The Optimal Menu of Tests

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February, 2024

Abstract

In many situations, decision-makers depend on tests to inform their choices. I consider a decision-maker that has access to a set of feasible tests and, prior to making a decision, requires a privately informed agent to choose a test from a menu. By offering a menu, the decision-maker can use the agent's choice as an additional source of information. The decision-maker must accept or reject the agent. The agent always wants to be accepted, while the decision-maker wants to accept only a subset of types. First, I show that the decision-maker does not benefit from commitment in this context. I use this result to show in several economic environments when the decision-maker benefits from offering a choice of test. When the domain of feasible tests contains a most informative test, I give necessary and sufficient conditions for when only the dominant test is offered for any prior and when a dominated test is always part of the optimal menu. I also show when the decision-maker benefits from a menu whenever types are multidimensional or tests vary in their difficulty.
JEL codes: D82, D83

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1 Introduction

In many economic settings, decision-makers (DMs) rely on tests to guide their actions. Universities use standardised tests as part of their admission process, firms interview job candidates before they hire them and regulators test products prior to authorisation. In these examples, the DM is deciding whether to accept an agent and his preferences depend on some private information held by the agent: the ability of the student, the productivity of the candidate or the quality of the product. Ideally, the DM would want to use a fully revealing test, but this is often not feasible and thus the ability to learn from the test outcome alone is limited. However, there is an additional channel the DM can learn from: he can offer a menu of tests and let the agent *choose* which test to take. The DM can then use the agent's choice as an additional source of information.

Depending on the setting, constraints on the testing ability can take many different forms. For instance, a hiring firm is constrained by the amount of time and resources it can allocate to the selection process; most universities have to use externally provided tests for their admission procedures; and medicine regulatory agencies face both technological and ethical constraints when authorising new drugs.

In this paper, I study the DM's optimal design of a menu of tests and provide conditions under which the DM learns from both the test choice and the outcome. I first show that the DM does not benefit from committing to a strategy for any arbitrary domains of feasible tests. I then apply this result to natural economic environments and provide conditions under which the DM benefits from offering more than one test and when he does not. I consider the following three structures on the feasible tests: the DM has access to a test more informative than all the others, tests are ordered by their difficulty and each test can only identify one dimension of the agent's private information.

More specifically, I study a DM who has to accept or reject a privately informed agent. While the DM wants to accept a subset of types (the A-types) and reject the others (the R-types), the agent always wants to be accepted. The domain of feasible tests is an exogenously given set of Blackwell experiments. The DM designs a menu of tests, a subset of the feasible tests, from which the agent chooses one. For example, a university can let a student decide whether he takes a test or how many reference letters to submit in its admission procedure. A regulator can let a pharmaceutical company design the clinical trials when authorising a new drug. After observing, the choice of test and its result, the DM decides whether or not to accept.

In Theorem 1, I show that regardless of whether the DM is able commit ex-ante to an acceptance rule, the optimal menu and strategies remain the same. Moreover, I provide an optimisation problem based on the problem with commitment that delivers equilibrium strategies. This result holds for arbitrary type structures and domains of feasible tests. In addition, I find that it is without loss to consider menus with as many tests as A-types. This implies that if there is only one type the DM wants to accept, it is sufficient to consider menus with a single test and the only information used is the test outcome. Theorem 1 follows from a max-min representation of the DM's problem where the maximiser chooses the DM and A-types' strategy and the minimiser the R-types' strategy.

In Section 4, I use Theorem 1 to determine which tests are part of the optimal menu in three natural economic environments. In Section 4.1, I consider a domain of feasible tests containing a dominant one, in the sense of Blackwell's (1953) informativeness order. One example of this environment is a university considering whether to allow students to opt out of an admission test like an interview or a standardised test when applying. This is effectively offering a menu containing the admission test and an uninformative test. In the same context, universities sometimes allow students to submit an additional, optional, reference letter. The more reference letters, the more informative the test is.

The first result in that environment is that the dominant test is always part of an optimal menu. In the example above, it means that the university should always allow students to take the admission test or submit the optional reference letter. I then examine the case of binary tests. I provide necessary and sufficient conditions on the dominant test under which a dominated test is part of the optimal menu for any prior. When tests are binary, types can be ordered by how likely they are to generate a high signal in the most informative test. I say that a test is *enclosed* if the DM wants to accept at least the worst and the best performer on the test. I show that the optimal menu includes a strictly less informative test for any prior if and only if the dominant test is enclosed. On the other hand, an optimal menu contains only the most informative test for any prior if and only if the dominant test is single-peaked. This corresponds to the DM willing to accept either only high types or only intermediate types, as measured by their performance on the test.

In the case where the tests generate more than two signals, the results extend as follows. If there exists a subset of signals where single-peakness is violated, there exists a less informative test that is part of the optimal menu for some prior. On the other hand, if the environment is one-dimensional, in the sense that all the tests satisfy the monotone likelihood ratio property and the DM wants to accept any type above a threshold, only the most informative test is offered.

These conditions capture the idea that the DM benefits from offering a dominated test whenever he thinks that there are sufficiently many A-types that do not perform well on the dominant test. For example, in the context of university admissions, it is common that some parts of the selection process is optional: universities often allow candidates to submit optional essays, recommendation letters or tests. Results of Section 4.1 can explain why universities would prefer to do that: if there is a concern that some good students would not perform well on the optional parts, offering a choice could be in fact optimal.[1]

[1]This could be the case if the optional part is only informative about some dimension of interest, e.g., the optional test/essay is a technical task but the university also cares about the student's creativity (or vice-versa). Another possibility is that some categories of the population is systematically rated lower in the dominant test because of e.g., access to education

From a technical point of view, I show that a singleton menu is optimal by showing the existence of a unique equilibrium where only the dominant test is chosen. In a sense, these results extend *unravelling* type arguments (e.g., Milgrom, 1981, 2008) where instead of unravelling to the most revealing evidence, the strategies unravel to the dominant, but possibly noisy, test. The equilibrium forces concentrate choices on the most informative test but potentially reveal *less* information than if types would choose different tests.

In Section 4.2, I consider a one-dimensional environment where feasible tests are ordered by their difficulty, instead of how informative they are. For example, a regulator protecting consumer welfare could mandate labelling to inform consumers about a product characteristics and decide which certifiers, varying by their stringency, a firm can use.[2] Or the DM could be a regulator deciding how hard a compliance test is before authorising a product. To this end, I define and characterise a new notion of difficulty to compare experiments. This notion captures that varying the difficulty of a test changes which types are better identified: a more difficult test is informative when it is passed, as only high types are likely to produce a high grade but it is less informative when it is failed. I show again that a singleton menu is optimal. In this case, it is possible to sustain an equilibrium with more than one test but satisfying the agent's incentive constraints is so demanding that the DM is better off offering only one test. Going back to the consumer product market application, this implies that, when it comes to mandatory labelling, firms should not have a choice of how difficult a certification is.

In the last environment considered, I turn to a multidimensional setting. For example, a hiring firm could be guided by two considerations, the candidate's technical and managerial skills and focus the interview on either dimension. More generally, I assume that the agent's type has two components and each test is informative about only one of them.[3] Offering tests for both dimensions allows A-types that perform badly in one dimension to select the test where they perform best. I show that the optimal menu contains both tests for any prior if and only if the DM wants to accept any type that performs well in at least one dimension. This would be the case if the hiring firm would want to hire a candidate with high technical skills but no managerial skills and vice-versa. On the other hand, if the firm cares about both dimension simultaneously, then, for some priors, it uses only one test.

Finally, I consider three extensions to the baseline model. The first one is to allow for cheap-talk communication on top of the test choice. I can show that in this case, each A-type announces his type deterministically and each R-type pretends to be an A-type. Moreover, I show that the DM can benefit from combining testing and communication. However, in the second extension, I introduce costless unobserved effort influencing the outcome of the test and show that it greatly reduces the role

or socio-economic background. The intuition for the results also echoes the recommendation from universities to favour "*quality over quantity*" when choosing whether to submit an optional recommendation letter (see e.g., Saviano, 2020, for a blogpost from an NYU admission officer).

[2]For example, the EU mandates energy labels on products such as washing machines, light bulbs or tyres. Varying the difficulty could correspond to varying the threshold to get a "high grade".

[3]The results extend easily to more than two dimensions.

that communication can play. Finally, I consider the possibility of going beyond menus and allow for general mechanisms where a mechanism can randomly allocate a type to a test. I show that the DM benefits from having access to a randomisation device and provide a characterisation of the optimal mechanism in terms of another max-min problem (Theorem 3). I also show that in this context, the DM does not benefit from committing to a strategy.

Relation to the literature

This paper studies a model of information provision, by the choice of test, by a privately informed agent. In contrast to most of the literature on strategic disclosure using hard evidence (e.g., Grossman, 1981; Milgrom, 1981; Dye, 1985; Milgrom, 2008), I allow for arbitrary stochastic tests.[4] Introducing stochastic tests allows me to study previously unexplored economic environments – in particular environments containing a most informative test and tests ordered by their difficulty – and to identify new conditions on tests to determine the optimal menu. The results that pertain to these environments also shed light on practices in selection procedures in higher education or labelling in consumer product markets.

As such, this paper mainly relates to the literature on strategic disclosure and mechanism design with evidence, the literature on information design without commitment and to papers studying the optimal design of tests or selection procedures. Formally, my model is most closely related to Glazer and Rubinstein (2006). They also study a problem where an agent wants to persuade a DM to accept him but in their model, the agent can only present deterministic evidence about his type. They characterise the optimal mechanism that maps evidence to a decision and show that the outcome can be implemented without commitment (see also Hart et al., 2017, for similar results with other payoff structures; Sher, 2011; Ben-Porath et al., 2021). They also show that with deterministic evidence, the optimal decision rule is deterministic. I extend their analysis in two ways. First, Theorem 1 generalises their result on commitment to arbitrary testing technology and my characterisation result also applies in their setup. I also show that the optimal decision rule is no longer deterministic when tests are stochastic. Second, I use the characterisation to prove general results on which test is included in the optimal menu depending on the properties of the feasible tests. Glazer and Rubinstein (2004) study a related problem where the agent communicates with the DM who, based on the communication, verifies one dimension of a multidimensional type.

More generally, this paper relates to the mechanism design with evidence literature (e.g., Green and Laffont, 1986; Bull and Watson, 2007; Deneckere and Severinov, 2008; Koessler and Perez-Richet, 2019; Forges and Koessler, 2005; Kartik and Tercieux, 2012; Strausz, 2017). In Section 6.3, I charac-

[4] Hard evidence is a particular kind of test that takes a deterministic form: the agent can provide evidence that he belongs to a certain subset of types. Another difference with modelling information with evidence is that, in my language, not all types can participate in all tests. I discuss the relation between these two modelling approaches in more details in Section 2.2.

terise the optimal mechanism and unlike most of that literature, I allow for arbitrary domain of feasible tests that include non-deterministic tests.[5] The payoff structure assumed in this paper is commonly used in this literature, e.g., in Glazer and Rubinstein (2004); Glazer and Rubinstein (2006) and special cases of Ben-Porath et al. (2019); Ben-Porath et al. (2021). Ben-Porath et al. (2021) show a similar result on the no value of commitment.

An important focus of the literature on strategic disclosure is finding conditions under which all information is revealed in equilibrium, see e.g., Grossman (1981), Milgrom (1981), Lipman and Seppi (1995), Giovannoni and Seidmann (2007), Hagenbach et al. (2014) or Carroll and Egorov (2019). In my model, if full information is possible, it is optimal, but I also characterise the optimal choice of test when full information is not attainable. In Proposition 9, I provide the necessary and sufficient conditions for full payoff-relevant information revelation.

The other branch of literature my paper relates to is information design without sender commitment. In these papers, the agent and the DM correspond to the sender and the receiver. In particular, this paper is closer to models characterising receiver-optimal tests where the sender can choose which test to take. In Rosar (2017) and Harbaugh and Rasmusen (2018), the receiver designs a test where a privately informed agent can either take the test, possibly at a cost, or take an uninformative test. In these papers, the receiver flexibly designs a test *given* that the sender has a choice. In my paper, the receiver designs the choice, i.e., the menu, given the restrictions on the feasible tests.

Other papers consider the receiver-optimal design of tests where the sender's action is partially observed or unobserved, e.g., DeMarzo et al. (2019), Deb and Stewart (2018), Perez-Richet and Skreta (2022) or Ball (2021) (note that Perez-Richet and Skreta, 2022, also consider observable actions). In a model of dynamic task allocation, Deb and Stewart (2018) share a similar concern about whether the optimal task should be the most informative. The key strategic friction in their model is that the agent can shirk on the task. In my paper, the design of the optimal menu also has to take into account the strategy of the sender, however unobservable actions fundamentally changes the sender's incentives and thus how information is revealed. I discuss in Section 2.2 which results would still apply if the outcome of the tests depends on the agent's unobserved actions and provide a framework and additional results in Section 6.2 where this possibility is allowed.[6]

Finally, there are numerous papers concerned with design of tests and selection procedures whose themes overlap with this paper's. In the context of college admissions in the US, Dessein et al. (2023) show that test-optional policies cannot be justified from an informational point of view. A key as-

[5] For examples of mechanism design paper with non-deterministic tests, see Ball and Kattwinkel (2022) and Ben-Porath et al. (2021).

[6] There are also papers studying sender-optimal tests when the sender cannot fully commit to reporting the test correctly, e.g., Nguyen and Tan (2021), Lipnowski et al. (2022) or Koessler and Skreta (2022). In Boleslavsky and Kim (2018) and Perez-Richet et al. (2020), the sender can commit but there is a third agent whose effort determines respectively the state of the world or the Blackwell experiment actually performed.

sumption in their model is that, if the university uses a test-optional policy, students observe their test score before deciding to submit them, in contrast to this paper. Whether the test result is observable depends on the application. This would be the case if test scores are automatically sent to universities,[7] they use interviews or the content of a reference letter is not revealed to the student. On the other hand, some popular standardised tests like the SAT or ACT do not automatically send test scores to universities. Krishna et al. (2022) model the university selection process as a contest and show that coarsening grades can be Pareto efficient. Ely et al. (2021) study the optimal allocation of tests from a restricted set to agents with observable characteristics. Dasgupta (2023) study the design of a test when the agent's private information is its belief about the underlying state of the world.

2 Model

There is a decision-maker (DM) and an agent. The agent has a type $\theta \in \Theta$, $|\Theta| < \infty$, with a common prior $\mu \in \Delta\Theta$. The prior is always full support unless specified otherwise. The set of types is partitioned in two: $\Theta = A \cup R$, $A \cap R = \emptyset$. The type is private information of the agent. The DM must take an action $a \in \{0, 1\}$, accept or reject. The utilities of the DM and the agent are $v(a, \theta) = a\big(\mathbb{1}[\theta \in A] - \mathbb{1}[\theta \in R]\big)$ and $u(a, \theta) = a$. That is, the DM wants to accept agents in A and reject agents in R. The agent always wants to be accepted. The analysis is virtually unchanged by allowing for DM's utility functions of the form $v(a, \theta) = a\nu(\theta)$ for some $\nu : \Theta \to \mathbb{R}$ (I explain why in appendix B).

There is a finite exogenous set of test $T \subseteq \Pi := \{\pi : \Theta \to \Delta X\}$, where X is some finite signal space. The conditional probabilities of test t are $\pi_t(\cdot|\theta)$. The set T captures the restriction on the DM's testing capacity. He can only perform one test from that set. A menu of test is a subset of the feasible tests, $\mathcal{M} \subseteq T$.

The timing of the game is as follows.

1. The agent learns his type θ.
2. The DM chooses a menu $\mathcal{M} \subseteq T$.
3. The agent chooses a test from the menu, denoted by $\sigma : \Theta \to \Delta\mathcal{M}$.
4. A signal x is drawn according to $\pi_t(\cdot|\theta)$.
5. The DM chooses an action based on the realised test choice and outcome, the acceptance probability denoted by $\alpha : \mathcal{M} \times X \to [0, 1]$.

[7]Some examples are the Law National Aptitude, Mathematics Admissions test or the Sixth Term Examination Paper Mathematics tests used for undergraduate admissions in the UK or the Dental Admission Test and Optometry Admission Test in the US.

