Managing the expectations of buyers with reference-dependent preferences

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Abstract

I consider a model of monopoly pricing where a firm makes a price offer to a buyer with reference-dependent preferences without being able to commit to it. The reference point is the ex-ante probability of trade and the buyer exhibits an attachment effect: the higher his expectations to buy, the higher his willingness-to-pay. When the buyer's valuation is private information, a unique equilibrium exists where the firm plays a mixed strategy and its profits are the same as in the reference-independent benchmark. The equilibrium always entails inefficiencies: even as the firm's information converges to complete information, it mixes on a non-vanishing support and the probability of no trade is greater than zero. Finally, I show that when the firm can design a test about the buyer's valuation, it can do strictly better than in the reference-independent benchmark by leveraging the uncertainty generated by a noisy test.

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The purpose of this paper is to study a monopoly pricing model where the buyer exhibits a specific type of reference-dependent preferences. I consider a buyer that values a good more when he expects to own it. For instance, a buyer expecting to own a specific car or house can get emotionally attached to it and finds it then harder to walk away from an offer. A job applicant, expecting to be employed, can build expectations about the prospects of a different lifestyle or social status, which can reduce his willingness to refuse an offer. More generally, this attachment effect is an "expectation-based endowment effect" and is a prediction of expectation-based reference-dependent preferences (Kőszegi and Rabin, 2006).

When can a firm benefit from facing a buyer with an attachment effect and what is its consequence on a firm's pricing strategy? To study this question, I adapt the monopoly model of Heidhues and Kőszegi (2014). In their model, a firm sells an indivisible good to a buyer by making a take-it-or-leave-it offer. To capture the attachment effect, the buyer's willingness-to-pay (WTP) increases linearly in the ex-ante probability of trading. Following the literature on expectation-based reference-dependent preferences, the buyer plays a Preferred Personal Equilibrium: when setting expectations, he correctly anticipates the firm's strategy and his own action and selects the most favourable plan of action.[1]

Importantly, in Heidhues and Kőszegi (2014), the firm can commit to a possibly random price offer distribution. They show that the optimal strategy is to randomise over prices. The low prices in the support ensure that the buyer expects to buy with positive probability. This increases the WTP through the attachment effect. The higher prices in the support exploit this higher WTP to increase profits. However, this strategy is not consistent with equilibrium behaviour when the firm cannot commit to a random price strategy.

In this paper, I characterise the firm's pricing strategy when it cannot commit to it. This model has two main features. First, like in Heidhues and Kőszegi (2014), the demand is endogenous: the probability of buying and hence the buyer's WTP depends on the firm's strategy. Second, unlike Heidhues and Kőszegi (2014), the firm has a commitment problem. There is a tension between offering low prices to induce expectations to buy and high prices to take advantage of a higher WTP. I characterise the equilibrium strategy under three different information environments: (1) the valuation is the buyer's private information, (2) the buyer's valuation is common knowledge, and (3) the firm can learn about the valuation.

Proposition 2 characterises the unique equilibrium of the game when valuations are private information. In equilibrium, the firm chooses the mixed strategy such that the resulting de-

[1] See for example Heidhues and Kőszegi (2014), Kőszegi and Rabin (2009), Herweg and Mierendorff (2013), Rosato (2016) or Macera (2018) for papers using this selection.

mand is unit-elastic on the support. That is, it creates the demand that makes it indifferent between any price on the support. This preserves the incentives for the mixed strategy and solves the firm's commitment problem. However, to induce this endogenous unit-elastic demand, different types must trade with different probability. This means that the equilibrium entails inefficiencies: the probability of no trade is always strictly greater than zero.

Moreover, the firm does not benefit from the attachment effect and equilibrium profits are independent of its strength. Indeed, types below the support of the mixed strategy only face prices above their valuation in equilibrium. By the PPE requirement, they cannot expect to buy and behave as if they have no attachment effect. This implies that the profits from the lowest price on the support must be the same as in the reference-independent benchmark[2], pinning down equilibrium profits. The commitment problem created by the attachment effect has thus two effects here: the firm must use random prices to overcome it and it does not benefit from having buyers with reference-dependent preferences.

In section 2.1, I characterise the firm's pricing strategy when it knows the buyer's valuation in two different ways, and get contrasting results. First, I use the incomplete information characterisation to study convergence to complete information. As the distribution over types concentrates on a singleton, the equilibrium strategy converges to a mixed strategy on a non-vanishing support and the limit probability of no trade is bounded away from zero. This follows from the incomplete information characterisation. In order to create a unit-elastic demand, the firm induces a large variation in the trading probability of almost identical types. This results in a positive probability of no trade. This problem is more severe for a stronger attachment effect: the probability of trading converges to zero as the attachment effect grows large.

However, there is a discontinuity at the limit: when the valuation is common knowledge, no equilibrium exists. Indeed, it is impossible for the firm to overcome its commitment problem. Any pricing strategy where the buyer is willing to accept increases his WTP above the price played in equilibrium.[3]

In Heidhues and Kőszegi (2014), the firm can commit to a price distribution and the valuation is common knowledge. In contrast, I consider a firm that cannot commit to a price distribution and the buyer's valuation is private information. I show that inefficiencies are a general feature of this model and that the firm does not benefit from facing a buyer with an attachment

[2]Throughout the paper, the reference-independent model refers to an equivalent model with no attachment effect, i.e., the WTP is the same as the valuation.

[3]Existence issues with PPE in strategic settings were already pointed out by Dato et al. (2017).

effect. Moreover, an equilibrium does not exist when the valuation is common knowledge. This shows that the commitment assumption is key to both existence and benefiting from the attachment effect. On the other hand, my model also predicts random prices like Heidhues and Kőszegi (2014).

The first two sets of results looked at extreme information structures: either incomplete information, where the firm does not benefit from the attachment effect or complete information, where no equilibrium exists. In section 3, I look at intermediate case where the firm is partially informed about the buyer's valuation. Specifically, I allow the firm to design a test to learn about the buyer's valuation. The idea behind this section is to explore whether the firm can use the buyer's expectations about his performance on the test to take advantage of the attachment effect. For example, suppose that a firm designs a screening process before making a wage offer to a candidate.[4] The candidate will use his performance during the screening process to assess what are his chances of getting a high wage. If the candidate believes he did well on the test designed by the firm, he will expect high wages and thus a high probability of accepting the job. If the candidate exhibits an attachment effect, this would weaken his bargaining position: he would be willing to accept lower wage offers to avoid the disappointment of not being employed. This could then be used by the firm to offer lower wages.

In this new environment, the firm first designs a publicly observed test, privately observes a signal realisation then makes an offer. In Proposition 6, I show that the firm can be better off with a noisy test. Intuitively, when the test is noisy, the firm can create random price offers like in Heidhues and Kőszegi (2014). At the same time, given the uncertainty generated by the test, the firm can credibly offer low prices after a low signal and high prices after a high signal. This last section shows that in the presence of an attachment effect, a monopolist has an incentive to design imperfect tests. This allows the buyer to credibly entertain the idea that he will get a low price. The firm can then use these expectations to make higher profits. In Proposition 7, I characterise the firm's optimal testing strategy.

Relation to the literature

This paper is part of the literature that studies the implication of rational expectations as the reference point in reference-dependent preferences, following Kőszegi and Rabin (2006). The two closest papers in this literature are Heidhues and Kőszegi (2014) and Eliaz and

[4]The model is set up as a buyer-seller interaction. It can be easily rewritten as firm making a wage offer to a candidate with unknown outside option.

Spiegler (2015).

Eliaz and Spiegler (2015) look at a more abstract model that nests the complete information environment with commitment of this paper as a special case. They show that uniqueness of the PE can be guaranteed through a first-order stochastic dominance property that is useful in this paper. This paper differs from the existing literature by not allowing the firm to commit to a price distribution. I show how to characterise the equilibrium pricing strategy by adapting the result of Eliaz and Spiegler (2015). I also show that an imperfect learning strategy can provide a foundation for the stochastic pricing strategy without commitment. Rosato (2016) also studies a monopoly pricing model where the uncertainty is used to exploit expectation-based reference-dependent preferences. There, the monopolist commits to the limited availability of substitutes to induce the expectations of buying.

The last section is related to the literature on optimal disclosure with a behavioural audiences as it is concerned with the design of the information environment with non-standard preferences, see e.g., Lipnowski and Mathevet (2018); Lipnowski et al. (2020); Levy et al. (2020). In particular, Karle and Schumacher (2017) study a model where a monopolist posts a public price as well as discloses a signal of the valuation of an initially uninformed buyer with expectation-based reference-dependent preferences. The firm benefits from imperfect disclosure when a low valuation is pooled with a high valuation. The buyer then expects to buy at the price posted and thus develops an attachment towards to good. In contrast, I consider a perfectly informed buyer and it is the firm that learns about the valuation. This has two implications. First, the price offered depend on the signal observed, so there is variation in price. Second, inducing expectations to buy is not enough for the firm to benefit from the attachment effect as this happens when prices are relatively low. So the firm must induce both high and low prices to benefit from it.[5]

Finally, there are links to the literature on optimal learning and price discrimination. Bergemann et al. (2015) characterise all the combination of consumers' surplus and monopoly profit after some learning of the firm. I depart from their framework by introducing reference-dependent preferences. Where in their model the optimal learning strategy is to perfectly learn the valuation, introducing reference-dependent preferences incentivises the firm to create a stochastic environment. Roesler and Szentes (2017) and Condorelli and Szentes (2020)

[5] Karle and Schumacher (2017) also show that the monopolist does not benefit from committing to its pricing strategy, unlike this paper. The type of commitment is however different: they consider a setting where the firm can commit to not change the price after the buyer has set expectations but they do not allow commitment to a random price strategy.

look at environments where an agent designs an optimal learning strategy taking into account the effect of information acquisition on the other agent's strategy. Here, the firm designs its optimal learning strategy taking into account its effect on the buyer's preferences.

1 The model

There is one firm and one buyer. The firm makes a take-it-or-leave-it offer $p \in \mathbb{R}$ for an indivisible good that the buyer can either accept, $a = 1$, or reject, $a = 0$. The buyer has a reference-point $r \in \{0, 1\}$ and an exogenous valuation v. His payoffs are

$$u(p, v, a|r) = a(v - p) - \lambda \cdot v \cdot r(1 - a),$$

where $r = 1$ stands for "expecting to accept", and $r = 0$ for "expecting to reject". Here, the buyer "pays" a penalty λv whenever he rejects an offer he was expecting to accept. Like in Kőszegi and Rabin (2006), I allow the reference point to be stochastic. The reference point is then $q \in [0, 1]$ which stands for the probability of accepting. The utility of buyer v is written as

$$u(p, v, a|q) = q \cdot \big(a(v - p) - \lambda v \mathbb{1}[a = 0]\big) + (1 - q) \cdot a(v - p).$$

The firm's payoff is

$$\pi(p, a) = a \cdot p.$$

The buyer knows v. The firm only knows that $v \sim G$, where G denotes a cdf. It admits a strictly positive density g on the support $V = [\underline{v}, \overline{v}]$, $\underline{v} \geq 0$. I use γ to denote the probability measure associated with G: for any measurable set A, $Pr[v \in A] = \gamma(A)$. I will often refer to a valuation v as the buyer's type. I assume that there is a positive surplus with any type and so the assumption that the firm has no cost is a normalisation.

Buyer's behaviour Given his valuation v and his reference point q, the buyer's payoffs from accepting and refusing at price p are

$$u(p, v, a = 1|q) = v - p,$$
$$u(p, v, a = 0|q) = 0 - \lambda v \cdot q.$$

Therefore, he optimally plays a cutoff strategy: he accepts an offer p if and only if $p \leq v + \lambda v q$.[6] I denote the buyer's optimal strategy by $a^*(p, v|q) = \mathbb{1}[p \leq v + \lambda v q]$.

Following Kőszegi and Rabin (2006), the buyer forms his reference point based on the correct expectations of trading. I assume that the buyer first learns his type, then forms his expectations based on the price distribution F. The reference point is thus formed after learning his own type but before the price realisation. Therefore, different types can have different expectations of trading. A Personal Equilibrium (PE) is a reference point q such that the probability of trading is consistent with the optimal strategy given the reference point.

Definition 1. *Given a price distribution F, $(Q_v)_v$ is a profile of Personal Equilibria if for each $v \in V$, Q_v satisfies*

$$Q_v = \int_{\mathbb{R}} a^*(p, v|Q_v) dF(p)$$

and $a^*(p, v|Q_v) = \mathbb{1}[p \leq v + \lambda v Q_v] \in \arg\max u(p, v, a|Q_v)$.

In a PE, the buyer with valuation v correctly anticipates how his expectations change his strategy and how his strategy changes his expectations. The PE Q_v depends on the type but also on the distribution over prices. Therefore, the buyer's behaviour will depend directly on the firm's strategy.

The expected utility of type v, for a given PE Q_v and price distribution $F(p)$ is

$$W(v|F, Q_v) = \int_{-\infty}^{v+\lambda v Q_v} (v - p) dF(p) + \int_{v+\lambda v Q_v}^{+\infty} -\lambda v Q_v \, dF(p).$$

For any prices in $(-\infty, v + \lambda v Q_v]$, the buyer accepts the offer and gets a utility $v - p$. For prices larger than $v + \lambda v Q_v$, the buyer rejects the offer and gets a loss of $-\lambda v Q_v$.

Because there can be multiple PEs, I assume the buyer plays his Preferred Personal Equilibrium (PPE). The PPE is the Personal Equilibrium that gives the highest expected utility (Kőszegi and Rabin, 2006, 2007).

Definition 2. *Given a price distribution F, $(Q_v^*)_v$ is a profile of Preferred Personal Equilibria if for each $v \in V$, $Q_v^* \in \arg\max_{Q_v \in PE} W(v|F, Q_v)$.*

This (Personal) equilibrium selection is common in the literature using Personal Equilibria.[7] Its motivation is based on an introspection interpretation of the PE. The buyer can entertain

[6] Here, I assume that, when indifferent, the buyer accepts the price offer. Allowing for different strategies when indifferent could change the PPE outcome. However, one can show that it would not change the equilibrium strategies in this paper. Therefore, to simplify the exposition, I omit this possibility.

