line, change the order of integration on the second and ignore that we must have $\mathbb{1}[x \geq v]$ on the third.

Let $d(x) := \int_{V^*(x)} \frac{1}{v} dG$ and using that $\mu(x) = \int_{x}^{\max\{x,\overline{v}\}} \frac{1}{v} dG$, the resulting, relaxed constraint is

$$
(d(s) - d(s+\epsilon))s\,\epsilon + \int_{\underline{s}}^{s} x(d(x) - d(x+\epsilon))dx \geq \int_{\underline{s}}^{s} \epsilon\,(d(x+\epsilon) - \mu(x))dx
$$

Then, remember that the constraint $p^*(v) = \min\{p : v - p = -\lambda v F(p|v)\}$ is equivalent to $v - p^*(v) = -\lambda v F(p^*(v)|v)$ and for all $p < p^*(v)$, $F(p|v) > \frac{p-v}{\lambda v}$ (Proposition 1). Relax it by only requiring $v - p^*(v) = -\lambda v F(p^*(v)|v)$ and for all $p < p^*(v)$, $F(p|v) \geq \frac{p-v}{\lambda v}$.

It is now optimal to set $F(s|v) = \frac{s-v}{\lambda v}$ for all $s \leq p^*(v)$, i.e., we choose a censored commitment distribution. This operation does not modify the relaxed PPE requirement nor the relaxed obedience constraints but improves profits. The firm's problem becomes

$$\sup_{\Sigma, d\in L^1(\Sigma)} \int_\Sigma x\,\frac{d(x)-\mu(x)}{\lambda}dx$$

s.t. for all $s\in\Sigma$ and $\epsilon>0$,

$$(d(s)-d(s+\epsilon))s\,\epsilon+\int_{\underline{s}}^{s} x(d(x)-d(x+\epsilon))dx \geq \int_{\underline{s}}^{s}\epsilon\,(d(x+\epsilon)-\mu(x))dx \tag{5}$$

$$d(s)\in[\int_s^{\max\{s,\overline{v}\}}\frac{1}{v}dG,\int_{\max\{\underline{v},\frac{s}{1+\lambda}\}}^{\overline{v}}\frac{1}{v}dG] \text{ for all } s\in\Sigma \tag{6}$$

$$d(\underline{s})=\int_{\underline{s}}^{\max\{\underline{s},\overline{v}\}}\frac{1}{v}dG;\ d \text{ non-increasing.} \tag{7}$$

where $d(s)\in[\int_s^{\max\{s,\overline{v}\}}\frac{1}{v}dG,\int_{\max\{\underline{v},\frac{s}{1+\lambda}\}}^{\overline{v}}\frac{1}{v}dG]$ comes from $p^*(v)\in[v,v+\lambda v]$, $d(\underline{s})=\int_{\underline{s}}^{\max\{\underline{s},\overline{v}\}}\frac{1}{v}dG$ follows from Lemma 10 and d non-increasing follows from the definition of $V^*(s)$, i.e., increasing the price necessarily decreases the mass of types willing to accept. $\square$

I am going to do a little detour now and focus on a set of obedience constraints and deviations that are bounded away from zero. Specifically, obedience constraints only need to hold for all $s\in\Sigma^i=[\underline{s}+\frac{1}{i},\overline{s}]$ and $\epsilon\in E^i=[\frac{1}{i},\overline{v}(1+\lambda)]$ for some $i\in\mathbb{N}_0$. Note that $\underline{s}\notin\Sigma^i$ and $0\notin E^i$. Furthermore $\Sigma^i\subseteq\Sigma^{i+1}$ and $E^i\subseteq E^{i+1}$.