The solution concept is weak Perfect Bayesian Equilibrium. There are generally many equilibria and I focus on the DM-preferred equilibrium.[8] For example, for any menu where two tests are offered, there are always equilibria where all types choose the same test, akin to a babbling equilibrium in a cheap-talk model. This selection is also motivated by the nature of the results in this paper. I focus on possibility and impossibility results: when is it optimal to offer more than one test and when it is not. Looking at the DM-preferred equilibrium allows me to establish when it is *possible* to benefit from offering more than one test and when it is *impossible*.

Notation: For any α, denote the probability of type θ to be accepted in test t by $p_t(\alpha;\theta) \equiv \sum_x \alpha(t,x)\pi_t(x|\theta)$. I also write $(\alpha, \sigma) \in \text{wPBE}(\mathcal{M})$ if there is a belief $\tilde{\mu}$ such that $(\alpha, \sigma, \tilde{\mu})$ is a weak PBE in the subgame when the menu is $\mathcal{M}$.

The optimal design of menu solves

$$V = \max_{\mathcal{M}\subseteq T} \max_{\alpha,\sigma} \sum_{\theta\in A} \mu(\theta) \sum_{t\in\mathcal{M}} \sigma(t|\theta)p_t(\alpha;\theta) - \sum_{\theta\in R} \mu(\theta) \sum_{t\in\mathcal{M}} \sigma(t|\theta)p_t(\alpha;\theta)$$
$$\text{s.t. } (\alpha,\sigma) \in \text{wPBE}(\mathcal{M})$$

The inner maximisation problem selects, for a fixed menu, the DM and agent strategy to maximise the DM's payoff for a fixed menu, under the constraint that they are equilibrium strategies. The outer maximisation problem selects the best possible menu for the DM. I say that the DM's payoffs are better in menu $\mathcal{M}$ than in $\mathcal{M}'$ if the DM's payoffs in the DM-preferred equilibrium in $\mathcal{M}$ is higher than in $\mathcal{M}'$.

Test restriction: The exogenous set of tests T can capture different constraints on DM's testing capacity. It could be a purely technological constraint, e.g., when choosing amongst standardised tests, universities can only choose from an exogenously given set of tests from test providers. The constraint can also be on some properties of the tests that can be used, e.g., $T \subset \{\pi : \pi \text{ has the MLRP}\}$. Finally, it could come from a capacity constraint in the information processing/acquisition abilities of the DM, e.g., a limited number of sample sizes a researcher can collect, a maximum number of reference letters a university can process or there could be a cost function associated with each experiment $C : \Pi \to \mathbb{R}$ and a maximum cost the DM can pay $c \in \mathbb{R}$, $T \subset \{\pi : c \geq C(\pi)\}$.

2.1 Example: Opting out of an admission test

Suppose a university uses some test for university admission and that there are three types of students: $A = \{A1, A2\}$ and $R = \{R1\}$. Consider the testing set $T = \{t, \emptyset\}$ where $\emptyset$ is an uninformative test.

[8] The results would be exactly the same if I would take DM-preferred Sequential Equilibrium (Kreps and Wilson, 1982) as my solution concept. I comment on this in more detail in the discussion of Theorem 1.

The test t is described by $X = \{x_0, x_1\}$ and

$$\pi_t(x|A1) = \begin{cases} 1/2 & \text{if } x = x_0 \\ 1/2 & \text{if } x = x_1 \end{cases} \qquad \pi_t(x|R1) = \begin{cases} 1/3 & \text{if } x = x_0 \\ 2/3 & \text{if } x = x_1 \end{cases}$$

$$\pi_t(x|A2) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \end{cases}$$

For the purpose of this example, suppose that $\mu(A1) < \frac{2}{3}\mu(R1) < \mu(A2)$. The lessons from this example can be extended to arbitrary priors.

This example can be interpreted as follows. The test t is a test a university uses to get information about students, like an interview or a standardised test. The signal x_1 represents a high grade and x_0 a low grade. A common concern about these tests is that they can be too easily gamed by unobserved test preparation or fail to identify good students in some categories of the population (see e.g., Academic Senate UC, 2020, for a report on standardised tests in the University of California system). The parametrisation of the test t captures this phenomenon. While A_2 and R_1 are naturally ordered, in the sense that A_2 is more likely to have a good grade than R_1, A_1 corresponds to a type of student that the university wants to accept but generates a lower grade than R_1. The prior could be obtained based on some other information provided by the student, e.g., after having observed transcripts, cover letter and socio-economic background. This parametrisation however still captures that a high grade is predictive of a high quality student, in line with the finding from the education literature that standardised tests are predictive of good educational outcome even after conditioning on various observables such as socio-economic background or race. Adding $\emptyset$ to the menu allows the student to opt out from the admission test. Another interpretation of this example is that the test is a recommendation letter and adding $\emptyset$ makes that recommendation letter optional. The parametrisation then captures the possibility that $A1$ is a good student but with no connection to professors or teachers that could write a good letter for them.

When only t is offered: The information structure and prior deliver the following best response when only t is offered,

$$\alpha(x,t) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \end{cases}$$

The acceptance probabilities of each types are then

$$p_t(\alpha; R1) = 2/3 \qquad p_t(\alpha; A1) = 1/2 \qquad p_t(\alpha; A2) = 1$$

When both t and $\emptyset$ are offered: Consider the equilibrium with the following strategies of the agent:

$$\sigma(\emptyset|R1) = \frac{\mu(A1)}{\mu(R1)} \qquad \sigma(\emptyset|A1) = 1 \qquad \sigma(t|A2) = 1$$

The student $R1$ mixes between the two tests, t and $\emptyset$, whereas $A1$ chooses $\emptyset$ with probability one and $A2$ chooses t with probability one. If all types play a pure strategy, it is not possible to maintain an equilibrium where both tests are chosen. If it is the case, there is a test that is only chosen by an A-type and in equilibrium the DM must accept with probability one after any signal in that test. Thus R_1 mixes in equilibrium to make the menu $\{t, \emptyset\}$ credible.

Given the agent's strategy, the DM's strategy after t remains the same as before. When the DM observes $\emptyset$, he is indifferent between accepting and rejecting. By accepting with probability $2/3$, $R1$ is indifferent between $\emptyset$ and t. The resulting acceptance probabilities are

$$\mathbb{E}[p(\alpha; R1)] = 2/3 \qquad p_{\emptyset}(\alpha; A1) = 2/3 \qquad p_t(\alpha; A2) = 1$$

Types $R1$ and $A2$ have the same acceptance probabilities as before but $A1$ is accepted with strictly higher probability. Therefore, allowing to opt out strictly increases the DM's payoffs. Intuitively, type A1 is a student that is poorly identified by the admission test. Giving the option of opting out allows that student to differentiate himself from the other types without benefiting the bad student R1.

2.2 Discussion

Effort: The outcome of the test is independent of the agent's action. The model would go unchanged if effort is costless and observable as it could be deterred with off-path beliefs. I explore the possibility for costless and unobservable effort in Section 6.2. Note also that if signals are ordered and the DM uses a cutoff strategy, as in many natural applications, a reasonable assumption on effort would be that the higher the effort, the likelier a high signal. In this case, the agent would always have an incentive to provide high effort. See Deb and Stewart (2018) and Ball and Kattwinkel (2022) for models that takes into account both asymmetric information and moral hazard in a model of testing.

Relation to models with evidence: The model can be interpreted as a generalisation of models with evidence. The idea of these models is that each type is endowed with a set of messages that only a subset of types can send. Formally, an evidence structure is a correspondence $E : \Theta \rightrightarrows M$ for some finite set of messages M. Thus type θ can only send messages in $E(\theta)$. We can capture these models in the following way. The set of feasible test has $X = \{x_1, x_0\}$ and for all $m \in M$, $\pi_m(x_1|\theta) = 1 \Leftrightarrow \theta \in E^{-1}(m)$. Thus a test m perfectly reveals whether θ is in $E^{-1}(m)$ or in $\Theta \setminus E^{-1}(m)$. In a model with evidence, a type θ can never reveal he is in $\Theta \setminus E^{-1}(m)$ for a message $m \notin E(\theta)$.

However, in the testing model, we can always incentivise any type to not choose such a test by setting $\alpha(x_0, m) = 0$ for all m. This strategy could be justified because (x_0, m) would always be off-path. Alternatively, we can set this restriction on α directly and Theorem 1 would still hold.

3 Characterisation of the optimal menu

An important step in the characterisation of the optimal menu is to show that the we can use the case where the DM can commit to a strategy to characterise the DM-preferred equilibrium. I also show that in the DM-preferred equilibrium, the A-types play a pure strategy.

Abusing notation, I will abstract momentarily from the choice of menu and let $\sigma : \Theta \to \Delta T$, i.e., the agent's strategy is a choice over all feasible tests and $\alpha : X \times T \to [0, 1]$ the strategy of the DM at any given test. Denote by $\sigma^{\Theta'} = \big(\sigma(\cdot|\theta)\big)_{\theta \in \Theta'}$ for any $\Theta' \subseteq \Theta$ the collection of strategies for a subset of types. I also use the following notation for the DM's payoffs:

$$v(\alpha, \sigma^A, \sigma^R) \equiv \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \sigma(t|\theta) p_t(\alpha; \theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \sigma(t|\theta) p_t(\alpha; \theta').$$

I call the game with commitment the game where the DM can commit to a decision rule α before observing the choice of test and the outcome. I say that the DM does not benefit from commitment if the payoffs from the DM-preferred equilibrium are the same in the game with commitment and in the original game.

I first observe in the following lemma that the DM's value from the game with commitment is obtained from a max-min problem.

Lemma 1. *The DM's payoff in the DM-preferred equilibrium in the game with commitment is*

$$\max_{\alpha, \sigma^A} \min_{\sigma^R} v(\alpha, \sigma^A, \sigma^R). \tag{1}$$

All proofs are relegated to the appendix.

To understand Lemma 1 better, first note that we can rewrite (1) as

$$\max_{\alpha} \big[\max_{\sigma^A} \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \sigma(t|\theta) p_t(\alpha; \theta) - \max_{\sigma^R} \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \sigma(t|\theta) p_t(\alpha; \theta') \big].$$

That is, it corresponds to the problem of finding the best strategy α when the agent chooses a test to maximise his payoffs given α. Because the interest of the A-types is fully aligned with the DM's, they try to maximise his payoffs whereas the R-types have opposite interests and thus try to minimise the

DM's payoffs. Unlike the original DM problem, the DM can commit to α and thus it does not need to be a best-reply.

We can use this characterisation to get the following result for the game without commitment.

Theorem 1. *For any* $(\alpha, \sigma^A) \in \arg\max_{\tilde{\alpha}, \tilde{\sigma}^A} \min_{\tilde{\sigma}^R} v(\tilde{\alpha}, \tilde{\sigma}^A, \tilde{\sigma}^R)$ *and* $\sigma^R \in \arg\min_{\tilde{\sigma}^R} \max_{\tilde{\alpha}} v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R)$*, an optimal menu is*

$$\mathcal{M} = \cup_{\theta \in A} \operatorname{supp} \sigma(\cdot|\theta),$$

and $(\alpha, \sigma^A, \sigma^R)$ *are the strategies in the corresponding DM-preferred equilibrium.*

Moreover,

- *The DM does not benefit from commitment,*
- *There exists a DM-preferred equilibrium where* σ^A *is in pure strategies and therefore* $|\mathcal{M}| \leq |A|$.

Theorem 1 provides two important tools to characterise the optimal menu. The first one is to show that there is no value of commitment and to provide a maximisation problem that gives exactly which strategies are used in the equilibrium *without commitment*. This is is a powerful tool to test equilibria and to establish whether a test should be part of an optimal menu. Indeed, it is not necessary to compare equilibria across menus to establish that an equilibrium (α, σ) is not optimal. It is enough to find an alternative DM and A-types strategy $(\tilde{\alpha}, \tilde{\sigma}^A)$ such that

$$\min_{\sigma^R} v(\alpha, \sigma^A, \sigma^R) < \min_{\sigma^R} v(\tilde{\alpha}, \tilde{\sigma}^A, \sigma^R)$$

to show that (α, σ) does not constitute an optimal equilibrium without having to worry whether $(\tilde{\alpha}, \tilde{\sigma}^A)$ is part of an equilibrium strategy. Importantly, it also allows to establish that a test is part of an optimal menu in equilibrium, even when it is not strictly optimal to add this test. This will be important in the next section where I will show that some tests are part of the optimal menu and then rely on their properties to characterise equilibria.

I interpret this result as a hierarchy over sources of learning. The DM has two sources of information, the "hard information" from the test results and the endogenously created information from the choice of test. When the DM can commit to a strategy, he can "sacrifice" payoffs from the test result by not best replying, in order to create separation of types through the test choice. By showing that the DM always best replies, even when he can commit, I show that he should always prioritise the hard information over creating endogenous information through the test choice.

That commitment has no value in this game comes from the max-min structure of the characterisation. Because a minimax theorem holds, this implies that the order of moves do not matter in this game: the

DM has the same payoffs if he moves first or last.[9]

The second tool is to establish that the size of the menu is bounded by the number of A-types. This limits the number of tests we need to consider. An immediate corollary is also that if there is only one type the DM would like to accept, an optimal menu is to use only one test. In particular, this result shows that in a binary state environment, the optimal mechanism uses only one test, no matter what the available set of test is.

Corollary 1. *Suppose* $|A| = 1$. *Then for any* T, *there is an optimal menu that uses only one test.*

Ben-Porath et al. (2021) also show that there is no value of commitment in a setup where preferences take a similar form as here.[10] The advantage of Theorem 1 is that it also provides a maximisation problem that delivers equilibrium strategies. Knowing that there is no value of commitment to α would for example not be enough to establish that it is without loss to consider pure strategies from A-types or to establish that a test is part of an optimal menu in equilibrium when it is not strictly optimal to add it.

Sequential Equilibrium. If the solution concept is DM-preferred Sequential Equilibrium (SE) (Kreps and Wilson, 1982), Theorem 1 would also hold. If all tests have full support, then all signals are on-path and weak PBE and SE coincide. If some tests do not have full support, then I can always assume that the trembling of R-types is more likely than the trembling of A-types. Then, the DM's off-path beliefs after the pair (t, x) are that the type is an A-type if the support of A- and R-types do not coincide and that the type is an R-type otherwise. This guarantees that if an A-type finds it profitable to deviate the problem without commitment then he would also find it profitable in the problem with commitment.

4 Economic environments

4.1 Environments with Blackwell dominant test

It is common in applications that the DM has access to a most informative test. This can be because the choice is simply between a test and opting out of the test like in the admission test example. It can also come from the structure of the constraints. For example, the DM could have a time budget to conduct an interview. The more time the interview takes, the more informative it is. Another possibility is that

[9]The minimax theorem does not hold for the max-min problem if both α and σ^A are allowed to vary. But I only use the minimax theorem holding σ^A fixed.

[10]The functional form they identify, semi-aligned preferences, is more general than what I assume here. The proof of Theorem 1 extends to semi-aligned preferences.

the DM can easily make a test less informative by simply not conducting part of the test, e.g., if a test is composed of a series of questions, the DM can ignore some of them.

I will use Blackwell's (1953) notion of informativeness.

Definition 1 (Blackwell (1953))**.** *A test t is more informative than t', $t \succeq t'$, if there is function $\beta : X \times X \to [0,1]$ such that for all $x' \in X$, $\sum_x \beta(x,x')\pi_t(x|\theta) = \pi_{t'}(x'|\theta)$ for all $\theta \in \Theta$ and for all $x \in X$, $\sum_{x'} \beta(x,x') = 1$.*

I call a test t a dominant test if $t \succeq t'$ for all $t' \in T$. If a test is more informative than another then in any decision problem, i.e., a pair of utility function and a prior, using the more informative test yields higher expected utility (Blackwell, 1953). A first important fact we will record here is that if there is a most informative test, then it is part of an optimal menu.

Lemma 2. *If there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$, then there is an optimal menu that includes t.*

This lemma follows from the max-min characterisation of Theorem 1 and the properties of dominant tests. Indeed, if we find a menu where the dominant test t is not included, we can modify the DM's strategy such that one A-type is accepted with higher probability than the test he is choosing, say t', and all R-types are accepted with lower probability than in t'. Then this constitutes a deviation or a new maximum in the max-min problem. Note that if we only knew that commitment had no value, we would not be guaranteed that t is part of an optimal menu.