[7] See e.g., Heidhues and Kőszegi (2014), Herweg and Mierendorff (2013), Rosato (2016) or Macera (2018).

multiple expectations of trading but cannot fool himself: his reference point must be correct given his optimal behaviour. Then, if he can "choose" amongst multiple reference points, he would choose the one with the highest expected utility.

It will be useful to think of the PE or PPE as the cutoff price it generates.

Definition 3. *Given F a distribution over prices and PE Q_v, the PE cutoff price of type $v \in V$ is $\hat{p}(v) = v + \lambda v Q_v$. Given PPE Q_v^*, the PPE cutoff price is $p^*(v) = v + \lambda v Q_v^*$.*

The PPE cutoff price determines buyer v's willingness-to-pay (WTP). Note also that we have $Q_v^* = F(p^*(v))$, as buyer v accepts any price below $p^*(v)$. In the rest of the paper, the valuation refers to a buyer's type v and his willingness-to-pay to his PPE cutoff price, $p^*(v)$.

Given a profile of PPE $(Q_v^*)_v$, let $V^*(p) = \{v \in V : p \leq p^*(v)\}$ be the set of types accepting price p. Define $v^*(p) = \inf\{v : v \in V^*(p)\}$, the lowest type in $V^*(p)$.

Equilibrium The firm's expected profits given the profile of PPE $(Q_v^*)_v$ are $\mathbb{E}[\pi(p)|(Q_v^*)_v] = p\,\gamma(V^*(p))$. I can now define an equilibrium in this model.

Definition 4. *A profile of strategy and reference points $\big(F(p), (Q_v^*)_v\big)$ is an equilibrium if for each $v \in V$, Q_v^* is type v's PPE given F and for each $p \in \operatorname{supp} F$, $p \in \arg\max_{\tilde{p}} \mathbb{E}[\pi(\tilde{p})|(Q_v^*)_v]$.*

In equilibrium, each buyer v forms his expectations based on on the firm's equilibrium strategy and his type and the firm's strategy is a best response to the buyers' PPEs.

1.1 Comments

Utility function The utility function I use allows me to capture an attachment effect in the simplest possible way. With this utility function, the agent with valuation v pays a penalty λv weighted by the probability of accepting q when he does not accept the offer. The original specification of Kőszegi and Rabin (2006) allows for a reference point that depends both on the distribution over consumption and price paid. Here, the utility function is similar to ones used in the literature with loss-aversion in one dimension only.[8] Having loss-aversion in one dimension only allows to cleanly isolate an effect of the reference-dependent preferences, for example aversion to price increases or in this case, the attachment effect.

[8] For example, section 4.1 in Heidhues and Kőszegi (2014), Herweg and Mierendorff (2013), Carbajal and Ely (2016), Rosato (2020) or Spiegler (2012).

The commitment assumption Whether commitment to a random pricing strategy is a reasonable assumption depends on the situation considered. When a patient firm post publicly observed price, the commitment assumption can be justified by the incentives of a firm to develop a certain reputation for some price distribution. On the other hand, in many settings, prices are not directly observed. This is the case for example for goods or services that are the outcome of some bargaining or not often traded such as houses, cars or jobs. In this case, the take-it-or-leave-it bargaining structure captures a bargaining process where the firm has all the bargaining power.

1.2 Characterisation of the PPE

Proposition 1 establishes two properties of the PPE. First, the PPE cutoff is the smallest of the PE cutoffs. Second, it establishes that if $p^*(v)$ is the PPE cutoff then $F(p)$ must lie strictly above $\frac{p-v}{\lambda v}$ on $(-\infty, p^*(v))$. The proof of Proposition 1 also establishes existence of the PPE.

Proposition 1. *For a fixed type v and distribution F, these three statements are equivalent:*

- $p^*(v)$ *is PPE cutoff price*
- $p^*(v) = \min\{p : F(p) = \frac{p-v}{\lambda v}\}$
- $v - p^*(v) = -\lambda v F\big(p^*(v)\big)$ *and for all* $p < p^*(v)$, $F(p) > \frac{p-v}{\lambda v}$

Moreover a PPE exists.

Proof. Fix a type v. First, I show that the PPE cutoff is the lowest PE cutoff. Fix two PE, Q_1, Q_2 and their respective PE cutoffs, p_1, p_2. Then,

$$v - p_1 = -\lambda v F(p_1)$$
$$\text{and } v - p_2 = -\lambda v F(p_2).$$

Note that we have $F(p_1) \neq F(p_2)$, for otherwise $p_1 = p_2$. The expected utility at PE Q_i is

$$W(v|Q_i) = \int_{-\infty}^{p_i} (v-p)dF(p) + \int_{p_i}^{+\infty} -\lambda v F(p_i) dF(p).$$

Using the equality defining the cutoff, p_1 is preferred to p_2 if and only if

$$\int_{-\infty}^{p_1} (v-p)dF(p) + (1-F(p_1))(v-p_1) \geq \int_{-\infty}^{p_2} (v-p)dF(p) + (1-F(p_2))(v-p_2)$$
$$\Leftrightarrow \int_{p_1}^{p_2} p\,dF(p) \geq (1-F(p_1))p_1 - (1-F(p_2))p_2$$
$$\Leftrightarrow p_2F(p_2) - p_1F(p_1) - \int_{p_1}^{p_2} F(p)dp \geq (1-F(p_1))p_1 - (1-F(p_2))p_2$$
$$\Leftrightarrow p_2 - p_1 \geq \int_{p_1}^{p_2} F(p)dp,$$

where I obtain the third line by integrating by part. Because $F(p_1) \neq F(p_2)$, this is satisfied if and only if $p_1 < p_2$.

Now let $\tilde{p} = \inf\{p:\ F(p) \leq \frac{p-v}{\lambda v}\}$. If F is continuous at $\tilde{p}$, then $\inf\{p:\ F(p) \leq \frac{p-v}{\lambda v}\} = \min\{p:\ F(p) \leq \frac{p-v}{\lambda v}\}$. If F is not continuous at $\tilde{p}$, then because it is non-decreasing, $\lim_{p\nearrow\tilde{p}} F(p) < F(\tilde{p})$, which then contradicts that $\tilde{p} = \inf\{p:\ F(p) \leq \frac{p-v}{\lambda v}\}$.

Therefore, F is continuous at $\tilde{p}$ and $\min\{p:\ F(p) \leq \frac{p-v}{\lambda v}\}$ exists. Because F is continuous at $\tilde{p}$, it also implies that $\min\{p:\ F(p) \leq \frac{p-v}{\lambda v}\} = \min\{p:\ F(p) = \frac{p-v}{\lambda v}\}$ which establishes existence of the PPE and the first equivalence.

We can now show $p^*(v) = \min\{p:\ v-p = -\lambda v F(p)\} \Leftrightarrow F(p) > \frac{p-v}{\lambda v}$ for all $p < p^*(v)$ and $F(p^*(v)) = \frac{p^*(v)-v}{\lambda v}$.

($\Rightarrow$) Suppose $p^*(v)$ is a PPE and we have some $\hat{p} < p^*(v)$ with $F(\hat{p}) \leq \frac{\hat{p}-v}{\lambda}$. Because we have established that $\min\{p:\ F(p) \leq \frac{p-v}{\lambda v}\} = \min\{p:\ F(p) = \frac{p-v}{\lambda v}\}$, we get a contradiction.

($\Leftarrow$) If for all $p < p^*(v)$, $F(p) > \frac{p-v}{\lambda v}$, then there are no other PE smaller $p^*(v)$. $\square$

The PPE cutoff price is the smallest of the PE cutoff prices because for any distribution, this cutoff is weakly above the valuation v. Therefore, the lowest PE cutoff minimises trade when $p > v$, i.e., when the buyer has a negative utility. The condition that $F(p) > \frac{p-v}{\lambda v}$ when $p < p^*(v)$ was introduced by Eliaz and Spiegler (2015). This property is similar to the characterisation of first-order stochastic dominance albeit on only part of the support. In particular, it implies that for any F implementing PPE cutoff $p^*(v)$,

$$\int_{-\infty}^{p^*(v)} x\,dF(x) < \int_{v}^{p^*(v)} x \cdot \frac{1}{\lambda v} dx.$$

This observation will be useful in section 3 when we will design the firm's optimal testing strategy. Figure 1 illustrates graphically how to determine the PPE cutoff using Proposition 1.

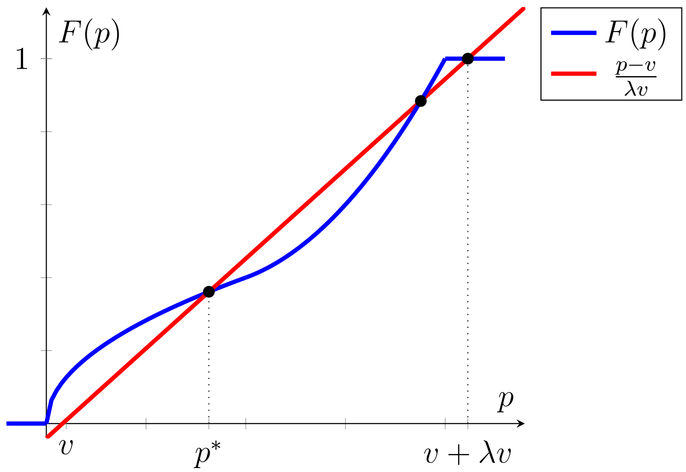

Figure 1: Each intersection of the blue and red curve is a PE. The lowest intersection, p^*, is the PPE.

2 Incomplete information

In this section, I first characterise the equilibrium when the valuation is private information. In section 2.1, I study the equilibrium when the set of types converges to a singleton and when the valuation is common knowledge.

To simplify the analysis in this section, I restrict attention to strictly concave reference-independent profits.

Assumption 1. *The function* $p(1 - G(p))$ *is strictly concave.*

Assumption 1 is made to simplify the exposition of the paper. The results of this section extend qualitatively to more general distributions but the exact characterisation of the equilibrium differs.

Proposition 2 characterises the unique equilibrium of the game. It shows that the firm plays a mixed strategy, the equilibrium demand is unit-elastic and the equilibrium profits are the same as in a reference-independent model.

Denote by $\pi^* = \max_p p(1-G(p))$ and $p_{ind} = \arg\max_p p(1-G(p))$, the equilibrium profits and prices of the reference-independent benchmark.

Proposition 2. *There is a unique equilibrium* $\left(F, (Q_v^*)_v\right)$. *In equilibrium,*

- *The firm plays the mixed strategy*

$$F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)},$$

with $\operatorname{supp} F = [p_{ind}, \overline{p}]$, $\overline{p} := F(\overline{p}) = 1$.

- *If* $p \in \operatorname{supp} F$, *then* $\gamma(V^*(p)) = \frac{\pi^*}{p}$, *i.e., the demand is unit-elastic on the support.*

- *The equilibrium profits are* $\pi^* = \max_p p(1 - G(p))$.

The proof is in appendix A.

Proposition 2 illustrates how the firm's commitment problem constrains its behaviour and how it can solve it. First, any pure strategy is not credible. To see why, suppose the firm plays a pure strategy p. Then by the PPE requirement, all the types $v \geq p$ accept the offer and all types $v < p$ refuse it. This means that their WTPs are $p^*(v) = v + \lambda v$ for $v \geq p$ and $p^*(v) = v$ for $v < p$. The firm has then a profitable deviation to a higher price as all types that accept p are willing to pay strictly more than p.

The firm can play a mixed strategy only if it is indifferent between any price on the support, i.e., the demand is unit-elastic on the support. Therefore, the firm's mixed strategy induces expectations of trading such that the resulting distribution over WTP is unit-elastic.

Moreover, the firm does not benefit from facing buyers with reference-dependent preferences. The buyers whose valuation is below the support know they will only face prices higher than their valuation. Because they play a PPE, they cannot expect to trade and their WTP is equal to their valuation. Therefore, they behave like players with no attachment effect. When offering the lowest price on the support, only buyers with valuation above that price accept, exactly like in a reference-independent model. By the indifference condition, these must be the equilibrium profits. However, some types do end up buying the good at a price above their valuations with some probability. The exploitation of the buyers' attachment effect is compensated by higher probability of no trade when offering high prices.

2.1 (Almost) Complete information

In this subsection, I look at the behaviour of equilibrium objects when the distribution over valuations converges to a singleton and when the valuation is common knowledge.

In what follows, I look at a sequence of games where the only varying primitive is the prior distribution G. Therefore, abusing notation, I will identify a sequence of games with a sequence of prior distributions. Denote by $\xrightarrow{\mathcal{D}}$ convergence in distribution and δ_v the Dirac measure on v.

Proposition 3. *Let $v > 0$. Take a sequence of games $\{G_i\}_{i=0}^{\infty}$ such that $G_i \xrightarrow{\mathcal{D}} \delta_v$. All other primitives of the model are fixed.*

Then, equilibrium profits converge to v and the firm's equilibrium strategy converges in distribution to $F_\infty(p) = \frac{p-v}{\lambda v}$ with $\operatorname{supp} F_\infty = [v, v + \lambda v]$.

Moreover, the limit probability of trade is

$$\frac{1}{\lambda} \log \frac{v + \lambda v}{v}.$$

Proof. **Limit distribution and profits:**

Let F_i be the equilibrium strategy given G_i, $\underline{p}_i = \min \operatorname{supp} F_i$, $\overline{p}_i = \max \operatorname{supp} F_i$ and $\pi_i^* = \underline{p}_i(1 - G_i(\underline{p}_i))$. From Proposition 2, $F_i(p) = \frac{p - G_i^{-1}\left(\frac{p-\pi_i^*}{p}\right)}{\lambda G_i^{-1}\left(\frac{p-\pi_i^*}{p}\right)}$ for all $p \in [\underline{p}_i, \overline{p}_i]$. Using that $G_i^{-1}(x) \to v$ for each $x \in (0, 1)$, for each $p \in \mathbb{R}$,

$$F_i(p) \to \begin{cases} 0 & if \quad p < v \\ \frac{p-v}{\lambda v} & if \quad p \in [v, v + \lambda v] \\ 1 & if \quad p > v + \lambda v \end{cases},$$

and thus $F_i \xrightarrow{\mathcal{D}} F_\infty$.