Let K^i be the set of functions satisfying the constraints (5), (6) and (7) for any $s\in\Sigma^i$, $\epsilon\in E^i$ and K be the set of functions satisfying these constraints for any $s\in\Sigma$ and $\epsilon>0$. Similarly, define OB^i as the set of functions satisfying the relaxed obedience constraints (5) for any $s\in\Sigma^i$ and $\epsilon\in E^i$ and define OB for any $s\in\Sigma$ and $\epsilon\in E=[0,\overline{v}(1+\lambda)]$. Define

$$\Gamma=\{\phi\in L^1(\Sigma):\text{ satisfying (6) and (7)}\}$$

where $L^1(\Sigma)$ is the set of measurable function from Σ to $\mathbb{R}$. Note that $K^i=OB^i\cap\Gamma$. Finally, let Lip be the set of Lipschitz continuous functions (not necessarily with the same Lipschitz constant). Endow the spaces defined above with the L^1-norm.

The following lemma correspond to step 3(a) in the plan of the proof. I argue that we can look at the limit of a problem where each d is Lipschitz continuous and increasingly many deviations are allowed.

Lemma 12. *Let* $\pi(d)=\int_\Sigma x\,\frac{d(x)-\mu(x)}{\lambda}dx$. *Then,*

$$\sup_{d\in K}\pi(d)\leq\lim_{i\to\infty}\sup_{d\in K^i}\pi(d)=\lim_{i\to\infty}\sup_{d\in K^i\cap Lip}\pi(d)$$

Proof. **1.** $\lim_{i\to\infty}\sup_{d\in K^i}\pi(d)$ exists. This follows from $K^{i+1}\subseteq K^i$, therefore $\sup_{d\in K^{i+1}}\pi(d)\leq \sup_{d\in K^i}\pi(d)$. Moreover, $\sup_{d\in K^i}\pi(d)\geq 0$ as choosing $d(x)=\mu(x)$ is always possible for any i. Thus, the limit exists.

2. $\lim_{i\to\infty}\sup_{d\in K^i}\pi(d)\geq \sup_{h\in K}\pi(d)$. For each i, $K\subseteq K^i$, therefore, $\sup_{d\in K^i}\pi(d)\geq \sup_{d\in K}\pi(d)$ for each i.

3. $\lim_{i\to\infty}\sup_{d\in K^i}\pi(d)=\lim_{i\to\infty}\sup_{d\in K^i\cap Lip}\pi(d)$

To prove this identity, I will show that $K^i\cap Lip$ is a dense subset of K^i. Because $\pi(d)$ is continuous in d in the L^1-norm, then $\sup_{d\in K^i}\pi(d)=\sup_{d\in K^i\cap Lip}\pi(d)$.

This part is in three steps. Step 1: show that $\Gamma\cap Lip$ is dense in Γ. Step 2: show that int(K^i) is non-empty in Γ. Step 3: Using that Lipschitz continuous functions are dense in int(K^i) because it is open and int$(K^i)\subseteq\Gamma$, and convexity of K^i, show that any function in K^i can be approximated by a function in $K^i\cap Lip$.

Step 1: $\Gamma\cap Lip$ **is dense in** Γ

Take $\phi\in\Gamma$. Define

$$\phi_n(x)=\begin{cases}\mu(\underline{s})+\frac{\phi_n(\underline{s}+1/n)-\mu(\underline{s})}{1/n}(x-\underline{s}) & if \quad x\in[\underline{s},\underline{s}+1/n)\\ n\int_{x-1/n}^{x}\phi(z)dz & if \quad x\geq \underline{s}+1/n\end{cases}$$

ϕ_n is differentiable everywhere but at one point, $\underline{s}+1/n$, and its derivative is bounded by n therefore Lipschitz continuous and $\phi_n\in\Gamma$.

We have to show that

$$\lim_{n\to\infty}\int_{\underline{s}}^{\overline{s}}|\phi_n(x)-\phi(x)|dx=0$$

Focusing on $x \geq \underline{s} + 1/n$[15].