As we have seen in the admission test example in Section 2.1, it can be optimal to add a strictly less informative in the optimal menu. I show now when the example of Section 2.1 generalises or does not. That is, I provide conditions under which it is optimal to include a dominated test and under which it isn't.

I first focus on binary signals environment, $X = \{x_0, x_1\}$. When signals are binary, we can define a linear order over types for each test. In this order, types are ranked by their likelihood of generating signal x_1: $\theta \geq_t \theta' \Leftrightarrow \pi_t(x_1|\theta) \geq \pi_t(x_1|\theta')$.[11] I characterise when including a dominated test is optimal or not as a function of the two properties of the dominant test based on this order.

Definition 2. *A test t is single-peaked if there are $\theta_1, \theta_2 \in A$ such that $A = \{\theta : \theta_1 \leq_t \theta \leq_t \theta_2\}$.*

Suppose we interpret the binary signal environment as a pass-fail test and x_1 as a pass grade. Single-peakness is satisfied in a natural specification where all A-types have a higher performance on the test than any R-type. But single-peakness also allows for A-types to have an intermediate performance on the test or to be the worst performers. A test t is not single-peaked whenever it is possible to find

[11] Given that tests are binary, this is equivalent to ordering type by the likelihood ratio, $\frac{\pi(x_1|\theta)}{\pi(x_0|\theta)}$.

$A_1, A_2 \in A$ and $R_1 \in R$ such that $A_1 <_t R_1 <_t A_2$. This was for example the case in the admission test example in Section 2.1. I call the tests where this condition is always satisfied enclosed.

Definition 3. *A test t is enclosed if there are $\theta_1, \theta_2 \in A$ such that $\theta_1 <_t \theta <_t \theta_2$ for any $\theta \neq \theta_1, \theta_2$.*

In appendix B, I provide a microfoundation for these two classes of tests when the DM cares about two dimensions of the agent's private information but the dominant test is only informative about one. Note as well that some tests are neither single-peaked nor enclosed.

We get the following results.

Proposition 1. *Let $X = \{x_0, x_1\}$. Suppose there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$ and that $\theta \neq_t \theta'$ for all $\theta, \theta' \in \Theta$.*

The singleton menu $\{t\}$ is optimal for any μ $\Leftrightarrow$ The test t is single-peaked.

The assumption that $\theta \neq_t \theta'$ for all $\theta, \theta' \in \Theta$ simply allows to state the result in a more convenient way.

The proof is in two steps. First, from Lemma 2, the most informative test is part of an optimal menu. I then show that whenever the dominant test is single-peaked, if the test is chosen, the equilibrium is unique and all types choose only the dominant test. The key argument in the analysis is noting that $p_t(\alpha;\theta) - p_{t'}(\alpha;\theta)$ is single-crossing in θ with respect to the order $\geq_t$, for any α. When preferences are single-peaked, we can use the single-crossing condition and properties of tests satisfying the monotone likelihood ratio property to show that there is a unique equilibrium where only t is chosen.

One can also interpret this result as an extension of unravelling arguments (e.g., Milgrom, 1981) to show when equilibrium forces select the dominant test. Unlike previous work on full disclosure in games with evidence, the resulting equilibrium might be less informative than if different types could be forced to select different tests. This is because the dominant test is not fully revealing and so an outside observer might learn more from the separation of types across different tests.

On the other hand, if the test is not single-peaked, there is a prior where offering even a completely uninformative test with the most informative test is strictly better for the DM. To illustrate, consider three types $A_1, A_2 \in A$ and $R_1 \in R$ such that $A_1 <_t R_1 <_t A_2$. Suppose the prior is such that if only t is offered, the DM accepts after x_1 and rejects after x_0. The DM can then offer an uninformative test where the probability of being accepted makes R_1 indifferent but is strictly preferred by A_1. This constitutes a deviation in the problem with commitment. This reasoning can be used to show that including a less informative test is always beneficial whenever the dominant test t is enclosed.

Proposition 2. *Let $X = \{x_0, x_1\}$. Suppose there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$ and that $\theta \neq_t \theta'$ for all $\theta, \theta' \in \Theta$.*

The DM's payoffs are higher in the menu $\{t, t'\}$ than in $\{t\}$ for any μ and $t' \in T \Leftrightarrow$ The test t is enclosed.

Proposition 1 and Proposition 2 show that whenever the dominant test is *not* single-peaked, there always exist a prior where including a dominated test is optimal. And when the dominant test is enclosed, it is always optimal to include a dominated test. Note that the dominated test could be a completely uninformative test.

The ideas of Proposition 1 and Proposition 2 can be partially extended to more than two signals. First, if all tests satisfy the monotone likelihood ratio property and the DM only wants to accept types above a threshold, the optimal menu is to only offer the dominant test.

Proposition 3. *Suppose $\Theta, X \subset \mathbb{R}$, $A = \{\theta : \theta > \overline{\theta}\}$ for some $\overline{\theta}$ and all tests in T have full-support and the monotone likelihood ratio property: for $\theta > \theta'$,*

$$\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \text{ is increasing in } x.$$

If there is $t \succeq t'$ for all $t' \in T$, then, the menu $\{t\}$ is optimal.

Again this result holds by showing a single-crossing difference property on the acceptance probability. Intuitively, the reason is that more informative tests send relatively higher signals for higher types. So if a low type chooses the most informative test, the higher types must also choose that one. This prevents any pooling of A-types and R-types on two different tests. Combined with Lemma 2 that guarantees the inclusion of the dominant test, we get our result. This result would hold using weaker information order like Lehmann (1988) or some weakening of it. The key property delivering the result is the single-crossing condition described above.

If it is possible to find two signals, x, x', two A-types A_1, A_2 and one R-type, R_1 such that $\frac{\pi_t(x|A_1)}{\pi_t(x'|A_1)} < \frac{\pi_t(x|R_1)}{\pi_t(x'|R_1)} < \frac{\pi_t(x|A_2)}{\pi_t(x'|A_2)}$, then there is a test t' strictly less informative than t and a prior such that offering $\{t, t'\}$ is better for the DM than just offering $\{t\}$.

Proposition 4. *Let t be a test. Suppose there are two signals $x, x' \in X$, types $A_1, A_2 \in A$ and $R_1 \in R$ such that*

$$\frac{\pi_t(x|A_1)}{\pi_t(x'|A_1)} < \frac{\pi_t(x|R_1)}{\pi_t(x'|R_1)} < \frac{\pi_t(x|A_2)}{\pi_t(x'|A_2)}.$$

There is a prior μ and a test $t' \prec t$ such that the DM's payoffs are higher in the menu $\{t, t'\}$ than in $\{t\}$.

Intuitively, if we interpret x as a high signal, the A-type A_1 sends relatively low signals. Suppose that the prior is such that, if only t is offered, x is accepted and x' is not. In a sense, it means that in the test t, type R_1 performs better than A_1 on the signals x, x'. It is then beneficial for the DM to include a test that pools signals x, x' together. In that new test, type A_1 can choose the coarsened test where the superior performance of type R_1 is less important than in the original test.

4.2 Environments with tests ordered by their difficulty

In many economic environments, the DM does not necessarily have access to a most informative test but can vary the difficulty to pass a test. This is for example the case for a regulator that can decide how demanding a certification test is. Like in Proposition 1 and Proposition 3, I show that the optimal menu is a singleton.

I first formalise the notion of difficulty of a test as follows.

Definition 4 (Difficulty environment)**.** *An environment is a Difficulty environment if $\Theta \subset \mathbb{R}$, $A = \{\theta : \theta > \overline{\theta}\}$ for some $\overline{\theta}$, $X = \{x_0, x_1\}$, $T \subset \mathbb{R}$, all tests have full-support, satisfy the monotone likelihood ratio property and for all t $> t'$, and $\theta > \theta'$,*

$$\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}, \textit{for } x = x_0, x_1.$$

If $t > t'$, I will say that t is harder than t'. Importantly, a harder test is not more informative. To understand this better, let $\mu(\cdot|x, t)$ be a posterior belief after observing signal x in test t and $\succeq_{FOSD}$ denote the first-order stochastic dominance order. The monotone likelihood ratio property implies $\mu(\cdot|t, x_1) \succeq_{FOSD} \mu(\cdot|t, x_0)$, a higher signal is "good news" about the type (Milgrom, 1981). The last property in the definition further implies $\mu(\cdot|t, x) \succeq_{FOSD} \mu(\cdot|t', x)$, and this property is tight as formalised below.

Proposition 5. *Test t is harder than t$'$ if and only if $\mu(\cdot|t, x) \succeq_{FOSD} \mu(\cdot|t', x)$ for all prior μ (including non full support).*

The prior in Proposition 5 is allowed to be non-full support because the proof for necessity is for the case of a binary type, as in the proof of necessity of Proposition 1 in Milgrom (1981). Intuitively, the characterisation shows that a pass grade shifts beliefs more towards higher type in a harder test and a fail grade shifts more beliefs towards lower types in an easy test. Or put differently, the harder a test the more informative it is about a high type when there is a pass-grade whereas an easier test is informative about the low types when the test is failed. Figure 1 illustrates this graphically. As an example, if $\Theta \subset (0, 1)$ and $\pi_t(x_1|\theta) = \theta^t$ we are in a Difficulty environment.

Proposition 6. *In a Difficulty environment, a singleton menu is optimal.*

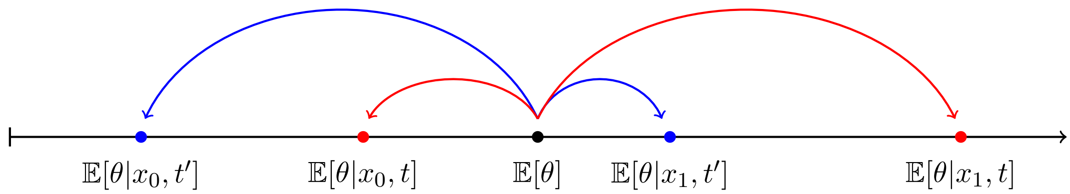


Figure 1: Illustration of posterior means for two tests, $t > t'$. The good news signal x_1 shifts the posterior towards a higher posterior mean in the harder test. The bad news signal x_0 shifts the posterior towards lower posterior mean in the easier test.

Like Proposition 1 and Proposition 3, Proposition 6 illustrates how incentive constraints shape the size of the optimal menu. In the case of a single-peaked dominant test, the equilibrium when the dominant test is offered is unique and only that test is chosen. Here, it is possible to construct an equilibrium where more than one test is chosen in equilibrium. However, the DM strategy needed to sustain that equilibrium is such that he is better off offering only one test.

The proof proceeds in two steps. First, I show that there are at most two tests in the optimal menu and if there are two tests, the harder test must be more lenient that the easy test. Intuitively, no one would choose a hard test if on top of that the requirements to be accepted harder would be harder to fulfil. In particular, I show that after the hard test, the DM must accept with some probability after a fail signal and in the easy test, reject with positive probability after a pass grade.

This means that to maintain incentives to select both tests, the DM only reacts to the least informative signal from the test: in the hard test after a fail grade, in the easy test after a pass grade. This in turn implies that it would be better for the DM to use only one test and reject after a fail grade and accept after a pass grade.

4.3 Bidimensional environment

In this subsection, I apply the tools of Theorem 1 to study environments with bidimensional types. The analysis here can be easily extended to more than two dimensions. I assume that the DM has access to tests that are only informative about one dimension and the preference of the DM is monotonic along each dimension.

Definition 5. *An environment is bidimensional if* $\Theta = \Theta_1 \times \Theta_2 \subset \mathbb{R}^2$*,* $X \subset \mathbb{R}$ *and* $T = \{t_1, t_2\}$ *such that for* $i = 1, 2$*,*

- *if* $\theta \in A$*, then for all* $\theta' \geq \theta$*,* $\theta' \in A$

- *t_i has full support and for all $\theta_i > \theta_i'$,*

$$\frac{\pi_{t_i}(x|\theta_i,\theta_j)}{\pi_{t_i}(x|\theta_i',\theta_j)} \text{ is strictly increasing in } x \text{ for any } \theta_j \in \Theta_j$$

- *$\pi_{t_i}(x|\theta_i,\theta_j) = \pi_{t_i}(x|\theta_i,\theta_j')$ for all $\theta_j,\theta_j' \in \Theta_j$ and $x \in X$*

This class of test technology is related to Glazer and Rubinstein (2004) and Carroll and Egorov (2019) who also study test allocation in multidimensional environments although the results have a different focus.[12] The first condition captures the idea that a higher type is always better for the DM. The second and third condition captures the idea that each test is only informative about one dimension and that a higher signal corresponds to a higher type in that dimension. I restrict attention to the case where only two tests, one for each dimension, are available to the DM.

In this environment, whether the DM wants to offer a menu containing both tests depends crucially on his preferences. In particular, I give a necessary and sufficient condition on the preferences such that a full menu is optimal for any prior, i.e., a menu containing both tests. Let $\overline{\theta}_i = \max \Theta_i$.

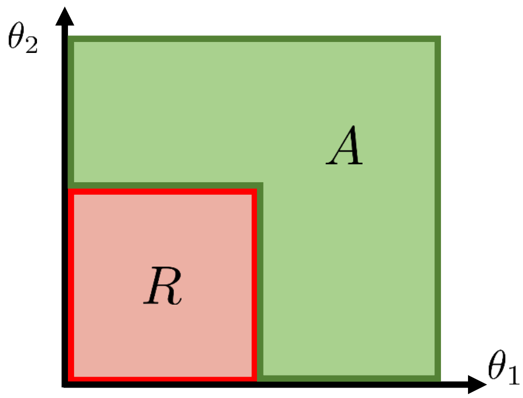
Proposition 7. *Suppose we are in a bidimensional environment. Offering a menu $\{t_1,t_2\}$ is strictly optimal for any prior if and only if*

$$\text{for } i = 1,2,\ (\overline{\theta}_i,\theta_j) \in A, \text{ for all } \theta_j \in \Theta_j. \tag{2}$$

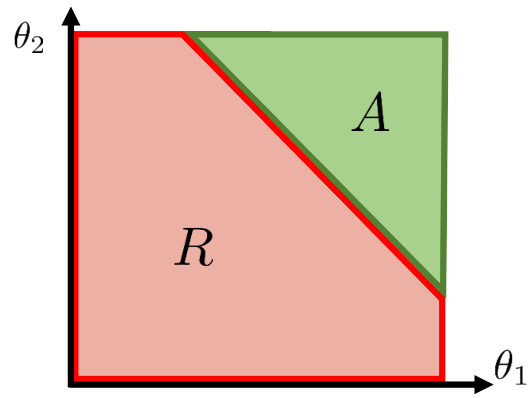
The proof of Proposition 7 works by showing that a deviation from a single test menu is always profitable when condition (2) is satisfied and constructs a prior under which there are no profitable deviations when the condition is not satisfied.

Figure 2 illustrates the condition of Proposition 7 with $\Theta \subset [0,1]^2$. In Figure 2a, the DM wants the agent's type to be high enough in at least one dimension. Then the DM always prefers to offer a full menu to the agent. On the other hand, in Figure 2b, the DM does not want to accept a type that is high in only one dimension. In this case, for some prior, the DM only wants to offer one test. This happens when after any deviation from the singleton menu any A-type is mimicked by too many R-types that cannot be distinguished from him. Finally, note that condition (2) is not related to the complementarity or substitutability of the two dimensions. The key condition is that the highest types in both dimensions is an A-type.

[12] Both papers allow for possibly random mechanisms but restrict attention to full revelation of the dimension tested. Glazer and Rubinstein (2004) introduced this environment and characterised the optimal mechanism. Carroll and Egorov (2019) study when the DM can achieve full learning in a setup with more general payoffs. Their condition for full learning is not related to the condition for optimality of having both tests in the menu in Proposition 7.



(a) Full menu always optimal.



(b) Full menu not always optimal.

Figure 2: Illustration of DM's preferences for Proposition 7.

5 Sufficient conditions for test inclusion

In this section, I study in more details the notion of efficient allocation of tests to the agent's types. I show that a sufficient condition to include a test in the optimal menu is if it is good at differentiating one A-type from all the R-types. This captures a notion of a test tailored for the A-type.