Profits converge to v as $\max_p \, p(1 - G_i(p)) \to v$.

Probability of trade: Denote the probability of trading at price p by $\phi(p)$. This probability is pinned down by the indifference condition:

$$\pi_i^* = p\phi_i(p).$$

The probability of trading is thus $\int_{\mathbb{R}} \phi_i(p) f_i(p) \, dp$ where f_i is the density of F_i. It is easy to verify that $f_i(p) \to \frac{1}{\lambda v}$ for all $p \in [v, v + \lambda v]$ and $\phi_i(p) f_i(p)$ is uniformly bounded. Using the dominated convergence theorem, we get that

$$\int_{\mathbb{R}} \phi_i(p) f_i(p) \, dp \to \int_v^{v+\lambda v} \frac{v}{p} \cdot \frac{1}{\lambda v} dp = \frac{1}{\lambda} \log \frac{v + \lambda v}{v}.$$

$\square$

As the distribution of types converges to the singleton v, the firm's strategy converges to a uniform distribution on $[v, v + \lambda v]$. The profits, on the other hand, are always equal to the reference-independent benchmark, $\pi^* = v$ in the limit. As the mass of types accumulate on v, the support does not converge to a singleton. Even though the interval of valuations could become arbitrarily small, the interval of potential WTP stays large: any $p \in [v, v + \lambda v]$ can be a PPE cutoff. The firm still needs to mix to create the endogenous demand that makes it indifferent on the support. This means that even as we converge to complete information, a large amount of uncertainty is needed to guarantee an equilibrium. This variation leads to a strictly positive probability of no trade in the limit.

Moreover, $\frac{1}{\lambda} \log \frac{v+\lambda v}{v}$, the probability of trade, decreases in the attachment effect λ and converges to 0 as $\lambda \to \infty$. Intuitively, a higher λ makes the buyer more vulnerable to exploitation but also increases the firm's commitment problem. The firm must lower the probability of trading to compensate for the higher demand induced by a higher λ. This can also be seen from the indifference condition: as profits converge to v for any λ, the higher prices must be compensated for by a higher probability of rejection.

Finally, I note that the limit strategy is arbitrarily close to the one the firm would use if it could commit to a price distribution.

Proposition 4 (Heidhues and Kőszegi, 2014)**.** *The solution to the commitment problem:*

$$\sup_{F \in \Delta \mathbb{R}} \int_{-\infty}^{p^*} p \, dF(p)$$

subject to p^* is PPE, is

$$\frac{\lambda + 2}{2} \cdot v.$$

The distribution that attains this profit converges to $F(p) = \frac{p-v}{\lambda v}$.

Proof. See Heidhues and Kőszegi (2014), section 4.1 or Eliaz and Spiegler (2015). □

The firm chooses a price distribution that maximises its profits amongst all the distribution that implement trade with probability one. By FOSD interpretation of the PPE (Proposition 1), this is done by choosing a price distribution as close as possible to $\frac{p-v}{\lambda v}$, see figure 2 for an illustration.

In contrast, if the value of v is common knowledge and there is no commitment, no equilibrium exists.

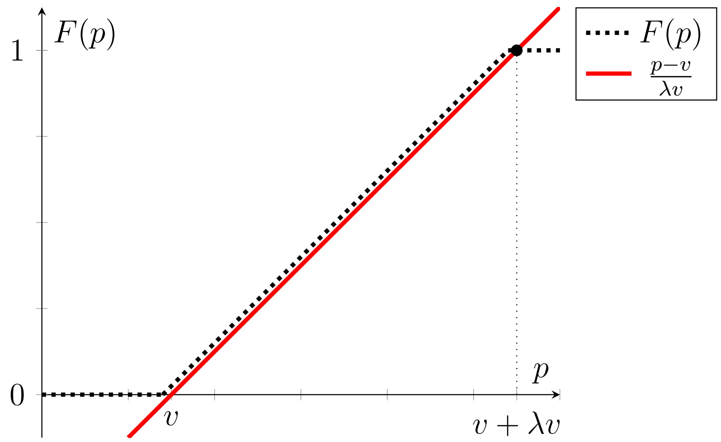

Figure 2: Almost optimal distribution over prices.

Proposition 5. *When v is common knowledge and $v > 0$, no equilibrium exists.*

Proof. For any $p^*(v)$, there is a unique best-response of the firm, which is to offer $p^*(v)$. Let p be the equilibrium price, i.e., the equilibrium strategy is $F(\tilde{p}) = \mathbb{1}[\tilde{p} \geq p]$.

If $p \leq v$, then there is a unique PE cutoff $p^*(v) = v + \lambda v F(v + \lambda) = v + \lambda v$. There is a profitable deviation to $p' = v + \lambda v$.

If $p > v$, then $p^*(v) = v + \lambda v F(v) = v$ is a PE cutoff and also the smallest PE cutoff. By Proposition 1, it is the PPE cutoff and thus there is no trade in equilibrium. Then, there is a profitable deviation to any $p' \in (0, v]$. $\square$

The key tension is that the firm wants to take advantage of the attachment effect. However, given the PPE requirement and a deterministic price, the buyer's WTP is only higher than his valuation if the price offered is below his valuation. An equilibrium with no trade is also impossible because any price below the valuation will be accepted for any PPE.

We obtain the non-existence result for the same reason there cannot be a pure strategy equilibrium in the incomplete information environment. Playing a pure strategy shifts the demand if the offer is accepted. Unlike the incomplete information environment, the firm cannot play a mixed strategy because it is facing only one type. Thus, it cannot solve its commitment problem.

Relation to other non-existence results: Dato et al. (2017) have already observed that in games where players are constrained to play a PPE, an equilibrium does not always exist.

They note that with binary actions, a PPE strategy never entails mixing. Therefore, if the equilibrium requires mixed strategies, these strategies can never be the players' PPE, even though they could be PEs. Here, the mechanism for non-existence is different as the equilibrium relaxing the PPE constraint would not be in mixed strategies. Instead, it occurs because the buyer's PPE price cutoff is always bounded away from the price offered.

Azevedo and Gottlieb (2012) show that games with prospect theory preferences can suffer from equilibrium existence issues in a game where a risk-neutral firm offers a gamble to an agent. In their case, they observe that for an exogenous reference point and some conditions on the value and probability weighting functions, there exists a bet with arbitrarily low expected value that an agent with prospect theory preferences is willing to accept. Our analysis differ both in the choice of reference-point as well as the choice of payoff function – they allow for probability weighting as well as restrict attention to gain-loss value function. In the model considered here, the payoffs are always bounded so the mechanism for non-existence is also different.

3 Intermediate case: Testing the valuation

We have so far looked at "extreme" information structures, either complete information or complete lack of information. In this section, I allow the firm to collect additional information on the buyer's valuation before setting a price. In Proposition 6, I first establish that having only partial information about the buyer's valuation can be beneficial for the firm. In Proposition 7, I characterise the firm's preferred testing strategy and its profits.

In many bilateral trade settings, one party can gather information about the other before making an offer. A leading example is job applications where an employer designs a screening process before making an offer to a candidate.[9] In the process, he collects information about the candidate's productivity and outside option. These can be explicit tests through an assessment centre or asking to submit certain documents like a CV or recommendation letters. The question being asked in this section is whether a firm can use a combination of the screening process and a candidate's attachment effect to offer lower wages.

This section would also be relevant to settings where a seller can gather information about

[9]The model is cast as a buyer-seller interaction but it can be rewritten as an firm-candidate interaction where the candidate has private information regarding his outside option and the firm makes take-it-or-leave-it wage offer.

the buyer's valuation. For example, a car dealer could ask questions to the consumer about his valuation.

I focus on $G \sim U[\underline{v}, \overline{v}]$. This is with loss of generality and I explain after the discussion of Proposition 7 why we need to restrict attention to these distributions. All functions and sets are assumed to be measurable.

Test: Let S be a set of signals with $\mathbb{R} \subset S$. A test is a mapping from types to distributions over signals $F : V \to \Delta(S)$. Denote by $F(s|v)$ the distribution of s conditional on v. Abusing notation, $F(v, s)$ is the joint distribution of (v, s) induced by $F(s|v)$ and $G(v)$.

Players' information: The test is common knowledge but only the firm observes the signal realisation. The valuation v is still privately known.

Players' strategy: A strategy for the firm is $P : S \to \Delta(\mathbb{R})$, a mapping from signals to distributions over prices.

The assumption regarding the players' information and strategies correspond to a setting where the seller can commit to a test or where the test is publicly observable. This would be the case in a job application setting where the candidate can observe the selection process he goes through. An important assumption is also that the candidate does not observe the signal realisation, e.g., he does not know the result of the test, does not observe his recommendation letters or an interview is made opaque to make it hard to interpret. Alternatively, this information could be revealed to him as long as his reference-point is set before learning the outcome of the test. The case of public signals is discussed at the end of the section.

The PE and PPE are defined in the same way as before. I assume that the buyer forms expectations after having observed his type and the test. Given the test (F, S) and strategy P, a reference point Q_v is type v's PE if

$$Q_v = \int_S \int_{\mathbb{R}} \mathbb{1}[p \leq v + \lambda v Q_v] \, dP(p|s) dF(s|v),$$

and the expected utility of type v given (F, S), P and Q_v is

$$W(v|Q_v) = \int_S \int_{\mathbb{R}} \Big(\mathbb{1}[p \leq v + \lambda v Q_v](v - p) + \mathbb{1}[p > v + \lambda v Q_v](-\lambda v Q_v) \Big) dP(p|s) dF(s|v).$$

Then, Q^*_v is a PPE if $Q^*_v \in \arg\max_{Q_v \in PE} W(v|Q_v)$.

The firm's ex-ante payoffs are

$$\mathbb{E}[\pi(P)|(Q_v^*)_v] = \int_V \int_S \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, dP(p|s)dF(s|v)dG(v).$$

Definition 5. *Fix a test F. An equilibrium is a profile $(P, (Q_v^*)_v)$ such that*

- $P \in \arg\max \mathbb{E}[\pi(\cdot)|(Q_v^*)_v]$,
- *and for all $v \in [\underline{v}, \overline{v}]$, Q_v^* is type v's PPE.*

In equilibrium, the buyer plays according to his PPE based on the test and the firm's strategy. The firm best replies to the buyers' PPE based on its information.

In the reference-independent model, the firm would perfectly learn the buyer's valuation and offer the valuation. This would also be the limit profits of the firm if the test would be arbitrarily close to fully revealing the valuation (Proposition 3).[10] The profits in this case would be $\int_V v\, dG$. A test is completely noisy if for almost all s and v, $Pr_F[s|v] < 1$ and there is no unique v such that $s \in \operatorname{supp} F(\cdot|v)$. That is, a test is completely noisy if no type sends a signal deterministically and any signal realisation does not reveal any type.

Proposition 6. *There is a completely noisy test (F, S) and an equilibrium $(P, (Q_v^*)_v)$ such that firm's profits are strictly greater than $\int_V v\, dG$.*

The proof is in appendix C.

Proposition 6 shows that the firm benefits from not fully learning the buyer's type. Coming back to the case of screening processes for job candidates, Proposition 6 suggests that firms could benefit from designing noisy or opaque screening procedures. This would let candidates entertain the idea that they would get a good wage offer and thus weaken their bargaining position. However, not all completely noisy tests achieve higher profits. In Proposition 7, I characterise this test and show that it is optimal amongst all tests.

A completely noisy test uses the two types of uncertainty it generates to credibly exploit the buyer's attachment effect. First, the buyer is uncertain about which signal he generated and therefore which types he is pooled with. At low signals, the firm offers low prices, inducing expectations to buy. Then, at higher signals, the firm offers higher prices, taking advantage of the higher WTP. From the buyer's perspective, he is facing random prices like in Heidhues

[10] Remember that if the firm perfectly learns the valuation, then there is no equilibrium in the resulting game.

and Kőszegi (2014). Second, the firm uses the uncertainty it has about the type to credibly offer low prices after a low signal, despite facing some buyers willing to accept higher prices.

The next step is to characterise the firm's preferred test. Like in Bergemann and Morris (2016), we can focus on tests that generate action recommendations compatible with the PPE requirement. The distribution $\delta_s(p)$ denotes the Dirac measure.

Lemma 1. *Consider a test F and an equilibrium $\big(P, (Q_v^*)_v\big)$ in F. Then, there exists a test $\tilde{F}$ with support $\cup_{s\in S} \operatorname{supp} P(\cdot|s))$ and an equilibrium $\big(\delta_s, (Q_v^*)_v\big)$ in that test such that each player gets the same payoffs as in the original equilibrium.*

Proof. See appendix B $\square$

Lemma 1 holds because the only thing that matters for the buyer's PPE is the distribution over prices given his type. Therefore, a standard revelation principle argument holds. If after two different signals, the firm offers the same price, we can modify the test to "merge" these two signals. This will not change the distribution over actions and thus all PPEs are preserved.

The firm's problem is to find the supremum profits and the test that attains it:

$$\sup_F \int_{V\times S} \mathbb{1}[v \in V^*(s)]\, s\, dF(v,s) \tag{1}$$

$$\text{s.t. } p^*(v) = \min\{p:\ v - p = -\lambda v\, F(p|v)\}, \tag{2}$$

for all $S' \subseteq S$,

$$\int_{V\times S'} \mathbb{1}[v \in V^*(x)]\, x\, dF(v,x) \geq \int_{V\times S'} \mathbb{1}[v \in V^*(\tilde{P}(x))]\, \tilde{P}(x) dF(v,x) \tag{3}$$

for all $\tilde{P} : S \to \mathbb{R}$

We need to solve for the supremum because the set of constraints is not closed. In particular, a sequence of distributions $F_i(\cdot|v) \to F(\cdot|v)$ that induce $p^*(v) = p'$ for each i can have $p' \neq \min\{p:\ v - p = -\lambda v\, F(p|v)\}$.[11] The firm chooses distribution over prices that will determine its profits and the WTP of each type. The constraint (2) pins down the WTP of each buyer. The obedience constraint (3) ensures that the firm is willing to follow almost all price recommendations. I say that a test is *admissible* if it satisfies the constraints (3) where the WTP are pinned down by (2). Obedience constraints are required to hold for any subset

[11] For example, the distribution $F(s|v) = \frac{1}{2} + \frac{s - v - \frac{\lambda v}{2}}{\lambda v + \epsilon}$ induces a PPE cutoff $p^*(v) = v + \lambda v/2$ for all $\epsilon > 0$ but a PPE cutoff $p^*(v) = v$ for $\epsilon = 0$.