$$
\begin{aligned}
&\int_{\underline{s}+1/n}^{\overline{s}} |n \int_{x-1/n}^{x} \phi(z)dz - \phi(x)|dx \\
&\leq \int_{\underline{s}+1/n}^{\overline{s}} n \int_{x-1/n}^{x} |\phi(z) - \phi(x)|dzdx \\
&= \int_{\underline{s}+1/n}^{\overline{s}} n \int_{-1/n}^{0} |\phi(x+y) - \phi(x)|dydx \\
&= \int_{-1/n}^{0} n \int_{\underline{s}+1/n}^{\overline{s}} |\phi(x+y) - \phi(x)|dxdy \\
&\leq \sup\{\int_{\underline{s}+1/n}^{\overline{s}} |\phi(x+y) - \phi(x)|dx \,:\, y \in [-1/n, 0]\}
\end{aligned}
$$

For simplicity, extend the domain to the real line and set $\phi(x) = 0$ when $x \notin [\underline{s}, \overline{s}]$. Let $\psi_m \in C_c(\mathbb{R})$, the set of continuous function in $\mathbb{R}$ with compact support, with $\psi_m \rightarrow_{L^1} \phi$. By the Heine-Cantor theorem, any ψ_m is uniformly continuous. We have for all m,

$$
\begin{aligned}
&\lim_{y\to 0} \int_{\mathbb{R}} |\phi(x+y) - \phi(x)|dx \\
&\leq \lim_{y\to 0} \int_{\mathbb{R}} |\phi(x+y) - \psi_m(x+y)|dx + \int_{\mathbb{R}} |\psi_m(x+y) - \psi_m(x)|dx + \int_{\mathbb{R}} |\psi_m(x) - \phi(x)|dx \\
&\leq 2 \int_{\mathbb{R}} |\psi_m(x) - \phi(x)|dx
\end{aligned}
$$

where $\lim_{y\to 0} \int_{\mathbb{R}} |\psi_m(x+y) - \psi_m(x)|dx = 0$ holds because ψ_m is uniformly continuous. Therefore, taking $m \to \infty$, $\lim_{y\to 0} \int_{\mathbb{R}} |\phi(x+y) - \phi(x)|dx = 0$. In turn, it means that $\sup\{\int_{\underline{s}+1/n}^{\overline{s}} |\phi(x+y) - \phi(x)|dx \,:\, y \in [-1/n, 0]\} \to 0$ as $n \to \infty$.

Now for $x \in [\underline{s}, \underline{s}+1/n)$, because $|\phi_n(x)|$ and $|\phi(x)|$ are bounded as $n \to \infty$, $\lim_{n\to\infty} \int_{\underline{s}}^{\underline{s}+1/n} |\phi_n(x) - \phi(x)|dx = 0$.

Therefore, $\Gamma \cap Lip$ is dense in Γ.

Step 2: Non-empty interior of $\Gamma \cap OB^i$ in Γ

Take $d(x) = \mu(x)$. It is easy to check that $h \in K^i = \Gamma \cap OB^i$. Define $z(s, \epsilon) = \int_{\underline{s}}^{s}(x + \epsilon)(d(x) - d(x+\epsilon))dx$ and $\underline{z} = \min_{s,\epsilon} z(s, \epsilon)$. Note that we have $\underline{z} > 0$ because d is strictly decreasing on parts of its domain and $\underline{s} \notin \Sigma^i$ and $0 \notin E^i$.

[15] I would like to thank user fourierwho of StackExchange for this proof.

Now take $\phi(x) \in \Gamma$ with $\int_\Sigma |\phi(x) - d(x)|dx \leq \eta$, $\eta > 0$. I will show that

$$(\phi(s) - \phi(s+\epsilon))s\epsilon + \int_{\underline{s}}^{s} x(\phi(x) - \mu(x))dx \geq \int_{\underline{s}}^{s} (x+\epsilon)(\phi(x+\epsilon) - \mu(x))dx$$

for all $\epsilon \in E^i$, $s \in \Sigma^i$ for η sufficiently small. Rearranging the obedience constraint,

$$\begin{aligned}(\phi(s) - \phi(s+\epsilon))s\epsilon + \int_{\underline{s}}^{s} x(\phi(x) - \mu(x))dx + \int_{\underline{s}}^{s} (x+\epsilon)(d(x) - d(x+\epsilon)) \\ \geq \int_{\underline{s}}^{s} (x+\epsilon)(\phi(x+\epsilon) - d(x+\epsilon))dx\end{aligned}$$

