Optimal menu of tests

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Abstract

I study the optimal design of menus of tests. Prior to taking a binary decision, accept or reject a privately informed agent, a decision-maker (DM) can perform one test from a restricted set. For example, the restriction can come from information processing or technological constraints. The DM wants to accept a subset of types whereas the agent always wants to be accepted. Instead of choosing the test himself, the DM let the agent choose a test from a menu. The choice itself then serves as an additional dimension for information revelation. I characterise when a menu is optimal and show that the DM does not benefit from committing to an action. Using this result, I characterise the optimal menu when the DM has a most informative test. I give conditions on the DM's preferences under which the DM wants or does not want to include a less informative test in the menu. I also characterise the optimal menu when types are multidimensional or when tests vary in their difficulty.

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1 Introduction

In many economic settings, decision-makers (DMs) rely on tests to guide their action. Universities use standardised tests as part of their admission process, firms interview job candidates before they hire them and regulators test products prior to authorisation. In these examples, the DM is trying to learn some private information held by an agent: the ability of the student, the productivity of the candidate or the quality of a product. Ideally, the DM would want to set up a fully revealing test, but his testing capacity is usually constrained and he cannot perform any test he wants. In this case, he can offer a *menu* of tests and let the agent choose which test to pass. The DM can then use the agent's choice as an additional source of information. In this paper, I study how the DM can optimally design a menu of tests when his testing capacity is constrained.[1]

If the DM has access to a rich set of tests, the classic unravelling argument of Grossman (1981) and Milgrom (1981) shows that using a menu is a powerful tool and can even lead to full information revelation. However, the constraints faced by the DM depend a lot on the application and in general full information revelation cannot be obtained. For example, a hiring firm can be constrained by the amount of time and resources it can allocate to the selection process. Most universities have to use externally provided tests for their admission procedures like the SAT or the GRE. Medicine regulatory agencies face both technological and ethical constraints when authorising new drugs.

In this paper, I develop general tools to characterise the optimal menu of tests without imposing any restriction on the set of available tests. I then apply these tools to natural economic applications and determine which tests are part of the optimal menu and how it depends on

[1]There are many instances where menus of tests are used in practice. In clinical trials, pharmaceutical firms design the studies themselves to convince the regulator. Universities sometimes allow students to opt out from standardised tests in their admission process. Further from economics, music conservatory generally allow candidates to choose which piece they are going to play when they audition. I also show that in some cases the optimal menu is a singleton, rationalising the absence of a menu.

their properties and the DM's preferences. In particular, I characterise the optimal menu when tests vary in their informativeness, difficulty and in multidimensional environments.

I consider a decision-maker who has to make a binary decision, accept or reject. His payoffs depend on an agent's private type. While the DM wants to accept a subset of types (the A-types) and reject the others (the R-types), the agent always wants to be accepted. Prior to taking the decision, the DM can require the agent to take one test modelled as a Blackwell experiment. The test is chosen from an exogenous set that can capture various constraints such as information acquisition, institutional or technological constraints. A menu is a subset of the available tests. Based on the test choice and outcome, the DM accepts or rejects.

The first step in the analysis is to provide a characterisation of the optimal menu for arbitrary type structures and tests available. In Theorem 1, I show that the optimal menu and strategies are the outcome of an auxiliary zero-sum game. This result greatly simplifies the analysis. Rather than comparing equilibria from different menus to determine the optimal one, it is enough to find an equilibrium in one auxiliary game. In that game, A-types choose a test while R-types choose an A-type to mimic. The R-types still want to maximise their probability of being accepted but the A-types' payoffs are modified to align them with the DM's. They maximise their probability of being accepted while being penalised whenever an R-type that mimics them is accepted. I show that the tests chosen in that game correspond to the optimal menu. Moreover, I show that if the DM could commit, the optimal menu and strategies would be exactly the same as without commitment.

I can also use Theorem 1 to show that the number of tests in the optimal menu is bounded by the number of A-types. Therefore, if there is only one A-type, like in binary types models, there is always an optimal menu with only one test. This is true without making any assumptions on the set of available tests.

Using Theorem 1, I characterise which tests are part of the optimal menu in specific economic applications. In the first application, I consider a DM whose set of tests contains a dominant one, in the sense of Blackwell (1953)'s informativeness. I provide conditions under which a dominated test is part of the optimal menu. One example of this environment is universities considering whether to allow students to opt out from standardised tests like the SAT when applying. This is effectively offering a menu with the SAT and an uninformative test. In Lemma 1, I first show that the most informative test is always part of an optimal menu. I then characterise the optimal menu when tests are pass-fail tests. In this case, types can be ordered by how likely they are to send the pass signal. I show that the optimal menu is to only offer the most informative test for any prior if, and only if, the DM's payoff is single-peaked with respect to that order.[2] On the other hand, the optimal menu always includes a strictly less informative test if the DM's payoff is single-dipped.[3] Thus including a less informative test can be optimal when the most informative test does not test all relevant dimensions or only tests a proxy of the relevant dimension.

In the case where there are more than two signals, the results extend in the following way. If there exists a subset of signals where single-peakness is violated, there exists a less informative test that is part of the optimal menu for some prior. On the other hand, if the environment is one-dimensional, in the sense that all the tests satisfy the monotone likelihood ratio property and the DM wants to accept any type above a threshold, only the most informative test is offered.[4] Lemma 1 guarantees that the most informative test is part of the menu. I also show that whenever it is part of the menu, it is the unique test chosen in equilibrium. Thus incentive constraints impose that the optimal menu is a singleton menu.

I illustrate this point further in another one-dimensional environment where tests are ordered

[2]This corresponds to the DM only willing to accept either only high type, only low types or only intermediate types, as measured by their performance on the test.

[3]This corresponds to the DM wanting to exclude intermediate types, as measured by their performance on the test.

[4]This result is in terms of Blackwell informativeness (Blackwell, 1953) but also works with weaker notion of informativeness like Lehmann (1988)'s or some weakening of it.

by their difficulty. For example, the DM could be a regulator deciding how hard a compliance test is before authorising a product. The testing technology is a set of pass-fail tests and varying the difficulty of a test changes which types it identifies better. A more difficult test is informative when it is passed, as only high types are likely to produce a high grade but it is less informative when it is failed. In this case, I show that it is possible to sustain an equilibrium with more than one test. However, the DM's strategy needed to maintain that equilibrium is such that he is better off offering only one test. Thus the optimal menu is also a singleton.

The applications in one-dimensional environments show that offering a non-singleton menu is not always optimal, despite the additional information it can potentially deliver. I then turn to multidimensional environments. For example, a hiring firm could care about technical and managerial skills and specialise the interview on either dimension. More generally, I assume that the agent's type has two components and each test is informative about only one of them.[5] Offering tests for both dimensions allows A-types that perform badly in one dimension to select the test where they perform best. I show that the optimal menu contains both tests whenever the DM cares about each dimension separately. This would be the case if the hiring firm would be happy to hire a candidate with high technical skills but no managerial skills and vice-versa. On the other hand, if the firm cares about both dimension simultaneously, then for some priors, it uses only one test.

In Section 5, I move beyond specific applications and give a general condition on the DM's preferences and tests available that guarantees that a test is part of an optimal menu. I develop a criterion relaxing Blackwell's order that captures the idea that some tests are more adapted to some A-types. It describes how good a test is at differentiating types, restricting attention to one A-type and all the R-types. If a test is dominant in that sense, then it is included in the

[5] The results extend easily to more than two dimensions.

optimal menu. Intuitively, it shows that what matters for a test is its relative informativeness about one A-type versus all the R-types and not its "absolute" informativeness. I also show the necessary and sufficient condition on tests for the DM to never make a mistake.

Finally, in Section 6, I show that the model can be easily extended to allow for communication. I model communication as an additional cheap-talk message on top of the test choice. For example, it could be a cover letter where the candidate can freely communicate with the DM when applying for a job or to university. A characterisation as in Theorem 1 also holds. I show that in this case, each A-type announces his type while the R-types pretend to be some A-types. Furthermore, I show that when there is communication, it is irrelevant for the outcome of the game who chooses the test, the DM or the agent. These results generalise those of Glazer and Rubinstein (2004) and expand those of Carroll and Egorov (2019).

Relation to the literature

The idea that the a DM can benefit from the endogenous choice of information of a privately informed agent is the foundation of the literature on strategic disclosure (Grossman, 1981; Milgrom, 1981, 2008). The idea of using hard evidence was also taken to mechanism design (Green and Laffont, 1986; Bull and Watson, 2007; Deneckere and Severinov, 2008).[6] The main modelling difference between this paper and the literature using evidence is that I model the information provision through Blackwell experiments. When providing evidence, an agent is proving that he belongs to a subset of types. So in a sense evidence are deterministic tests. The second, and more important, difference is that in evidence games, the agent cannot show evidence he does not have. Thus, in my language, not all types can participate in all tests.

[6] See also Kartik and Tercieux (2012), Koessler and Perez-Richet (2019) and Strausz (2017).

An important focus of the literature of strategic disclosure is finding conditions under which all information is revealed in equilibrium, see e.g., Grossman (1981), Milgrom (1981), Giovannoni and Seidmann (2007) or Hagenbach et al. (2014). In my context, if full information is possible, it is optimal for the DM. In Proposition 9, I provide the necessary and sufficient conditions for full information revelation in this model but also characterise the optimal choice of test when full information is not attainable.

In mechanism design, several papers provide conditions for a revelation principle to hold, e.g., Green and Laffont (1986), Bull and Watson (2007), Deneckere and Severinov (2008) or Strausz (2017). Many applied papers assume one of these conditions to characterise the optimal mechanism, for example, Hart et al. (2017) or Ben-Porath et al. (2019). But then the answer to which evidence is used is built in the assumption that provides the revelation principle: each type presents the evidence associated with its type. Other papers do not assume anything about the evidence structure like Glazer and Rubinstein (2006) or Sher (2011) but they do not characterise which evidence is presented in equilibrium. In this paper, I characterise the optimal menu of tests for an arbitrary family of tests. I then use this general characterisation to understand which tests are part of the optimal menu and how it depends on their properties.

This paper is also related to the literature on communication with costless verification (e.g., Glazer and Rubinstein, 2004; Carroll and Egorov, 2019; Dziuda and Salas, 2018).[7] This project differs in at least two ways from these papers. First, they consider verification problems, that is the problem of finding the optimal test to use after an observable message from the agent. Glazer and Rubinstein (2004) and Carroll and Egorov (2019) consider the optimal verification mechanism for a fixed testing technology. In the baseline model, I do not allow for communication and the agent chooses the test, not the DM. Thus the only endogenous

[7]There is also a literature on costly verification, e.g., Townsend (1979), Ben-Porath et al. (2014), Erlanson and Kleiner (2020), Halac and Yared (2020) or Li (2020).

information that can be transmitted is through the choice of test.

The second main difference is that I consider an arbitrary type structure and testing technology. On the other hand, Glazer and Rubinstein (2004) and Carroll and Egorov (2019) consider the verification of one dimension of a multidimensional type. Dziuda and Salas (2018) characterise the structure of communication when lies can be detected with some probability but does not consider any design element.

Some papers cited above find that commitment does not play any role. Theorem 2 extends the result of Glazer and Rubinstein (2004) to a more general environment allowing for arbitrary type structure and testing technology.

Finally, some papers consider the optimal design of tests. Harbaugh and Rasmusen (2018) and DeMarzo et al. (2019) consider the design of tests when a privately informed agent produces the information and has the option of not disclosing anything. Zik and Weksler (2022) study the optimal choice of test prior to a signalling game. They show that the receiver might prefer a less informative test because of strategic consideration. Motivated by the COVID-19 pandemic, Ely et al. (2021) study the optimal allocation of tests from a restricted set to agents with observable characteristics. Deb and Stewart (2018) study the dynamic choice of test in the presence of asymmetric information and moral hazard, i.e., where both effort and the agent's type affect the outcome of the test. They derive conditions for using the most informative test available.

2 Model

There is a DM and an agent. The agent has a type $\theta \in \Theta$, $|\Theta| < \infty$, with a common prior $\mu(\theta)$. The set of types is partitioned in two: $\Theta = A \cup R$, $A \cap R = \emptyset$. The type is private information of the agent. The DM must take an action $a \in \{0, 1\}$, accept or reject. The utilities of the DM and the agent are $v(a, \theta) = a\big(\mathbb{1}[\theta \in A] - \mathbb{1}[\theta \in R]\big)$ and $u(a, \theta) = a$. That is the DM wants to accept agents in A and reject agents in R. The agent always wants to be accepted. The analysis is virtually unchanged by allowing for DM's utility functions of the form $v(a, \theta) = a\nu(\theta)$ for some $\nu : \Theta \to \mathbb{R}$.

There is an exogenous set of test $T \subseteq \Pi \equiv \{\pi : \Theta \to \Delta X\}$, where X is some finite signal space. I assume that $|T| < \infty$. The conditional probabilities of test t are $\pi_t(\cdot|\theta)$. The exogenous set of tests can capture different constraints on the set of tests. For example, it could come from a capacity constraints in the information processing/acquisition abilities of the DM, $T \subset \{\pi : c \geq C(\pi)\}$ for some cost function C. The constraint can also be on some properties of the tests that can be used $T \subset \{\pi : \pi \text{ has the MLRP}\}$. Finally, it could be a purely technological constraint, e.g., when choosing amongst standardised test, there is only a limited set of existing tests $T = \{\text{SAT}, \emptyset\}$.