S' to take into account that some signal realisations might be zero probability events and have no well-defined density.

To fix ideas, I first characterise the firm's first-best solution, i.e., ignoring the obedience constraints. It is equivalent to allow the firm to commit to a distribution over prices, like in Heidhues and Kőszegi (2014). Proposition 4 in the previous section has thus already characterised the first-best solution.

Claim 1. *The first-best solution is to take a sequence $\{F_i\}$ such that*

- $F_i(s|v) \to \frac{s-v}{\lambda v}$ *with* $\operatorname{supp} F_i \to [v, v+\lambda v]$ *for all v and* $S_i \to [\underline{v}, \overline{v}(1+\lambda)]$.
- *For each i, for each v, $p^*(v) = v + \lambda v$ and there is trade with probability one.*

I call the distribution $\frac{s-v}{\lambda v}$ on $[v, v+\lambda v]$ the commitment distribution. The commitment distribution and its modification defined below will be important for the characterisation of the optimal test.

Definition 6 (Censored commitment distribution)**.** *F is the cdf of a censored commitment distribution if there exists a $\tilde{p} \in [v, v+\lambda v]$ and $\tilde{F}(s|v) < \frac{s-v}{\lambda v}$ such that*

$$F(s|v) = \begin{cases} 0 & if \quad s < v \\ \frac{s-v}{\lambda v} & if \quad s \in [v, \tilde{p}] \\ \tilde{F}(s|v) & if \quad s > \tilde{p} \end{cases} .$$

The censored commitment distribution behaves like the commitment distribution for prices below $\tilde{p}$ but stays below $\frac{s-v}{\lambda v}$ for higher prices. The commitment distribution is a censored commitment distribution with $\tilde{p} = v + \lambda v$.

Proposition 7. *Assume $v \sim U[\underline{v}, \overline{v}]$ and let $\hat{v}(s) = \min\{\overline{v}, s\}$. The firm's supremum profits are*

$$\pi^* = \int_{\underline{v}}^{\overline{v}(1+\lambda)} \frac{\min\left\{\hat{v}(s) - \underline{v},\ s\log\frac{(1+\lambda)\hat{v}(s)}{s}\right\}}{\lambda\Delta v} ds,$$

and there exists a sequence of admissible tests $\{F_i\}$ that approximate the firm's supremum profits such that

- *For each v, $F_i(\cdot|v)$ converges to a censored commitment distribution.*
- *The sequence F_i converges to a completely noisy test.*

- *There is downward distortion: the probability of trading is increasing v.*

In the limit, we have $v^*(s) = \max\{\hat{v}(s) \exp\left(\frac{-\hat{v}(s)+\underline{v}}{s}\right), \frac{s}{1+\lambda}\}$ and $p^*(\cdot)$ is the inverse of $v^*(\cdot)$.

The proof is in appendix C.

Proposition 7 characterises the firm's preferred test and supremum profits. As in Proposition 6, the preferred test is a completely noisy test. Moreover, higher types are more likely to trade and also face, and accept, higher prices compared to their valuation. In our job screening example, it means that less productive candidates (the equivalent of high valuation buyers) are more likely to suffer from their attachment towards the job as they are more likely to receive relatively better offers compared to their outside option.

The firm can credibly follow the equilibrium strategy because it uncertain about which type it is facing. However, there are limits to the uncertainty the firm can generate. For example, the firm cannot pool the lowest type in the support with even lower types. Therefore, this type always expects prices higher than his WTP. By the PPE requirement, he must have a WTP equal to his valuation. In turn, this means that he must trade with probability 0. This logic can be extended to more types: relatively low valuations can be pooled with fewer lower types. The firm can take advantage of their attachment effect but not fully. Their probability of trade is then smaller than one.

This argument can also be seen from the action recommendation approach. In the first-best solution, each type trades with probability one and therefore $p^*(v) = v + \lambda v$. In particular, $p^*(v) \geq \underline{v} + \lambda \underline{v}$ for all v. Moreover, the signal space is $S = [\underline{v}, \overline{v}(1+\lambda)]$. This solution does not respect the obedience constraints because for any signal in $[\underline{v}, \underline{v}(1+\lambda))$, there would be a profitable deviation to $\underline{v}(1+\lambda)$. To satisfy the obedience constraints, the firm decreases the WTP of low types to reduce the incentives to increase the price at low signals. But this means that the probability of trading of low types must be less than one. Hence, the firm generates inefficiencies in the form of downward distortion. The optimal way the firm generates the downward distortion is by using censored commitment distributions. These are the distribution over signals that generate the largest profit for a given WTP.

As in the incomplete information environment, the firm generates inefficiencies to maintain the credibility of its own strategy. In both cases, the firm must manage the buyers' expectations to ensure that it is willing to follow its strategy. Unlike the incomplete information environment, the firm can now make more profits than in the reference-independent benchmark. In Proposition 2, the firm can only make the reference-independent profits because types below the support are always expecting prices higher than their valuation. Here, the support of types conditional on the signal is not common knowledge anymore. Therefore, only the lowest type in the prior distribution expects to face prices higher than his valuation.

The case of non-uniform priors The results derived in this section do not hold for all prior distributions over valuations. The proof of Proposition 7 relies on solving for local obedience constraints and then check that global constraints hold as well. This approach works for some distributions, including uniform distributions, but not all.

Public signals Consider a model where the signal realisation is public and the buyer's reference point is set after having observed the signal realisation. In this different environment, the firm's information is common knowledge. Thus, after each signal realisation, we are back to the environment of section 2. The optimal test for the firm is then to take an arbitrarily fine partition of the type space and play the equilibrium of Proposition 2 in each element of the partition.

The profits are the same as in the reference-independent benchmark but like in section 2, there is a positive probability of no-trade despite the near-complete information. Another difference is that the WTP is no longer monotonic in the valuation: in each element of the partition $[v, v+\epsilon)$, the support of the mixed strategy is $\approx [v, v+\lambda v]$ and the WTP vary on an interval $\approx [v, v+\lambda v)$. Therefore, some types might be on higher elements of the partition, but have a lower WTP.

4 Conclusion

In this paper, I study a model of monopoly pricing where the buyer has expectation-based reference-dependent preferences, focusing on an attachment effect. The model has two main features. The expectation-based reference point renders the demand an endogenous object. The PPE requirement creates a commitment problem for the firm.

On a theoretical level, this model offers two main lessons. First, uncertainty can help overcome the firm's commitment problem. In all the environments studied, the firm must manage the buyers' expectations, and thus the demand, to maintain a credible strategy. In the incomplete information environment, the firm needs the uncertainty to induce a unit-elastic demand.

For its optimal testing strategy, the firm uses the uncertainty to create obedient distributions over prices. While it can deliver equilibrium existence or credible price distributions, using uncertainty necessarily entails inefficiencies. Furthermore, a higher λ, associated with a stronger commitment problem, implies a higher probability of no trade.

The other recurring theme is the impossibility for the firm to exploit the low types. This follows from the buyers' rational expectations as a low type always anticipate prices above his valuation and therefore cannot expect to buy in a PPE. The consequence was particularly stark in the incomplete information model where it made the profits the same as in the reference-independent model. In the optimal testing environment, it generated downward distortions.

One issue put aside in this paper is the possibility for the buyer to experience gain-loss utility in the money dimension as well. This modification would change the characterisation of the PPE. For example, a PE price cutoff would not be determined by the probability of reaching the cutoff but also on the expected loss in price and thus a result like Proposition 1 would not hold.

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A Proof of Proposition 2

Proof. **Preliminary lemmas:**

The first lemmas guarantee the good behaviour of two equilibrium objects, F, the firm's strategy, and $p^*(v)$ the WTP of each buyer as a function of his type.

Lemma 2. *In any equilibrium, $p^*(v)$ is strictly increasing.*

Proof. Take $v_1 < v_2$. Let $p^*(v_1) = p_1$ and $p^*(v_2) = p_2$ be their PPE cutoffs. Assume that $p_1 \geq p_2$. Because p_1, p_2 are PE cutoffs,

$$v_1 - p_1 = -\lambda v_1 F(p_1), \tag{4}$$

$$v_2 - p_2 = -\lambda v_2 F(p_2). \tag{5}$$

Clearly, $p_1 = p_2$ cannot hold. We either have $v_1 - p_2 \geq -\lambda v_2 F(p_2)$ or $v_1 - p_2 < -\lambda v_2 F(p_2)$. In the first case, we have

$$v_2 - p_2 > v_1 - p_2 \geq -\lambda v_2 F(p_2),$$

contradicting equation (5). In the second case, $F(p_2) < \frac{p_2 - v_1}{\lambda v_2}$ and $p_1 > p_2$ contradict Proposition 1. $\square$

Lemma 3. *Let F be an equilibrium strategy. If $p \in \operatorname{supp} F$, then there exists $v \in V$ such that $p^*(v) = p$.*

Proof. Assume not: there is a $p \in \operatorname{supp} F$ and no v such that $p^*(v) = p$. First, if $p \in \operatorname{supp} F$, then $p^*(v) \geq p$ for some v, for otherwise the firm makes zero profits. The firm can always make strictly positive profits by offering a price $p < \overline{v}$. This would be accepted by all types $v \in [p, \overline{v}]$ because $p^*(v) \geq v$ in any PPE. Because $p^*(\cdot)$ is strictly increasing, this implies that there is a v such that $p^*(\cdot)$ is not continuous at v and $p \in [\lim_{x \searrow v} p^*(x), \lim_{x \nearrow v} p^*(x)]$. By continuity of G, $\gamma(V^*(p)) = \gamma(V^*(p^*(v)))$. But then both $p^*(v), p \in \operatorname{supp} F$ but they give different profits, a contradiction. $\square$

Lemma 4. *Any equilibrium strategy F is continuous.*

Proof. Assume not. Let $\tilde{p}$ be a point of discontinuity of F. If $\tilde{p}$ is a PE cutoff for some v, then $F(\tilde{p}) = \frac{\tilde{p} - v}{\lambda v}$. Using the upper semicontinuity of F and continuity of $\frac{p - v}{\lambda v}$, there exists $p' < \tilde{p}$ such that $F(p') < \frac{p' - v}{\lambda v}$. By Proposition 1, $\tilde{p}$ cannot be a PPE cutoff of v. This contradicts Lemma 3. $\square$

Lemma 5. *In any equilibrium, $p^*(v)$ is continuous.*

Proof. Assume there exists a point of discontinuity $\tilde{v}$, i.e., $p_1 \equiv \lim_{v \nearrow \tilde{v}} p^*(v) < \lim_{v \searrow \tilde{v}} p^*(v) \equiv p_2$. We have that $F(p^*(v)) = \frac{p^*(v)-v}{\lambda v}$ and F is continuous, therefore,

$$F(p_1) = \lim_{v \nearrow \tilde{v}} \frac{p^*(v) - v}{\lambda v} < \lim_{v \searrow \tilde{v}} \frac{p^*(v) - v}{\lambda v} = F(p_2).$$

We can then find $\tilde{p} \in (p_1, p_2)$ such that $\tilde{p} \in \operatorname{supp} F$ and there exist no v such that $p^*(v) = \tilde{p}$. This contradicts Lemma 3. $\square$

Lemma 4 rules out pure strategies for the firm. It shows that if the firm puts strictly positive mass at one point of the support, it creates a discontinuity in the demand exactly at that point. Then, it wants to take advantage of it.

Lemma 2 and Lemma 5 also imply that we can think of $v^*(p) = \inf\{v : p^*(v) \geq p\}$ as the inverse of $p^*(v)$: for any p in the support, $p^*(v^*(p)) = p$. Furthermore, $V^*(p) = \{v : p^*(v) \geq p\} = [v^*(p), \overline{v}]$ and the demand at any price $\gamma(V^*(p)) = 1 - G(v^*(p))$.

Let $\underline{p} = \min \operatorname{supp} F$ and $\overline{p} = \max \operatorname{supp} F$.

Profits from $\underline{p}$

The profits from $\underline{p}$ are $\underline{p}(1 - G(v^*(\underline{p}))$. Indeed, for any $v \leq p$, $F(v) = 0$. Therefore, $p^*(v) = v + \lambda F(v|v) = v$ is a PE cutoff. This being the smallest PE cutoff possible, it is the PPE cutoff by Proposition 1. Moreover, for any v, $p^*(v) \geq v$. Therefore, all types above $\underline{p}$ accepts it and all types below reject it, i.e., $v^*(\underline{p}) = \underline{p}$. Profits when offering $\underline{p}$ are then $\underline{p}\,(1 - G(\underline{p}))$. These must be the equilibrium profits.

Finding the equilibrium strategy

For any $p \in \operatorname{supp} F$, by indifference on the support,

$$\pi^* \equiv \underline{p}(1 - G(\underline{p})) = p(1 - G(v^*(p))).$$

Therefore,

$$v^*(p) = G^{-1}\left(\frac{p - \pi^*}{p}\right),$$

for all $p \in \operatorname{supp} F$. Since $\frac{p-\pi^*}{p} \in [0, 1)$ for all $p \geq \underline{p}$, the expression above is well-defined. The equilibrium strategy F must guarantee that a PE cutoff of $v^*(p)$ is $p^*(v^*(p)) = p$:

$$v^*(p) - p = -\lambda v^*(p) F(p) \Rightarrow F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)}, \tag{6}$$

using that $p^*(v^*(p)) = p$. Note that this discussion also implies that in equilibrium, $p^*(v) = \frac{\pi^*}{1-G(v)}$.