Take the LHS, we have

$$z(s,\epsilon) + (\phi(s) - \phi(s+\epsilon)s\epsilon + \int_{\underline{s}}^{s} x(\phi(x) - d(x))dx \geq \underline{z} - \eta\overline{s}$$

using that ϕ is non-increasing. The RHS gives

$$\int_{\underline{s}}^{s} (x+\epsilon)(\phi(x+\epsilon) - d(x+\epsilon))dx \leq \eta(\overline{s}+\epsilon)$$

Therefore, we need

$$\begin{aligned}\underline{z} - \eta\overline{s} &\geq \eta(\overline{s}+\epsilon) \\ \underline{z} &\geq (2\overline{s}+\epsilon)\eta\end{aligned}$$

which holds for all $s \in \Sigma^i$ and $\epsilon \in E^i$ for η small enough.

Step 3: $K^i \cap Lip$ **is dense in** K^i

First observe that int(K^i) is an open set in Γ in the metric space $(\Gamma, L^1$-norm). Therefore, int$(K^i) \cap Lip$ is dense in int(K^i).

Note that the set K^i is convex. This can be verified by simply summing over the relaxed obedience constraints. The properties of Γ are also maintained when taking convex combinations.

Take some $d \in$ int(K^i). Any function $\phi \in K^i$ can be approximated by a sequence of $\alpha^n d + (1-\alpha^n)\phi$ with the appropriate sequence of α^n. Moreover, any point in the sequence is in the interior of Γ.[16]

[16] To see this note that there exists $\eta > 0$ such that any $B_\eta(d) \subseteq K^i$. Take $\psi = \alpha d + (1-\alpha)\phi$. I will show that any $w \in B_{\eta\alpha}(\psi)$ is in K^i. First, define $z = d + \frac{w-\psi}{\alpha}$. Then, $|z - d| = |d + \frac{w-\psi}{\alpha} - d| < \alpha\frac{\eta}{\alpha} = \eta$. Therefore $z \in K^i$. Then, choosing $\beta = \alpha$, we have $w = \beta z + (1-\beta)\phi$ and thus $w \in K^i$.

Take any $\phi \in K^i$ and $\epsilon > 0$. Let $\phi^n = \alpha^n \phi + (1-\alpha^n)d$, such that $|\phi - \phi^n| < \epsilon/2$ for all $n \geq N$ for some $N \in \mathbb{N}$. Define also $\psi^n \in K^i \cap Lip$ such that $|\psi^n - \phi^n| < \epsilon/2$ for all n, using that $\phi^n \in$ int K^i. Therefore,

$$|\phi - \psi^n| \leq |\phi - \phi^n| + |\phi^n - \psi^n| < \epsilon/2 + \epsilon/2 = \epsilon$$

for $n \geq N$.

Now, given that $\pi(d) = \int_\Sigma x \frac{d(x)-\mu(x)}{\lambda} dx$ is continuous in the L^1-norm, we have established that $\sup_{d \in K^i} \pi(d) = \sup_{d \in Lip \cap K^i} \pi(d)$. $\square$

This lemma shows that taking the restricted set of constraints provides another upper bound to our problem. Furthermore, in the restricted problem, it is without loss to restrict attention to Lipschitz continuous functions.

This is now the last step from the plan of the proof. I look at increasingly smaller deviations as $i \to \infty$ to derive a condition on d that will form an upper bound on the profits.

Now, let's focus on $\lim_{i\to\infty} \sup_{d \in K^i \cap Lip} \pi(d) = \pi(d^*)$ (for some d^*). This implies that there exist a sequence $\{d^i\}$ with $d^i \in K^i \cap Lip$ such that $d^i \to_{L^1} d^*$. Let $\underline{\epsilon}_i = \min\, E^i$.