The DM chooses to post a menu of tests, that is a subset $M \subseteq T$. A strategy for the agent is a choice of test, $\sigma : \Theta \to \Delta M$. A strategy for the DM is a probability of accepting after observing the test chosen and the signal realisation, $\alpha : M \times X \to [0, 1]$. Beliefs of the DM are $\tilde{\mu} : M \times X \to \Delta\Theta$, a probability distribution over types given an observed test and signal realisation.

The solution concept is DM-preferred Perfect Bayesian Equilibrium.

I write $(\alpha, \sigma) \in \text{wPBE}(M)$ if there is a belief $\tilde{\mu}$ where $(\alpha, \sigma, \tilde{\mu})$ is a PBE when the menu is M.

The optimal design of menu solves

$$V = \max_{M \subseteq T} \max_{\sigma, \alpha} \sum_{\theta \in A} \mu(\theta) \sum_{t \in M} \sigma(t|\theta) \sum_{x} \alpha(t,x)\pi_t(x|\theta) - \sum_{\theta \in R} \mu(\theta) \sum_{t \in T'} \sigma(t|\theta) \sum_{x} \alpha(t,x)\pi_t(x|\theta)$$

$$\text{s.t. } (\alpha, \sigma) \in \text{wPBE}(M)$$

Notation: For any α, denote the probability of type θ to be accepted in test t by $\pi_t(\alpha|\theta) \equiv \sum_x \alpha(t,x)\pi_t(x|\theta)$.

2.1 Example: Opting out from SAT

Suppose a university uses some standardised test for university admission and that there are three types of students: $A = \{A1, A2\}$ and $R = \{R1\}$. Consider the testing set $T = \{t, \emptyset\}$ where $\emptyset$ is an uninformative test. The test t is described by $X = \{x_0, x_1\}$ and

$$\pi_t(x|A1) = \begin{cases} 1/2 & \text{if } x = x_0 \\ 1/2 & \text{if } x = x_1 \end{cases} \qquad \pi_t(x|R1) = \begin{cases} 1/3 & \text{if } x = x_0 \\ 2/3 & \text{if } x = x_1 \end{cases}$$

$$\pi_t(x|A2) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \end{cases}$$

Furthermore, suppose that $\mu(A1) < \frac{2}{3}\mu(R1) < \mu(A2)$.

This example can be interpreted in the following way. The test t is a standardised test the university uses to get information about students, like the SAT or ACT. However, the test is not very good at identifying some type of good student, $A1$. Indeed, this type has "lower grades" than $R1$ and $A2$. One reason could be that $A1$ did not learn how to do well on the test or that it does not test a dimension he is good on. Another reason could be that $R1$ got some special training that allows him to get good grades on the test despite not being "intrinsically" a good student. Offering $\emptyset$ allows students to opt out from the standardised test.

When only t is offered: The information structure and prior delivers the following best response when only t is offered,

$$\alpha(x,t) = \begin{cases} 0 & \text{if } x = x_0 \\ 1 & \text{if } x = x_1 \end{cases}$$

The acceptance probabilities of each types is then

$$\pi_t(\alpha|R1) = 2/3 \qquad \pi_t(\alpha|A1) = 1/2 \qquad \pi_t(\alpha|A2) = 1$$

When both t and $\emptyset$ are offered: Consider the equilibrium with the following strategies of the agent:

$$\sigma(\emptyset|R1) = \frac{\mu(A1)}{\mu(R1)} \qquad \sigma(\emptyset|A1) = 1 \qquad \sigma(t|A2) = 1$$

The student $R1$ mixes between the two tests, t and $\emptyset$, whereas $A1$ chooses $\emptyset$ with probability one and $A2$ chooses test t with probability one. Note that if the two A-types separate, then to maintain an equilibrium, $R1$ must mix. Otherwise, there is a test that is only chosen by an A-type and in equilibrium the DM must accept with probability one after any signal in that test.

Given the agent's strategies, the DM's strategy after t remains the same as before. When the DM observes $\emptyset$, he is indifferent between accepting and rejecting. He then mixes in a way to make $R1$ indifferent between $\emptyset$ and t: $\alpha(x, \emptyset) = 2/3$. The resulting acceptance probabilities are

$$\mathbb{E}[\pi(\alpha|R1)] = 2/3 \qquad\qquad \pi_\emptyset(\alpha|A1) = 2/3 \qquad\qquad \pi_t(\alpha|A2) = 1$$

Types $R1$ and $A2$ have the same acceptance probabilities as before but $A1$ is accepted with strictly higher probability. Therefore allowing to opt out strictly increases the DM's payoffs.

3 Characterisation of the optimal menu

In this section, I show that the value of the optimal menu is characterised by an equilibrium of a zero-sum game. I provide a sketch of the proof of Theorem 1 in Section 3.1.

Let $s : A \to \Delta T$ and $s' : R \to \Delta A$ and abusing notation, let $\alpha : T \times X \to [0,1]$ and

$$v(\alpha, s, s') \equiv \sum_{\theta \in A} \sum_t s(t|\theta) \Big[\mu(\theta)\pi_t(\alpha|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(\alpha|\theta') \Big]. \tag{1}$$

The function s can be interpreted as A-types choosing a test, s' as R-types choosing an A-type to mimic, α as the DM accepting the agent after a test and signal realisation. The function v is then the DM's expected payoffs from a distribution over tests induced by the pair (s, s'). I explain these objects in more detail in the discussion of Theorem 1.

Theorem 1. *The value of an optimal menu is*

$$V = \max_{\alpha,s} \min_{s'} v(\alpha, s, s') = \min_{s'} \max_{\alpha,s} v(\alpha, s, s')$$

A saddle point $\big((\alpha, s), s'\big)$ *of* v *such that* $s(\cdot|\theta)$ *is in pure strategies for all* $\theta \in A$ *exists and characterises an optimal menu and strategies:*

- *for* $\theta \in A : \sigma(t|\theta) = s(t|\theta)$

- *for* $\theta' \in R : \sigma(t|\theta') = \sum_{\theta \in A} s'(\theta|\theta') s(t|\theta)$

- *the DM's strategy is* α*.*

Moreover, the DM does not benefit from committing over α*.*

Proof. See appendix. □

Theorem 1 provides a characterisation of the optimal menu in terms of an auxiliary zero-sum game. The fact that an optimal menu is an equilibrium of *a* game gives us a powerful tool to test equilibria. Indeed, it is not necessary to compare equilibria accross menus to establish that a menu is not optimal. It is enough to find that $(\tilde{\alpha}, \tilde{s})$ such that

$$\min_{s'} v(\alpha, s, s') < \min_{s'} v(\tilde{\alpha}, \tilde{s}, s')$$

to show that (α, s, s') does not constitute an optimal menu without having to care whether $(\tilde{\alpha}, \tilde{s})$ are optimal.

To understand the structure of this game better, consider the zero-sum game presented at the interim stage for a fixed α. In that game, the payoffs of the A- and R-types for a strategy (s, s') are

$$\begin{aligned}
&\text{for } \theta \in A \text{ choosing } t,\ \mu(\theta)\pi_t(\alpha|\theta) - \sum_{\theta' \in R} \mu(\theta') s'(\theta|\theta') \pi_t(\alpha|\theta') \\
&\text{for } \theta' \in R \text{ choosing } \theta,\ \sum_t s(t|\theta) \pi_t(\alpha|\theta')
\end{aligned}$$

In the zero-sum game, each A-type is choosing a test and each R-type is choosing an A-type to mimic. The R-types are maximising their probability of being accepted like in the original problem. The A-type maximise a modified version of their utility where they maximise their probability of being accepted while being penalised every time a R-type mimics them and is accepted. The A-types' utility is thus modified to align it with the DM's payoffs.

The strategies of the zero-sum game induce a distribution over tests for each type. The A-types get the distribution over test they choose and the R-types the distribution of the A-types they choose to mimic. Theorem 1 shows that this distributions are actually the equilibrium strategies of the optimal menu game. Moreover, the A-types play a pure strategy.

To understand why choosing test t for type $\theta \in A$ in the zero-sum game delivers the right equilibrium behaviour in the original game, consider the following interpretation of the game. The payoffs of a type $\theta \in A$ can be understood as a gross payoff

$$\mu(\theta)\pi_t(\alpha|\theta),$$

corresponding to the payoffs in the original game and a net payoff

$$\mu(\theta)\pi_t(\alpha|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(\alpha|\theta').$$

The equilibrium behaviour of R-types means that the test they choose in equilibrium carries the largest negative term because they would choose a type $\theta \in A$ only if it maximises their probability of being accepted. That is, assuming a pure strategy from the A-types, if $s'(\theta|\theta') > 0$, then $\pi_t(\alpha|\theta') \geq \pi_{t'}(\alpha|\theta')$ for any t' chosen by some other A-type. Let's consider a deviation of that A-type θ to test t' when the equilibrium is (s, s'). Equilibrium behaviour gives us

$$\mu(\theta)\pi_t(\alpha|\theta) - \sum_{\theta'\in R}\mu(\theta')s'(\theta|\theta')\pi_t(\alpha|\theta') \geq \mu(\theta)\pi_{t'}(\alpha|\theta) - \sum_{\theta'\in R}\mu(\theta')s'(\theta|\theta')\pi_{t'}(\alpha|\theta')$$

$$\Rightarrow \mu(\theta)\big(\pi_t(\alpha|\theta) - \pi_{t'}(\alpha|\theta)\big) \geq \sum_{\theta'\in R}\mu(\theta')s'(\theta|\theta')\big(\pi_t(\alpha|\theta') - \pi_{t'}(\alpha|\theta')\big) \geq 0$$

where the last inequality comes from the equilibrium behaviour of the R-types. Thus the A-types choose the test that maximise their probability of being accepted. Intuitively, the test chosen in equilibrium is "the most expensive" amongst all the tests. This means that the gross payoffs from it must be the largest.

Theorem 1 also shows that commitment has no value. I interpret this result as a hierarchy over sources of learning. The DM has two sources of information, the "hard information" from the test results and the endogenously created information from the choice of test. When the DM can commit to a strategy, he can "sacrifice" payoffs from the test result by not best replying, in order to create separation of types through the test choice. By showing that the DM always best replies, even when he can commit, I show that he should always prioritise the hard information over creating endogenous information through the test choice.

That commitment has no value in this game comes from the zero-sum structure of the characterisation. Because a minimax theorem holds,[8] this implies that the order of moves do not matter in this game: the DM has the same payoffs if he moves first or last.

Finally, Theorem 1 gives an upper bound on the number of tests needed in an optimal menu. If A-types are playing a pure strategy and R-types only use tests A-types use, then the number of tests used is at most $|A|$.

Corollary 1. *The number of tests used in the optimal menu is at most* $|A|$.

[8]Note that classic minimax theorems like Von Neumann's or Sion's do not hold here. Instead, I rely on an equilibrium existence result in non-quasiconcave games (Baye et al., 1993) to show that the max-min equality holds.

An immediate corollary is also that if there is only one type the DM would like to accept an optimal menu is to use only one test. In particular, this results shows that in a binary state environment, the optimal mechanism uses only one test, no matter what the available set of test is.

Corollary 2. *Suppose $|A| = 1$. Then for any T, there is an optimal menu that uses only one test.*

3.1 Sketch of proof Theorem 1

To prove Theorem 1, I will first need to introduce *mechanisms*. A (direct) mechanism is a mapping $\tilde{\sigma} : \Theta \to \Delta T$, a function from types to distribution over tests. Suppose there is a designer that could design $\tilde{\sigma}$ to maximise the DM payoffs. The DM only observes the realised test and signal, thus the definition of his strategy is unchanged. The agent's strategy is now to report a type into the mechanism. The solution concept is still DM-preferred PBE. Standard arguments show that without loss of generality we can restrict attention to direct truthful mechanism. The designer's problem is

$$\begin{aligned}
\tilde{V} = \max_{\tilde{\sigma},\alpha} & \sum_{\theta \in A} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(\alpha|\theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(\alpha|\theta) \\
\text{s.t. } & \sum_t (\tilde{\sigma}(t|\theta) - \tilde{\sigma}(t|\theta'))\pi_t(\alpha|\theta) \geq 0 \text{ for all } \theta, \theta' \\
& \sum_t \tilde{\sigma}(t|\theta) = 1 \text{ for all } \theta \\
& \alpha \in BR(\tilde{\sigma})
\end{aligned}$$

where the first constraint is the agent's incentive compatibility constraint, the second is a feasibility constraint and $\alpha \in BR(\tilde{\sigma})$ means that the strategy α is a best-response to some beliefs consistent with the mechanism. We have $\tilde{V} \geq V$, i.e., the value of the optimal mechanism is larger than the value of the optimal menu, as imposing a menu is simply restricting the class of mechanism the designer could use.

The first part of the proof shows that $\tilde{V} = \max_{\alpha,s} \min_{s'} v(\alpha, s, s')$. The second part shows that the optimal mechanism can be implemented by posting a menu of tests.