Pinning down $\underline{p}$. For any $p < \underline{p}$, $F(p) = 0$. Therefore, $v^*(p) = p - \lambda v^*(p)F(p) = p$. In equilibrium, we must have

$$\pi^* \geq p(1 - G(p)).$$

For any $p > \underline{p}$, we have:

$$F(p) = \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)} \Rightarrow G^{-1}\left(\frac{p-\pi^*}{p}\right) = p - \lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)F(p) < p$$
$$\Leftrightarrow p(1 - G(p)) < \pi^*,$$

using that $F(p) > 0$ for $p > \underline{p}$. Therefore, we have $\underline{p} = \arg\max_p p(1 - G(p))$.[12]

F **is well-defined on the support**

I check here that $\frac{p-G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)}$ is a strictly increasing and positive function. For all $p \geq \underline{p}$,

$$p - G^{-1}\left(\frac{p-\pi^*}{p}\right) \geq 0 \Leftrightarrow G(p) \geq \frac{p-\pi^*}{p} \Leftrightarrow \pi^* \geq p(1 - G(p))$$

This is satisfied because $\pi^* = \max p(1 - G(p))$.

I now show that for all $p > \underline{p}$, the derivative of F is strictly positive. This follows from the following fact

1. For $p > \underline{p}$, $\frac{1-G(p)}{p} < g(p)$: by strict concavity of the profit function, the derivative is negative after the maximum.

Taking the derivative of F,

$$F'(p) \propto \frac{G^{-1}\left(\frac{p-\pi^*}{p}\right) - p\frac{\pi^*}{p^2}\frac{1}{g\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)}}{\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)^2} > \frac{G^{-1}\left(\frac{p-\pi^*}{p}\right) - \frac{\pi^*}{p}\frac{G^{-1}\left(\frac{p-\pi^*}{p}\right)}{1-G\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)}}{\left(G^{-1}\left(\frac{p-\pi^*}{p}\right)\right)^2} = 0,$$

using fact 1 to get the inequality and rearranging to get the equality.

Pinning down $\overline{p}$. We have to check that there exists a $\overline{p}$, such that $F(\overline{p}) = 1$. To do that, we will check that there exists p such that $F(p) = 1$. Note that $F(\underline{p}) = 0$ and $F(\overline{v} + \lambda\overline{v}) =$

[12] $\underline{p}$ is well-defined by the strict concavity of $p(1 - G(p))$.

$\frac{\overline{v}+\lambda\overline{v}-G^{-1}\left(\frac{\overline{v}+\lambda-\pi^*}{\overline{v}+\lambda\overline{v}}\right)}{\lambda G^{-1}\left(\frac{\overline{v}+\lambda\overline{v}-\pi^*}{\overline{v}+\lambda\overline{v}}\right)} > 1$ (rearranging and using that $G(\overline{v}) = 1$). Therefore, by continuity of F, there exists, $\underline{p} < p < \overline{v} + \lambda\overline{v}$ such that $F(p) = 1$. Then, $\overline{p} := F(\overline{p}) = 1$.[13]

Preferred Personal Equilibrium The last step is to check that the PE cutoffs pinned down by equation (6) are PPE cutoffs. This follows from the fact that the PE pinned down by equation (6) is unique:

$$\begin{aligned} \frac{p - G^{-1}\left(\frac{p-\pi^*}{p}\right)}{\lambda G^{-1}\left(\frac{p-\pi^*}{p}\right)} &\geq \frac{p-v}{\lambda v} \\ \Leftrightarrow G(v) &\geq \frac{p-\pi^*}{p} \\ \Leftrightarrow p &\leq \frac{\pi^*}{1-G(v)} = p^*(v). \end{aligned}$$

Hence, it is also a PPE cutoff. $\square$

B Proof of Lemma 1

The firm's strategy and test as defined are Markov kernels. For simplicity, for any measurable space $(X, \mathcal{X})$, I simply write X. A mapping $Q : X \times Y \to [0, 1]$ is a Markov kernel if (i) for any measurable $A \subseteq Y$, $Q(A|\cdot)$ is measurable and (ii) for any $x \in X$, $Q(\cdot|x)$ is a probability measure. I will make repeated use of composition of Markov kernels. Let $Q : X \times Y \to [0, 1]$ and $P : Y \times Z \to [0, 1]$ be two Markov kernels. Then the composition of $P \circ Q : X \times Z \to [0, 1]$ defined as

$$(P \circ Q)(A|x) = \int_Y P(A|y)dQ(y|x) \text{ for all measurable } A \subseteq Z \text{ and } x \in X$$

is a Markov kernel. Furthermore, for all bounded measurable $f : Z \to \mathbb{R}$,

$$\int f(z)d(P \circ Q)(z|x) = \int\int f(z)dP(z|y)dQ(y|x)$$

See e.g., Bauer (1996), chapter VIII, §36.

Proof. Start with a test (F, S) and an equilibrium $\left(P, (Q_v^*)_v\right)$. We are going to construct a new test $(\tilde{F}, \tilde{S})$ and an equilibrium $\left(\delta_s, (Q_v^*)_v\right)$ such that all players get the same payoffs. We

[13]Because F is strictly increasing, $\overline{p}$ is well-defined.

can construct the Markov kernel $\tilde{F} : V \times \tilde{S} \to [0,1]$, where $\tilde{S} = \cup_{s \in S} \operatorname{supp} P(s) \subseteq \mathbb{R}$ as $\tilde{F} = P \circ F$.

Let's first verify that the PPE do not change. Fix a v.

$$\begin{aligned}
Q_v &= \int_S \int_{\mathbb{R}} \mathbb{1}[p \leq v + \lambda v Q_v] dP(p|s) dF(s|v) \\
&= \int_{\tilde{S}} \mathbb{1}[p \leq v + \lambda v Q_v] d\tilde{F}(p|v) \\
&= \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[p \leq v + \lambda v Q_v] d\delta_{\tilde{s}}(p) d\tilde{F}(\tilde{s}|v),
\end{aligned}$$

and $$\begin{aligned}
W_{(F,S)}(v|Q_v) &= \int_S \int_{\mathbb{R}} \Big(\mathbb{1}[p \leq v + \lambda v Q_v](v - p) + \mathbb{1}[p > v + \lambda v Q_v](-\lambda v Q_v) \Big) dP(p|s) dF(s|v) \\
&= \int_{\tilde{S}} \Big(\mathbb{1}[p \leq v + \lambda v Q_v](v - p) + \mathbb{1}[p > v + \lambda v Q_v](-\lambda v Q_v) \Big) d\tilde{F}(p|v) \\
&= \int_{\tilde{S}} \int_{\mathbb{R}} \Big(\mathbb{1}[p \leq v + \lambda v Q_v](v - p) + \mathbb{1}[p > v + \lambda Q_v](-\lambda v Q_v) \Big) d\delta_{\tilde{s}}(p) d\tilde{F}(\tilde{s}|v) \\
&= W_{(\tilde{F},\tilde{S})}(v|Q_v).
\end{aligned}$$

This shows that the set of PE and the payoff from each of them does not change under the new test and equilibrium. Moving to the firm to the firm's payoffs, we get similarly

$$\begin{aligned}
\mathbb{E}_{(F,S)}[\pi(P)|(Q_v^*)_v] &= \int_V \int_S \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, dP(p|s) dF(s|v) dG \\
&= \int_V \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, d\delta_{\tilde{s}}(p) d\tilde{F}(\tilde{s}|v) dG \\
&= \mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\delta_{\tilde{s}})|(Q_v^*)_v].
\end{aligned}$$

Note that the integrand is bounded because $\mathbb{1}[v \in V^*(p)] = 0$ for $p > \overline{v}(1+\lambda)$ and offering a negative is a strictly dominated action. The last step is to check that any deviation from $\delta_{\tilde{s}}(p)$ is suboptimal in the new test. I show that from any strategy in $(\tilde{F}, \tilde{S})$, we can construct a strategy in (F, S) that yields the same payoff. Let $\tilde{P} : \tilde{S} \times \mathbb{R} \to [0,1]$ a strategy in $(\tilde{F}, \tilde{S})$.

Define the Markov kernel $P' : S \times \tilde{S} \to [0,1]$ as $P' = \tilde{P} \circ P$. Then,

$$\begin{aligned}
\mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\tilde{P})|(Q^*_v)_v] &= \int_V \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, d\tilde{P}(p|\tilde{s}) d\tilde{F}(\tilde{s}|v) dG(v) \\
&= \int_V \int_S \int_{\tilde{S}} \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, d\tilde{P}(p|\tilde{s}) dP(\tilde{s}|s) dF(s|v) dG(v) \\
&= \int_V \int_S \int_{\mathbb{R}} \mathbb{1}[v \in V^*(p)]\, p\, dP'(p|s) dF(s|v) dG(v) \\
&\leq \mathbb{E}_{(F,S)}[\pi(P)|(Q^*_v)_v] = \mathbb{E}_{(\tilde{F},\tilde{S})}[\pi(\delta_{\tilde{s}})|(Q^*_v)_v].
\end{aligned}$$

□

C Proof of Proposition 6 and Proposition 7

Plan of the proof:

1. Relax the problem by requiring that obedience constraint holds on intervals of signals $[\underline{s}, s]$ for all s and only upward deviation, $\tilde{P}(x) = x + \epsilon$ for all $\epsilon > 0$.

2. Use that $F(s|v) > \frac{s-v}{\lambda v}$ for all $s < p^*(v)$ to relax the obedience constraints and make them only depend on a new object $h(x) := \int_{V^*(x)} \frac{1}{v} dG$. If the obedience constraints depend only h, then it is optimal to choose a censored commitment distribution, using the FOSD interpretation of the PPE (Proposition 1). We are left with optimising over h.

3. Look only at local deviations, i.e., $\epsilon \to 0$, to get an integral inequality that pins down h.

   (a) Because h is not necessarily Lipschitz continuous, which is needed for the operation described above, I construct a sequence of relaxed problems with a smaller set of relaxed obedience constraints where deviations are bounded away from 0. For each problem, I show that it is without loss to focus on Lipschitz continuous h.

   (b) Then, I look at the limit of these problems with Lipschitz continuous h and focusing on the smallest possible deviation in each element of the sequence to derive a condition on h.

4. The resulting supremum problem with the condition from local relaxed obedience constraints gives an upper bound on profits.

5. I show that there exists a sequence of tests respecting the obedience constraints converging to the upper bound.

6. Proof of Proposition 6: Compare the characterised payoffs to the full information benchmark to show that it *strictly* dominates it.

Proof. Let S denote from now on the support of $F(s)$ the marginal distribution over signals. Based on Lemma 1, it can be any subset of $[\underline{v}, \overline{v}(1+\lambda)]$. Let $\underline{s} = \inf S$ and $\overline{s} = \sup S$.

Lemma 6. *For any F, $V^*(\underline{s}) = [\underline{s}, \overline{v}]$.*

Proof. By definition of PPE and $\underline{s}$, for all $v < \underline{s}$, $p^*(v) = v - \lambda v F(v|v) = v < \underline{s}$, using that $F(v|v) = 0$.

On the other hand, $p^*(v) \geq v$, therefore, $p^*(v) \geq \underline{s}$ for all $v \geq \underline{s}$. Thus, $V^*(\underline{s}) = [\underline{v}, \overline{v}]$. $\square$

Define $h(s) := \int_{V^*(s)} \frac{1}{v} dG$. The following lemma states that there exists a relaxation of the original problem where the constraints only depend on h, not on the test.

Lemma 7. *The following problem is a relaxation of the firm's problem:*

$$\sup_{S,\, h \in L^1(S)} \int_S x \frac{h(x) - \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, x\}}{x}}{\lambda} dx$$

s.t. for all $s \in S$ and $\epsilon > 0$,

$$(h(s) - h(s+\epsilon))s\,\epsilon + \int_{\underline{s}}^{s} x(h(x) - h(x+\epsilon))dx \geq \int_{\underline{s}}^{s} \epsilon\,(h(x+\epsilon) - \frac{\log G(x)}{\Delta v})dx \quad (7)$$

$$h(s) \in [\frac{1}{\Delta v} \log \frac{\max\{\overline{v}, s\}}{s}, \frac{1}{\Delta v} \log \left(\frac{\overline{v}}{\max\{\underline{v}, \frac{s}{1+\lambda}\}} \right)] \text{ for all } s \in S \quad (8)$$

$$h(\underline{s}) = \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, \underline{s}\}}{\underline{s}}; \; h \text{ non-increasing.} \quad (9)$$

Proof. First, focus on the following subset of obedience constraints: for all $s \in S$,

$$\int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x)]\, x\, dF(v,x) \geq \int_{V \times [\underline{s}, s]} \mathbb{1}[v \in V^*(x+\epsilon)]\, (x+\epsilon)\, dF(v,x) \text{ for all } \epsilon > 0$$