Because each d^i is monotonic and uniformly bounded, by Helly's selection theorem, there exists a subsequence $\{d^{i_k}\}$ such that $d^{i_k}(s) \to d^*(s)$ for all $s \in$ int Σ. Let's focus on that subsequence and rename its elements: $\{d^k\}_{k=0}^\infty$. This implies that for each $s \in$ int Σ, for all $\eta > 0$, there exists $P(s,\eta) \in \mathbb{N}$ such that $|d^*(s) - d^k(s)| < \eta$ for all $k \geq P(s,\eta)$ and there exists $M(\eta) \in \mathbb{N}$ such that $\int_\Sigma |d^*(s) - d^k(s)| ds < \eta$ for all $k \geq M(\eta)$. Note also that $\int_\Sigma |d^i(x) - d^k(x)| dx < \eta$ for all $k, i \geq M(\eta/2)$.

Finally, d^* being the limit of monotone function, it is monotone and thus continuous almost everywhere. Therefore, wherever d^* is continuous, there exists $N(s,\eta) \in \mathbb{N}$ such that $|d^*(s) - d^*(s + \underline{\epsilon}_i)| < \eta$ for all $i \geq N(s,\eta)$.

Fix $\eta > 0$ and $s > \underline{s}$ where d^* is continuous. Define $i = \max\{\frac{1}{s}, N(s,\eta/3)\}$. Then, for all $k > k^*(s,\eta) \equiv \max\{i, P(s,\eta/3), P(s+\underline{\epsilon}_i, \eta/3)\}$, we have

$$\begin{aligned} |d^k(s) - d^k(s+\underline{\epsilon}_k)| &\leq |d^k(s) - d^k(s+\underline{\epsilon}_i)| \\ &\leq |d^k(s) - d^*(s)| + |d^*(s) - d^*(s+\underline{\epsilon}_i)| + |d^*(s+\underline{\epsilon}_i) - d^k(s+\underline{\epsilon}_i)| \\ &< \eta/3 + \eta/3 + \eta/3 = \eta \end{aligned}$$

using that $\underline{\epsilon}_i > \underline{\epsilon}_k$ on the first line. Therefore, for all $k > \max\{k^*(s,\eta), M(\eta/2)\}$,

$$(d^k(s) - d^k(s+\underline{\epsilon}_k)s + \int_{\underline{s}}^{s} x(d^k(x) - \mu(x))dx \\ \geq \int_{\underline{s}}^{s} (x+\underline{\epsilon}_k)(d^k(x+\underline{\epsilon}_k) - \mu(x))dx$$

$$\Rightarrow \; s\eta + \int_{\underline{s}}^{s} x(d^i(x) - \mu(x))dx + s\eta \\ \geq \int_{\underline{s}}^{s} (x+\underline{\epsilon}_k)(d^i(x+\underline{\epsilon}_k) - \mu(x))dx - (s+\underline{\epsilon}_k)\eta,$$

where I have added and subtracted $\int_{\underline{s}}^{s} xd^i(x)dx$ and $\int_{\underline{s}}^{s}(x+\underline{\epsilon}_k)d^i(x+\underline{\epsilon}_k)dx$.

We can rearrange the constraint and let $k \to \infty$ (which implies $\underline{\epsilon}_k \to 0$),

$$(3s+\underline{\epsilon}_k)\eta + \int_{\underline{s}}^{s} x\,\frac{d^i(x) - d^i(x+\underline{\epsilon}_k)}{\underline{\epsilon}_k}dx \geq \int_{\underline{s}}^{s} d^i(x+\underline{\epsilon}_k) - \mu(x)dx$$

$$\text{letting } k \to \infty, \quad 3s\eta + \int_{\underline{s}}^{s} -x\frac{\partial d^i}{\partial x}dx \geq \int_{\underline{s}}^{s} d^i(x) - \mu(x)dx$$

where we used the dominated convergence theorem, using that $|d^i|$ is bounded and $\frac{d^i(x)-d^i(x+\underline{\epsilon}_k)}{\underline{\epsilon}_k}$ is bounded by Lipschitz continuity. Integrating by part, we get

$$3s\eta - [d^i(x)x]_{\underline{s}}^{s} + \int_{\underline{s}}^{s} d^i(x)dx \geq \int_{\underline{s}}^{s} d^i(x)dx - \int_{\underline{s}}^{s} \mu(x)dx$$

$$d^i(s) \leq \frac{\int_{\underline{s}}^{s} \mu(x)dx + d^i(\underline{s})\underline{s}}{s} + 3\eta$$