To show that $\tilde{V} = \max_{\alpha,s} \min_{s'} v(\alpha, s, s')$, I characterise the optimal mechanism when the DM commits to α. The designer's problem becomes

$$\begin{aligned}\tilde{V}(\alpha) = \max_{\tilde{\sigma}} \sum_{\theta \in A} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(\alpha|\theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(\alpha|\theta) \\ \text{s.t. } \sum_t (\tilde{\sigma}(t|\theta) - \tilde{\sigma}(t|\theta'))\pi_t(\alpha|\theta) \geq 0 \text{ for all } \theta, \theta' \\ \sum_t \tilde{\sigma}(t|\theta) = 1 \text{ for all } \theta\end{aligned}$$

The notation $\tilde{V}(\alpha)$ indicates the designer's problem when the DM has committed to the strategy α.

This is a linear program and verifying complementary slackness conditions shows that $\tilde{V}(\alpha) = \max_s \min_{s'} v(\alpha, s, s')$. As in the statement of the theorem, the pair (s, s') characterises an optimal mechanism $\tilde{\sigma}$ by setting $\tilde{\sigma}(t|\theta) = s(t|\theta)$ for $\theta \in A$ and $\tilde{\sigma}(t|\theta') = \sum_{\theta \in A} s(t|\theta)s'(\theta|\theta')$.

The value of the DM if he could commit to α is $\max_\alpha \tilde{V}(\alpha)$. Now notice that

$$\max_\alpha \tilde{V}(\alpha) = \max_\alpha \max_s \min_{s'} v(\alpha, s, s') = \max_{\alpha,s} \min_{s'} v(\alpha, s, s').$$

Using a result from Baye et al. (1993) on the existence of Nash equilibrium in non-quasiconcave games, $\max_\alpha \tilde{V}(\alpha)$ is attained by (α^*, s^*, s'^*) such that

$$v(\alpha, s, s'^*) \leq v(\alpha^*, s^*, s'^*) \leq v(\alpha^*, s^*, s') \text{ for all } \alpha, s, s'$$

This in turn implies that α^* is a best-response to the mechanism implied by (s^*, s'^*) as $v(\alpha^*, s^*, s'^*) \geq v(\alpha, s^*, s'^*)$ for all α. Therefore the best-response constraint of the original problem would be satisfied if we would not impose it. This proves that the DM would not benefit from committing to α if he could offer a mechanism.

The second part shows that the optimal mechanism can be implemented by posting a menu of tests. The way the proof proceeds is by showing that there is $(\alpha^*, s^*) \in \arg\max \min_{s'} v(\alpha, s, s')$, where s^* is a pure strategy for all $\theta \in A$. If this is the case, then we can take the menu of tests as the support of tests in the optimal mechanism. Each type $\theta \in A$ is better off choosing "his" test as choosing another one would violate the incentive compatibility constraints. Types $\theta' \in R$ possibly have a randomised allocation but they are indifferent between any tests they are allocated to. Indeed, their randomised allocation corresponds to a mixed strategy in the auxiliary game where they are maximising their probability of being accepted, just like in the menu-game.

To understand why s^* must be a pure strategy, note that given s'^*, the DM and types in $\theta \in A$ must choose α and s to maximise $v(\alpha, s, s'^*)$. If the A-types are willing to mix, they must be indifferent between all the tests in the support for a fixed α^*. This α^* is itself a best-response to (s^*, s'^*). Choosing a pure strategy in the support of s^* allows then the DM to re-optimise over α and get a higher payoff for both the DM and the A-types.

4 Applications

4.1 Optimal menu with Blackwell dominant test

It is common in applications that the DM has access to a most informative test. This can be because the choice is simply between a test and opting out of the test like in the SAT example. It can also come from the structure of the constraints. For example, the DM could have a time budget to conduct an interview. The more time the interview takes, the more informative it is. It can also be that the DM can combine multiple tests together, making it more informative. Suppose that only two tests are technologically feasible to test the agent. Taking both tests is more informative to take either test individually. Another possibility is that the DM can easily make a test less informative by simply not conducting part of the test. If a test is composed of a series of questions, the DM can ignore some of them.

The main notion of informativeness in economic theory is the one of Blackwell (1953).

Definition 1 (Blackwell (1953))**.** *A test t is more informative than t', $t \succeq t'$, if there is function $\beta : X \times X \to [0,1]$ such that for all $x' \in X$, $\sum_x \beta(x,x')\pi_t(x|\theta) = \pi_{t'}(x'|\theta)$ for all $\theta \in \Theta$ and for all $x \in X$, $\sum_{x'} \beta(x,x') = 1$.*

I call a test t a dominant test if $t \succeq t'$ for all $t' \in T$. If a test is more informative than another then in any decision problem, i.e., a pair of utility function and a prior, using the more informative test yields higher expected utility. A first important fact we will record here is that if there is a most informative test, then it is part of an optimal menu.

Lemma 1. *If there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$, then there is an optimal menu that includes t.*

Proof. See appendix. □

This lemma follows from the zero-sum characterisation of Theorem 1 and the properties of dominant test. Indeed, if we find a menu where the dominant test t is not included, we can modify the DM's strategy such that one A-type is accepted with higher probability than the test he is choosing, say t', and all R-types are accepted with lower probability than in t'. Then this A-type has a profitable deviation to t.

As we have seen in the SAT example in Section 2.1, it can be optimal to add a strictly less informative in the optimal menu. I first focus on binary signals environment, $X = \{x_0, x_1\}$. Let t be the most informative test in T. When signals are binary, we can order the types by their likelihood of sending signal x_1: $\theta \geq_t \theta' \Leftrightarrow \pi_t(x_1|\theta) \geq \pi_t(x_1|\theta')$.[9] I characterise the optimal menu for different payoff function of the DM.

Definition 2. *The DM's preferences are single-peaked given the order $\geq$ on Θ if there is $\theta_1, \theta_2 \in A$ such that $A = \{\theta : \theta_1 \leq \theta \leq \theta_2\}$.*

Preferences are single-peaked if the DM only wants to either only accept high types, only low types or only intermediate types, where the order is determined by the performance of types on the test. Preferences are not single-peaked whenever it is possible to find $A_1, A_2 \in A$ and $R_1 \in R$ such that $A_1 <_t R_1 <_t A_2$. This was for example the case in the SAT example in Section 2.1.

We get the following characterisation.

Proposition 1. *Let $X = \{x_0, x_1\}$. Suppose there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$ and let $\geq_t$ on Θ be the order implied by t.*

The singleton menu $\{t\}$ is optimal for any $\mu \Leftrightarrow$ the DM's preferences are single-peaked given $\geq_t$.

[9] Note that given that tests are binary, this is equivalent to ordering type by the likelihood ratio, $\frac{\pi(x_1|\theta)}{\pi(x_0|\theta)}$.

From Lemma 1, the most informative test is part of the optimal menu. Whenever the DM's preferences are single-peaked, if the most informative test is included in the menu, the unique resulting equilibrium is one where all types choose the most informative test. The key argument in the analysis is noting that $\pi_t(\alpha|\theta) - \pi_{t'}(\alpha|\theta)$ is single-crossing in θ with respect to the order $\geq_t$, for any α. When preferences are single-peaked, we can use the single-crossing condition and properties of tests satisfying the monotone likelihood ratio property to show that there is a unique equilibrium where only t is chosen.

On the other hand, if the preferences are not single-peaked, there is a prior where offering even a completely uninformative test with the most informative test is strictly better for the DM. To illustrate, consider three types $A_1, A_2 \in A$ and $R_1 \in R$ such that $A_1 <_t R_1 <_t A_2$. Suppose the prior is such that if only t is offered, the DM accepts after x_1 and rejects after x_0. Using the likelihood ratio order, the DM can offer an uninformative test where the probability of being accepted makes R_1 indifferent but is strictly preferred by A_1. This constitutes a deviation in the zero-sum game. This reasoning can be used to show that including a less informative test is always beneficial whenever the DM's payoff is *single-dipped*: there is $\theta_1, \theta_2 \in A$ such that $R = \{\theta : \theta_1 <_t \theta <_t \theta_2\}$.

Proposition 2. *Let $X = \{x_0, x_1\}$. Suppose there is $t \in T$ such that $t \succeq t'$ for all $t' \in T$ and let $\geq_t$ on Θ be the order implied by t.*

If the DM's preferences are single-dipped given $\geq_t$, then, for any $t' \in T$, the DM's payoffs are higher in the menu $\{t, t'\}$ than in $\{t\}$ for any μ.

The ideas of Proposition 1 and Proposition 2 can be partially extended to more than two signals. First, if all tests satisfy the monotone likelihood ratio property and the DM only wants to accept types above a threshold, the optimal menu is to only offer the most informative test.

Proposition 3. *Suppose $\Theta, X \subset \mathbb{R}$, $A = \{\theta : \theta > \overline{\theta}\}$ for some $\overline{\theta}$ and all tests in T have* *full-support and the monotone likelihood ratio property: for $\theta > \theta'$,*

$$\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \text{ is increasing in } x.$$

If there is $t \succeq t'$ for all $t' \in T$, then, the menu $\{t\}$ is optimal.

Again this result holds by showing a single-crossing difference property on the acceptance probability. Intuitively, the reason is that more informative tests send relatively higher signals for higher types. So if a low type chooses the most informative test, the higher types must also choose that one. This prevents any pooling of A-types and R-types on two different tests. Combined with Lemma 1 that guarantees the inclusion of the dominant test, we get our result. Note also that this result would hold using weaker information order like Lehmann (1988) or some weakening of it. The key property delivering the result is the single-crossing condition described above.

If it is possible to find two signals, x, x', two A-types A_1, A_2 and one R-type, R_1 such that $\frac{\pi_t(x|A_1)}{\pi_t(x'|A_1)} < \frac{\pi_t(x|R_1)}{\pi_t(x'|R_1)} < \frac{\pi_t(x|A_2)}{\pi_t(x'|A_2)}$, then there is a test t' strictly less informative than t and a prior such that offering $\{t, t'\}$ is better for the DM than just offering $\{t\}$.

Proposition 4. *Let t be a test. Suppose there are two signals $x, x' \in X$, types $A_1, A_2 \in A$ and $R_1 \in R$ such that*

$$\frac{\pi_t(x|A_1)}{\pi_t(x'|A_1)} < \frac{\pi_t(x|R_1)}{\pi_t(x'|R_1)} < \frac{\pi_t(x|A_2)}{\pi_t(x'|A_2)}.$$

There is a prior μ and a test $t' \prec t$ such that the DM's payoffs are higher in the menu $\{t, t'\}$ than in $\{t\}$.

Intuitively, if we interpret x as a high signal, the A-type A_1 sends relatively low signals. Suppose that the prior is such that, if only t is offered, x is accepted and x' is not. In a sense, it means that in the test t, type R_1 performing better than A_1 on the signals x, x'. It is then beneficial for the DM to include a test that pools signals x, x' together. In that new test, type A_1 can choose the coarsened test where the superior performance of type R_1 is less important than in the original test.

The proof of Proposition 4 actually uses the following criterion to determine whether a less informative is part of the optimal menu. It gives condition to include coarsened version of a test.

Definition 3. *A test t is a coarsening of test t' if there is a partition of X, $\{X_i\}$, such that for all $\theta \in \Theta$,*

$$\pi_t(x_i|\theta) = \sum_{x \in X_i} \pi_{t'}(x|\theta) \quad \textit{for some } x_i \in X_i$$
$$\pi_t(x'|\theta) = 0 \quad \textit{for all } x' \in X_i,\ x' \neq x_i$$

The idea of a coarsening is that it pools all the signal in one element of the partition X_i on one signal x_i. The test t' is more informative than t as any strategy under t can be implemented under t'. I say that a test pools signals in X' if the partition is $\{X', \{x\} : x \notin X'\}$. Let $z^+ = \max\{0, z\}$.

Proposition 5. *Let $\alpha(x,t)$ be the optimal strategy when only test t is used. If there is $\tilde{\alpha} \in [0,1]$ and $X' \subseteq X$ such that*

$$\sum_{\theta \in A} \sum_{x \in X'} \mu(\theta)\big[(\tilde{\alpha} - \alpha(x,t))\pi_t(x|\theta)\big]^+ \geq \sum_{\theta' \in R} \sum_{x \in X'} \mu(\theta')\big[(\tilde{\alpha} - \alpha(x,t))\pi_t(x|\theta')\big]^+$$

then it is optimal to include a coarsened version of t that pools signals in X'.

Proof. See appendix. □

This result is a direct application of the zero-sum game of Theorem 1. It considers using the same strategy as in test t for the coarsened test but for the coarsened signal in X' where it uses $\tilde{\alpha}$. The condition then boils down to checking for a profitable deviation. The intuition for Proposition 5 is the same as in Proposition 4. The set X' identifies a set of signals where some A-types are performing worse than R-types. Offering a test that coarsens signals in X' creates a profitable deviation for these A-types.

4.2 Optimal menu with tests ordered by their difficulty

In many economic environments, the DM does not necessarily have access to a most informative test but can vary the difficulty to pass a test. This is for example the case for a regulator that can decide how demanding a certification test. Like in Proposition 1 and Proposition 3, I show that the optimal menu is a singleton.

I first formalise the notion of more difficult test as follows.