Definition 6. *Fix $\theta \in A$. Test t θ-dominates t', $t \succeq_\theta t'$, if there is $\beta : X \times X \to [0, 1]$ such that for all $x' \in X$*

$$\sum_x \beta(x, x')\pi_t(x|\theta) \leq \pi_{t'}(x'|\theta)$$

$$\textit{for all } \theta' \in R, \quad \sum_x \beta(x, x')\pi_t(x|\theta') \geq \pi_{t'}(x'|\theta')$$

$$\textit{for all } x \in X, \quad \sum_{x'} \beta(x, x') \leq 1$$

To understand this definition better, compare it to Blackwell's (1953) informativeness order. It requires the existence of a function β such that for all $x' \in X$, $\sum_x \beta(x, x')\pi_t(x|\theta) = \pi_{t'}(x'|\theta)$ for all $\theta \in \Theta$ and for all $x \in X$, $\sum_{x'} \beta(x, x') = 1$. The key difference is that we restrict attention to one A-type and all the R-types. This captures the idea the test θ-dominant test is tailored to differentiate θ from each R-type. The second difference is that it requires only inequalities whereas the Blackwell order requires equalities. This is because we are fixing the utility function we are interested in, unlike in Blackwell (1953).

If a type $\theta \in A$ has a $\succeq_\theta$-dominant test, then this test is used in an optimal menu. This shows that an important property of tests is not so much how good they are at differentiating types, but how good they are at differentiating one type the DM wants to accept from all the types he wants to reject.

Proposition 8. *Suppose there is $t \in T$ and $\theta \in A$ such that $t \succeq_\theta t'$ for all $t' \in T$, then t is part of an optimal menu.*

The stronger notion of a test able to differentiate some $\theta \in A$ from all R-types is if $\operatorname{supp} \pi_t(\cdot|\theta) \cap \Big(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta')\Big) = \emptyset$. If each type in A has such a test, then the principal never makes a mistake. This condition is also necessary.

Proposition 9. *The principal's expected payoff is $\sum_{\theta \in A} \mu(\theta)$ if and only if for all $\theta \in A$, there exists $t \in T$ such that*

$$\operatorname{supp} \pi_t(\cdot|\theta) \cap \Big(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta')\Big) = \emptyset$$

Here, the principal just needs for each type he wants to accept a test where he can discriminate between that type and the R-types. Then he can offer a menu of tests where each A-type self selects into the test that discriminates him from the R-types. The actual learning only happens by observing the test selected and the testing technology serves as a detriment to deviations from R-types. The argument is then similar to an unravelling argument à la Milgrom (1981) and Grossman (1981). These are not fully revealing tests but tests that allow to perfectly discriminate *one* A-type from all the R-types. But it could be a very noisy tests for the other A-types.

6 Extensions and discussions

In this section, I explore three ways the baseline model can be extended. First, I show the consequences of adding cheap-talk communication to the choice of test. In the context of an environment with a Blackwell dominant test, cheap-talk can benefit the DM. But these benefits disappear once the possibility of unobserved effort is introduced, the second extension. Finally, I show that the DM benefits from using randomised mechanisms that map a private message of the agent to a possibly random allocation over tests.

6.1 Communication

I consider here the possibility of adding a communication channel on top of the test choice. There is now a finite set C of output messages with $|C| \geq |A|$ and a strategy is a mapping $\sigma : \Theta \to \Delta(T \times C)$. Note that all the general results from the previous sections go through as from any finite set T one can create another T' that duplicates each test $|C|$ times. I call this variant of the model the *menu game with communication*.

In line with Theorem 1, each A-type chooses a message-test pair deterministically and each R-type mixes over some A-types message-test pair. Moreover, I show that when communication is added, each type in A announces his type, thus maximally differentiating himself, and each R-type pretends to be an A-type.

Theorem 2. *If communication is allowed, the same construction as Theorem 1 holds. Moreover, there is a DM-preferred equilibrium where each A-type reports his own type.*

Theorem 2 shows that the results extend naturally to an environment where communication is allowed. Because the DM could commit to a strategy, he can always guarantee each A-type at least as much as he would have if he would pool with another A-type. This guarantees that there is a solution to the problem with commitment where he separates from the other A-types.

Communication can play an important economic role. For example, if the set of feasible tests contains a Blackwell dominant test and communication is allowed, the DM only uses the dominant test.

Proposition 10. *Suppose there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$ and communication is allowed. Then an optimal menu is $\{t\}$.*

In the optimal menu, each type chooses t and each A-type communicates his type. Note that Proposition 1 established that whenever tests are binary, $t \succeq t'$ for all $t' \in T$ and t is single-peaked, an optimal menu is also to only offer $\{t\}$. In particular, Proposition 1 holds also if $t \sim t'$ for some tests t' as would be the case in the game with communication. This shows that whenever the dominant test is single-peaked, communication cannot strictly improve payoffs. This connects to results from Silva (2022) and Weksler and Zik (2022) that show conditions under which communication is useful when the receiver has access to a test. In particular, Silva (2022) studies a model with the same payoffs and finds that failure of monotonicity in the test makes communication valuable. I show more generally that failure of single-peakness can make communication valuable.

The example below illustrates how the DM can benefit from communication in our admission test example.

Example from Section 2.1 revisited: Suppose that we allow for communication in the admission test example (Section 2.1). In this case, all types choose test t, type $A1$ and $A2$ communicate their type and $R1$ mixed over the two messages:

$$\sigma(A1, t|R1) = \frac{3\mu(A1)}{4\mu(R1)} \qquad \sigma(A1, t|A1) = 1 \qquad \sigma(A2, t|A2) = 1$$

The DM accepts only after the signal x_1 after the message test pair $(A2, t)$ and mixes after x_1 and accepts after x_1 in the message test pair $(A1, t)$:

$$\alpha(x, A1, t) = \begin{cases} 1 & \text{if } x = x_0 \\ 1/2 & \text{if } x = x_1 \end{cases} \qquad \alpha(x, A2, t) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \end{cases}$$

The probability of acceptance of each type is now:

$$\mathbb{E}[p(\alpha; R1)] = 2/3 \qquad p_{A1,t}(\alpha; A1) = 3/4 \qquad p_{A2,t}(\alpha; A2) = 1$$

Note that the DM must now use a different decision rule for the same test. In particular, the meaning of the signals is different: a "high grade" after message $A1$ is x_0 whereas a "high grade" after message $A2$ is x_1. I explore the consequence of this observation in the next section where I introduce effort.

6.2 Effort and moral hazard

An assumption we have maintained throughout this article is that the agent cannot influence the outcome of the test once it is chosen. But in many instances, the agent can do that through some unobserved effort. I will focus on costless, unobserved effort. I show that it can be optimal to offer a dominated test, even in the presence of communication. The notation in this section explicitly allows for communication given the focus of some the results. While not being stated and proven, Theorem 1 can be extended to allow for effort as modelled here.

Define the correspondence $e : T \times \Theta \rightrightarrows \Theta$. The set $e(t, \theta)$ denotes the set of types the type θ can copy in test t. If type θ copies θ' in test t, then type θ generates a distribution over signals $\pi_t(\cdot|\theta')$. I call this effort as it will be the leading interpretation once I put more structure on e. The story I have in mind is that the set $e(t, \theta)$ captures the set of types type θ can copy by putting less effort into the test.

The agent strategy is now $\sigma : \Theta \to \Delta(T \times C \times \Theta)$ with the restriction that $\sigma(t, c, \theta'|\theta) > 0$ only if $\theta' \in e(t, \theta)$. The payoff of type θ from choosing (t, c, θ') for a given α is $p_{t,c}(\alpha; \theta') = \sum_x \alpha(t, c, x)\pi_t(x|\theta')$. The DM's payoffs become, abusing notation,

$$\begin{aligned} v(\alpha, \sigma^A, \sigma^R) \equiv \sum_{\theta \in A} \sum_{t,x} \sum_{\theta' \in e(t,\theta)} \mu(\theta)\sigma(t, c, \theta'|\theta)\alpha(t, c, x)\pi_t(x|\theta') \\ - \sum_{\theta \in R} \sum_{t,x} \sum_{\theta' \in e(t,\theta)} \mu(\theta)\sigma(t, c, \theta'|\theta)\alpha(t, c, x)\pi_t(x|\theta') \end{aligned}$$

To illustrate the consequences of unobserved effort, I will focus on the following setup which I will refer to as the *effort environment*. All tests are binary, $X = \{L, H\}$. As in Section 4.1, we can define for each test t, the order $\geq_t$ such that $\theta \geq_t \theta'$ if and only if $\pi_t(H|\theta) \geq \pi_t(H|\theta')$. The correspondence e is defined as $e(t, \theta) = \{\theta' : \theta' \leq_t \theta\}$. In the context of a university admission test, it means that any student can achieve the distribution of grades of a worse performing student. Denote by $\underline{\theta}_t$ a type such that $\underline{\theta}_t \leq_t \theta$ for all θ. In this new environment, signals have an explicit meaning and signals H and L stand for high and low.

The main consequence of introducing effort in this way is that the DM's strategy must be increasing in the signal. If for a given pair (t, c), we have $\alpha(L, t, c) > \alpha(H, t, c)$ then all types choosing (t, c) have a strict incentive to copy $\underline{\theta}_t$. The other consequence is that for any types choosing the test t, the acceptance probability has to be increasing with respect to $\geq_t$. This makes communication a much less powerful tool in this context, as the following result illustrates. I say that an equilibrium has *productive communication* if some types choose the same test t and send distinct messages c, c' and the DM's strategy varies with the message, $\alpha(t, c, \cdot) \neq \alpha(t, c', \cdot)$.

Proposition 11. *Let $\emptyset$ denote an uninformative test. In the effort environment, for any equilibrium with productive communication where the support of the agent's strategy is $\mathcal{C} \times \mathcal{M} \subseteq C \times T$, there exist payoff-equivalent equilibria where*

- *the support of the agent's strategy is $\{c, c'\} \times \mathcal{M}$*
- *and where the support of the agent's strategy is $\{c\} \times \big(\mathcal{M} \cup \{\emptyset\}\big)$*

Proposition 11 shows that in the effort environment, any productive communication cannot do better than allowing to choose an uninformative test and have no communication. The proof of Proposition 11 shows that in any equilibrium where there is productive communication, we can focus on an equilibrium with only two messages. One message has a meaning "I will put full effort in the test" and the agent indeed does so. The other message has meaning "I will put no effort into the test" and all types sending that message copy $\underline{\theta}_t$. After the "no effort" message, the DM knows that the signal observed carries no information. So this is equivalent to offer an uninformative test. It is worth noting that this argument holds even if different tests define different orders $\geq_t$.

Proposition 11 also helps understand the role of communication in the model without effort *and* with communication. Remember that in this case, whenever $t \succeq t'$ for all $t' \in T$, it is optimal to only offer test $\{t\}$ and use the cheap-talk messages to differentiate types (Proposition 10). But that required that after some message c, the DM accepts with higher probability after a low signal L than after the high signal H. In the presence of effort, this cannot be an equilibrium strategy as all types sending message c have a strict incentive to copy $\underline{\theta}_t$. Thus the role of communication without effort is to "flip" the meaning of messages making the low signal "good news". This channel is shut down with effort and the DM cannot do better than offering an uninformative test.

Unobserved effort therefore reopens the possibility of benefiting from a dominated test.

Observation 1. *In the effort environment with communication, it can be optimal to include a strictly dominated test in the menu.*

The following example provides a situation where a menu consisting of two tests $\{t, t'\}$ with $t \succ t'$ is strictly better than offering $\{t, \emptyset\}$ where $\emptyset$ denotes an uninformative test. An important feature of this example is that the orders over types defined by the tests differ, i.e., $\geq_t \neq \geq_{t'}$. It remains an open question whether including a strictly dominated test can be optimal if $\geq_t = \geq_{t'}$ for all $t, t' \in T$.

Example 1. Let $\Theta = \{A1, A2, R1, R2\}$ with $A = \{A1, A2\}$. The prior on types is uniform, $\mu(\theta) = 1/4$. The test t is described by

$$\pi_t(x|A1) = \begin{cases} 1/4 & \text{if } x = H \\ 3/4 & \text{if } x = L \end{cases}, \qquad \pi_t(x|R1) = \begin{cases} 1/3 & \text{if } x = H \\ 2/3 & \text{if } x = L \end{cases},$$

$$\pi_t(x|A2) = \begin{cases} 1 & \text{if } x = H \\ 0 & \text{if } x = L \end{cases}, \qquad \pi_t(x|R2) = \begin{cases} 3/4 & \text{if } x = H \\ 1/4 & \text{if } x = L \end{cases}.$$

Note that we have $A1 <_t R1 <_t R2 <_t A2$.

The test t' is the following garbling of t: $\pi_{t'}(H|\theta) = \epsilon\pi_t(H|\theta) + \pi_t(L|\theta)$ for some $\epsilon \in (0, 1)$:

$$\pi_{t'}(H|A1) = \frac{3+\epsilon}{4}, \qquad \pi_{t'}(H|R1) = \frac{2+\epsilon}{3},$$
$$\pi_{t'}(H|A2) = \epsilon, \qquad \pi_{t'}(H|R2) = \frac{1+\epsilon}{4}.$$

In this case, $A1 >_{t'} R1 >_{t'} R2 >_{t'} A2$.

We can interpret the example as follows. The DM is a university that cares about technical skills and creative skills and these two skills are anti-correlated: the more creative a type is, the less technical it is. Types $A2$ and $R2$ are two technical types and $A1$ and $R1$ are two creative types. The test t is good at identifying technical types whereas test t' is good at identifying creative types. However, the "creative" test t' is less precise than the "technical" test t.

If only t is offered and all types put effort then the DM best replies by only accepting after H. This in turn makes it a best-reply for all types to copy themselves, i.e., to put full effort in the test. Then the DM's payoffs are

$$\pi(t) = \frac{1}{4} \cdot (1 + 1/4 - 1/3 - 3/4).$$

The DM can do better by offering t and allowing to opt out, $\emptyset$. In the optimal menu and strategies, the DM accept with probability $1/3$ after observing $\emptyset$ and only accepts after H after observing t. Types

$A1, R1$ choose to opt out and types $A2, R2$ choose t and copy themselves. Because $\mu(A1) = \mu(R1)$, the DM's strategy is a best-reply.[13] The DM's payoffs are

$$\pi(t, \emptyset) = \frac{1}{4} \cdot (1 + 1/3 - 1/3 - 3/4) > \pi(t).$$

Now let's consider the situation where the DM offers both t and t' and the following strategies: types $A1, R1$ choose t' and copy themselves, types $A2, R2$ choose t and copy themselves and the DM accepts only after H in both tests. Given the uniform prior, this delivers equilibrium strategies. The DM's payoffs are now

$$\pi(t, t') = \frac{1}{4} \cdot (1 + \frac{3+\epsilon}{4} - \frac{2+\epsilon}{3} - 3/4) > \pi(t, \emptyset).$$

6.3 Mechanism and randomisation

So far we have restricted attention to menus of tests. But the DM could potentially use a more elaborate *mechanism* to allocate tests to agents, possibly randomly. I define a mechanism $\tau : \Theta \to \Delta T$, a possibly random mapping from type report to distribution over tests. A strategy for the DM remains a mapping from test allocation and signal realisation to an acceptance decision, $\alpha : T \times X \to [0, 1]$, and the agent's strategy is a mapping from type to type report. I assume that the DM cannot observe the type report but we can naturally extend the mechanism τ to allow for output messages in the spirit of Section 6.1. Standard revelation principle arguments show that it is without loss of generality to restrict attention to type reports.