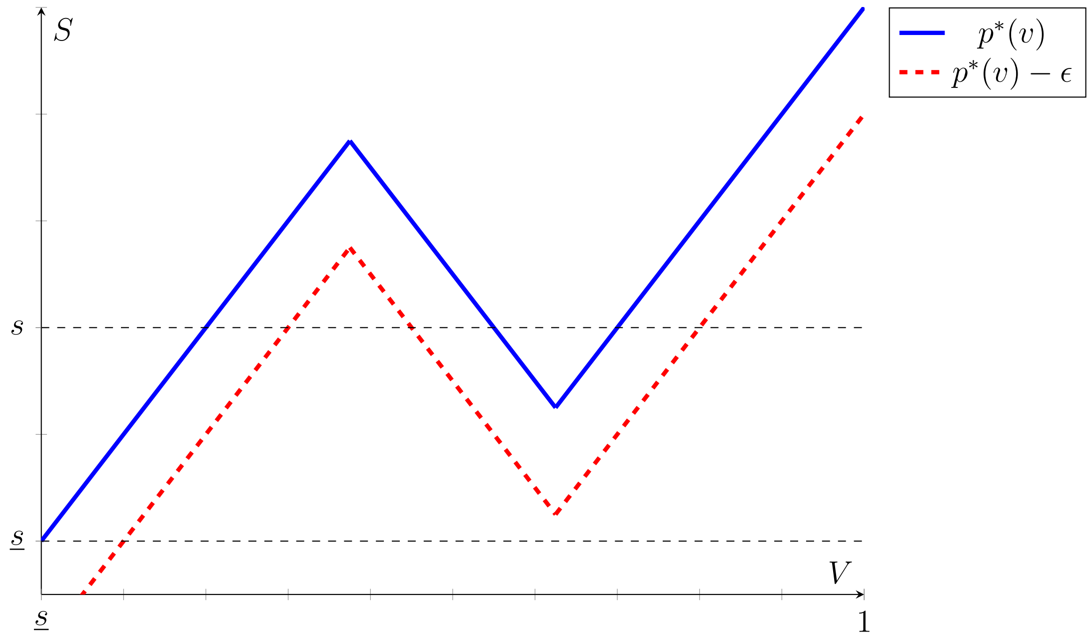

Figure 3: Area of integration with $V^*(x) = \{v : p^*(v) \geq x\}$ and $V^*(x+\epsilon) = \{v : p^*(v)-\epsilon \geq x\}$

Noting that $V^*(x+\epsilon) \subseteq V^*(x)$, we can rearrange the relaxed obedience constraint as

$$\int_{V\times[\underline{s},s]} \mathbb{1}[v \in V^*(x) \setminus V^*(x+\epsilon)]\, x\, dF(v,x) \geq \int_{V\times[\underline{s},s]} \mathbb{1}[v \in V^*(x+\epsilon)]\, \epsilon\, dF(v,x)$$

This is equivalent to (see figure 3 for an illustration)

$$\int_{V^*(s)\setminus V^*(s+\epsilon)} \int_{p^*(v)-\epsilon}^{s} x\, dF(x|v) dG + \int_{V\setminus V^*(s)} \int_{p^*(v)-\epsilon}^{p^*(v)} x\, dF(x|v) dG$$
$$\geq \int_{V^*(s+\epsilon)} \int_{\underline{s}}^{s} \epsilon\, dF(x|v) dG + \int_{V\setminus V^*(s+\epsilon)} \int_{\underline{s}}^{p^*(v)-\epsilon} \epsilon\, dF(x|v) dG$$

I will now use repeatedly the FOSD interpretation of the PPE (Proposition 1): $F(x|v) > \frac{x-v}{\lambda v}$ for $x < p^*(v)$ and $F(p^*(v)|v) = \frac{p^*(v)-v}{\lambda v}$.

Take the RHS first.

$$
\begin{aligned}
&\int_{V^*(s+\epsilon)}\int_{\underline{s}}^{s}\epsilon\,dF(x|v)dG+\int_{V\setminus V^*(s+\epsilon)}\int_{\underline{s}}^{p^*(v)-\epsilon}\epsilon\,dF(x|v)dG\\
&\geq\int_{V^*(s+\epsilon)}\int_{\underline{s}}^{s}\epsilon\,\mathbb{1}[x\geq v]\frac{1}{\lambda v}dx\,dG+\int_{V\setminus V^*(s+\epsilon)}\int_{\underline{s}}^{p^*(v)-\epsilon}\mathbb{1}[x\geq v]\epsilon\,\frac{1}{\lambda v}dx\,dG\\
&=\int_{\underline{s}}^{s}\int_{V^*(x+\epsilon)}\mathbb{1}[x\geq v]\epsilon\frac{1}{\lambda v}dG\,dx\\
&\geq\int_{\underline{s}}^{s}\frac{\epsilon}{\lambda}\Big[\int_{V^*(x+\epsilon)}\frac{1}{v}\,dG-\int_{[x,\max\{x,\overline{v}\}]}\frac{1}{v}\,dG\Big]dx
\end{aligned}
$$

using the FOSD property on the second line, changing the order of integration in the third and using that $1\geq\gamma(V^*(x+\epsilon))+\gamma([0,x])-\gamma\big(V^*(x+\epsilon)\cap[0,x]\big)$ on the last.

Now focusing on the LHS,

$$
\begin{aligned}
&\int_{V^*(s)\setminus V^*(s+\epsilon)}\int_{p^*(v)-\epsilon}^{s}x\,dF(x|v)dG+\int_{V\setminus V^*(s)}\int_{p^*(v)-\epsilon}^{p^*(v)}x\,dF(x|v)dG\\
&\leq\underbrace{\int_{V^*(s)\setminus V^*(s+\epsilon)}\int_{p^*(v)-\epsilon}^{s}x\,dF(x|v)dG}_{I}\\
&+\underbrace{\int_{V^*(s)\setminus V^*(s+\epsilon)}\int_{p^*(v)-\epsilon}^{s}\mathbb{1}[x\geq v]x\frac{1}{\lambda v}\,dx\,dG+\int_{V\setminus V^*(s)}\int_{p^*(v)-\epsilon}^{p^*(v)}x\,dF(x|v)dG}_{II}
\end{aligned}
$$

where the inequality simply follows from adding a positive term on the second line.

$$
\begin{aligned}
I&\leq\int_{V^*(s)\setminus V^*(s+\epsilon)}s\big(F(s|v)-F(p^*(v)-\epsilon|v)\big)dG\\
&\leq\int_{V^*(s)\setminus V^*(s+\epsilon)}s\big(\frac{p^*(v)-v}{\lambda v}-\frac{p^*(v)-\epsilon-v}{\lambda v}\big)dG=\frac{s\epsilon}{\lambda}\int_{V^*(s)\setminus V^*(s+\epsilon)}\frac{1}{v}dG
\end{aligned}
$$

using that $s\geq x$ and the FOSD property.

$$
\begin{aligned}
II&\leq\int_{V^*(s)\setminus V^*(s+\epsilon)}\int_{p^*(v)-\epsilon}^{s}\mathbb{1}[x\geq v]x\frac{1}{\lambda v}\,dx\,dG+\int_{V\setminus V^*(s)}\int_{p^*(v)-\epsilon}^{p^*(v)}\mathbb{1}[x\geq v]x\frac{1}{\lambda v}\,dx\,dG\\
&=\int_{\underline{s}}^{s}\int_{V^*(x)\setminus V^*(x+\epsilon)}\mathbb{1}[x\geq v]x\frac{1}{\lambda v}\,dG\,dx\\
&\leq\int_{\underline{s}}^{s}\int_{V^*(x)\setminus V^*(x+\epsilon)}x\frac{1}{\lambda v}\,dG\,dx
\end{aligned}
$$

where I use the FOSD property on the first line, change the order of integration on the second and ignore that we must have $\mathbb{1}[x \geq v]$ on the third.

Using that $h(x) = \int_{V^*(x)} \frac{1}{v} dG$, he resulting, relaxed constraint is

$$(h(s) - h(s+\epsilon))s\,\epsilon + \int_{\underline{s}}^{s} x(h(x) - h(x+\epsilon))dx \geq \int_{\underline{s}}^{s} \epsilon\,(h(x+\epsilon) - \frac{\log \frac{\max\{\overline{v},x\}}{x}}{\Delta v})dx$$

Then, remember that the constraint $p^*(v) = \min\{p : v - p = -\lambda v F(p|v)\}$ is equivalent to $v - p^*(v) = -\lambda v F(p^*(v)|v)$ and for all $p < p^*(v)$, $F(p|v) > \frac{p-v}{\lambda v}$ (Proposition 1). Relax it by only requiring $v - p^*(v) = -\lambda v F(p^*(v)|v)$ and for all $p < p^*(v)$, $F(p|v) \geq \frac{p-v}{\lambda v}$.

It is now optimal to set $F(s|v) = \frac{s-v}{\lambda v}$ for all $s \leq p^*(v)$, i.e., we choose a censored commitment distribution. This operation does not modify the relaxed PPE requirement nor the relaxed obedience constraints but improves profits. The firm's problem becomes

$$\sup_{S,\, h \in L^1(S)} \int_S x\,\frac{h(x) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}}{\lambda}dx$$

s.t. for all $s \in S$ and $\epsilon > 0$,

$$(h(s) - h(s+\epsilon))s\,\epsilon + \int_{\underline{s}}^{s} x(h(x) - h(x+\epsilon))dx \geq \int_{\underline{s}}^{s} \epsilon\,(h(x+\epsilon) - \frac{\log \frac{\max\{\overline{v},x\}}{x}}{\Delta v})dx$$

$$h(s) \in [\frac{1}{\Delta v}\log\frac{\max\{\overline{v}, s\}}{s}, \frac{1}{\Delta v}\log\Big(\frac{\overline{v}}{\max\{\underline{v}, \frac{s}{1+\lambda}\}}\Big)] \text{ for all } s \in S$$

$$h(\underline{s}) = \frac{1}{\Delta v}\log\frac{\max\{\overline{v}, \underline{s}\}}{\underline{s}};\ h \text{ non-increasing.}$$

where $h(s) \in [\frac{1}{\Delta v}\log\frac{\max\{\overline{v},s\}}{s}, \frac{1}{\Delta v}\log\Big(\frac{\overline{v}}{\max\{\underline{v},\frac{s}{1+\lambda}\}}\Big)]$ comes from $p^*(v) \in [v, v+\lambda v]$, $h(\underline{s}) = \frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}}$ follows from Lemma 6 and h non-increasing follows from the definition of $V^*(s)$, i.e., increasing the price necessarily decreases the mass of types willing to accept. □

I am going to do a little detour now and focus on a set of obedience constraints and deviations that are bounded away from zero. Specifically, obedience constraints only need to hold for all $s \in S^i = [\underline{s} + \frac{1}{i}, \overline{s}]$ and $\epsilon \in E^i = [\frac{1}{i}, \overline{v}(1+\lambda)]$ for some $i \in \mathbb{N}_0$. Note that $\underline{s} \notin S^i$ and $0 \notin E^i$. Furthermore $S^i \subseteq S^{i+1}$ and $E^i \subseteq E^{i+1}$.

Let K^i be the set of functions satisfying the constraints (7), (8) and (9) for any $s \in S^i$, $\epsilon \in E^i$ and K be the set of functions satisfying these constraints for any $s \in S$ and $\epsilon > 0$. Similarly, define OB^i as the set of functions satisfying the relaxed obedience constraints (7) for any

$s \in S^i$ and $\epsilon \in E^i$ and define OB for any $s \in S$ and $\epsilon \in E = [0, \overline{v}(1+\lambda)]$. Define

$$\Gamma = \{\phi \in L^1(S) : \text{ satisfying (8) and (9)}\}$$

where $L^1(S)$ is the set of measurable function from S to $\mathbb{R}$. Note that $K^i = OB^i \cap \Gamma$. Finally, let Lip be the set of Lipschitz continuous functions (not necessarily with the same Lipschitz constant). Endow the spaces defined above with the L^1-norm.

Lemma 8. *Let* $\pi(h) = \int_S x \, \frac{h(x) - \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, x\}}{x}}{\lambda} dx$. *Then,*

$$\sup_{h \in K} \pi(h) \leq \lim_{i \to \infty} \sup_{h \in K^i} \pi(h) = \lim_{i \to \infty} \sup_{h \in K^i \cap Lip} \pi(h)$$

Proof. **1.** $\lim_{i \to \infty} \sup_{h \in K^i} \pi(h)$ exists. This follows from $K^{i+1} \subseteq K^i$, therefore $\sup_{h \in K^{i+1}} \pi(h) \leq \sup_{h \in K^i} \pi(h)$. Moreover, $\sup_{h \in K^i} \pi(h) \geq 0$ as choosing $h(x) = \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, x\}}{x}$ is always possible for any i. Thus, the limit exists.

2. $\lim_{i \to \infty} \sup_{h \in K^i} \pi(h) \geq \sup_{h \in K} \pi(h)$. For each i, $K \subseteq K^i$, therefore, $\sup_{h \in K^i} \pi(h) \geq \sup_{h \in K} \pi(h)$ for each i.

3. $\lim_{i \to \infty} \sup_{h \in K^i} \pi(h) = \lim_{i \to \infty} \sup_{h \in K^i \cap Lip} \pi(h)$

To prove this identity, I will show that $K^i \cap Lip$ is a dense subset of K^i. Because $\pi(h)$ is continuous in h in the L^1-norm, then $\sup_{h \in K^i} \pi(h) = \sup_{h \in K^i \cap Lip} \pi(h)$.

This part is in three steps. Step 1: show that $\Gamma \cap Lip$ is dense in Γ. Step 2: show that int(K^i) is non-empty in Γ. Step 3: Using that Lipschitz continuous functions are dense in int(K^i) because it is open and int$(K^i) \subseteq \Gamma$, and convexity of K^i, show that any function in K^i can be approximated by a function in $K^i \cap Lip$.

Step 1: $\Gamma \cap Lip$ **is dense in** Γ

Take $\phi \in \Gamma$. Define

$$\phi_n(x) = \begin{cases} \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, \underline{s}\}}{\underline{s}} + \frac{\phi_n(\underline{s}+1/n) - \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, \underline{s}\}}{\underline{s}}}{1/n}(x - \underline{s}) & if \quad x \in [\underline{s}, \underline{s} + 1/n) \\ n \int_{x-1/n}^{x} \phi(z) dz & if \quad x \geq \underline{s} + 1/n \end{cases}$$

ϕ_n is differentiable everywhere but at one point, $\underline{s} + 1/n$, and its derivative is bounded by n therefore Lipschitz continuous and $\phi_n \in \Gamma$.