Definition 4 (Difficulty environment)**.** *An environment is a Difficulty environment if* $\Theta \in \mathbb{R}$, $A = \{\theta : \theta > \overline{\theta}\}$ *for some* $\overline{\theta}$, $X = \{x_0, x_1\}$, $T \subset \mathbb{R}$, *all tests have full-support, satisfy the monotone likelihood ratio property and for all* $t > t'$, *and* $\theta > \theta'$,

$$\frac{\pi_t(x_1|\theta)}{\pi_t(x_1|\theta')} \geq \frac{\pi_{t'}(x_1|\theta)}{\pi_{t'}(x_1|\theta')} \quad \textit{and} \quad \frac{\pi_t(x_0|\theta)}{\pi_t(x_0|\theta')} \geq \frac{\pi_{t'}(x_0|\theta)}{\pi_{t'}(x_0|\theta')}$$

If $t > t'$, I will say that t is harder than t'. To understand the last condition better, let $\mu(\cdot|x,t)$ be a posterior belief after observing signal x in test t. The monotone likelihood ratio property implies $\mu(\cdot|t, x_1) \succeq_{FOSD} \mu(\cdot|t, x_0)$, a higher signal is "good news" about the type (Milgrom, 1981). The last property in the definition further implies $\mu(\cdot|t, x) \succeq_{FOSD} \mu(\cdot|t', x)$. That means that a pass grade shifts beliefs more towards higher type in a harder test and a fail grade shifts more beliefs towards lower types in an easy test. Or put differently, the harder a test the more informative it is about a high type when there is a pass-grade whereas an easier test is informative about the low types when the test is failed. As an example, if $\Theta \subset (0, 1)$ and $\pi_t(x_1|\theta) = \theta^t$ we are in a Difficulty environment.

Proposition 6. *In a Difficulty environment, a singleton menu is optimal.*

Proof. See appendix. □

Like Proposition 1 and Proposition 3, Proposition 6 illustrates how incentive constraints shape the size of the optimal menu. In the case of the single-peaked preferences with dominant test, the equilibrium when the most informative test is offered is unique and only that test is chosen. Here, it is possible to construct an equilibrium where more than one test is chosen in equilibrium. However, the DM strategy needed to sustain that equilibrium is such that he is better off offering only one test.

The proof proceeds in two steps. First, I show that there are at most two tests in the optimal menu and if there are two tests, the harder test must be more lenient that the easy test. In particular, I show that after the hard test, the DM must accept with some probability after a fail signal and in the easy test, reject with positive probability after a pass grade.

This means that to maintain incentives to select both tests, the DM only reacts to the least informative signal from the test: in the hard test after a fail grade, in the easy test after a pass grade. This in turn implies that it would be better for the DM to use only one test and reject after a fail grade and accept after a pass grade.

4.3 Bidimensional environment

In this subsection, I apply the tools of Theorem 1 to study environments with bidimensional types. The analysis here can be easily extended to more than two dimensions. I assume that the DM has access to tests that can only reveal one dimension and the preference of the DM have some monotonicity along each dimension.

Definition 5. *An environment is bidimensional if* $\Theta = \Theta_1 \times \Theta_2 \subset \mathbb{R}^2$, $X \subset \mathbb{R}$ *and* $T = \{t_1, t_2\}$ *such that for* $i = 1, 2$,

- *for all* $\theta_j \in \Theta_j$, *there is* $\overline{\theta}_i$ *such that* $(\theta_i, \theta_j) \in A$ *for all* $\theta_i \geq \overline{\theta}_i$
- t_i *has full support and for all* $\theta_i > \theta_i'$,

$$\frac{\pi_{t_i}(x|\theta_i, \theta_j)}{\pi_{t_i}(x|\theta_i', \theta_j)} \text{ is strictly increasing in } x \text{ for any } \theta_j \in \Theta_j$$

- $\pi_{t_i}(x|\theta_i, \theta_j) = \pi_{t_i}(x|\theta_i, \theta_j')$ *for all* $\theta_j, \theta_j' \in \Theta_j$ *and* $x \in X$

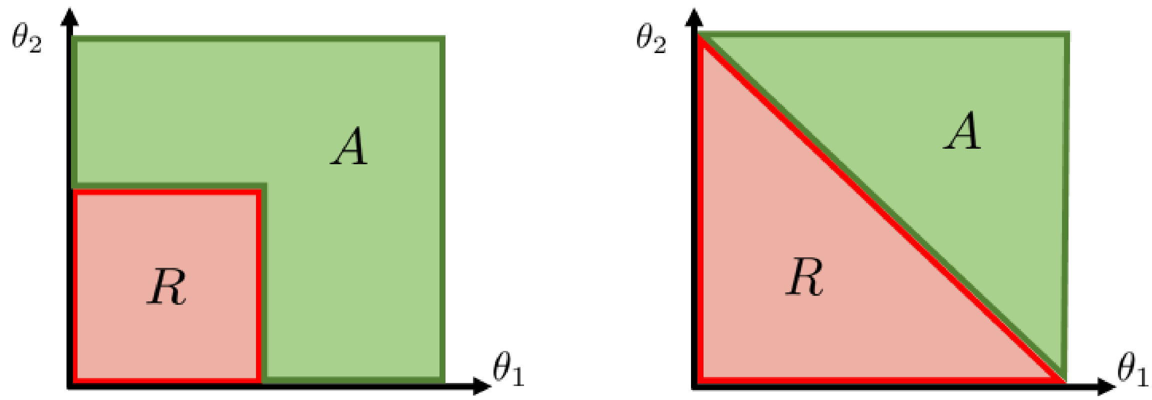
The first condition captures the idea that a higher type is always better for the DM. The second and third condition captures the idea that each test is only informative about one dimension and that a higher signal corresponds to a higher type in that dimension.

In this environment, whether the DM wants to offer a menu depends crucially on his preferences. In particular, I give a necessary and sufficient condition on the preferences such that a menu is optimal for any prior. Let $\theta_i^+ = \max \Theta_i$.

Proposition 7. *Suppose we are in a bidimensional environment. Offering a menu* $\{t_1, t_2\}$ *is strictly optimal for any prior if and only if*

$$\text{for } i = 1, 2,\ (\theta_i^+, \theta_j) \notin R, \text{ for all } \theta_j \in \Theta_j. \tag{2}$$

The proof of Proposition 7 works by showing that a deviation from a single test menu is always profitable when condition (2) is satisfied and constructs a prior under which there are no profitable deviations when the condition is not satisfied.



(a) $A = \{\theta : \max\{\theta_1, \theta_2\} \geq \bar{\theta}\}$, full menu always optimal.

(b) $A = \{\theta : \theta_1 + \theta_2 > 1\}$, full menu not always optimal.

Figure 1: Illustration of DM's preferences for Proposition 7.

Figure 1 illustrates the condition of Proposition 7 with $\Theta \subset [0,1]^2$. In Figure 1a, the DM wants the agent's type to be high enough in at least one dimension, $A = \{\theta : \max\{\theta_1, \theta_2\} \geq \bar{\theta}\}$. He thus only cares about each dimension individually. Then the DM always prefers to offer a full menu to the agent. On the other hand, in Figure 1b, the DM wants the agent's type to be high in both dimensions, $A = \{\theta : \theta_1 + \theta_2 > 1\}$. In particular, "being good" in only one dimension is not enough to be accepted. In this case, for some prior, the DM only wants to offer one test. This happens when after any deviation from the singleton menu any A-type is mimicked by too many R-types that cannot be distinguished from him.

5 Sufficient conditions for test inclusion

In this section, I study in more details the notion of efficient allocation of tests to the agent's types. I show that a sufficient condition to include a test in the optimal menu is if it is good at differentiating one A-type from all the R-types. This captures a notion of a test tailored for the A-type.

Definition 6. *Fix $\theta \in A$. Test t θ-dominates t', $t \succeq_\theta t'$, if there is $\beta : X \times X \to [0,1]$ such* *that for all* $x' \in X$

$$\sum_x \beta(x,x')\pi_t(x|\theta) \leq \pi_{t'}(x'|\theta)$$

$$\textit{for all } \theta' \in R, \quad \sum_x \beta(x,x')\pi_t(x|\theta') \geq \pi_{t'}(x'|\theta')$$

$$\textit{for all } x \in X, \quad \sum_{x'} \beta(x,x') \leq 1$$

To understand this definition better, compare it to Blackwell (1953)'s informativeness order. It requires the existence of a function β such that for all $x' \in X$, $\sum_x \beta(x,x')\pi_t(x|\theta) = \pi_{t'}(x'|\theta)$ for all $\theta \in \Theta$ and for all $x \in X$, $\sum_{x'} \beta(x,x') = 1$. The key difference is that we restrict attention to one A-type and all the R-types. This captures the idea the test θ-dominant test is tailored to differentiate θ from each R-type. The second difference is that it requires only inequalities whereas the Blackwell order requires equalities. This is because we are fixing the utility function we are interested in, unlike in Blackwell (1953).

If a type $\theta \in A$ has a $\succeq_\theta$-dominant test, then this test is used in an optimal menu. This shows that an important property of tests is not so much how good they are at differentiating types, but how good they are at differentiating one type the DM wants to accept from all the types he wants to reject.

Proposition 8. *Suppose there is* $t \in T$ *and* $\theta \in A$ *such that* $t \succeq_\theta t'$ *for all* $t' \in T$*, then* t *is part of an optimal menu.*

Proof. See appendix. $\square$

The stronger notion of a test able to differentiate some $\theta \in A$ from all R-types is if $\operatorname{supp} \pi_t(\cdot|\theta) \cap \left(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta') \right) = \emptyset$. If each type in A has such a test, then the principal never makes a mistake. This condition is also necessary.

Proposition 9. *The principal's expected payoff is $\sum_{\theta \in A} \mu(\theta)$ if and only if for all $\theta \in A$, there exists $t \in T$ such that*

$$\operatorname{supp} \pi_t(\cdot|\theta) \cap \left(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta') \right) = \emptyset$$

Here, the principal just needs for each type he wants to accept a test where he can discriminate between that type and the R-types. Then he can offer a menu of tests where each A-type self selects into the test that discriminates him from the R-types. The actual learning only happens by observing the test selected and the testing technology serves as a detriment to deviations from R-types. The argument is then similar to an unravelling argument à la Milgrom (1981) and Grossman (1981). These are not fully revealing tests but tests that allow to perfectly discriminate *one* A-type from all the R-types. But it could be a very noisy tests for the other A-types.

6 Extension: Communication

I consider here the possibility of adding a communication channel on top of the test choice. I will also relate my results to those of Glazer and Rubinstein (2004) and Carroll and Egorov (2019). There is now a finite set M of output messages with $|M| \geq |A|$ and a strategy is a mapping $\sigma : \Theta \to \Delta(T \times M)$. Note that all the results from the previous sections go through as from any finite set T one can create another T' that duplicate each test $|M|$ times. I call this variant of the model the menu game with communication.

In line with Theorem 1, each A-type chooses a message-test pair deterministically and each R-type mixes over some A-types message-test pair. Moreover, I show that when communication is added, each type in A announces his type, thus maximally differentiating himself, and each R-type pretends to be an A-type.

Theorem 2. *If communication is allowed, the same construction as Theorem 1 holds. Moreover, there is a DM-preferred equilibrium where each A-type reports his own type.*

Proof. See appendix. $\square$

Theorem 2 shows that the results extend naturally to an environment where communication is allowed. Because the DM could commit to a strategy, he can always guarantee each A-type at least as much as he would have if he would pool with another A-type. This guarantees that there is an equilibrium where he separates from the other A-types.

Note that because each A-type uses a different message and does not mix over tests, the test chosen does not contain any information: $\mu(\theta|m,t) = \mu(\theta|m)$. Thus all the information revealed by the test is through the signal realisations and not the test choice.

In the remainder of this section, I will connect the results developed in this model to the existing literature, and in particular to Glazer and Rubinstein (2004). Consider the following model generalising the one of Glazer and Rubinstein (2004). They consider a model of persuasion and verification where the agent sends a message and the DM chooses a test and a decision based on the message. Formally, the DM designs a mechanism defined by $\tau : M \to \Delta(T \times [0,1]^X)$, that is a mechanism commits to a test and a decision for each test and signal realisation for each message. A strategy for the agent is $\delta : \Theta \to \Delta M$. The solution concept is weak Perfect Bayesian Equilibrium. In Glazer and Rubinstein (2004), the state space is some multidimensional set and each test in T perfectly reveals one dimension. I will call the mechanism τ a GR-mechanism.

One of the results of Glazer and Rubinstein (2004) is that the outcome of the optimal mechanism τ can be implemented without commitment in a PBE of the following game: the agent chooses a message in M, based on the message, the DM chooses a test and based on the signal realisation and test, the DM accepts or rejects the agent. If the outcome of the optimal GR-mechanism is the same as the one of the game above, I will say that it is credible. I will call that game a GR-game.

The fundamental difference between the Glazer and Rubinstein (2004) model and the one we have studied so far is that it is now the DM that chooses the test and not the agent. But as we will see, if we allow for communication in the menu of test model, this distinction does not matter anymore.