The DM's problem is now to maximise his expected payoff subject to incentive-compatibility constraints. In the baseline model, the DM cannot commit to its strategy α. Let $BR(\tau) := \{\alpha : \mathbb{E}_{\alpha,\tau}[v(a, \theta)] \geq \mathbb{E}_{\alpha',\tau}[v(a, \theta)]$, for all $\alpha'\}$, the set of best-reply when the mechanism is τ. The DM's problem is

$$\begin{aligned} \max_{\tau, \alpha} \; & \sum_{\theta \in A} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha; \theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tau(t|\theta) p_t(\alpha; \theta) \\ \text{s.t.} \; & \sum_t (\tau(t|\theta) - \tau(t|\theta')) p_t(\alpha; \theta) \geq 0 \text{ for all } \theta, \theta' \\ & \alpha \in BR(\tau) \end{aligned}$$

The first constraint is the incentive compatibility constraint of type θ deviating to θ' and the second constraint ensures that the DM best replies to the information revealed by the output of the mecha-

[13] Note that because $\mu(A1) = \mu(R1)$, the DM could mix with any probability in $[1/3, 3/4]$ and maintain the same strategy for the agent and same payoffs for himself.

nism.[14]

By using a mechanism, the agent can be randomly allocated to different tests without being indifferent between them. On the other, when restricting attention to menus, the agent has to be indifferent between tests if he randomises over tests. In the following example (inspired by Glazer and Rubinstein, 2004), I show that access to a mechanism can strictly improve the DM payoffs.

Example 2 (Randomised allocation)**.** Suppose there are six types $\Theta = \{(\theta_1, \theta_2, \theta_3) : \theta_i = 0, 1,\ 1 \leq \theta_1 + \theta_2 + \theta_3 \leq 2\}$ and $A = \{(\theta_1, \theta_2, \theta_3) : \theta_1 + \theta_2 + \theta_3 = 2\}$. The DM has access to three tests, each perfectly revealing one dimension: $T = \{1, 2, 3\}$ with $\pi_t(x = \theta_t|\theta) = 1$. The prior μ is uniform. At the optimum, the DM accepts when the signal is equal to 1. The optimal mechanism τ allocates each A-type with probability $1/2$ to each test where their dimension is equal to 1. Each R-type is allocated with probability $1/2$ to the dimension where it has value 1 and $1/4$ in the other dimensions.

This mechanism accepts A-types with probability one and accepts R-types with probability $1/2$. In particular, the mechanism randomises the allocation of R-types over tests they are not indifferent between. If the DM could only use a menu, the R-types would always choose the test that reveals their dimension equal to one. Thus any menu and strategy that accepts R-types with probability $1/2$ must also accept A-types with probability $1/2$.

I now show that the optimal mechanism can also be characterised by a max-min problem and the DM does not benefit from commitment.

To set up the characterisation of the optimal mechanism, let $s : A \to \Delta T$ and $m : R \to \Delta A$ and abusing notation, let $\alpha : T \times X \to [0, 1]$ and

$$\begin{aligned} v(\alpha, s, m) &\equiv \sum_{\theta \in A} \sum_{t \in T} s(t|\theta) \Big[\mu(\theta) p_t(\alpha; \theta) - \sum_{\theta' \in R} \mu(\theta') m(\theta|\theta') p_t(\alpha; \theta')\Big] \\ &= \sum_{\theta \in A} \sum_{t \in T} \mu(\theta) s(t|\theta) p_t(\alpha; \theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{\theta \in A} m(\theta|\theta') \sum_{t \in T} s(t|\theta) p_t(\alpha; \theta') \end{aligned} \tag{3}$$

The function s can be interpreted as A-types choosing a test, m as R-types choosing an A-type to mimic, α as the DM accepting the agent after a test and signal realisation. The function v is then the DM's expected payoffs from a distribution over tests induced by the pair (s, m). I explain these objects in more detail in the discussion of Theorem 3.

Theorem 3. *The value of an optimal mechanism is*

$$V = \max_{\alpha, s} \min_{m} v(\alpha, s, m) \tag{4}$$

[14]This definition does not put constraints on off-path optimality but because any strategy is best-reply to some beliefs, this guarantees that satisfying $BR(\tau)$ will lead to a weak PBE.

For any $(\alpha, s) \in \arg\max_{\tilde{\alpha},\tilde{s}} \min_{\tilde{m}} v(\tilde{\alpha}, \tilde{s}, \tilde{m})$ *and* $m \in \arg\min_{\tilde{m}} \max_{\tilde{\alpha}} v(\tilde{\alpha}, s, \tilde{m})$*, an optimal mechanism is*

- *for* $\theta \in A : \tau(t|\theta) = s(t|\theta)$
- *for* $\theta' \in R : \tau(t|\theta') = \sum_{\theta\in A} m(\theta|\theta')\tau(t|\theta)$
- *the DM's strategy is* α

Moreover, the DM does not benefit from committing to α*.*

Theorem 3 provides another characterisation of the optimal mechanism in terms of a max-min problem. As in Theorem 1, the no value of commitment follows from the max-min structure of the characterisation. To understand the structure of this max-min problem better, consider the objective function v for a fixed α. This can be interpreted as a zero-sum game where where the maximiser, the A-types, chooses $s : A \to \Delta T$ and the minimiser, the R-types, chooses $m : R \to \Delta A$. The payoffs of a given A-type θ choosing test t and a given R-type, θ', choosing an A-type $\tilde{\theta}$ can be expressed as:

$$\begin{aligned}
&\text{for } \theta \in A \text{ choosing } t,\ \mu(\theta)p_t(\alpha;\theta) - \sum_{\theta'\in R} \mu(\theta')m(\theta|\theta')p_t(\alpha;\theta') \\
&\text{for } \theta' \in R \text{ choosing } \tilde{\theta},\ \mu(\theta')\sum_t s(t|\tilde{\theta})p_t(\alpha;\theta') - \sum_{\theta\in A,t} \mu(\theta)s(t|\theta)p_t(\alpha;\theta)
\end{aligned}$$

In the payoffs of the R-type, his strategy, the choice of $\tilde{\theta}$, only affects the first part of the payoffs. So the R-type is effectively trying to maximise his probability of being accepted. On the other hand, the A-type maximise a modified version of their utility where they maximise their probability of being accepted while being penalised every time a R-type mimics them and is accepted. The A-types' utility is thus modified to align it with the DM's payoffs. The induced distribution over tests determines the optimal mechanism when strategy α is used.

7 Conclusion

I study the design of optimal menus of tests. Menus allow the DM to have an additional dimension for information revelation by letting types separate across different tests. However, the DM faces two constraints to maintain separation: the agent incentive constraints and his own best-reply constraints. I show in Theorem 1 that this problem is equivalent to solving the DM's problem without best-reply constraints through a max-min characterisation. This characterisation stems from the structure of preferences and is thus applicable in a wide range of mechanism design settings, beyond choices of tests.

Throughout the paper, I provide conditions under which equilibrium forces limit whether the DM can benefit from offering a menu as well as use the commitment result to show when including a test in the menu is optimal. An important technical observation is that single-crossing conditions on the acceptance probability play a key role to have a singleton menu. While single-crossing conditions are usually used to maintain separation in signalling and screening models, in this case separation reveals too much information through the choice. This in turn makes it impossible to maintain the incentives to separate in the first place.

Finally, putting more structure on the feasible tests forces us to think about what are reasonable and economically meaningful restrictions on tests. I have considered three possibilities in this paper and introduced a new order on experiments to characterise the notion of difficulty. Developping more ways of restricting feasible tests would allow to explore more trade-offs between experiments and further enrich the literature on test design.

References

Academic Senate UC (2020), Report of the uc academic council standardized testing task force (sttf), Technical report.

Ball, I. (2021), 'Scoring strategic agents', *arXiv preprint arXiv:1909.01888* .

Ball, I. and Kattwinkel, D. (2022), 'Probabilistic verification in mechanism design'.

Ben-Porath, E., Dekel, E. and Lipman, B. L. (2019), 'Mechanisms with evidence: Commitment and robustness', *Econometrica* **87**(2), 529–566.
URL: *https://onlinelibrary.wiley.com/doi/abs/10.3982/ECTA14991*

Ben-Porath, E., Dekel, E. and Lipman, B. L. (2021), 'Mechanism design for acquisition of/stochastic evidence'.

Blackwell, D. (1953), 'Equivalent comparisons of experiments', *The Annals of Mathematical Statistics* **24**(2), 265–272.

Boleslavsky, R. and Kim, K. (2018), 'Bayesian persuasion and moral hazard', *Available at SSRN 2913669* .

Bull, J. and Watson, J. (2007), 'Hard evidence and mechanism design', *Games and Economic Behavior* **58**(1), 75–93.

Carroll, G. and Egorov, G. (2019), 'Strategic communication with minimal verification', *Econometrica* **87**(6), 1867–1892.

Dasgupta, S. (2023), 'Optimal test design for knowledge-based screening'.

Deb, R. and Stewart, C. (2018), 'Optimal adaptive testing: informativeness and incentives', *Theoretical Economics* **13**(3), 1233–1274.
URL: *https://econtheory.org/ojs/index.php/te/article/view/2914/0*

DeMarzo, P. M., Kremer, I. and Skrzypacz, A. (2019), 'Test design and minimum standards', *American Economic Review* **109**(6), 2173–2207.
URL: *https://www.aeaweb.org/articles?id=10.1257/aer.20171722*

Deneckere, R. and Severinov, S. (2008), 'Mechanism design with partial state verifiability', *Games and Economic Behavior* **64**(2), 487–513.

Dessein, W., Frankel, A. and Kartik, N. (2023), 'Test-optional admissions'.

Dye, R. A. (1985), 'Disclosure of nonproprietary information', *Journal of accounting research* pp. 123–145.

Ely, J., Galeotti, A., Jann, O. and Steiner, J. (2021), 'Optimal test allocation', *Journal of Economic Theory* **193**, 105236.

Forges, F. and Koessler, F. (2005), 'Communication equilibria with partially verifiable types', *Journal of Mathematical Economics* **41**(7), 793–811.

Giovannoni, F. and Seidmann, D. J. (2007), 'Secrecy, two-sided bias and the value of evidence', *Games and Economic Behavior* **59**(2), 296–315.
URL: *https://www.sciencedirect.com/science/article/pii/S0899825606001047*

Glazer, J. and Rubinstein, A. (2004), 'On optimal rules of persuasion', *Econometrica* **72**(6), 1715–1736.

Glazer, J. and Rubinstein, A. (2006), 'A study in the pragmatics of persuasion: a game theoretical approach', *Theoretical Economics* **1**(4), 395–410.
URL: *https://econtheory.org/ojs/index.php/te/article/view/20060395/0*

Green, J. R. and Laffont, J.-J. (1986), 'Partially verifiable information and mechanism design', *The Review of Economic Studies* **53**(3), 447–456.

Grossman, S. J. (1981), 'The informational role of warranties and private disclosure about product quality', *The Journal of Law and Economics* **24**(3), 461–483.

Hagenbach, J., Koessler, F. and Perez-Richet, E. (2014), 'Certifiable pre-play communication: Full disclosure', *Econometrica* **82**(3), 1093–1131.

Harbaugh, R. and Rasmusen, E. (2018), 'Coarse grades: Informing the public by withholding information', *American Economic Journal: Microeconomics* **10**(1), 210–35.
URL: *https://www.aeaweb.org/articles?id=10.1257/mic.20130078*

Hart, S., Kremer, I. and Perry, M. (2017), 'Evidence games: Truth and commitment', *American Economic Review* **107**(3), 690–713.

Kartik, N. and Tercieux, O. (2012), 'Implementation with evidence', *Theoretical Economics* **7**(2), 323–355.
URL: *https://onlinelibrary.wiley.com/doi/abs/10.3982/TE723*

Koessler, F. and Perez-Richet, E. (2019), 'Evidence reading mechanisms', *Social Choice and Welfare* **53**(3), 375–397.

Koessler, F. and Skreta, V. (2022), 'Informed information design'.

Kreps, D. M. and Wilson, R. (1982), 'Sequential equilibria', *Econometrica* pp. 863–894.

Krishna, K., Lychagin, S., Olszewski, W., Siegel, R. and Tergiman, C. (2022), Pareto improvements in the contest for college admissions, Working Paper 30220, National Bureau of Economic Research.
URL: *http://www.nber.org/papers/w30220*

Lehmann, E. L. (1988), 'Comparing location experiments', *The Annals of Statistics* **16**(2), 521–533.

Lipman, B. L. and Seppi, D. J. (1995), 'Robust inference in communication games with partial provability', *Journal of Economic Theory* **66**(2), 370–405.

Lipnowski, E., Ravid, D. and Shishkin, D. (2022), 'Persuasion via weak institutions', *Journal of Political Economy* **130**(10), 2705–2730.

Milgrom, P. (2008), 'What the seller won't tell you: Persuasion and disclosure in markets', *Journal of Economic Perspectives* **22**(2), 115–131.

Milgrom, P. R. (1981), 'Good news and bad news: Representation theorems and applications', *The Bell Journal of Economics* pp. 380–391.

Nguyen, A. and Tan, T. Y. (2021), 'Bayesian persuasion with costly messages', *Journal of Economic Theory* **193**, 105212.

Perez-Richet, E. and Skreta, V. (2022), 'Test design under falsification', *Econometrica* **90**(3), 1109–1142.

Perez-Richet, E., Vigier, A. and Bizzotto, J. (2020), 'Information design with agency'.

Persico, N. (2000), 'Information acquisition in auctions', *Econometrica* **68**(1), 135–148.

Rockafellar, R. T. (2015), *Convex Analysis*, Princeton University Press.
URL: *https://doi.org/10.1515/9781400873173*

Rosar, F. (2017), 'Test design under voluntary participation', *Games and Economic Behavior* **104**, 632–655.

Saviano, J. (2020), 'Recommendations: The optional, third letter'.
URL: *https://meet.nyu.edu/advice/optional-third-letter-of-recommendation/*

Sher, I. (2011), 'Credibility and determinism in a game of persuasion', *Games and Economic Behavior* **71**(2), 409–419.
URL: *https://www.sciencedirect.com/science/article/pii/S0899825610000953*

Silva, F. (2022), 'Information transmission in persuasion models with imperfect verification', *Available at SSRN 3728325* .

Strausz, R. (2017), Mechanism Design with Partially Verifiable Information, Rationality and Competition Discussion Paper Series 45, CRC TRR 190 Rationality and Competition.
URL: *https://ideas.repec.org/p/rco/dpaper/45.html*

Weksler, R. and Zik, B. (2022), 'Job market cheap talk'.

A Omitted proofs

A.1 Proof of Lemma 1

The problem the DM needs to solve when committing to α is

$$\max_{\alpha} \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \tilde{\sigma}(t|\theta) p_t(\alpha;\theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \tilde{\sigma}(t|\theta') p_t(\alpha;\theta')$$
$$\text{s.t. } \tilde{\sigma}(\cdot|\theta) \in \arg\max \sum_{t \in T} \sigma(t|\theta) p_t(\alpha;\theta), \quad \text{for all } \theta \in \Theta$$

Therefore, $\sum_{t \in T} \tilde{\sigma}(t|\theta) p_t(\alpha;\theta) = \max_{\sigma(\cdot|\theta)} \sigma(t|\theta) p_t(\alpha;\theta)$ for all θ. We can plug this is in the DM's maximisation problem to obtain

$$\max_{\alpha} \max_{\sigma^A} \min_{\sigma^R} \sum_{\theta \in A} \mu(\theta) \sum_{t \in T} \sigma(t|\theta) p_t(\alpha;\theta) - \sum_{\theta' \in R} \mu(\theta') \sum_{t \in T} \sigma(t|\theta') p_t(\alpha;\theta')$$

where the $\min$ is obtained because of the minus sign.

A.2 Proof of Theorem 1

Let $(\alpha, \sigma^A) \in \arg\max_{\tilde{\alpha}, \tilde{\sigma}^A} \min_{\tilde{\sigma}^R} v(\tilde{\alpha}, \tilde{\sigma}^A, \tilde{\sigma}^R)$ and $\sigma^R \in \arg\min_{\tilde{\sigma}^R} \max_{\tilde{\alpha}} v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R)$. Note that σ^A is fixed in the min-max problem. I will now show that these strategies are equilibrium strategies.

Because the order of maximisation does not matter, $\sigma^A \in \arg\max \min v(\alpha, \tilde{\sigma}^A, \tilde{\sigma}^R)$. Moreover, $\sigma^A \in \arg\max v(\alpha, \tilde{\sigma}^A, \sigma_1^R) \Leftrightarrow \sum_{t \in T} \sigma(t|\theta) p_t(\alpha;\theta) \geq \sum_{t \in T} \tilde{\sigma}(t|\theta) p_t(\alpha;\theta)$ for all $\theta \in A$ and $\tilde{\sigma}^A$. This last expression does not depend σ^R. Therefore, $\sigma^A \in \arg\max v(\alpha, \tilde{\sigma}^A, \sigma^R)$.