We have to show that

$$\lim_{n \to \infty} \int_{\underline{s}}^{\overline{s}} |\phi_n(x) - \phi(x)| dx = 0$$

Focusing on $x \geq \underline{s} + 1/n$[14].

$$\begin{aligned}
&\int_{\underline{s}+1/n}^{\overline{s}} |n \int_{x-1/n}^{x} \phi(z)dz - \phi(x)|dx \\
&\leq \int_{\underline{s}+1/n}^{\overline{s}} n \int_{x-1/n}^{x} |\phi(z) - \phi(x)|dzdx \\
&= \int_{\underline{s}+1/n}^{\overline{s}} n \int_{-1/n}^{0} |\phi(x+y) - \phi(x)|dydx \\
&= \int_{-1/n}^{0} n \int_{\underline{s}+1/n}^{\overline{s}} |\phi(x+y) - \phi(x)|dxdy \\
&\leq \sup\{\int_{\underline{s}+1/n}^{\overline{s}} |\phi(x+y) - \phi(x)|dx \; : \; y \in [-1/n, 0]\}
\end{aligned}$$

For simplicity, extend the domain to the real line and set $\phi(x) = 0$ when $x \notin [\underline{s}, \overline{s}]$. Let $\psi_m \in C_c(\mathbb{R})$, the set of continuous function in $\mathbb{R}$ with compact support, with $\psi_m \to_{L^1} \phi$. By the Heine-Cantor theorem, any ψ_m is uniformly continuous. We have for all m,

$$\begin{aligned}
&\lim_{y \to 0} \int_{\mathbb{R}} |\phi(x+y) - \phi(x)|dx \\
&\leq \lim_{y \to 0} \int_{\mathbb{R}} |\phi(x+y) - \psi_m(x+y)|dx + \int_{\mathbb{R}} |\psi_m(x+y) - \psi_m(x)|dx + \int_{\mathbb{R}} |\psi_m(x) - \phi(x)|dx \\
&\leq 2 \int_{\mathbb{R}} |\psi_m(x) - \phi(x)|dx
\end{aligned}$$

where $\lim_{y \to 0} \int_{\mathbb{R}} |\psi_m(x+y) - \psi_m(x)|dx = 0$ holds because ψ_m is uniformly continuous. Therefore, taking $m \to \infty$, $\lim_{y \to 0} \int_{\mathbb{R}} |\phi(x+y) - \phi(x)|dx = 0$. In turn, it means that $\sup\{\int_{\underline{s}+1/n}^{\overline{s}} |\phi(x+y) - \phi(x)|dx \; : \; y \in [-1/n, 0]\} \to 0$ as $n \to \infty$.

Now for $x \in [\underline{s}, \underline{s}+1/n)$, because $|\phi_n(x)|$ and $|\phi(x)|$ are bounded as $n \to \infty$, $\lim_{n \to \infty} \int_{\underline{s}}^{\underline{s}+1/n} |\phi_n(x) - \phi(x)|dx = 0$.

Therefore, $\Gamma \cap Lip$ is dense in Γ.

Step 2: Non-empty interior of $\Gamma \cap OB^i$ in Γ

Take $h(x) = \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, x\}}{x}$. It is easy to check that $h \in K^i = \Gamma \cap OB^i$. Define $z(s, \epsilon) = \int_{\underline{s}}^{s} (x + \epsilon)(h(x) - h(x + \epsilon))dx$ and $\underline{z} = \min_{s,\epsilon} z(s, \epsilon)$. Note that we have $\underline{z} > 0$ because h is strictly increasing on parts of its domain and $\underline{s} \notin S^i$ and $0 \notin E^i$.

[14] I would like to thank user fourierwho of StackExchange for this proof.

Now take $\phi(x) \in \Gamma$ with $\int_S |\phi(x) - h(x)|dx \leq \eta$, $\eta > 0$. I will show that

$$(\phi(s)-\phi(s+\epsilon))s\epsilon+\int_{\underline{s}}^{s} x(\phi(x)-\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx \geq \int_{\underline{s}}^{s}(x+\epsilon)(\phi(x+\epsilon)-\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx$$

for all $\epsilon \in E^i$, $s \in S^i$ for η sufficiently small. Rearranging the obedience constraint,

$$(\phi(s)-\phi(s+\epsilon))s\epsilon+\int_{\underline{s}}^{s} x(\phi(x)-\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx+\int_{\underline{s}}^{s}(x+\epsilon)(h(x)-h(x+\epsilon))$$
$$\geq \int_{\underline{s}}^{s}(x+\epsilon)(\phi(x+\epsilon)-h(x+\epsilon))dx$$

Take the LHS, we have

$$z(s,\epsilon)+(\phi(s)-\phi(s+\epsilon)s\epsilon+\int_{\underline{s}}^{s} x(\phi(x)-h(x))dx \geq \underline{z}-\eta\overline{s}$$

using that ϕ is non-increasing. The RHS gives

$$\int_{\underline{s}}^{s}(x+\epsilon)(\phi(x+\epsilon)-h(x+\epsilon))dx \leq \eta(\overline{s}+\epsilon)$$

Therefore, we need

$$\underline{z}-\eta\overline{s} \geq \eta(\overline{s}+\epsilon)$$
$$\underline{z} \geq (2\overline{s}+\epsilon)\eta$$

which holds for all $s \in S^i$ and $\epsilon \in E^i$ for η small enough.

Step 3: $K^i \cap Lip$ is dense in K^i

First observe that int(K^i) is an open set in Γ in the metric space $(\Gamma, L^1$-norm). Therefore, int$(K^i) \cap Lip$ is dense in int(K^i).

Note that the set K^i is convex. This can be verified by simply summing over the relaxed obedience constraints. The properties of Γ are also maintained when taking convex combinations.

Take some $h \in$ int(K^i). Any function $\phi \in K^i$ can be approximated by a sequence of $\alpha^n h + (1-\alpha^n)\phi$ with the appropriate sequence of α^n. Moreover, any point in the sequence is in the interior of Γ.[15]

[15] To see this note that there exists $\eta > 0$ such that any $B_\eta(h) \subseteq K^i$. Take $\psi = \alpha h + (1-\alpha)\phi$. I will show that any $w \in B_{\eta\alpha}(\psi)$ is in K^i. First, define $z = h + \frac{w-\psi}{\alpha}$. Then, $|z-h| = |h + \frac{w-\psi}{\alpha} - h| < \alpha\frac{\eta}{\alpha} = \eta$. Therefore $z \in K^i$. Then, choosing $\beta = \alpha$, we have $w = \beta z + (1-\beta)\phi$ and thus $w \in K^i$.

Take any $\phi \in K^i$ and $\epsilon > 0$. Let $\phi^n = \alpha^n \phi + (1-\alpha^n)h$, such that $|\phi - \phi^n| < \epsilon/2$ for all $n \geq N$ for some $N \in \mathbb{N}$. Define also $\psi^n \in K^i \cap Lip$ such that $|\psi^n - \phi^n| < \epsilon/2$ for all n, using that $\phi^n \in$ int K^i. Therefore,

$$|\phi - \psi^n| \leq |\phi - \phi^n| + |\phi^n - \psi^n| < \epsilon/2 + \epsilon/2 = \epsilon$$

for $n \geq N$.

Now, given that $\pi(h) = \int_S x^{\frac{h(x) - \frac{\log \frac{\max\{\overline{v}, x\}}{x}}{\Delta v}}{\lambda}} dx$ is continuous in the L^1-norm, we have established that $\sup_{h \in K^i} \pi(h) = \sup_{h \in Lip \cap K^i} \pi(h)$. $\square$

This lemma shows that taking the restricted set of constraints provides another upper bound to our problem. Furthermore, in the restricted problem, it is without loss to restrict attention to Lipschitz continuous functions.

Now, let's focus on $\lim_{i \to \infty} \sup_{h \in K^i \cap Lip} \pi(h) = \pi(h^*)$ (for some h^*). This implies that there exist a sequence $\{h^i\}$ with $h^i \in K^i \cap Lip$ such that $h^i \to_{L^1} h^*$. Let $\underline{\epsilon}_i = \min E^i$.

Because each h^i is bounded and of bounded total variation, by Helly's selection theorem, there exists a subsequence $\{h^{i_k}\}$ such that $h^{i_k}(s) \to h^*(s)$ for all $s \in$ int S. Let's focus on that subsequence and rename its elements: $\{h^k\}_{k=0}^{\infty}$. This implies that for each $s \in$ int S, for all $\eta > 0$, there exists $P(s, \eta) \in \mathbb{N}$ such that $|h^*(s) - h^k(s)| < \eta$ for all $k \geq P(s, \eta)$ and there exists $M(\eta) \in \mathbb{N}$ such that $\int_S |h^*(s) - h^k(s)| ds < \eta$ for all $k \geq M(\eta)$. Note also that $\int_S |h^i(x) - h^k(x)| dx < \eta$ for all $k, i \geq M(\eta/2)$.

Finally, h^* being the limit of monotone function, it is monotone and thus continuous almost everywhere. Therefore, wherever h^* is continuous, there exists $N(s, \eta) \in \mathbb{N}$ such that $|h^*(s) - h^*(s + \underline{\epsilon}_i)| < \eta$ for all $i \geq N(s, \eta)$.

Fix $\eta > 0$ and $s > \underline{s}$ where h^* is continuous. Define $i = \max\{\frac{1}{s}, N(s, \eta/3)\}$. Then, for all $k > k^*(s, \eta) \equiv \max\{i, P(s, \eta/3), P(s + \underline{\epsilon}_i, \eta/3)\}$, we have

$$\begin{aligned}
|h^k(s) - h^k(s + \underline{\epsilon}_k)| &\leq |h^k(s) - h^k(s + \underline{\epsilon}_i)| \\
&\leq |h^k(s) - h^*(s)| + |h^*(s) - h^*(s + \underline{\epsilon}_i)| + |h^*(s + \underline{\epsilon}_i) - h^k(s + \underline{\epsilon}_i)| \\
&< \eta/3 + \eta/3 + \eta/3 = \eta
\end{aligned}$$

using that $\underline{\epsilon}_i > \underline{\epsilon}_k$ on the first line. Therefore, for all $k > \max\{k^*(s,\eta), M(\eta/2)\}$,

$$(h^k(s) - h^k(s+\underline{\epsilon}_k)s + \int_{\underline{s}}^{s} x(h^k(x) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx \\ \geq \int_{\underline{s}}^{s}(x+\underline{\epsilon}_k)(h^k(x+\underline{\epsilon}_k) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx$$

$$\Rightarrow\ s\eta + \int_{\underline{s}}^{s} x(h^i(x) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx + s\eta \\ \geq \int_{\underline{s}}^{s}(x+\underline{\epsilon}_k)(h^i(x+\underline{\epsilon}_k) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x})dx - (s+\underline{\epsilon}_k)\eta$$

We can rearrange the constraint and let $k \to \infty$ (which implies $\underline{\epsilon}_k \to 0$),

$$(3s+\underline{\epsilon}_k)\eta + \int_{\underline{s}}^{s} x\frac{h^i(x) - h^i(x+\underline{\epsilon}_k)}{\underline{\epsilon}_k}dx \geq \int_{\underline{s}}^{s} h^i(x+\underline{\epsilon}_k) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx$$

$$\text{letting } k\to\infty,\quad 3s\eta + \int_{\underline{s}}^{s} -x\frac{\partial h^i}{\partial x}dx \geq \int_{\underline{s}}^{s} h^i(x) - \frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx$$

where we used the dominated convergence theorem, using that $|h^i|$ is bounded and $\frac{h^i(x)-h^i(x+\underline{\epsilon}_k)}{\underline{\epsilon}_k}$ is bounded by Lipschitz continuity. Integrating by part, we get

$$3s\eta - [h^i(x)x]_{\underline{s}}^{s} + \int_{\underline{s}}^{s} h^i(x)dx \geq \int_{\underline{s}}^{s} h^i(x)dx - \int_{\underline{s}}^{s}\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx$$

$$h^i(s) \leq \frac{\int_{\underline{s}}^{s}\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx + (\frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}})\underline{s}}{s} + 3\eta$$

Using that $h^i(\underline{s}) = \frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}}$.

Then, we can take a sequence of $\eta \to 0$, and thus $i \to \infty$, and we get for each s where h^* is continuous

$$h^*(s) = \lim_{\eta\to 0} h^i(s) \leq \lim_{\eta\to 0}\frac{\int_{\underline{s}}^{s}\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx + (\frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}})\underline{s}}{s} + 3\eta \\ = \frac{\int_{\underline{s}}^{s}\frac{1}{\Delta v}\log\frac{\max\{\overline{v},x\}}{x}dx + (\frac{1}{\Delta v}\log\frac{\max\{\overline{v},\underline{s}\}}{\underline{s}})\underline{s}}{s}$$

This holds for any s where $h^*(s)$ is continuous.

Therefore, we get another upper bound on the firm's problem.

$$\sup_{h \in Lip} \int_{\underline{s}}^{\overline{s}} x \frac{h(x) - \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, x\}}{x}}{\lambda} dx$$

$$\text{s.t. for all } s \in S' : \ h(s) \leq \frac{\int_{\underline{s}}^{s} \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, x\}}{x} dx + (\frac{1}{\Delta v} \log \frac{\max\{\overline{v}, \underline{s}\}}{\underline{s}}) \underline{s}}{s}$$

$$h(s) \in [\frac{1}{\Delta v} \log \frac{\max\{\overline{v}, s\}}{s}, \frac{1}{\Delta v} \log \Big(\frac{\overline{v}}{\max\{\underline{v}, \frac{s}{1+\lambda}\}} \Big)]$$

$$h(\underline{s}) = \frac{1}{\Delta v} \log \frac{\max\{\overline{v}, \underline{s}\}}{\underline{s}}$$

for some $S' \subseteq S$ such that $\mu(S') = \mu(S)$, where $\mu(\cdot)$ is the Lebesgue measure. This is solved by setting $\underline{s} = \underline{v}$, $\overline{s} = \overline{v}(1+\lambda)$ and $h(s) = \min\{ \frac{\int_{\underline{v}}^{s} \log \frac{\max\{\overline{v}, x\}}{x} dx + \underline{v} \log \frac{\overline{v}}{\underline{v}}}{s \Delta v}, \ \frac{1}{\Delta v} \log \Big(\frac{\overline{v}}{\max\{\underline{v}, \frac{s}{1+\lambda}\}} \Big) \}$. Calculating the integral shows that if $\underline{v} \geq \frac{s}{1+\lambda}$, then $\frac{\int_{\underline{v}}^{s} \log \frac{\max\{\overline{v}, x\}}{x} dx + \underline{v} \log \frac{\overline{v}}{\underline{v}}}{s \Delta v} \leq \frac{1}{\Delta v} \log \Big(\frac{\overline{v}}{\underline{v}} \Big)$. We can thus simplify this expression further

$$h(s) = \min\{ \frac{\int_{\underline{v}}^{s} \log \frac{\max\{\overline{v}, x\}}{x} dx + \underline{v} \log \frac{\overline{v}}{\underline{v}}}{s \Delta v}, \ \frac{1}{\Delta v} \log \Big(\frac{\overline{v}(1+\lambda)}{s} \Big) \}$$

□

Showing the upper bound is achievable

I will now show that there exists a sequence of tests satisfying the constraints and such that the profits converge to the upper bound.