Proposition 10. *The outcome of a GR-mechanism is credible for any T. Moreover, its outcome coincides with the menu game with communication.*

Proof. See appendix. □

This proposition generalises the commitment result of Glazer and Rubinstein (2004) to an arbitrary testing technology and type structure. Moreover, it shows that when there is communication, who chooses the test is not important. To understand this better, let us first note the dual role of test choice in the model without communication. In this case, the test is used both to communicate to the DM and to provide evidence which type the agent is. When we add communication on top of the menu of test, all the communication is through the cheap-talk message and the test is only used to provide evidence about the type.

Now consider the zero-sum game characterisation of the optimal menu, and in particular the payoffs of the A-types. Remember that in the zero-sum game, the A-types were maximising the DM's payoffs. Combined with the fact that the test choice does not carry additional information, we can let the DM choose it. If it was optimal for A-types to choose test t after message m, it will also be for the DM.

Carroll and Egorov (2019) study a similar model as Glazer and Rubinstein (2004), multidimensional types with the testing technology revealing one dimension, but with a different agent payoff function. They study under which condition on the agent's payoffs there is full information revelation. They show that when there is full information revelation and some technical conditions are satisfied, the mechanism can be implemented by having the agent choosing the test, a parallel result to Proposition 10. Thus I show that the equivalence result they have also applies to other environments and is not a feature of full information revelation and their testing technology.

7 Concluding remarks

I study the design of optimal menus of tests. Menus allow the DM to have an additional dimension for information revelation as well as allow for a more efficient allocation of tests to the agent's types. I provide a characterisation of the optimal menu in terms of an auxiliary zero-sum game. One advantage of this characterisation is that it does not rely on any structure on types or tests. While proving this result, I also show that the characterisation holds for a general class of mechanisms allocating agent to tests.

In applications, I show that using a menu can be a powerful tool, and even a dominated test, in the Blackwell sense, can be part of the optimal menu. However, this channel also has limits and I show that in natural economic environments the optimal menu is a singleton. Interestingly, all the results also hold when the DM can commit to an action. I interpreted this result as a hierarchy over information sources: even when the DM can "artificially" create a menu by taking suboptimal action, he is better off using a menu only when he can best reply to the information revealed.

Results for the optimality of the inclusion of some tests, like Proposition 5 and Proposition 8, reveal an interesting asymmetry between types. They are comparing properties of a test or acceptance probability of one or some A-type to those of all the R-types. This asymmetry between the R-types and the A-types is due to their different incentives to separate as reflected by their strategy in the auxiliary zero-sum game. While an A-type wants to be singled-out by choosing a different test, the R-types want to "hide behind" A-types and only choose to mimic them. Thus an optimal test should give these incentives.

Finally, I show that adding a communication channel links the current model to existing models in the literature and generalises their results. Adding the communication highlights the role of tests when there is no communication. Without communication, the tests also serve as a communication channel. When communication is allowed, the test choice does not add any information beyond the test results. The DM is thus as well off choosing the test himself following the cheap-talk message.

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A Omitted proofs

A.1 Proof of Theorem 1

The plan of the proof is the following. First, I characterise the optimal mechanism, where a mechanism maps an input message to a distribution over tests. Because an equilibrium of the menu game can be implemented by a mechanism, the payoffs from the optimal mechanism are weakly greater than the payoffs from any optimal menu. In the second part of the proof, I show that the optimal mechanism can be implemented by posting a menu. In the proof, I will refer to a distribution over test as an allocation.

By standard arguments, a direct truthful mechanism is without loss of generality. A direct mechanism is a mapping $\tilde{\sigma} : \Theta \to \Delta T$. The designer's problem is

$$\begin{aligned}
\tilde{V} = \max_{\tilde{\sigma},\alpha} & \sum_{\theta \in A} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(\alpha|\theta\theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(\alpha|\theta) \\
\text{s.t.} & \sum_t (\tilde{\sigma}(t|\theta) - \tilde{\sigma}(t|\theta'))\pi_t(\alpha|\theta) \geq 0 \text{ for all } \theta, \theta' \\
& \sum_t \tilde{\sigma}(t|\theta) = 1 \text{ for all } \theta \\
& \alpha \in BR(\tilde{\sigma})
\end{aligned}$$

The first constraint is the incentive compatibility constraint of type θ deviating to θ', the second guarantees that an allocation is well-defined and the last constraint ensures that the DM best replies to the information revealed by the output of the mechanism.

Note that any equilibrium in the menu game is incentive compatible and therefore a solution to $\tilde{V}$ gives weakly higher expected payoffs to the DM.

If the DM could commit over a strategy α, his problem would be

$$\begin{aligned}
\tilde{V}(\alpha) = \max_{\tilde{\sigma}} \ & \sum_{\theta \in A} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(\alpha|\theta) - \sum_{\theta \in R} \mu(\theta) \sum_t \tilde{\sigma}(t|\theta)\pi_t(\alpha|\theta) \\
\text{s.t. } & \sum_t (\tilde{\sigma}(t|\theta) - \tilde{\sigma}(t|\theta'))\pi_t(\alpha|\theta) \geq 0 \text{ for all } \theta, \theta' \\
& \sum_t \tilde{\sigma}(t|\theta) = 1 \text{ for all } \theta
\end{aligned}$$

We have that $\max_\alpha V\tilde{(\alpha)} \geq \tilde{V}$ as the DM could always commit to the strategy used to get $\tilde{V}$.

Show that $\tilde{V}(\alpha) = \max_s \min_{s'} v(\alpha, s, s')$ where v is defined in (1).

The dual problem of $\tilde{V}(\alpha)$ is

$$\begin{aligned}
& \min_{y_{\theta,\theta'}, z_\theta} \sum_\theta z_\theta \\
\text{s.t. } & \text{for } \theta \in A, t: \ -\pi_t(\alpha|\theta) \sum_{\theta'} y_{\theta,\theta'} + \sum_{\theta'} \pi_t(\alpha|\theta') y_{\theta',\theta} + z_\theta \geq \mu(\theta)\pi_t(\alpha\theta) \\
& \text{for } \theta \in R, t: \ -\pi_t(\alpha|\theta) \sum_{\theta'} y_{\theta,\theta'} + \sum_{\theta'} \pi_t(\alpha|\theta') y_{\theta',\theta} + z_\theta \geq -\mu(\theta)\pi_t(\alpha\theta) \\
& y_{\theta,\theta'} \geq 0, \ z_\theta \in \mathbb{R}
\end{aligned}$$

Note that $y_{\theta,\theta'}$ is the dual variable associated to the IC constraint of type θ deviating to θ' and z_θ the dual variable associated with the feasibility constraint of type θ.

I will show that for any α, the solution to $\tilde{V}(\alpha)$ can be characterised by an equilibrium of the zero-sum game by verifying that this solution is feasible and satisfy the complementary slackness conditions. To this end I will

1. Guess values for $\tilde{\sigma}, y, z$.

2. Verify that the guessed variables satisfy the constraints of their respective problem, i.e., are feasible.

3. Verify complementary slackness conditions.

If variables are feasible and satisfy complementary slackness then they are optimal (see e.g., Bertsimas and Tsitsiklis, 1997, Theorem 4.5).

Take an equilibrium of the zero-sum game fixing α, (s, s'), i.e., $s \in \arg\max \min_{s'} v(\alpha, \tilde{s}, s')$ and $s' \in \arg\min \max_s v(\alpha, s, \tilde{s}')$.

Guess

- $y_{\theta,\theta'} = 0$ for $\theta \in A$
- $y_{\theta',\theta} = 0$ for $\theta', \theta \in R$
- $y_{\theta',\theta} = \mu(\theta')s'(\theta|\theta')$ for $\theta' \in R$, $\theta \in A$
- $z_{\theta'} = 0$ for $\theta' \in R$
- $z_\theta = \mu(\theta)\pi_{t^\theta}(\alpha|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_{t^\theta}(\alpha|\theta')$ for some $t^\theta \in \operatorname{supp} s(\cdot|\theta)$ for $\theta \in A$
- $\tilde{\sigma}(t|\theta) = s(t|\theta)$ for $\theta \in A$
- $\tilde{\sigma}(t|\theta') = \sum_{\theta \in A} s'(\theta|\theta')s(t|\theta)$ for $\theta' \in R$

Feasibility in the dual problem: Plugging in these guessed values in the constraints of the dual problem, we get for the constraints $(\theta \in R, t)$,

$$-\pi_t(\alpha|\theta) \sum_{\theta' \in A} \mu(\theta)s'(\theta'|\theta) \geq -\mu(\theta)\pi_t(\alpha|\theta)$$

which holds with equality because $\sum_{\theta' \in A} s'(\theta'|\theta) = 1$.

For the constraints $(\theta \in A, t)$, plugging in the guessed values gives

$$\mu(\theta)\pi_{t^\theta}(\alpha|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_{t^\theta}(\alpha|\theta') \geq \mu(\theta)\pi_t(\alpha|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(\alpha|\theta')$$

which holds because (s, s') is an equilibrium of the zero-sum game and thus t^θ maximises this expression.

Feasibility in the primal problem: The solution $\tilde{\sigma}$ is positive and satisfies $\sum_t \tilde{\sigma}(t|\theta) = 1$ for all θ. We are left to check that it satisfies the IC constraints. Note that any allocation is either the allocation of an A-type or a convex combination of allocations of A-types.

First, I show that the IC constraints of A-types are satisfied. Because (s, s') is an equilibrium of the auxiliary game, any $\theta \in A$ must be weakly worse off mimicking another A-type, $\tilde{\theta}$, in the auxiliary game:

$$\sum_t s(t|\theta)\Big[\mu(\theta)\pi_t(\alpha|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(\alpha|\theta')\Big] \geq \sum_t s(t|\tilde{\theta})\Big[\mu(\theta)\pi_t(\alpha|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(\alpha|\theta')\Big]$$
$$\Leftrightarrow \mu(\theta)\sum_t (s(t|\theta) - s(t|\tilde{\theta}))\pi_t(\alpha|\theta) \geq \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\sum_t (s(t|\theta) - s(t|\tilde{\theta}))\pi_t(\alpha|\theta')$$

Note that the LHS is the IC constraint of θ deviating to $\tilde{\theta}$ and the RHS is positive. Indeed, whenever $\sum_t (s(t|\theta) - s(t|\tilde{\theta}))\pi_t(\alpha|\theta') < 0$, we have $s'(\theta|\theta') = 0$. Therefore the IC constraints of an A-type deviating to an A-type are satisfied. Because the $\tilde{\sigma}(t|\theta')$ for $\theta' \in R$ is a convex combination of A-type allocation, all the IC constraints of A-types are satisfied.

For the IC constraint of R-types, note that any R-type is indifferent between reporting his type and reporting an A-type he is mimicking in the zero-sum game. He also weakly prefers reporting his own type over an A-type he is not mimicking. Thus there are no deviations to A-types. Because any other allocation of an R-type is a convex combination of allocation of A-types, no R-type is willing to report another R-type.

Complementary slackness conditions: Complementary slackness conditions are: if a variable in the primal or dual problem is strictly positive, then the corresponding constraint must be binding.

The dual variables y is strictly positive if and only if $\theta \in A$, $\theta' \in R$ and $s'(\theta|\theta') > 0$. The corresponding IC constraint is θ' deviating to θ. But in that case the IC constraint binds as mimicking θ maximises the probability of being accepted in the zero-sum game and thus θ' gets the same expected probability of being as if he would get θ's distribution.

On the other hand the dual constraints are only slack for $(\theta \in A, t)$ such that

$$\mu(\theta)\pi_{t^\theta}(\alpha|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_{t^\theta}(\alpha|\theta') > \mu(\theta)\pi_t(\alpha|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(\alpha|\theta')$$

In this case $\tilde{\sigma}(t|\theta) = 0$ as $s(t|\theta) = 0$. Therefore, the complementary slackness conditions are satisfied and we have characterised an optimal mechanism when the DM commits to α.

Remember that $v(\alpha, s, s') = \sum_t \sum_{\theta \in A} s(t|\theta)\left[\mu(\theta)\pi_t(\alpha|\theta) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(\alpha|\theta')\right]$ and note that it is the DM's payoff in the induced mechanism. Thus, we can express the value of an optimal mechanism with commitment to α, $\tilde{V}(\alpha) = \max_s \min_{s'} v(\alpha, s, s')$. The optimal value of the DM, when he can commit is therefore $\max_\alpha \max_s \min_{s'} v(\alpha, s, s') = \max_{\alpha,s} \min_{s'} v(\alpha, s, s')$.

Show that a saddle-point of v exists and the DM does not benefit from commitment in the optimal mechanism.

Consider the two-players game where player one chooses (s, α) to maximise v and player two chooses s' to maximise $-v$. This game satisfies the condition for the existence of a NE in Baye et al. (1993). Indeed, a sufficient condition for the existence of NE is that (1) strategy spaces are a subset of $\mathbb{R}^m$, (2) v is continuous in all arguments, (3) v is linear in one player's strategy and (4) there are two players. Condition (2) guarantees diagonal transfer continuity (see Proposition 2 in Baye et al., 1993), conditions (2) and (3) guarantee diagonal transfer quasi-concavity (see Proposition 1(e) in Baye et al., 1993). Together this implies the conditions stated in Theorem 1 in Baye et al. (1993). (For complete definitions see the paper.)