Similarly, $\alpha \in \arg\max \min v(\tilde{\alpha}, \sigma^A, \tilde{\sigma}^R)$. Because v is linear in both α and σ^R, (α, σ^R) is a saddle-point of $v(\cdot, \sigma^A, \cdot)$ by the minimax theorem. As for σ^A, $\sigma^R \in \arg\min v(\alpha, \sigma^A, \tilde{\sigma}^R) \Leftrightarrow \sum_{t \in T} \sigma(t|\theta) p_t(\alpha;\theta) \geq \sum_{t \in T} \tilde{\sigma}(t|\theta) p_t(\alpha;\theta)$ for all $\theta \in R$ and $\tilde{\sigma}^R$. Therefore all strategies are best-reply.

Beliefs on-path are formed using Bayes' rule and off-path beliefs are chosen to justify the off-path actions of α (this is always possible to find as any action is best-reply to some beliefs).

Note also that we can without loss of generality take σ^A to be in pure strategy as it is maximises a linear function.

A.3 Proof of Lemma 2

Because $t \succeq t'$ implies $t \succeq_\theta t'$ for some $\theta \in A$, Lemma 2 is a corollary of Proposition 8 proven below.

A.4 Proof of Proposition 1 and Proposition 2

Suppose that test t is single-peaked. Suppose there is a menu with both t, t'. Take $A_1, A_2 \in A$ with $A_1 <_t A_2$ and because the signal's labels are arbitrary, suppose without loss of generality that A_1 chooses t' and A_2 chooses t in some equilibrium. Let α denote the DM equilibrium strategy in this equilibrium.

Because $t \succeq t'$, there is $\beta : X \times X \to [0,1]$ such that $\pi_{t'}(\tilde{x}|\theta) = \beta(x,\tilde{x})\pi_t(x|\theta) + \beta(x',\tilde{x})\pi_t(x'|\theta)$ and $\sum_x \beta(\tilde{x}, x) = 1$ for $\tilde{x} = x, x'$. Type $\theta \in \Theta$ prefers test t' over t if

$$\begin{aligned}&\alpha(x_1,t')\Big(\beta(x_1,x_1)\pi_t(x_1|\theta) + \beta(x_0,x_1)\pi_t(x_0|\theta)\Big)\\ &+\alpha(x_0,t')\Big(\beta(x_1,x_0)\pi_t(x_1|\theta) + \beta(x_0,x_0)\pi_t(x_0|\theta)\Big) - \alpha(x_1,t)\pi_t(x_1|\theta) - \alpha(x_0,t)\pi_t(x_0|\theta) \geq 0\end{aligned}$$

Note that this expression is striclty monotonic in θ. Indeed, if $\pi_t(x_0|\theta) > 0$, then dividing by $\pi_t(x_0|\theta)$ gives

$$\begin{aligned}\alpha(x_1,t')\Big(\beta(x_1,x_1)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} + \beta(x_0,x_1)\Big) + \alpha(x_0,t')\Big(\beta(x_1,x_0)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} + \beta(x_0,x_0)\Big)\\ - \alpha(x_1,t)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} - \alpha(x_0,t)\end{aligned}$$

which is linear in $\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)}$, a strictly increasing function of θ (recall that $\theta \neq_t \theta'$ for all θ, θ'). If $\pi_t(x_0|\theta) = 0$, then $\theta = \max \Theta$ where the $\max$ is taken with respect to $\geq_t$.

To have A_1 choose t' and A_2 choose t, it must be strictly decreasing in θ, i.e.,

$$\alpha(x_1,t')\beta(x_1,x_1) + \alpha(x_0,t')\beta(x_1,x_0) - \alpha(x_1,t) < 0 \tag{5}$$

A necessary condition for (5) to hold is that $\alpha(x_1,t) > 0$. Note the strict monotonicity also implies that there is $\overline{\theta} \in A$ such that any $\theta > \overline{\theta}$ prefers t and any $\theta \leq \overline{\theta}$ prefers t'. Let $A^+ = \{\theta \in A : \theta >_t \overline{\theta}\}$ and $R^+ = \{\theta \in R : \theta >_t \theta', \text{ for all } \theta' \in A\}$. But because only types in $A^+ \cup R^+$ choose t, the likelihood ratios $\frac{\pi_t(x_1|\theta)}{\pi_t(x_1|\theta')} < \frac{\pi_t(x_0|\theta)}{\pi_t(x_0|\theta')}$ for any $\theta \in A^+$, $\theta' \in R^+$ and $\alpha(x_1,t) > 0$ imply that $\alpha(x_0,t) = 1$ (Milgrom, 1981).

But then no type ever prefer t' over t. Indeed, the condition to prefer t' over t,

$$\Big(\alpha(x_1,t')\beta(x_1,x_1)+\alpha(x_0,t')\beta(x_1,x_0)-\alpha(x_1,t)\Big)\pi_t(x_1|\theta) \\ \geq \Big(1-\alpha(x_1,t')\beta(x_0,x_1)-\alpha(x_0,t')\beta(x_0,x_0)\Big)\pi_t(x_0|\theta)$$

is never satisfied as the LHS is strictly negative because (5) must hold and the RHS is positive because $\beta(x_0,x_1)+\beta(x_0,x_0)=1$ and $\alpha(\tilde{x},t')\leq 1$, $\tilde{x}=x_1,x_0$.

Thus there cannot be an equilibrium where another test than t is chosen.

Suppose that test t is enclosed.

Suppose $(\tilde{\alpha},\tilde{\sigma}^A)\in \arg\max\min_{\sigma^R} v(\alpha,\sigma^A,\sigma^R)$ with $\tilde{\sigma}(t|\theta)=1$ for all $\theta\in A$. Because there is only one test chosen in the induced equilibrium, it is enough to only consider $\tilde{\alpha}(\cdot,t)$ in pure strategy.

Suppose the prior is such that when only t is offered, x_0 is rejected and x_1 is accepted. Let $\underline{\theta}=\min\{\theta\in R\}$ where the $\min$ is taken with respect to $\geq_t$.

Then consider the following deviation: take some $t'\neq t$ and let $\alpha(x,t')=\pi_t(x_1|\underline{\theta})$ for all $x\in X$ and $\alpha=\tilde{\alpha}$ otherwise. Because t is enclosed, there is $\theta\in A$ such that $\pi_t(x_1|\theta)<\pi_t(x_1|\underline{\theta})$ and for all $\theta'\in R$, $\pi_t(x_1|\theta')\geq\pi_t(x_1|\underline{\theta})$. Let $\sigma(t'|\theta)=1$ for that type and $\sigma=\tilde{\sigma}$ otherwise. This deviation is strictly profitable, i.e., $\min_{\sigma^R} v(\tilde{\alpha},\tilde{\sigma}^A,\sigma^R)<\min_{\sigma^R} v(\alpha,\sigma^A,\sigma^R)$. The case where the prior is such that $\tilde{\alpha}(x_0,t)=1$ and $\tilde{\alpha}(x_1,t)=0$ is treated in the same way.

Finally, suppose the prior is such that $\tilde{\alpha}(x_1,t)=\tilde{\alpha}(x_0,t)\in\{0,1\}$ when only t is offered. This means that the DM does not react to information. Let $\alpha(x,t')=\tilde{\alpha}(x,t)$ for some $t'\neq t$ and $\sigma(t'|\theta)=1$ for some $\theta\in A$ and $\sigma=\tilde{\sigma}$ otherwise. We get $\min_{\sigma^R} v(\tilde{\alpha},\tilde{\sigma}^A,\sigma^R)=\min_{\sigma^R} v(\alpha,\sigma^A,\sigma^R)$, so it is also a solution.

Suppose that test t is not single-peaked.

In this case, it is possible to find $A_1,A_2\in A$ and $R_1\in R$ such that $A_1<_t R_1<_t A_2$. Let $\mu(\theta)\approx 0$ for $\theta\neq A_1,A_2,R_1$ and be such that x_0 is rejected and and x_1 is accepted when only t is offered. Because t is informative, there is always such prior. Then from the reasoning above the menu $\{t,t'\}$ is strictly better for the DM than $\{t\}$ when only focusing on A_1,A_2,R_1 have positive probability. But because $\mu(\theta)\approx 0$ for $\theta\neq A_1,A_2,R_1$, then the menu $\{t,t'\}$ remains strictly better than $\{t\}$ whatever the behaviour of the other types.

Suppose that test t is not enclosed.

If test t is not enclosed, then suppose without loss of generality that there is $R_1\in R$ such that $R_1>_t\theta$ for any $\theta\in\Theta$ (otherwise, simply change the roles of x_1 and x_0). Take some $A_1\in A$ such that $A_1 >_t R_1$. (We have a strict inequality because we assumed that all types generate different distribution over signals.) Suppose that for $\theta \neq A_1, R_1$, $\mu(\theta) \approx 0$. An argument analogue to the proof that single-peakness implies that only t is chosen in equilibrium holds.

A.5 Proof of Proposition 3

Proof. Note that in an MLRP environment, the strategy of the DM takes the form of a cutoff strategy. For each test t, there is $x_t \in X$ such that $\alpha(x,t) = 0$ for $x < x_t$, $\alpha(x,t) = 1$ for $x > x_t$ and $\alpha(x_t,t) \in [0,1]$. From Lemma 2, we know that there is an optimal menu containing the Blackwell most informative test. Because all tests are MLRP and the DM's payoffs satisfy single-crossing condition, the Lehmann order is well-defined and the Blackwell order implies the Lehmann order (Lehmann, 1988; Persico, 2000). Let $\succeq^a$ denote the Lehmann order.

The Lehmann order is defined on continuous information structure. But as outlined in Lehmann (1988), we can always make our conditional probabilities continuous by adding independent uniform between each signal. Let's assume, without loss of generality, that $X = \{1, ..., n\}$. The new distribution over signal is $\tilde{y}|\theta = \tilde{x}|\theta - u$ where $u \sim$U$[0,1]$. Denote by F_t the cdf associated with the new information structure.

We have that $t \succeq^a t'$ if $y^*(\theta, y) \equiv F_t(y^*|\theta) = F_{t'}(y|\theta)$ is nondecreasing in θ for all y (Lehmann, 1988). In particular, this condition implies that if $F_t(y|\theta') \leq (<) F_{t'}(y'|\theta')$ then $F_t(y|\theta) \leq (<) F_{t'}(y'|\theta)$ for all $\theta > \theta'$.

Let α be the optimal strategy and x_t be the cutoff signal associated to each test. To each $\big(\alpha(\cdot, t), x_t\big)$ we can associate a $y_t \equiv x_t - \alpha(x_t, t)$.

If t is part of an optimal menu, it must be that there is some $\theta' \in R$ such that $p_t(\alpha; \theta') \geq p_{t'}(\alpha; \theta')$ for all t'. Or put differently, $F_t(y_t|\theta') \leq F_{t'}(y_{t'}|\theta')$ for all t'. But then $F_t(y_t|\theta) \leq F_{t'}(y_{t'}|\theta)$ for all t' and all $\theta > \theta'$, in particular all $\theta \in A$. Therefore all type in A prefer test t as well and there is an solution of the max-min problem where all types in $\theta \in A$ choose t. (If there is an A-type that is indifferent between t and t' then all types in R must be indifferent or prefer t' so choosing t is an equilibrium strategy for such A-type.) $\square$

A.6 Proof of Proposition 4

Suppose that t is the only test used in the optimal menu. Define a test t' such that for all $\theta \in \Theta$,

$$\pi_{t'}(x|\theta) = \sum_{\tilde{x}=x,x'} \pi_t(\tilde{x}|\theta)$$
$$\pi_{t'}(\tilde{x}|\theta) = \pi_t(\tilde{x}|\theta), \text{ for all } \tilde{x} \neq x, x'$$

The test t' pools signals x and x' together and is otherwise identical to t. We have $t \succ t'$ as any strategy under t' can be replicated under t.

Let μ be such that for all $\theta \neq A_1, A_2, R_1$, $\mu(\theta) \approx 0$ and such that when only t is chosen, the DM's best-reply is $\alpha(x,t) = 1$ and $\alpha(x',t) = 0$.

In the problem with commitment, consider the deviation $\tilde{\alpha}$ such that $\tilde{\alpha}(x,t') = \frac{\pi_t(x|A_1)}{\pi_t(x|A_1)+\pi_t(x'|A_1)} + \epsilon$ for some small $\epsilon > 0$ and $\tilde{\alpha} = \alpha$ otherwise.

This implies that $p_t(\alpha; A_1) < p_{t'}(\tilde{\alpha}; A_1)$ but for ϵ small enough $p_t(\alpha;\theta) > p_{t'}(\tilde{\alpha};\theta)$ for $\theta = R_1, A_2$. Therefore A_1 is accepted with strictly higher probability and the other types with the same probability as they choose the same test. If the prior on other types is sufficiently small, the deviation is still strictly profitable for the DM.

A.7 Proof of Proposition 5

($\Rightarrow$) The proof is similar to the one in Milgrom (1981). Denote by $G_t(\cdot|x)$ the cdf of posterior beliefs after signal x in test t. For all $\theta > \theta'$,

$$\mu(\theta)\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \mu(\theta)\frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$$

Take some $\theta^* \geq \theta'$. Summing over θ, we get

$$\sum_{\theta>\theta^*} \mu(\theta)\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \sum_{\theta>\theta^*} \mu(\theta)\frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$$

Inverting and summing over θ', we get

$$\frac{\sum_{\theta^*\geq\theta'} \mu(\theta')\pi_t(x|\theta')}{\sum_{\theta>\theta^*} \mu(\theta)\pi_t(x|\theta)} \leq \frac{\sum_{\theta^*\geq\theta'} \mu(\theta')\pi_{t'}(x|\theta')}{\sum_{\theta>\theta^*} \mu(\theta)\pi_{t'}(x|\theta)}$$

which implies

$$\frac{G_t(\theta^*|x)}{1-G_t(\theta^*|x)} \leq \frac{G_{t'}(\theta^*|x)}{1-G_{t'}(\theta^*|x)} \quad \Rightarrow \quad G_t(\theta^*|x) \leq G_{t'}(\theta^*|x)$$

($\Leftarrow$) Take $\theta > \theta'$ and let $\mu(\theta) + \mu(\theta') = 1$ (i.e., there is probability zero on other types). If $\mu(\cdot|t,x) \succeq_{FOSD} \mu(\cdot|t',x)$, then we have $\mu(\theta|t,x) \geq \mu(\theta|t',x)$ and $\mu(\theta'|t,x) \leq \mu(\theta'|t',x)$. By Bayes' rule this implies $\frac{\pi_t(x|\theta)}{\pi_{t'}(x|\theta)} \geq 1$ and $\frac{\pi_t(x|\theta')}{\pi_{t'}(x|\theta')} \leq 1$. This gives $\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$.

A.8 Proof of Proposition 6

The way this proof proceeds is by fixing a menu and dividing tests in two categories: (1) those for which $\alpha(x_0,\tilde{t}) \in (0,1)$ and $\alpha(x_1,\tilde{t}) = 1$ and (2) $\alpha(x_0,\tilde{t}) = 0$ and $\alpha(x_1,\tilde{t}) \in (0,1]$. I exclude the possibility that the DM always accepts or rejects after any signal as it would either be the only test chosen in equilibrium or never chosen. Then, I show that within each category, it is without loss of optimality to have at most one test. It is thus optimal to have at most two tests in the menu. The last part of the proof shows that the resulting menu is dominated by having only one test.

If there are two tests, $t > t'$ such that $\alpha(x_0,\tilde{t}) = 0$ and $\alpha(x_1,\tilde{t}) \in (0,1]$, I will show that,

$$p_t(\alpha;\theta') \geq p_{t'}(\alpha;\theta') \quad \Rightarrow \quad p_t(\alpha;\theta) \geq p_{t'}(\alpha;\theta) \text{ for all } \theta > \theta'$$

Take two tests such that $\alpha(x_0,\tilde{t}) = 0$, $t > t'$. Let α, α' denote their respective probability of accepting after x_1. Define $\alpha(\theta) \equiv \alpha(\theta)\pi_t(x_1|\theta) - \alpha'\pi_{t'}(x_1|\theta) = 0$. Rearranging, $\alpha(\theta) = \alpha'\frac{\pi_{t'}(x_1|\theta)}{\pi_t(x_1|\theta)}$. From our assumption on the difficulty environment, $\alpha(\theta)$ is decreasing in θ. If $p_t(\alpha;\theta') \geq p_{t'}(\alpha;\theta')$ for some θ' then $\alpha \geq \alpha(\theta')$. Then $\alpha \geq \alpha(\theta)$ for all $\theta > \theta'$.