Lemma 9. *There exists a test F_ϵ that satisfy the obedience constraints for $\epsilon > 0$ small enough and such that the profits converge to the upper bound as $\epsilon \to 0$.*

Proof. **Construct the test** Let $\Lambda = \{p : V \to [\underline{v}, \overline{v}(1+\lambda)] : p(v) \geq v, \ p(\underline{v}) = \underline{v}, \ K\text{-Lipschitz continuous and non-decreasing}\}$ for some $K > 1$. Endow that space with the L^1-norm. Construct the mapping $\Phi : \Lambda \to \Lambda$ as follows. Take $p \in \Lambda$ and define $\underline{p}(v)$ as

$$\frac{1}{(\lambda + \epsilon \frac{p(v) - v}{\underline{v}}) v} = \frac{\frac{p(v) - v}{\lambda v}}{p(v) - \underline{p}(v)} \Leftrightarrow \underline{p}(v) = \underline{v} \frac{\lambda v + \epsilon(p(v) - v)}{\lambda \underline{v} + \epsilon(p(v) - v)}$$

Note that if ϵ is small enough then $\underline{p}(v)$ is strictly increasing and it is continuous for any ϵ. Let $\hat{v}(s) = \min\{\underline{p}^{-1}(s), \overline{v}\}$ and $\tilde{v}(s) = \hat{v}(s) \exp \frac{-\hat{v}(s) + v}{s}$. One can check that $\tilde{v}$ is continuous with $\tilde{v}(\underline{v}) = \underline{v}$ and has its derivative uniformly bounded away from zero for any p. We can now define $\Phi : p(v) \to \min\{v(1+\lambda), \tilde{v}^{-1}(v)\} \in \Lambda$.

We know want to apply Schauder fixed point theorem: Every continuous self-map on a nonempty compact and convex subset of a normed linear space has a fixed point. (Ok, 2007, p. 469)

The function Φ is a composition of mappings that are continuous in the L^1-norm and is thus continuous. The set Λ is compact by applying Helly's selection theorem. Because Λ is of bounded variation, bounded and closed, by Helly's selection theorem any sequence in Λ admits a convergent subsequence that converges in Λ because it is closed. It is also convex. Finally, $L^1([0,1])$ is a normed linear space.

Let $p^*(v)$ be a fixed point of Φ. Abusing notation, let $\epsilon(s) = \epsilon\,\frac{s-\underline{v}}{\underline{v}}$. The test is

$$F_\epsilon(s|v) = \begin{cases} 0 & \text{if} \quad s < \underline{p}(v) \\ \frac{s-\underline{p}(v)}{(\lambda+\epsilon(s))v} & \text{if} \quad s \in [\underline{p}(v), p^*(v)] \\ \frac{s-v^*(s)}{\lambda v^*(s)} & \text{if} \quad s > p^*(v) \end{cases}$$

Note that $\underline{p}(v) \in [\frac{v(1+\epsilon)}{1+\epsilon v/\underline{v}}, v]$ because $p(v) \in [v, v(1+\lambda)]$ and thus $\hat{v}(s) \in [\min\{s, \overline{v}\}, \min\{\frac{s\underline{v}}{\underline{v}(1+\epsilon)-s\epsilon}, \overline{v}\}]$. Finally, v^* is simply the inverse of p^*.

Show it respects the obedience constraints Let $\pi(s,s') = \int_{v^*(s')}^{\overline{v}(s)} s' f_\epsilon(s|v)dv$. To satisfy the obedience constraints, we must have

$$s \in \arg\max_{s'} \pi(s,s')$$

Let's examine upward and downward deviations separately. First, upward deviation. We can write the profits from offering price s' at signal s as $\pi(s,s') = \frac{1}{\Delta v}\int_{v^*(s')}^{\hat{v}(s)} s' f_\epsilon(s|v)dv$. The derivative with respect s' is proportional to

$$\log\frac{\hat{v}(s)}{v^*(s')} - s'\frac{(v^*(s')'}{v^*(s')}$$

Setting the derivative equal to 0 when $s' = s$, we get $\log\frac{\hat{v}(s)}{v^*(s)} - s\frac{(v^*(s)'}{v^*(s)} = 0$. This is always satisfied when $v^*(s) = \hat{v}(s)\exp\frac{-\hat{v}(s)+\underline{v}}{s}$. When, $v^*(s) = \frac{s}{1+\lambda}$, we need

$$\log\frac{\hat{v}(s)(1+\lambda)}{s} - s\frac{1/(1+\lambda)}{s/(1+\lambda)} \leq 0$$

Note that $v^*(s) = \frac{s}{1+\lambda}$ when $\frac{s}{1+\lambda} \geq \hat{v}(s)\exp\frac{-\hat{v}(s)+\underline{v}}{s}$. Using that $\hat{v}(s) \leq \frac{s\underline{v}}{\underline{v}(1+\epsilon)-s\epsilon}$, we get

$$\log\frac{\hat{v}(s)(1+\lambda)}{s} \leq \frac{\hat{v}(s)-\underline{v}}{s} \leq \frac{\frac{s\underline{v}}{\underline{v}(1+\epsilon)-s\epsilon} - \underline{v}}{s} \leq 1, \text{ for } \epsilon > 0.$$

Then observe that for $s' > s$,

$$\log \frac{\hat{v}(s)}{v^*(s')} - s'\frac{(v^*(s')'}{v^*(s')} \leq \log \frac{\hat{v}(s')}{v^*(s')} - s'\frac{(v^*(s')'}{v^*(s')} \leq 0$$

because $\hat{v}(\cdot)$ is increasing. Therefore, there is no profitable upward deviation.

Now, for downward deviations, assuming $v^*(s) > s/(1+\lambda)$, the payoffs are proportional to

$$\int_{v^*(s)}^{\hat{v}(s)} \frac{s'}{(\lambda+\epsilon(s))v} dv + \int_{v^*(s')}^{v^*(s)} s'\Big(\frac{s-v^*(s)}{\lambda v^*(s)}\Big)' dv$$

Note that $\left(\frac{s-v^*(s)}{\lambda v^*(s)}\right)' = \frac{v^*(s)-(v^*(s))'s}{\lambda v^*(s)^2}$. Note also that $\frac{\lambda+\epsilon(s)}{\lambda} \cdot \frac{v^*(s)-(v^*(s))'s}{v^*(s)} \leq 1$ for ϵ small enough for all s as

$$\begin{aligned}
\frac{\lambda+\epsilon(s)}{\lambda} \cdot \frac{v^*(s)-(v^*(s))'s}{v^*(s)} &= \frac{\lambda+\epsilon(s)}{\lambda} \cdot (1 - \frac{(v^*(s))'s}{v^*(s)}) \\
&= \frac{\lambda+\epsilon(s)}{\lambda} \cdot (1 - \log \hat{v}(s)/v^*(s)) \\
&= \frac{\lambda+\epsilon(s)}{\lambda} \cdot (1 - \frac{\hat{v}(s)-\underline{v}}{s}) \\
&\leq \frac{\lambda+\epsilon\, \frac{s-\underline{v}}{\underline{v}}}{\lambda} \cdot (1 - \frac{\min\{s,\overline{v}\}-\underline{v}}{s}) \leq 1
\end{aligned}$$

where I have use that $\hat{v}(s) \geq \min\{s, \overline{v}\}$. The last inequality is satisfied for all s for ϵ small enough. Taking the derivative and evaluating it at $s' = s$, we have

$$\begin{aligned}
\frac{1}{\lambda+\epsilon(s)} \log \frac{\hat{v}(s)}{v^*(s)} - (v^*(s))'s\frac{v^*(s)-(v^*(s))'s}{\lambda v^*(s)^2} \geq 0 \\
\Leftrightarrow \log \frac{\hat{v}(s)}{v^*(s)} - \frac{\lambda+\epsilon(s)}{\lambda} \cdot \frac{v^*(s)-(v^*(s))'s}{v^*(s)} \cdot \frac{(v^*(s))'s}{v^*(s)} \geq \log \frac{\hat{v}(s)}{v^*(s)} - \frac{(v^*(s))'s}{v^*(s)} = 0
\end{aligned}$$

using that $\frac{\lambda+\epsilon(s)}{\lambda} \cdot \frac{v^*(s)-(v^*(s))'s}{v^*(s)} \leq 1$ for ϵ small enough for all s.

We are then left to check that the profit function is concave when $s' < s$. Take the derivative with respect to s' twice, we need to show that

$$\begin{aligned}
-f(s|v)\,(v^*(s'))' - f(s|v)\,(v^*(s'))' - f(s|v)(v^*(s'))''\,s' \leq 0 \\
\Leftrightarrow -2(v^*(s'))' - (v^*(s'))''\,s' \leq 0
\end{aligned}$$

To see this is verified, note that $h(s)s = \int_{\underline{v}}^{s} \log \frac{\overline{v}}{\hat{v}(t)} dt + h(\underline{v})\underline{v}$. Taking the derivative twice on both sides, we get, $2h'(s) + sh''(s) = -\frac{\hat{v}'(s)}{\hat{v}(s)} \leq 0$. Given that $h(s) \propto \int_{v^*(s)}^{\overline{v}} \frac{1}{v} dv$, this means that

$$-2\frac{(v^*(s))'}{v^*(s)} + s\frac{-(v^*(s))''v^*(s) + (v^*(s))'^2}{v^*(s)^2} \leq 0$$

Rearranging, we get

$$-2(v^*(s'))' - (v^*(s'))'' s' \leq -\frac{s(v^*(s))'^2}{v^*(s)} \leq 0$$

If $v^*(s) = s/(1+\lambda)$, there are no profitable downward deviation as $f(s|v) = 0$ for all $v \leq v^*(s)$.

As $\epsilon \to 0$, profits converge to the upper bound derived in the previous section. In the limit, we also have $\hat{v}(s) = \min\{s, \overline{v}\}$.

$\square$

Simple calculation shows that if $h(s) = \min\{\frac{\int_{\underline{v}}^{s} \log \frac{\max\{\overline{v},x\}}{x} dx + \underline{v} \log \frac{\overline{v}}{\underline{v}}}{s\Delta v}, \frac{1}{\Delta v} \log \frac{\overline{v}(1+\lambda)}{s}\}$, the upper bound is the profits stated in Proposition 7:

$$\int_{\underline{v}}^{\overline{v}(1+\lambda)} s \frac{h(s) - \frac{1}{\Delta v} \log \frac{\max\{\overline{v},s\}}{s}}{\lambda} ds = \int_{\underline{v}}^{\overline{v}(1+\lambda)} \frac{\min\left\{\hat{v}(s) - \underline{v},\ s \log \frac{(1+\lambda)\hat{v}(s)}{s}\right\}}{\lambda \Delta v} ds.$$

Showing that the profits are strictly greater than full information:

The profits for the limit of fully revealing tests is

$$\begin{aligned}
\int_{\underline{v}}^{\overline{v}} \frac{v}{\Delta v} dv &= \int_{\underline{v}}^{\overline{v}} \int_{v}^{v(1+\lambda)} \frac{1}{\lambda \Delta v} ds dv \\
&= \int_{\underline{v}}^{\overline{v}(1+\lambda)} \int_{\max\{\underline{v}, \frac{s}{1+\lambda}\}}^{\hat{v}(s)} \frac{1}{\lambda \Delta v} dv\, ds \\
&= \int_{\underline{v}}^{\overline{v}(1+\lambda)} \frac{\hat{v}(s) - \max\{\underline{v}, \frac{s}{1+\lambda}\}}{\lambda \Delta v} ds.
\end{aligned}$$

Clearly, $\hat{v}(s) - \max\{\underline{v}, \frac{s}{1+\lambda}\} \leq \hat{v}(s) - \underline{v}$ and observe that for $s < \overline{v}(1+\lambda)$,

$$\begin{aligned}
s \log \frac{(1+\lambda)\hat{v}(s)}{s} &= \int_{\frac{s}{1+\lambda}}^{\hat{v}(s)} \frac{s}{v} dv \\
&> \int_{\frac{s}{1+\lambda}}^{\hat{v}(s)} dv \geq \hat{v}(s) - \max\{\underline{v}, \frac{s}{1+\lambda}\},
\end{aligned}$$

where the first inequality follows from $\frac{s}{v} > 1$ when $v \in [\frac{s}{1+\lambda}, \hat{v}(s))$. Note that there must be a strictly positive measure of signals such that $\min\left\{\hat{v}(s) - \underline{v},\ s \log \frac{(1+\lambda)\hat{v}(s)}{s}\right\} = s \log \frac{(1+\lambda)\hat{v}(s)}{s}$ as $\hat{v}(\overline{v}(1+\lambda)) - \underline{v} > 0$ and $\overline{v}(1+\lambda) \log \frac{(1+\lambda)\hat{v}(\overline{v}(1+\lambda))}{\overline{v}(1+\lambda)} = 0$ and all functions are continuous.

Therefore,

$$\int_{\underline{v}}^{\overline{v}(1+\lambda)} \frac{\min\left\{\hat{v}(s)-\underline{v},\ s\log\frac{(1+\lambda)\hat{v}(s)}{s}\right\}}{\lambda\Delta v}ds > \int_{\underline{v}}^{\overline{v}} \frac{v}{\Delta v}dv.$$