Therefore there is (α^*, s^*, s'^*) such that

$$v(\alpha, s, s'^*) \leq v(\alpha^*, s^*, s'^*) \leq v(\alpha^*, s^*, s')$$

for all α, s, s' and $v(\alpha^*, s^*, s'^*) = \max_{\alpha,s} \min_{s'} v(\alpha, s, s')$.

Notice that $v(\alpha, s^*, s'^*) \leq v(\alpha^*, s^*, s'^*)$ for all α. Because v is the DM's expected utility and s^*, s'^* induce the optimal mechanism, $\alpha^* \in BR(\tilde{\sigma}^*)$ where $\tilde{\sigma}^*$ is the mechanism induced by (s^*, s'^*).

Show that an optimal mechanism can be implemented by posting a menu.

Lemma 2. *Take a saddle-point of v, $\big((\alpha, s), s'\big)$. If $s(\cdot|\theta)$ is in pure strategy for all $\theta \in A$, then the optimal mechanism $\tilde{\sigma}$ with DM strategy α is implementable by posting a menu where the strategies are*

- *for $\theta \in A : \sigma(t|\theta) = s(t|\theta)$*
- *for $\theta' \in R : \sigma(t|\theta') = \sum_{\theta \in A} s'(\theta|\theta') s(t|\theta)$*
- *the DM's strategy is α.*

Moreover, the DM does not benefit from committing to α.

Proof. Note that the strategies σ are the same as the outcome of the optimal mechanism $\tilde{\sigma}$ when the DM strategy is α.

Optimal mechanism is implementable with a menu.

The menu posted by the DM is $M = \cup_{\theta \in A} \operatorname{supp} s(\cdot|\theta)$. To prove the result, we simply need to show that the pair (σ, α) is a PBE in the game when the menu M is posted. Let t^θ be the test chosen by type $\theta \in A$.

The incentive compatibility constraint of type $\theta \in A$ deviating to $\tilde{\theta} \in A$ in the optimal mechanism implies

$$\pi_{t^\theta}(\alpha|\theta) \geq \pi_{t^{\tilde{\theta}}}(\alpha|\theta)$$

for any $\tilde{\theta} \in A$. Thus $\theta \in A$ prefers t^θ to any other $t' \in M$.

The incentive compatibility constraint of type $\theta' \in R$ deviating to $\tilde{\theta} \in A$ in the optimal mechanism implies

$$\sum_t \sigma(t|\theta')\pi_t(\alpha|\theta') = \sum_t \sum_{\theta \in A} s'(\theta|\theta')s(t|\theta)\pi_t(\alpha|\theta') \geq \pi_{t^{\tilde{\theta}}}(\alpha|\theta)$$

which again implies that $\sum_t \sigma(t|\theta')\pi_t(\alpha|\theta') \geq \pi_{t'}(\alpha|\theta')$ for all $t' \in M$.

Because $\big((\alpha, s), s'\big)$ is a saddle-point of v,

$$v(\alpha, s, s') \geq v(\alpha', s, s')$$

for all α'. Because v is the DM's payoffs and (s, s') induce the strategies in the equilibrium of the menu game, the DM's strategy is a best-reply. Note that this holds on- and off-path.

On-path, beliefs are pinned down by the strategy σ, the tests π_t and the prior. Off-path, we can choose a belief $\tilde{\mu}(\cdot|t, x)$ such that $\alpha(t, x)$ is a best-reply to $\tilde{\mu}$.

This conclude the description of the PBE.

No benefit to commitment.

This follows from the fact that the DM does not benefit from commitment in the optimal mechanism and that the payoffs from the optimal menu without commitment are the same as in the optimal mechanism with commitment. Given that the payoffs from the optimal mechanism with commitment are always weakly higher than the optimal menu with commitment, the DM does not benefit from commitment to α in the optimal menu. $\square$

Lemma 3. *There is a saddle point of v, $\big((\alpha, s), s'\big)$ where $s(\cdot|\theta)$ is in pure strategy for all $\theta \in A$.*

Proof. The idea of the proof is the following. Take a saddle point (α, s, s') of v and suppose that s mixes over two tests for some $\theta \in A$. Then for $\tilde{s}$ that puts probability one on one test, we must have $v(\alpha, s, s') = v(\alpha, \tilde{s}, s')$. The idea is that under the new mechanism induced by $(\tilde{s}, s')$, there might be $\tilde{\alpha}$ such that $v(\alpha, \tilde{s}, s') < v(\tilde{\alpha}, \tilde{s}, s')$, contradicting that (α, s, s') is a saddle point. Intuitively, putting probability one on one test for $\theta \in A$ does not change his payoffs but allows the DM to reoptimise his strategy and increase payoffs. If all A-types have a "pure" allocation in the optimal mechanism, then a menu implementation is feasible: IC constraints imply that A-types prefer their tests over another A-type's test and any randomisation from R-types can be implemented in equilibrium as they maximise their probability of being accepted in the auxiliary game.

The bulk all the proof is devoted to show that generically, there is such $\tilde{\alpha}$ that *strictly* increase payoffs. This is done by showing that, generically, a type $\theta \in A$ is only willing to mix if the DM also mixes after some signal. The $\tilde{\alpha}$ that increases payoffs is one that only uses pure strategies. Then there is a pure strategy of A-types that increases v as they were only willing to mix if the DM also mixes after some signal. Thus we would have found a $(\tilde{\alpha}, \tilde{s})$ such that $v(\alpha, \tilde{s}, s') < v(\tilde{\alpha}, \tilde{s}, s')$. A convergence argument shows that A-types' strategies are also pure in the non-generic case.

We first are going to record properties of equilibrium strategies when mixed strategies are played. They characterise under which condition an A-type mixing implies that the DM is using a mixed strategy after some signal.

Lemma 4. *For any fixed α, if there is an equilibrium of the zero-sum game where some type $\theta \in A$ plays a mixed strategy, then there is also an equilibrium of the the zero-sum game (s, s') where*

- *either for all $t \in \operatorname{supp} s(\cdot|\theta)$, there is $x, x' \in \operatorname{supp} \cup_\theta \pi_t(\cdot|\theta)$ such that $\alpha(t, x) < 1$ and $\alpha(t, x') > 0$, i.e., some signals are accepted, some signals are not.*
- *or θ plays a pure strategy on a test t with $\pi_t(\alpha|\tilde{\theta}) = 0$ for all $\tilde{\theta} \in \Theta$.*

Proof. Fix α and take some $s^* \in \arg\max \min_{s'} v(\alpha, s, s')$, i.e., it is an equilibrium strategy of the zero-sum game. Suppose s^* is in mixed strategy for some $\theta \in A$.

If the DM accepts after all signals in all tests in the support of θ's strategy, then playing a pure strategy on one test for θ does not change the payoffs of any type on- and off-path and thus constitutes a new equilibrium. Similarly if the DM rejects after any signal in the support of θ's strategy.

Suppose there are two tests, t, t' in the support of $s^*(\cdot|\theta)$ where the DM accepts after all signals in t but not in the other test t'. Then I can show that θ deviating to a strategy $s(\cdot|\theta)$ putting probability zero on t can only increase v. Indeed, type $\theta \in A$ is indifferent between all test in the support of his strategy, if the behaviour of types in R does not change. On the other hand, the overall payoff of the R types can only decrease as they can only choose strategies that either were previously available or have a smaller probability of being accepted if they keep the same strategy. Therefore, $\min_{s'} v(\alpha, s^*, s') \leq \min_{s'} v(\alpha, s, s')$ and the new strategy s is also in $\arg\max \min_{s'} v(\alpha, s, s')$. Thus we can always find another equilibrium strategy where the tests that accept after all signals is not used.

If there is a test in the support of θ's strategy where the DM rejects after all signals, then a similar argument as above shows that playing that test with probability one also constitutes an optimal strategy for θ and therefore he can play a pure strategy in equilibrium. $\square$

We will now focus on equilibria where when a type $\theta \in A$ mixes, for all t in the support, some signals are accepted and some are rejected.

Continue to fix α and consider an equilibrium of the auxiliary game (s^*, s'^*) such that for some $\theta \in A$ and $t, t' \in T$, $s^*(t|\theta)$ and $s^*(t'|\theta) > 0$. Note that $s^* \in \arg\max_s \min_{s'} v(\alpha, s, s')$.

The following lemma shows that if there is an equilibrium in mixed strategy for some $\theta \in A$, it is possible to perturb the strategy of that A-type and obtain another equilibrium strategy.

Lemma 5. *Fix α. Suppose there is an equilibrium of the zero-sum game (s^*, s'^*) such that for some $\theta \in A$ and $t, t' \in T$, $s^*(t|\theta)$ and $s^*(t'|\theta) > 0$. Then there is some $\epsilon > 0$ such that the strategy $\tilde{s}$ that increases type θ's probability on t by ϵ and decreases it on t' by ϵ and keeps all the others' fixed is also an equilibrium strategy.*

Proof. Note that the type $\theta \in A$ is indifferent between s^* and $\tilde{s}$, i.e., $v(\alpha, s^*, s'^*) = v(\alpha, \tilde{s}, s'^*)$, and $s^* \in \arg\max_s \min_{s'} v(\alpha, s, s')$. We want to show that $\tilde{s} \in \arg\max_s \min_{s'} v(\alpha, s, s')$. For that it will be enough to show that $\min_{s'} v(\alpha, \tilde{s}, s') \geq \min_{s'} v(\alpha, s^*, s')$.

When evaluating $\max\min v(\alpha, s, s')$, take a selection of $\arg\min_{s'} v(\alpha, s, s')$ such that if $\sum_t s(t|\theta)\pi_t(\alpha|\theta') \geq \sum_t s(t|\tilde{\theta})\pi_t(\alpha|\theta')$ for all $\tilde{\theta} \in A$, then $s'(\theta|\theta') = 1$ (by linearity of v in s', this is without loss of generality).

Consider all the types $\theta' \in R$ such that $s'(\theta|\theta') \neq 1$ under the selection described above. That is all R-types that strictly prefer another $\tilde{\theta} \in A$ over θ. In that case, modifying θ's strategy by a small enough ϵ will not change the strategy of these R-types.

For the types $\theta' \in R$ such that $s'(\theta|\theta') = 1$, either they now strictly prefer another type or θ

is still their preferred type. In any case, the payoffs v is weakly increasing, as either s' does not change or some $\theta' \in R$ choose to mimic another $\tilde{\theta} \in A$, i.e., choose a strategy that was previously available. Thus the probability of being accepted must be weakly lower for these R-types. Therefore, $\tilde{s} \in \arg\max_s \min_{s'} v(\alpha, s, s')$. $\square$

We use Lemma 5 to show some properties of equilibrium strategies whenever $s^*(\cdot|\theta)$ is in mixed strategy.

Lemma 6. *Fix α. Suppose there is an equilibrium of the zero-sum game (s^*, s'^*) such that for some $\theta \in A$ and $t, t' \in T$, $s^*(t|\theta)$ and $s^*(t'|\theta) > 0$. Then for all $\theta' \in R$ such that $s'^*(\theta|\theta') > 0$, either $\pi_t(\alpha|\theta') = \pi_{t'}(\alpha|\theta')$ or they strictly prefer θ, $s'^*(\theta|\theta') = 1$.*

Proof. As the set of saddle points is a product set, s'^* must be a BR to $\tilde{s}$ as defined in Lemma 5 where $\theta \in A$ puts ϵ higher probability on t and ϵ lower probability on t'. One possibility is that the payoffs when facing s^* and $\tilde{s}$ do not change for some R-types, i.e., $\pi_t(\alpha|\theta') = \pi_{t'}(\alpha|\theta')$ and so the same strategy is best-response for them.

To show the other possibility, suppose that $\sum_{\bar{t}} s^*(\bar{t}|\theta)\pi_{\bar{t}}(\alpha|\theta') = \sum_{\bar{t}} s^*(\bar{t}|\tilde{\theta})\pi_{\bar{t}}(\alpha|\theta')$ for some $\tilde{\theta} \in A$ and $\pi_t(\alpha|\theta') \neq \pi_{t'}(\alpha|\theta')$. Then to have s'^* to be a best response $\tilde{s}$, it must be that the change to $\tilde{s}$ strictly increases the payoffs of θ', i.e., $\pi_t(\alpha|\theta') > \pi_{t'}(\alpha|\theta')$. But then consider the other strategy $\tilde{s}_1$ that puts additional probability ϵ_1 on t' and decreases the probability on t by ϵ_1. Using the same argument as in Lemma 5, $\tilde{s}_1 \in \arg\max_s \min_{s'} v(\alpha, s, s')$. The strategy s'^* must also be a best-response to $\tilde{s}_1$, which cannot be true if $\sum_t s^*(t|\theta)\pi_t(\alpha|\theta') = \sum_t s^*(t|\tilde{\theta})\pi_t(\alpha|\theta')$ for some $\tilde{\theta} \in A$ and $\pi_t(\alpha|\theta') > \pi_{t'}(\alpha|\theta')$. Therefore, $\sum_t s^*(t|\theta)\pi_t(\alpha|\theta') > \sum_t s^*(t|\tilde{\theta})\pi_t(\alpha|\theta')$ and thus θ' plays a pure strategy. $\square$

This means that if there is a mixed strategy over t, t' for some $\theta \in A$ in the zero-sum game, at least one of these two assertions are true (1) for some $\theta' \in R$, $\pi_t(\alpha|\theta') = \pi_{t'}(\alpha|\theta')$ or (2)

there exists a subset $Z \subseteq R$ such that $\mu(\theta)\pi_t(\alpha|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_t(\alpha|\theta') = \mu(\theta)\pi_{t'}(\alpha|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_{t'}(\alpha|\theta')$. Condition (1) was explicitly shown. Condition (2) is the indifference condition of type $\theta \in A$ when a subset $Z \subseteq R$ mimicks him. Lemma 6 tells us that if condition (1) does not hold then all R-types mimicking θ must play a pure strategy. If the set of types mimicking θ is Z, condition (2) gives the indifference condition.