In equilibrium, we must have that there is one $\theta' \in R$ that chooses t and thus for all $\theta \in A$, $p_t(\alpha;\theta) \geq p_{t'}(\alpha;\theta)$. Then there is an solution of the max-min problem where t' is never chosen.

A similar argument can be made for all tests where $\alpha(x_0,\tilde{t}) > 0$.

Thus we conclude that it is without loss of optimality that the optimal menu has at most two tests.

Suppose the optimal menu uses two tests, $t > t'$. I will now show that it must be that $\alpha(x_0,t) \in (0,1)$ and $\alpha(x_1,t') \in (0,1)$, i.e., the DM must accept in the hard test when there is a fail grade and only accept in the easy test if there is a pass grade. Suppose it is not the case and denote by α, α' their respective mixing probabilities. Define $\alpha(\theta) \equiv \alpha(\theta)\pi_t(x_1|\theta) - \alpha'\pi_{t'}(x_0|\theta) - \pi_{t'}(x_1|\theta) = 0$, which is equivalent to $\alpha(\theta) = \alpha'\frac{1}{\pi_t(x_1|\theta)} + (1-\alpha')\frac{\pi_{t'}(x_1|\theta)}{\pi_t(x_1|\theta)}$. Again from our assumptions, this is decreasing in θ. A type θ chooses t if $\alpha \geq \alpha(\theta)$. Thus if one $\theta \in A$ chooses t all $\theta \in R$ choose t and there is no pooling of A and R-types on t', or it is payoff equivalent to just offering t. Therefore, $\alpha(x_0, t) \in (0, 1)$ and $\alpha(x_1, t') \in (0, 1)$ for $t > t'$.

If the DM mixes, he must be indifferent and thus we have

$$\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta)\pi_t(x_0|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta')\pi_t(x_0|\theta') = 0$$
$$\sum_{\theta \in A} \mu(\theta)\sigma(t'|\theta)\pi_{t'}(x_1|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t'|\theta')\pi_{t'}(x_1|\theta') = 0$$

In the easy test, because the DM rejects with positive probability after x_1 and rejects for sure after x_0 (as he uses a cutoff strategy), his payoffs from t' is 0, i.e., he does as well as rejecting for sure.

In the hard test, he accepts with some probability after x_0 and thus his payoffs are

$$\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta'),$$

that is the payoffs he would get from accepting all types choosing t. Thus the overall payoffs from the menu is $\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta')$. Offering a menu is better than a singleton menu if this value is strictly greater than offering t and following the signal

$$\begin{aligned}\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta') &> \sum_{\theta \in A} \mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta' \in R} \mu(\theta')\pi_t(x_1|\theta') \\ &= \sum_{\theta \in A} \sigma(t|\theta)\mu(\theta)\pi_t(x_1|\theta) + \sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta) \\ &\quad - \sum_{\theta' \in R} \sigma(t|\theta')\mu(\theta')\pi_t(x_1|\theta') - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta')\end{aligned}$$

We can rearrange and use the indifference condition at (x_0, t) to get

$$0 > \sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta')$$

Using the indifference condition at (x_1, t'), we can replace 0 on the LHS and get

$$\begin{aligned}&\sum_{\theta \in A} \mu(\theta)\sigma(t'|\theta)\pi_{t'}(x_1|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t'|\theta')\pi_{t'}(x_1|\theta') \\ &\qquad > \sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta')\end{aligned}$$

But from the definition of the environment, for all $\theta > \theta'$,

$$\frac{\pi_t(x_1|\theta)}{\pi_t(x_1|\theta')} \geq \frac{\pi_{t'}(x_1|\theta)}{\pi_{t'}(x_1|\theta')}$$

which implies that $\mu(\theta|x_1, t) \succeq_{FOSD} \mu(\theta|x_1, t')$. Thus we get a contradiction.

A.9 Proof of Proposition 7

Suppose condition (2) holds.

Suppose that all types choose the same test testing dimension j. Take $(\tilde{\theta}_i, \tilde{\theta}_j) \in \arg\min_{\theta \in A} p_{t_j}(\alpha; \theta)$. Because $p_{t_j}(\alpha; \theta_i, \theta_j)$ is constant in θ_i, we have $(\overline{\theta}_i, \tilde{\theta}_j) \in \arg\min_{\theta \in \Theta} p_{t_j}(\alpha; \theta)$ as well and from condition (2), $(\overline{\theta}_i, \tilde{\theta}_j) \in A$. Consider the deviation in the problem with commitment to $(\tilde{\alpha}, \tilde{s})$ such that for t_i,

- $\tilde{\alpha}(\cdot, t_i)$ is set so that it has a cutoff structure and $p_{t_i}(\tilde{\alpha}|\overline{\theta}_i, \tilde{\theta}_j) = p_{t_j}(\alpha; \overline{\theta}_i, \tilde{\theta}_j) + \epsilon$ and $\tilde{\alpha}(\cdot, t_j) = \alpha(\cdot, t_j)$ otherwise.
- $\tilde{\sigma}(t_i|\overline{\theta}_i, \tilde{\theta}_j) = 1$ and $\tilde{\sigma}(\cdot|\theta) = \sigma(\cdot|\theta)$ otherwise.

Because the test t_i has the strict MLRP when restricting attention to dimension i, for all $\theta_i < \overline{\theta}_i$, $\min_{\theta \in \Theta} p_{t_j}(\alpha; \theta) \geq p_{t_i}(\tilde{\alpha}|\overline{\theta}_i, \theta_j) > p_{t_i}(\tilde{\alpha}|\theta_i, \theta_j)$ if ϵ is small enough. This means that no other type has an incentive to choose test i but $(\overline{\theta}_i, \tilde{\theta}_j) \in A$ is accepted with strictly higher probability. Thus the menu with only test j cannot be the optimal menu.

Suppose condition (2) does not hold.

If condition (2) is not satisfied, then there a dimension, say 1, and $\tilde{\theta}_2 \in \Theta_2$ such that $(\overline{\theta}_1, \tilde{\theta}_2) \in R$. By the monotonicity of payoffs in the bidimensional environment, this implies that $(\theta_1, \tilde{\theta}_2) \in R$ for all $\theta_1 \in \Theta_1$. Moreover, for all $\theta_2 < \tilde{\theta}_2$ and all $\theta_1 \in \Theta_1$, $(\theta_1, \theta_2) \in R$.

Now suppose μ is such that $\mu(\theta_1, \tilde{\theta}_2) > \sum_{\theta_2' \neq \theta_2} \mu(\theta_1, \theta_2')$ for all $\theta_1 \in \Theta_1$. And that $\mu(\theta_1, \theta_2) \approx 0$ for all $(\theta_1, \theta_2) \in R$ such that $\theta_2 > \tilde{\theta}_2$.

I am going to show that $\{t_2\}$ is optimal when t_1 fully reveals dimension 1. Because this test can replicate the strategies of any t_1, it is enough to prove our claim.

Suppose there is an optimal menu $\{t_1, t_2\}$. From our assumptions on μ, the DM follows a cutoff strategy after t_2. That's because his payoff is monotone along that dimension, ignoring $(\theta_1, \theta_2) \in R$ such that $\theta_2 > \tilde{\theta}_2$ whose prior probability is close to zero. So it does not upset the cutoff structure of the best-response. This implies that $p_{t_2}(\alpha; \theta_1, \theta_2) > p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$ for all $\theta_2 > \tilde{\theta}_2$ because the likelihood ratio is strictly increasing.

Suppose that some $(\theta_1, \tilde{\theta}_2)$ chooses t_1 with probability 1 in equilibrium. Because $\mu(\theta_1, \tilde{\theta}_2) > \sum_{\theta_2' \neq \theta_2} \mu(\theta_1, \theta_2')$ for all $\theta_1 \in \Theta_1$, it must be that the best-response is $\alpha(x = \theta_1, t_1) = 0$ (recall that t_1 fully reveals θ_1). Thus $p_{t_2}(\alpha; \theta_1, \theta_2) = 0$ for all $\theta_2 \in \Theta_2$, otherwise there is a profitable deviation. Either this contradicts the fact that the DM best replies or in equilibrium the DM rejects after all signals in every test. But then he is weakly better off only offering t_2.

Thus to have $\{t_1, t_2\}$ strictly better, it must be that all $(\theta_1, \tilde{\theta}_2)$ choosing t_1 mix in equilibrium. This means that $p_{t_1}(\alpha; \theta_1, \tilde{\theta}_2) = p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$. But by the cutoff structure of $\alpha(\cdot, t_2)$ and the strict MLRP assumption, we have $p_{t_2}(\alpha; \theta_1, \theta_2) > p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$ for all $\theta_2 > \tilde{\theta}_2$ and $p_{t_2}(\alpha; \theta_1, \theta_2) < p_{t_2}(\alpha; \theta_1, \tilde{\theta}_2)$ for all $\theta_2 < \tilde{\theta}_2$. Thus t_2 is strictly preferred for all $(\theta_1, \theta_2) \in A$. Thus choosing only $\{t_2\}$ is an optimal menu.

A.10 Proof of Proposition 8

Proof. I will first prove the following lemma. This result generalises the observation that the DM can find a strategy with lower type-I and -II errors when $t \succeq t'$ (Blackwell informativeness order) to $t \succeq_\theta t'$.

Lemma 3. *For any $t \succeq_\theta t'$ and $\alpha(\cdot, t')$, there is $\alpha(\cdot, t)$ such that*

$$\begin{aligned} &\sum_x \alpha(x,t)\pi_t(x|\theta) \geq \sum_x \alpha(x,t')\pi_{t'}(x|\theta) \\ \textit{for all } \theta' \in R, \quad &\sum_x \alpha(x,t)\pi_t(x|\theta') \leq \sum_x \alpha(x,t')\pi_{t'}(x|\theta') \end{aligned}$$

Proof. We can prove this lemma by using a theorem of the alternative (see e.g., Rockafellar (2015) Section 22). Only one of the following statement is true:

- There exists $\alpha(\cdot, t)$ such that

$$\begin{aligned} &\sum_x \alpha(x,t)\pi_t(x|\theta) \geq \sum_x \alpha(x,t')\pi_{t'}(x|\theta) \\ \text{for all } \theta' \in R, \quad &\sum_x \alpha(x,t)\pi_t(x|\theta') \leq \sum_x \alpha(x,t')\pi_{t'}(x|\theta') \\ \text{for all } x \in X, \quad &\alpha(x,t) \leq 1 \\ \text{for all } x \in X, \quad &\alpha(x,t) \geq 0 \end{aligned}$$

- There exists $z, y \geq 0$ such that

$$\text{for all } x \in X, \quad -z_\theta \pi_t(x|\theta) + \sum_{\theta' \in R} z_{\theta'} \pi_t(x|\theta') + y_x \geq 0 \tag{6}$$

$$-z_\theta \sum_{x'} \alpha(x', t') \pi_{t'}(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_{x'} \alpha(x', t') \pi_{t'}(x'|\theta') + \sum_{x'} y_{x'} < 0 \tag{7}$$

Take inequality (6) from the second alternative and multiply by $\beta(x, x')$ as described in Definition 6 and sum over $x \in X$:

$$-z_\theta \sum_x \beta(x, x') \pi_t(x|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_x \beta(x, x') \pi_t(x|\theta') + \sum_x \beta(x, x') y_x \geq 0$$

Because $t \succeq_\theta t'$, we get for all $x' \in X$,

$$-z_\theta \pi_{t'}(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \pi_{t'}(x'|\theta') + \sum_x \beta(x, x') y_x \geq 0$$

We can then multiply by $\alpha(x', t')$ and sum over $x' \in X$:

$$-z_\theta \sum_{x'} \alpha(x', t') \pi_t(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_{x'} \alpha(x', t') \pi_{t'}(x'|\theta') + \sum_{x,x'} \alpha(x', t') \beta(x, x') y_x \geq 0 \tag{8}$$

Because $\sum_{x'} \beta(x, x') \leq 1$ and $\alpha(x', t') \leq 1$ for all $x' \in X$, we have $\sum_{x,x'} \alpha(x', t') \beta(x, x') y_x \leq \sum_x y_x$. Therefore, the inequality (7) cannot hold and the first alternative holds. □

With this result in hand, we can now prove our result. Suppose that t is not part of the optimal menu. Thus we can find an solution of the max-min problem, $(\alpha, \sigma^A) \in \arg\max_{\alpha', \sigma^{A'}} \min_{\sigma^R} v(\alpha', \sigma^{A'}, \sigma^R)$ with $\sigma(t|\theta) = 0$ for all $\theta \in A$. Take a test t' used in the solution by some $\theta \in A$. Then from Lemma 3, we can construct a $(\tilde{\alpha}, \tilde{\sigma}^A)$ such that

- $p_t(\tilde{\alpha}; \theta) \geq p_{t'}(\alpha; \theta)$
- $p_t(\tilde{\alpha}; \theta') \leq p_{t'}(\alpha; \theta')$ for all $\theta' \in R$
- $\tilde{\sigma}(t|\theta) = 1$
- $\tilde{\sigma} = \sigma$ otherwise

This constitutes a solution to the max-min problem.

□

A.11 Proof of Proposition 9

Proof. ($\Leftarrow$) For each $\theta \in A$, let t_θ such that

$$\operatorname{supp} \pi_t(\cdot|\theta) \cap \bigg(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta') \bigg) = \emptyset$$

Then posting a menu $(t_\theta)_{\theta \in A}$ is optimal (eliminating duplicates if there are some). Each $\theta \in A$ chooses t_θ. For any strategy of $\theta' \in R$, the DM accepts after any $(x, t) \in \cup_{\theta:\sigma(t|\theta)=1} \operatorname{supp} \pi_t(\cdot|\theta)$ and rejects otherwise. This gives the DM and the A-types maximal payoffs and the R-types get rejected for any strategy they follow.

($\Rightarrow$) Suppose the DM's payoffs are maximal and there is $\theta \in A$ and for all $t \in T$ there is $\theta' \in R$ and $x \in X$ such that $\pi_t(x|\theta), \pi_t(x|\theta') > 0$. Then when θ chooses t out of the menu of tests, if θ' chooses t as well, at x, either the DM accepts θ' or rejects θ. Therefore, payoffs cannot be maximal. $\square$

A.12 Proof of Theorem 2

The only thing we need prove is that it is optimal to have a different message for each type $\theta \in A$, the rest follows from Theorem 1. Suppose it is not the case and take a solution (α, σ^A) of the max-min problem where the A-types play a pure strategy.

Then, there is $\theta_1, \theta_2 \in A$ and $(t, c) \in T \times C$ such that $\sigma(t, c|\theta_1) = \sigma(t, c|\theta_2) = 1$ (if they use a different test then we can also change the message and nothing is changed). Consider the alternative strategy α' where, for some unused (t, c') in the original mechanism, $\alpha'(t, c', x) = \alpha(t, c, x)$ for all $x \in X$ and $\alpha'(t'', c'', x) = \alpha'(t'', c'', x)$ for all other $(t'', c'') \in T \times C$ and all $x \in X$ otherwise. The new strategy α' is thus the same as α but makes sure that if the pair (t, c') is chosen, it uses the same actions as (t, c). Now consider the following strategy $\tilde{\sigma}^A$ in the auxiliary max-min problem, $\tilde{\sigma}(\cdot|\theta) = \sigma(\cdot|\theta)$ for $\theta \neq \theta_1$ and $\tilde{\sigma}(t, c'|\theta_1) = 1$. This gives the same value in the max-min proble as under (α, σ^A). Moreover, any deviations under α' gives the same payoff than under α. Therefore, $(\alpha', \tilde{\sigma}^A)$ is an solution to problem with commitment.