Then, we can take a sequence of $\eta \to 0$, and thus $i \to \infty$, and we get for each s where d^* is continuous

$$d^*(s) = \lim_{\eta\to 0} d^i(s) \leq \lim_{\eta\to 0} \frac{\int_{\underline{s}}^{s} \mu(x)dx + d^i(\underline{s})\underline{s}}{s} + 3\eta \\ = \frac{\int_{\underline{s}}^{s} \mu(x)dx + \underline{s}\int_{\underline{s}}^{\max\{\underline{s},\overline{v}\}} \frac{1}{v}dG}{s}$$

Using that $d^i(\underline{s}) = \int_{\underline{s}}^{\max\{\underline{s},\overline{v}\}} \frac{1}{v}dG$. This holds for any s where $d^*(s)$ is continuous.

Therefore, we get another upper bound on the firm's problem.

$$\sup_{\Sigma, d \in Lip} \int_{\Sigma} x \frac{d(x) - \mu(x)}{\lambda} dx$$

$$\text{s.t. for all } s \in \Sigma' : \; d(s) \leq \frac{\int_{\underline{s}}^{s} \mu(x)dx + \underline{s} \int_{\underline{s}}^{\max\{\underline{s},\overline{v}\}} \frac{1}{v} dG}{s}$$

$$d(s) \in [\int_{s}^{\max\{s,\overline{v}\}} \frac{1}{v} dG, \int_{\max\{\underline{v}, \frac{s}{1+\lambda}\}}^{\overline{v}} \frac{1}{v} dG] \text{ for all } s \in \Sigma$$

$$d(\underline{s}) = \int_{\underline{s}}^{\max\{\underline{s},\overline{v}\}} \frac{1}{v} dG$$

for some Σ' whose Lebesgue measure is equal to Σ. This is solved by setting $\Sigma = [\underline{v}, \overline{v}(1+\lambda)]$ and $d(s) = \min\{\frac{\int_{\underline{v}}^{s} \mu(x)dx + \underline{v} \int_{\underline{v}}^{\overline{v}} \frac{1}{v} dG}{s}, \int_{\max\{\underline{v}, \frac{s}{1+\lambda}\}}^{\overline{v}} \frac{1}{v} dG\}$.

We can also show that whenever $\underline{v} \geq \frac{s}{1+\lambda}$, $\frac{\int_{\underline{v}}^{s} \mu(x)dx + \underline{v} \int_{\underline{v}}^{\overline{v}} \frac{1}{v} dG}{s} \leq \int_{\underline{v}}^{\overline{v}} \frac{1}{v} dG$ as $\mu(x) = \int_{x}^{\max\{x,\overline{v}\}} \frac{1}{v} dG \leq \int_{\underline{v}}^{\overline{v}} \frac{1}{v} dG$. Therefore, the optimal d is

$$d(s) = \min\{\frac{\int_{\underline{v}}^{s} \mu(x)dx + \underline{v} \int_{\underline{v}}^{\overline{v}} \frac{1}{v} dG}{s}, \int_{\frac{s}{1+\lambda}}^{\overline{v}} \frac{1}{v} dG\}.$$

It remains to show that this corresponds indeed to candidate test we have derived earlier. This is verified noting that in the limit of ϵ-candidate tests, $V^*(x) = [v^*(x), \overline{v}]$ with $v^*(x) = \max\{\frac{s}{1+\lambda}, \tilde{v}(s)\}$ where $\tilde{v}(s)$ is the solution to

$$\tilde{v}'(s) = \frac{\tilde{v}(s)}{sg(\tilde{v}(s))} \int_{\tilde{v}(s)}^{\hat{v}(s)} \frac{g(v)}{v} dv, \text{ with } \tilde{v}(\underline{v}) = \underline{v}.$$