Consider the following condition:

Condition 1 (Mix)**.** *For any test $t \in T$, for all $\tilde{\theta} \in \Theta$, $\pi_t(\cdot|\tilde{\theta})$ has full support on X.*

For any two tests $t, t' \in T$, there are no subsets of signals $\emptyset \neq X_1, X_2 \subset X$ such that

- $\sum_{x \in X_1} \pi_t(x|\theta') - \sum_{x \in X_2} \pi_{t'}(x|\theta') = 0$ *for some* $\theta' \in R$
- *or for some $\theta \in A$ and $Z \subseteq R$,* $\sum_{x \in X_1} \mu(\theta)\pi_t(x|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_t(x|\theta') = \sum_{x \in X_2} \mu(\theta)\pi_{t'}(x|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_{t'}(x|\theta')$.

If condition (Mix) is satisfied then we cannot have $s^*(\cdot|\theta)$ in mixed strategy and $\alpha(t, x) \in \{0, 1\}$ for all on-path (t, x). Indeed, if we do have a mixed strategy, the full support assumption guarantees that if one type is accepted or rejected with probability one, then all types are. Then from Lemma 4, we can assume that $\pi_t(\alpha|\theta') \in (0, 1)$ for all $\theta' \in R$ in a mixed strategy equilibrium.

But in Lemma 6, we have proven that to have a mixed strategy, either $\pi_t(\alpha|\theta') = \pi_{t'}(\alpha|\theta')$ or each $\theta' \in R$ mimicking θ plays a pure strategy. The first possibility is not possible as $\pi_t(\alpha|\theta') \in (0, 1)$ and thus that would contradict the first condition in (Mix).

The second possibility from Lemma 6 is that there is a subset $Z \subseteq R$ that mimics θ and from the indifference condition, we get $\mu(\theta)\pi_t(\alpha|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_t(\alpha|\theta') = \mu(\theta)\pi_{t'}(\alpha|\theta) - \sum_{\theta' \in Z} \mu(\theta')\pi_{t'}(\alpha|\theta')$. Again, if $\alpha(t, x) \in \{0, 1\}$ for all on-path (t, x) and condition (Mix) holds, then we have a contradiction.

Now take a saddle point of v, $\big((\alpha^*, s^*), s'^*\big)$. Suppose that T satisfies condition (Mix). By linearity of v in α, it must be that $v(\alpha^*, s^*, s'^*) = v(\tilde{\alpha}, s^*, s'^*)$, where $\tilde{\alpha}$ sets $\tilde{\alpha}(t,x) = 0$ whenever $\alpha^*(t,x) \in (0,1)$ and the argument above guarantees such (t,x) exists. From condition (Mix), no $\theta \in A$ is willing to mix under $\tilde{\alpha}$, so there is $\tilde{s}$ that is a BR to $(\tilde{\alpha}, s'^*)$ such that $v(\tilde{\alpha}, \tilde{s}, s'^*) > v(\tilde{\alpha}, s^*, s'^*) = v(\alpha^*, s^*, s'^*)$, a contradiction. Therefore, if condition (Mix) is satisfied, $s^*(\cdot|\theta)$ is a pure strategy for all $\theta \in A$.

I now show that condition (Mix) is generic in the following sense. First, endow Π, the set of tests with signal space X, with the following metric, $d_2(\pi, \pi') \equiv \max_\theta \sqrt{\sum_x \big(\pi(x|\theta) - \pi'(x|\theta)\big)^2}$.[10]

Lemma 7. *For any T, either T satisfies (Mix) or for each $t \in T$, there is a sequence of tests $\pi_t^n \to \pi_t$ such that $T^n \equiv \{\pi_t^n\}_{t\in T}$ satisfy (Mix).*[11]

Proof. Remember that there is a finite number of types, tests and signals, thus there is a finite number of equalities to check to make sure that condition (Mix) is satisfied.

Suppose T does not satisfy (Mix). The first possibility is that there is $t \in T$ that does not satisfy full-support. In that case, for each $\tilde{\theta} \in \Theta$, take some $x \in X$ with $\pi_t(x|\tilde{\theta}) > 0$ and modify it by setting $\pi_t'(x|\tilde{\theta}) = \pi_t(x|\theta) - \epsilon$ and $\pi_t'(x'|\tilde{\theta}) = \pi_t(x'|\theta) + \frac{\epsilon}{|X|-1}$ for $x' \neq x$ for some $\epsilon > 0$ small enough.

Suppose that after this change, there are two tests $t, t' \in T$ such that for all $\emptyset \neq X_1, X_2 \subset X$ and for some $\theta' \in R$, $\pi_t(X_1|\theta') = \pi_{t'}(X_2|\theta')$. Consider the following modification, take $x \in X_1$, $x' \in X \setminus X_1$ and set $\pi_t'(x|\theta') = \pi_t(x|\theta') + \epsilon$ and $\pi_t'(x'|\theta') = \pi_t(x'|\theta') - \epsilon$ for some $\epsilon > 0$ small enough. We get $\pi_t(X_1|\theta') \neq \pi_{t'}(X_2|\theta')$ and that would be true for all $\epsilon' \in (0, \epsilon]$. Moreover, there is only a finite number of ϵ' that would create new equalities that would upset condition (Mix). Indeed, suppose that for some ϵ', there is $\emptyset \neq X_1', X_2' \subset X$

[10] Note that under this metric, two information structures inducing the same posterior beliefs can be "far" from each other. However, two information structures "close" to each other under this metric will have posterior beliefs "close" to each other. This does not play any role in the analysis.

[11] Note that by a small abuse of notation, T is here both a set of tests and an index set.

such that $\pi_t(X'_1|\theta') \neq \pi_{t'}(X'_2|\theta')$ but $\pi'_t(X'_1|\theta') = \pi_{t'}(X'_2|\theta')$, i.e., a new inequality is created. This means that $x \in X'_1$ and $x' \in X \setminus X'_1$ or vice-versa. But then for any $\epsilon'' < \epsilon'$, the equality would be upset again.

A similar modification can be done for any $\theta \in A$ by modifying $\pi_t(\cdot|\theta)$ in a similar way.

Because for each $\tilde{\theta} \in \Theta$, there is only a finite number of inequalities to check and a modification like the one described above is possible, we can find a sequence of $(\epsilon^n) \to 0$, where for each n the associated set T^n satisfies (Mix). As each modified test converges to the original t, we have $\pi_t^n \to \pi_t$ for all $t \in T$. $\square$

Suppose T does not satisfy (Mix). Take a sequence (T^n) converging to T as described in Lemma 7. For each $t \in T$, $\pi_t^n(\cdot|\theta) \to \pi_t(\cdot|\theta)$ where the convergence is in the Euclidian metric. To make the dependence of v with T explicit, I write $v(\alpha, s, s'; T)$. I will now show that the Nash Equilibrium correspondence of the zero-sum game satisfies the closed graph property and that allows to conclude that there is an s in pure strategies in an equilibrium of the zero-sum game at T.

Take a sequence (α^n, s^n, s'^n) of NE for each T^n such that s^n is in pure strategy for each $\theta \in A$.[12] Because (α^n, s^n, s'^n) is a bounded sequence of $\mathbb{R}^m$, it admits a converging subsequence. Let us focus on that converging subsequence to some (α^*, s^*, s'^*). If (s^n) is in pure strategy for all $\theta \in A$ and converges, it must be that it converges to a pure strategy for all $\theta \in A$.

I will now show that (α^*, s^*, s'^*) constitutes a NE at T thus proving there is a pure strategy for A-types in equilibrium at T. The function v is a bounded product and sum of continuous

[12]Note that the definitions of α and s depend on T but only as an index of tests and not on the test set itself. Because T and T^n have the same cardinality, the sequence is well-defined.

function and so it is jointly continuous in $(\alpha, s, s'; T)$. For each n,

$$
\begin{aligned}
& v(\alpha^n, s^n, s'^n; T^n) \geq v(\tilde{\alpha}, \tilde{s}, s'^n; T^n), \text{ for all } \tilde{\alpha}, \tilde{s} \\
\Rightarrow \ & v(\alpha^*, s^*, s'^*; T) \geq v(\tilde{\alpha}, \tilde{s}, s'^*; T), \text{ for all } \tilde{\alpha}, \tilde{s}, \text{by continuity of } v
\end{aligned}
$$

Therefore (α^*, s^*) is a best-response to s'^* at T. Similarly, we can show that s'^* is a best-response to (α^*, s^*) at T. This shows that (α^*, s^*, s'^*) constitutes a NE at T and establishes that there is an equilibrium with pure strategies for A-types when (Mix) is not satisfied. $\square$

A.2 Proof of Lemma 1

Because $t \succeq t'$ implies $t \succeq_\theta t'$ for some $\theta \in A$, Lemma 1 is a corollary of Proposition 8 proven below.

A.3 Proof of Proposition 1 and Proposition 2

Suppose the DM's preferences are single-peaked given $\geq_t$. Suppose there is a menu with both t, t'. Take $A_1, A_2 \in A$ with $A_1 < A_2$ and without loss of generality, suppose A_1 chooses t' and A_2 chooses t in some equilibrium. Let α denote the DM equilibrium strategy in this equilibrium.

Because $t \succeq t'$, there is $\beta : X \times X \to [0,1]$ such that $\pi_{t'}(\tilde{x}|\theta) = \beta(x, \tilde{x})\pi_t(x|\theta) + \beta(x', \tilde{x})\pi_t(x'|\theta)$ and $\sum_x \beta(\tilde{x}, x) = 1$ for $\tilde{x} = x, x'$. Type $\theta \in \Theta$ prefers test t' over t if

$$
\begin{aligned}
\alpha(x_1, t')\Big(\beta(x_1, x_1)\pi_t(x_1|\theta) + \beta(x_0, x_1)\pi_t(x_0|\theta)\Big) + \alpha(x_0, t')\Big(\beta(x_1, x_0)\pi_t(x_1|\theta) + \beta(x_0, x_0)\pi_t(x_0|\theta)\Big) \\
- \alpha(x_1, t)\pi_t(x_1|\theta) - \alpha(x_0, t)\pi_t(x_0|\theta) \geq 0
\end{aligned}
$$

This expression is monotonic in θ. If $\pi_t(x_0|\theta) > 0$, then dividing by $\pi_t(x_0|\theta)$ gives

$$\alpha(x_1,t')\Big(\beta(x_1,x_1)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} + \beta(x_0,x_1)\Big) + \alpha(x_0,t')\Big(\beta(x_1,x_0)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} + \beta(x_0,x_0)\Big) - \alpha(x_1,t)\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)} - \alpha(x_0,t)$$

which is linear in $\frac{\pi_t(x_1|\theta)}{\pi_t(x_0|\theta)}$, an increasing function of θ. If $\pi_t(x_0|\theta) = 0$, then $\pi_t(x_0|\theta') = 0$ for all $\theta' >_t \theta$ and the expression is constant.

To have A_1 choose t' and A_2 choose t, it must be strictly decreasing[13] in θ, i.e.,

$$\alpha(x_1,t')\beta(x_1,x_1) + \alpha(x_0,t')\beta(x_1,x_0) - \alpha(x_1,t) < 0 \tag{3}$$

A necessary condition for (3) to hold is that $\alpha(x_1,t) > 0$. Note that this also implies that there is $\overline{\theta} \in A$ such that any $\theta > \overline{\theta}$ prefers t and any $\theta \le \overline{\theta}$ prefers t'. Let $A^+ = \{\theta \in A : \theta >_t \overline{\theta}\}$ and $R^+ = \{\theta \in R : \theta >_t \theta', \text{ for all } \theta' \in A\}$. But because only types in $A^+ \cup R^+$ choose t, the likelihood ratios $\frac{\pi_t(x_1|\theta)}{\pi_t(x_1|\theta')} < \frac{\pi_t(x_0|\theta)}{\pi_t(x_0|\theta')}$ for any $\theta \in A^+$, $\theta' \in R^+$ and $\alpha(x_1,t) > 0$ imply that $\alpha(x_0,t) = 1$ (Milgrom, 1981).

But then no type ever prefer t' over t. Indeed, the condition

$$\Big(\alpha(x_1,t')\beta(x_1,x_1) + \alpha(x_0,t')\beta(x_1,x_0) - \alpha(x_1,t)\Big)\pi_t(x_1|\theta) \ge 1 - \alpha(x_1,t')\beta(x_0,x_1) - \alpha(x_0,t')\beta(x_0,x_0)\pi_t(x_0|\theta)$$

is never satisfied as the LHS is strictly negative because (3) must hold and the RHS is positive because $\beta(x_0,x_1) + \beta(x_0,x_0) = 1$ and $\alpha(\tilde{x},t') \le 1$, $\tilde{x} = x_1, x_0$.