A.13 Proof of Proposition 10

This follows from Lemma 2. If a Blackwell dominated test is chosen by an A-type, then we can introduce a message test pair (c, t) in the max-min problem that will make the A-type better off without making any R-type better off. This will create a new solution to max-min problem.

A.14 Proof of Proposition 11

I only consider equilibria where for any two messages, there is an x such that $\alpha(t,c,x) \neq \alpha(t,c',x)$. Otherwise, we can construct an equilibrium where these two messages are merged and all incentives are preserved. After any message c and test t, there are four possibilities.

(1) if $\alpha(L,t,c) > \alpha(H,t,c)$, all types sending message c have a strict incentive to copy $\underline{\theta}_t$ and this strategy is equilibrium behaviour only if $\sum_{\theta \in A} \mu(\theta)\sigma(t,c,\underline{\theta}_t|\theta) = \sum_{\theta \in R} \mu(\theta)\sigma(t,c,\underline{\theta}_t|\theta)$.

(2) if $0 < \alpha(L,t,c) = \alpha(H,t,c) < 1$, it must hold that $\sum_{\theta \in A} \sum_{\theta' \in e(t,\theta)} \mu(\theta)\sigma(t,c,\theta'|\theta) = \sum_{\theta \in R} \sum_{\theta' \in e(t,\theta)} \mu(\theta)\sigma(t,c,\theta'|\theta)$.

(3) if $0 < \alpha(L,t,c) < \alpha(H,t,c) < 1$,[15] then it is a strictly dominant strategy to copy itself for any type. It also has to be that $\sum_{\theta \in A} \mu(\theta)\sigma(t,c,\theta|\theta)\pi_t(x|\theta) = \sum_{\theta \in R} \mu(\theta)\sigma(t,c,\theta|\theta)\pi_t(x|\theta)$.

(4) if $0 \leq \alpha(L,t,c) < \alpha(H,t,c) \leq 1$, with at least one equality, then there are two possibilities:

(a) $0 = \alpha(L,t,c) < \alpha(H,t,c) \leq 1$ and (b) $0 < \alpha(L,t,c) < \alpha(H,t,c) = 1$.

Now note that there cannot be two messages c, c' that are in case (4) as one would message would strictly dominate the other for any type.

In both case (1) and (2), the acceptance probability is the same for each type. So if we have two messages in case (1) and (2) respectively, say c_1, c_2, either one message dominates the other or we can construct another equilibrium where all types that sent c_1 now send c_2 and copy $\underline{\theta}_t$. This would not change any player's payoffs or incentives because we have added an equal mass of A- and R-types.

If we have two messages c_3, c_4 that are in case (3) and (4), we can also construct a new equilibrium where all types that sent c_3 now send c_4 and copy themselves. This would not change any player's payoffs or incentives because we have added an equal mass of A- and R-types.

So we have now constructed an equilibrium that has at most two messages. After one message, say c_1, the acceptance probability is constant and the mass of R- and A-types choosing c_1 is the same (corresponding to case (1) or (2)). After the other message, say c_2, the acceptance probability is not constant. Note that if more than one test is used, the probability of being accepted after a message where the probability of being accepted is constant must be the same across tests, otherwise there is a strict incentive to deviate.

If this is the case, then, there is also an equilibrium that is payoff equivalent, where all types choose an uninformative test $\emptyset$ with the same probability they were sending message c_1 and the DM accepts after $\emptyset$ with the same probability he was accepting after (c_1, t). The strategies remain the same otherwise.

[15] We can exclude $\alpha(L,t,c) = \alpha(H,t,c) \in \{0,1\}$ as in any equilibrium where the DM benefits from communication, it would give either a strict incentive to all types to choose message c or a strict incentive not to choose it.

A.15 Proof of Theorem 3

The designer's problem is

$$\begin{aligned}
&\max_{\tau,\alpha} \sum_{\theta\in A}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) - \sum_{\theta\in R}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta)\\
&\text{s.t. } \sum_t (\tau(t|\theta)-\tau(t|\theta'))p_t(\alpha;\theta) \geq 0 \text{ for all } \theta,\theta'\\
&\qquad \alpha \in BR(\tau)
\end{aligned}$$

If the DM could commit over a strategy α, his problem would be

$$\begin{aligned}
\tilde{V}(\alpha) = &\max_{\tau} \sum_{\theta\in A}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) - \sum_{\theta\in R}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta)\\
&\text{s.t. } \sum_t (\tau(t|\theta)-\tau(t|\theta'))p_t(\alpha;\theta) \geq 0 \text{ for all } \theta,\theta'
\end{aligned}$$

Step 1: Show that $\tilde{V}(\alpha) = \max_s \min_m v(\alpha, s, m)$ where v is defined in (3).

To show this claim, I am going to relax the mechanism design problem by restricting attention to the IC constraints of R-types deviating to reporting an A-type:

$$\begin{aligned}
\tilde{V}(\alpha) = &\max_{\tau} \sum_{\theta\in A}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) - \sum_{\theta\in R}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta)\\
&\text{s.t. } \sum_t \tau(t|\theta)p_t(\alpha;\theta) \geq \max_{m(\cdot|\theta)} \sum_{\theta'\in A} m(\theta'|\theta)\sum_t \tau(t|\theta')p_t(\alpha;\theta), \text{ for all } \theta\in R
\end{aligned}$$

The IC constraints are written to express that reporting type θ for $\theta \in R$ is better than any other reporting strategy over the A-types.

Now note that if an IC constraint is slack at the optimum, we could improve the DM's payoff by setting $\tau(t|\theta) = \sum_{\theta'\in A} m^*(\theta'|\theta)\tau(t|\theta')$ where $m^* \in \arg\max_{m(\cdot|\theta)} \sum_{\theta'\in A} m(\theta'|\theta)\sum_t \tau(t|\theta')p_t(\alpha;\theta)$. That would reduce the probability of type $\theta \in R$ of being accepted and would not change any other constraints in the relaxed problem. Thus at the optimum,

$$\sum_t \tau(t|\theta)p_t(\alpha;\theta) \geq \max_{m(\cdot|\theta)} \sum_{\theta'\in A} m(\theta'|\theta)\sum_t \tau(t|\theta')p_t(\alpha;\theta).$$

We can plug this expression in the payoffs to get

$$\begin{aligned}\tilde{V}(\alpha) &= \max_{\tau}\sum_{\theta\in A}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) - \sum_{\theta\in R}\mu(\theta)\max_{m(\cdot|\theta)}\sum_{\theta'\in A} m(\theta'|\theta)\sum_t \tau(t|\theta')p_t(\alpha;\theta)\\ &= \max_{\tau}\min_{m}\sum_{\theta\in A}\mu(\theta)\sum_t \tau(t|\theta)p_t(\alpha;\theta) - \sum_{\theta\in R}\mu(\theta)\sum_{\theta'\in A} m(\theta'|\theta)\sum_t \tau(t|\theta')p_t(\alpha;\theta)\end{aligned}$$

Note that we can take out the max of the summation by the linearity of the expression in m and it becomes a min because of the minus sign. This expression also corresponds to v as defined in (3).

It remains to show that the solution of this relaxed mechanism is indeed optimal. Take $s \in \arg\max \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$ and $m \in \arg\min v(\alpha, s, \tilde{m})$ and define the optimal mechanism by

- $\tau(t|\theta) = s(t|\theta)$ for $\theta \in A$
- $\tau(t|\theta') = \sum_{\theta\in A} m(\theta|\theta')s(t|\theta)$ for $\theta' \in R$

Note that an outcome of this mechanism gives payoff weakly higher than $\max_{\tilde{s}} \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$ as $v(\alpha, s, m) \geq \min_{\tilde{m}} v(\alpha, s, \tilde{m})$. Thus if it is incentive-compatible, it must be actually equal to the upper bound $\max_{\tilde{s}} \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$.

Note that all allocations are either allocations of A-types or convex combinations of the A-types' allocations thus it is enough to check that no type has any incentive to report any A-type in the mechanism.

By definition of m,

$$v(\alpha, s, m) \geq v(\alpha, s, m'), \text{ for all } m'$$

Therefore, for each R-type, θ',

$$\sum_{\theta\in A} m(\theta|\theta')\sum_t \tau(t|\theta)p_t(\alpha;\theta') \geq \sum_t \tau(t|\tilde{\theta})p_t(\alpha;\theta'), \text{ for any } \tilde{\theta} \in A$$

For an A-type θ, consider the choice of choosing $\tilde{s}$ such as $\tilde{s}(\cdot|\theta) = s(\cdot|\tilde{\theta})$ for some $\tilde{\theta} \in A$ and the same otherwise. By definition of s,

$$v(\alpha, s, m) = \min_{\tilde{m}} v(\alpha, s, \tilde{m}) \geq \min_{\tilde{m}} v(\alpha, \tilde{s}, \tilde{m})$$

Rearranging,

$$\mu(\theta)\sum_t \big(s(t|\theta)-s(t|\tilde\theta)\big)p_t(\alpha;\theta) \geq \max_{\tilde m}\sum_{\theta'\in R}\mu(\theta')\sum_{\theta''\in A}\tilde m(\theta''|\theta')\sum_t s(t|\theta)p_t(\alpha;\theta') - \max_{\tilde m}\sum_{\theta'\in R}\mu(\theta')\sum_{\theta''\in A}\tilde m(\theta''|\theta')\sum_t \tilde s(t|\theta)p_t(\alpha;\theta')$$

where the min is transformed in max because of the negative sign. Note that the LHS is the IC constrain of type θ deviating to type $\tilde\theta$. The RHS is the difference payoff is the probability of the R-types of being accepted when they choose a mimicking strategy $\tilde m$. Note that the only difference between s and $\tilde s$, from their point of view is that there is weakly less choice of allocations to mimic as we have θ choosing the same allocation as $\tilde\theta$. Therefore it must be that the RHS is positive which implies that the LHS is as well.

Step 2: Show that the DM does not benefit from commitment in the optimal mechanism.

Take $(\alpha, s) \in \arg\max_{\tilde\alpha,\tilde s}\min_{\tilde m} v(\tilde\alpha,\tilde s,\tilde m)$ and $m \in \arg\min_{\tilde m}\max_{\tilde\alpha} v(\tilde\alpha, s,\tilde m)$. The α selected would be the optimal strategy when the DM can commit.

Note that because the order of maximisation does not matter, we also have $\alpha \in \arg\max_{\tilde\alpha}\min_{\tilde m} v(\tilde\alpha, s,\tilde m)$. Note that v is linear in $\tilde\alpha$ and $\tilde m$ and thus by the minimax theorem,

$$v(\alpha,s,m) \geq v(\alpha',s,m), \text{ for all } \alpha'$$
$$v(\alpha,s,m) \leq v(\alpha,s,m'), \text{ for all } m'$$

Thus α best-replies to the optimal mechanism when the DM can commit and m is also a best reply to (α, s), thus satisfying the condition for characterising the equilibrium in Step 1.

B Microfoundation for single-peaked and enclosed test

I provide here a simple microfoundation for a test to be single-peaked or enclosed. To do so, I first show how DM payoffs with arbitrary weights on accepting a type can be recasted in the model presented in Section 2. Assume that $v(a,\theta) = a\cdot\nu(\theta)$ where for some function ν. Define $\Theta^+ = \{\theta : \nu(\theta) > 0\}$ and $\Theta^- = \{\theta : \nu(\theta) \leq 0\}$. These sets will play the same role as A and R. Let $\tilde\mu(\theta) = \frac{\mu(\theta)|\nu(\theta)|}{\sum_{\theta'}\mu(\theta')|\nu(\theta')|}$. Note that $\tilde\mu$ is a well-defined probability distribution. We can rewrite the expected payoffs as

$$\mathbb{E}_\mu[a\cdot\nu(\theta)] = \sum_{\theta'}\mu(\theta')|\nu(\theta')|\cdot\mathbb{E}_{\tilde\mu}\left[a\cdot(\mathbb{1}[\theta\in\Theta^+]-\mathbb{1}[\theta\in\Theta^-])\right].$$

Rewriting the payoffs like this shows that even when putting arbitrary weights on the types, we can recover the original payoff specification by reinterpreting the prior. In the original model, the prior already served as a weight on accepting a given type. If we allow for arbitrary weights, we can always reinterpret the weight combined with the prior as a new prior distribution. Because the results presented in this paper hold independent of the prior, they all carry through.

We can rewrite our definition of the single-peaked and enclosed test for these more general payoffs.

Definition 1. *A test t is single-peaked if there are $\theta_1, \theta_2 \in \Theta^+$ such that $\Theta^+ = \{\theta : \theta_1 \leq_t \theta \leq_t \theta_2\}$.*

A test t is enclosed if there are $\theta_1, \theta_2 \in \Theta^+$ such that $\theta_1 <_t \theta <_t \theta_2$ for all $\theta \neq \theta_1, \theta_2$.

I now turn to the microfoundation. The idea is to augment the type space to two dimensions, $(\theta, \eta) \in \Theta \times H$. The test is only informative about one dimension, θ. Let the marginal distribution of θ be denoted by $\mu(\cdot)$ and the probability of η conditional on θ by $\lambda(\cdot|\theta)$. The test t does not depend on η: for any $x \in X$, $\theta \in \Theta$, $\pi_t(x|\theta, \eta) = \pi_t(x|\theta, \eta')$ for any $\eta, \eta' \in H$. I still assume that $v(a = 1, (\theta, \eta)) \in \{-1, 1\}$. I call this environment, the *microfoundation environment*. For example, the DM could be a university testing a student. The university cares about both technical skills, θ, and creative skills, η, but the test can only test technical skills. Another interpretation could be that θ represents test taking ability that is only imperfectly informative about the true dimension of interest, η. The goal is to give to give conditions on λ that guarantee that the test t is single-peaked or enclosed.

For each θ, let $A(\theta) = \{\eta : (\theta, \eta) \in A\}$ and $R(\theta) = \{\eta : (\theta, \eta) \in R\}$. We can define preferences on θ only by defining the payoff function for the DM as

$$v(a = 1, \theta; \lambda) = \sum_{\eta \in A(\theta)} \lambda(\eta|\theta) - \sum_{\eta \in R(\theta)} \lambda(\eta|\theta).$$

In this case, $\theta \in \Theta^+$ if and only $\sum_{\eta \in A(\theta)} \lambda(\eta|\theta) - \sum_{\eta \in R(\theta)} \lambda(\eta|\theta) > 0$. We can define the order $\geq_t$ by $\theta \geq_t \theta' \Leftrightarrow \pi_t(x_1|\theta, \eta) \geq \pi_t(x_1|\theta', \eta')$ for any η, η'.[16] Let $\underline{\theta}_t = \min \theta$ and $\overline{\theta}_t = \max \theta$.

We can find conditions on the conditional probability λ that guarantee that a test is enclosed or single-peaked for any μ.

Proposition 12. *In the microfoundation environment, a test t is enclosed if and only if*

$$\sum_{\eta \in A(\theta)} \lambda(\eta|\theta) - \sum_{\eta \in R(\theta)} \lambda(\eta|\theta) > 0, \textit{ for } \theta = \underline{\theta}_t, \overline{\theta}_t.$$

[16]This defines a complete order because the conditional probability do not vary with η, η'.

A test t is single-peaked if and only if there are θ_1, θ_2 such that

$$\sum_{\eta \in A(\theta)} \lambda(\eta|\theta) - \sum_{\eta \in R(\theta)} \lambda(\eta|\theta) > 0 \textit{ if and only if } \theta_1 \leq_t \theta \leq_t \theta_2.$$

For example, if we interpret θ and η as technical and creative skills, the test is enclosed if the DM believes that the mass of good types conditional on being the worst technical type is higher than the mass of bad types. On the other hand, if we think that θ represents test taking ability and η is the true variable of interest, the test t is single-peaked if the function $\sum_{\eta \in A(\theta)} \lambda(\eta|\theta) - \sum_{\eta \in R(\theta)} \lambda(\eta|\theta)$ is single-crossing in θ, which would be equivalent to assuming a strong form of correlation between θ and η. Note that, assuming (θ, η) are in $\mathbb{R}^2$, it could be that they are positively correlated yet fail the single-peaked condition and the test could even be enclosed. This is because enclosedness is a local property at the highest and lowest θ type. Finally, other than statistical dependency, there could be additional considerations like putting different weights on type-I and -II error depending on the variable θ.