We can check that $\int_{\tilde{v}(s)}^{\overline{v}} \frac{g(v)}{v} dv = \frac{\int_{\underline{v}}^{s} \mu(x)dx + \underline{v} \int_{\underline{v}}^{\overline{v}} \frac{1}{v} dG}{s}$ as it holds at $s = \underline{v}$ and differentiating on both sides yields the equation defining $\tilde{v}$. It is also clear that $\int_{\frac{s}{1+\lambda}}^{\overline{v}} \frac{1}{v} dG = \int_{v^*(s)}^{\overline{v}} \frac{1}{v} dG$ when $v^*(s) = \frac{s}{1+\lambda}$. □

C.3 Proof of Proposition 7

By Lemma 7 and Lemma 8, a limit of ϵ-candidate tests are obedient when $v \sim U[\underline{v}, \overline{v}]$ and converge to a censored commitment distribution for each v and is completely noisy. By Lemma 2, it must also solve the firm's optimal testing problem whenever it satisfies the obedient constraints.

Solving for equation (L-OB) when $v \sim U[\underline{v}, \overline{v}]$ gives $\tilde{v}(s) = \hat{v}(s)\exp\left(\frac{-\hat{v}(s)+\underline{v}}{s}\right)$. Therefore, we get that the limit of ϵ-candidate tests gives $v^*(s) = \max\{\hat{v}(s)\exp\left(\frac{-\hat{v}(s)+\underline{v}}{s}\right), \dfrac{s}{1+\lambda}\}$. Plugging this in the profit function gives

$$\pi^* = \int_{\underline{v}}^{\overline{v}(1+\lambda)} \int_{v^*(s)}^{\hat{v}(s)} s\frac{g(v)}{\lambda v} dvds = \int_{\underline{v}}^{\overline{v}(1+\lambda)} \frac{\min\left\{\hat{v}(s) - \underline{v},\ s\log\frac{(1+\lambda)\hat{v}(s)}{s}\right\}}{\lambda \Delta v} ds.$$

It remains to show that the probability of trade is increasing. To show this, we can show that

$$\frac{s - v^*(s)}{\lambda v^*(s)}$$

is increasing in s. This is the probability of trade of type $v^*(s)$. Because $v^*(s)$ is increasing in s, this is enough. When $v^*(s) = \frac{s}{1+\lambda}$, this function is constant. When $v^*(s) = \hat{v}(s)\exp\left(\frac{-\hat{v}(s)+\underline{v}}{s}\right)$, we can take the derivative to obtain

$$\begin{aligned}
\left(\frac{s - v^*(s)}{\lambda v^*(s)}\right)' &= \frac{v^*(s) - (v^*(s))'s}{\lambda v^*(s)^2} \\
&= \frac{1}{\lambda v^*(s)}\left(1 - \frac{(v^*(s))'s}{v^*(s)}\right) \\
&= \frac{1}{\lambda v^*(s)}\left(1 - \frac{\hat{v}(s) - \underline{v}}{s}\right) \geq 0.
\end{aligned}$$

C.4 Relaxing Lipschitz continuity and $\underline{v} > 0$

I use the assumption that g is Lipschitz continuous to show the existence of an ϵ-candidate test and $\underline{v} > 0$ to show the existence, in Lemma 8 and in Lemma 2. For Proposition 6, it is enough that there is an interval $I \subset V$ where g is Lipschitz continuous to create a test that achieves strictly higher profits than $\int_V vdG$ as we can always use a test that gives strictly higher payoffs on I and have perfect price discrimination on $V \setminus I$. For Proposition 7, a uniform distribution has a Lipschitz continuous density so the only binding assumption is that $\underline{v} > 0$. If $\underline{v} = 0$, we can always create a test that first partitions V in $\{\{0, \delta\}, \{\delta, \overline{v}\}\}$. In the element of the partition $\{\delta, \overline{v}\}$ a candidate test is optimal and obedient. Because δ is arbitrary, the supremum profits are also reached by using a candidate test on V by letting $\delta \to 0$.