Thus there is a unique equilibrium where only t is chosen.

[13] If all types are indifferent between t and t' then it is also an equilibrium to offer only t and the DM's payoffs are the same.

Suppose the DM's preferences are single-dipped given $\geq_t$.

Suppose $(\tilde{\alpha}, \tilde{s}) \in \arg\max \min_{s'} v(\alpha, s, s')$ with $\tilde{s}(t|\theta) = 1$ for all $\theta \in A$.

Suppose the prior is such that when only t is offered, x_0 is rejected and x_1 is accepted. Let $\underline{\theta} = \min\{\theta \in R\}$ where the $\min$ is taken with respect to $\geq_t$.

Then consider the following deviation: take some $t' \neq t$ and let $\alpha(x, t') = \pi_t(x_1|\underline{\theta})$ for all $x \in X$ and $\alpha = \tilde{\alpha}$ otherwise. Because preferences are single-dipped, there is $\theta \in A$ such that $\pi_t(x_1|\theta) < \pi_t(x_1|\underline{\theta})$ and for all $\theta' \in R$, $\pi_t(x_1|\theta') \geq \pi_t(x_1|\underline{\theta})$. Let $s(t'|\theta) = 1$ for that type and $s = \tilde{s}$ otherwise. This deviation is strictly profitable, i.e., $\min_{s'} v(\tilde{\alpha}, \tilde{s}) < \min_{s'} v(\alpha, s, s')$.

Suppose the prior is such that $\tilde{\alpha}(x_1, t) = \tilde{\alpha}(x_0, t) \in \{0, 1\}$ when only t is offered. This means that the DM does not react to information. Let $\alpha(x, t') = \tilde{\alpha}(x, t)$ for some $t' \neq t$ and $s(t'|\theta) = 1$ for some $\theta \in A$ and $s = \tilde{s}$ otherwise. We get $\min_{s'} v(\tilde{\alpha}, \tilde{s}) = \min_{s'} v(\alpha, s, s')$, so it is also an equilibrium.

Suppose that the DM's preferences are not single-peaked given $\geq_t$.

In this case, it is possible to find $A_1, A_2 \in A$ and $R_1 \in R$ such that $A_1 <_t R_1 <_t A_2$. Let $\mu(\theta) \approx 0$ for $\theta \neq A_1, A_2, R_1$ and be such that x_0 is rejected and and x_1 is accepted when only t is offered. Because t is informative, there is always such prior. Then from the reasoning above the menu $\{t, t'\}$ is strictly better for the DM than $\{t\}$ when only focusing on A_1, A_2, R_1 have positive probability. But because $\mu(\theta) \approx 0$ for $\theta \neq A_1, A_2, R_1$, then the menu $\{t, t'\}$ remains strictly better than $\{t\}$ whatever the behaviour of the other types.

A.4 Proof of Proposition 5

Proof. Suppose the DM only uses t and let t' be the coarsened version of t that pools signals in X'. Let $T = \{t, t'\}$. Let $\pi_{t'}(x'|\theta) = \sum_{x \in X'} \pi_t(x|\theta)$ for some $x' \in X'$.

Consider the deviation, $(\tilde{\alpha}, \tilde{s})$: $\tilde{\alpha}(x', t') = \tilde{\alpha}$ and $\tilde{\alpha}(x, \tilde{t}) = \alpha(x, \tilde{t})$ for $x \neq x'$, $\tilde{t} = t, t'$ and $\tilde{s}(t'|\theta) = 1$ if $\sum_{x \in X'} \tilde{\alpha}\pi_t(x|\theta) > \sum_{x \in X'} \alpha(x,t)\pi_t(x|\theta)$and $\tilde{s}(\cdot|\theta) = s(\cdot|\theta)$ otherwise. We want to show that

$$\min_{s'} v(\tilde{\alpha}, \tilde{s}, s') \geq \min_{s'} v(\alpha, s, s')$$
$$\Leftrightarrow \sum_{\theta \in A} \sum_{x \in X'} \mu(\theta) \left[(\tilde{\alpha} - \alpha(x,t))\pi_t(x|\theta)\right]^+ \geq \sum_{\theta' \in R} \sum_{x \in X'} \mu(\theta') \left[(\tilde{\alpha} - \alpha(x,t))\pi_t(x|\theta')\right]^+$$

which is exactly the condition in Proposition 5. Note that the strategy of the R-types is to mimick a type choosing t' iff $\sum_{x \in X'} \tilde{\alpha}\pi_t(x|\theta') > \sum_{x \in X'} \alpha(x,t)\pi_t(x|\theta')$. $\square$

A.5 Proof of Proposition 3

Proof. Note that in an MLRP environment, the strategy of the DM takes the form of a cutoff strategy. For each test t, there is $x_t \in X$ such that $\alpha(x,t) = 0$ for $x < x_t$, $\alpha(x,t) = 1$ for $x > x_t$ and $\alpha(x_t, t) \in [0,1]$. From Lemma 1, we know that there is an optimal menu containing the Blackwell most informative test. Because all tests are MLRP and the DM's payoffs satisfy single-crossing condition, the Lehmann order is well-defined and the Blackwell order implies the Lehmann order (Lehmann, 1988; Persico, 2000). Let $\succeq^a$ denote the Lehmann order.

The Lehmann order is defined on continuous information structure. But as outlined in Lehmann (1988), we can always make our conditional probabilities continuous by adding independent uniform between each signal. Let's assume, without loss of generality, that $X = \{1, ..., n\}$. The new distribution over signal is $\tilde{y}|\theta = \tilde{x}|\theta - u$ where $u \sim$U$[0,1]$. Denote by F_t the cdf associated with the new information structure.

We have that $t \succeq^a t'$ if $y^*(\theta, y) \equiv F_t(y^*|\theta) = F_{t'}(y|\theta)$ is nondecreasing in θ for all y. In particular, this condition implies that if $F_t(y|\theta') \leq (<) F_{t'}(y'|\theta')$ then $F_t(y|\theta) \leq (<) F_{t'}(y'|\theta)$

for all $\theta > \theta'$.

Let α be the optimal strategy and x_t be the cutoff signal associated to each test. To each $\big(\alpha(\cdot, t), x_t\big)$ we can associate a $y_t \equiv x_t - \alpha(x_t, t)$.

If t is part of an optimal menu, it must be that there is some $\theta' \in R$ such that $\pi_t(\alpha|\theta') \geq \pi_{t'}(\alpha|\theta')$ for all t'. Or put differently, $F_t(y_t|\theta') \leq F_{t'}(y_{t'}|\theta')$ for all t'. But then $F_t(y_t|\theta) \leq F_{t'}(y_{t'}|\theta)$ for all t' and all $\theta > \theta'$, in particular all $\theta \in A$. Therefore all type in A prefer test t as well and there is an equilibrium of the zero-sum game where all types in $\theta \in A$ choose t. (If there is an A-type that is indifferent between t and t' then all types in R must be indifferent or prefer t' so choosing t is an equilibrium strategy for such A-type.) $\square$

A.6 Proof of Proposition 6

I first show that if $t > t'$, then $\mu(\cdot|t, x) \succeq_{FOSD} \mu(\cdot|t', x)$ where $\succeq_{FOSD}$ denotes first-order stochastic dominance.

Proof. The proof is similar to the one in Milgrom (1981). Denote by $G_t(\cdot|x)$ the cdf of posterior beliefs after signal x in test t. For all $\theta > \theta'$,

$$\mu(\theta)\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \mu(\theta)\frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$$

Take some $\theta^* \geq \theta'$. Summing over θ, we get

$$\sum_{\theta > \theta^*} \mu(\theta)\frac{\pi_t(x|\theta)}{\pi_t(x|\theta')} \geq \sum_{\theta > \theta^*} \mu(\theta)\frac{\pi_{t'}(x|\theta)}{\pi_{t'}(x|\theta')}$$

Inverting and summing over θ', we get

$$\frac{\sum_{\theta^*\geq\theta'} \mu(\theta')\pi_t(x|\theta')}{\sum_{\theta>\theta^*} \mu(\theta)\pi_t(x|\theta)} \leq \frac{\sum_{\theta^*\geq\theta'} \mu(\theta')\pi_{t'}(x|\theta')}{\sum_{\theta>\theta^*} \mu(\theta)\pi_{t'}(x|\theta)}$$

which implies

$$\frac{G_t(\theta^*|x)}{1-G_t(\theta^*|x)} \leq \frac{G_{t'}(\theta^*|x)}{1-G_{t'}(\theta^*|x)} \quad \Rightarrow \quad G_t(\theta^*|x) \leq G_{t'}(\theta^*|x)$$

□

The way this proof proceeds is by fixing a menu and dividing tests in two categories: (1) those for which $\alpha(x_0,\tilde{t}) \in (0,1)$ and $\alpha(x_1,\tilde{t}) = 1$ and (2) $\alpha(x_0,\tilde{t}) = 0$ and $\alpha(x_1,\tilde{t}) \in (0,1]$. I exclude the possibility that the DM always accepts or rejects after any signal as it would either be the only test chosen in equilibrium or never chosen. Then, I show that within each category, it is without loss of optimality to have at most one test. It is thus optimal to have at most two tests in the menu. The last part of the proof shows that the resulting menu is dominated by having only one test.

If there are two tests, $t > t'$ such that $\alpha(x_0,\tilde{t}) = 0$ and $\alpha(x_1,\tilde{t}) \in (0,1]$, I will show that,

$$\pi_t(\alpha|\theta') \geq \pi_{t'}(\alpha|\theta') \quad \Rightarrow \quad \pi_t(\alpha|\theta) \geq \pi_{t'}(\alpha|\theta) \text{ for all } \theta > \theta'$$

Take two tests such that $\alpha(x_0,\tilde{t}) = 0$, $t > t'$. Let α, α' denote their respective probability of accepting after x_1. Define $\alpha(\theta) \equiv \alpha(\theta)\pi_t(x_1|\theta) - \alpha'\pi_{t'}(x_1|\theta) = 0$. Clearly, $\alpha(\theta) = \alpha' \frac{\pi_{t'}(x_1|\theta)}{\pi_t(x_1|\theta)}$. From our assumption on the difficulty environment, $\alpha(\theta)$ is decreasing in θ. If $\pi_t(\alpha|\theta') \geq \pi_{t'}(\alpha|\theta')$ for some θ' then $\alpha \geq \alpha(\theta')$. Then $\alpha \geq \alpha(\theta)$ for all $\theta > \theta'$.

In equilibrium, we must have that there is one $\theta' \in R$ that chooses t and thus for all $\theta \in A$,

$\pi_t(\alpha|\theta) \geq \pi_{t'}(\alpha|\theta)$. Then there is an equilibrium of the zero-sum game where t' is never chosen.

A similar argument can be made for all tests where $\alpha(x_0, \tilde{t}) > 0$.

Thus we conclude that it is without loss of optimality that the optimal menu has at most two tests.

Suppose the optimal menu uses two tests, $t > t'$. I will now show that it must be that $\alpha(x_0, t) \in (0, 1)$ and $\alpha(x_1, t') \in (0, 1)$, i.e., the DM must accept in the hard test when there is a fail grade and only accept in the easy test if there is a pass grade. Suppose it is not the case and denote by α, α' their respective mixing probabilities. Define $\alpha(\theta) \equiv \alpha(\theta)\pi_t(x_1|\theta) - \alpha'\pi_{t'}(x_0|\theta) - \pi_{t'}(x_1|\theta) = 0$, which is equivalent to $\alpha(\theta) = \alpha' \frac{1}{\pi_t(x_1|\theta)} + (1-\alpha')\frac{\pi_{t'}(x_1|\theta)}{\pi_t(x_1|\theta)}$. Again from our assumptions, this is decreasing in θ. A type θ chooses t if $\alpha \geq \alpha(\theta)$. Thus if one $\theta \in A$ chooses t all $\theta \in R$ choose t and there is no pooling of A and R-types on t', or it is payoff equivalent to just offering t. Therefore, $\alpha(x_0, t) \in (0, 1)$ and $\alpha(x_1, t') \in (0, 1)$ for $t > t'$.

If the DM mixes, he must be indifferent and thus we have

$$\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta)\pi_t(x_0|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta')\pi_t(x_0|\theta') = 0$$
$$\sum_{\theta \in A} \mu(\theta)\sigma(t'|\theta)\pi_{t'}(x_1|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t'|\theta')\pi_{t'}(x_1|\theta') = 0$$

In the easy test, because the DM rejects with positive probability after x_1 and rejects for sure after x_0 (as he uses a cutoff strategy), his payoffs from t' is 0, i.e., he does as well as rejecting for sure.

In the hard test, he accepts with some probability after x_0 and thus his payoffs are

$$\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta')$$

that is the payoffs he would get from accepting all types choosing t. Thus the overall payoffs from the menu is $\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta')$. Offering a menu is better than a singleton menu if this value is strictly greater than offering t and following the signal

$$\begin{aligned}\sum_{\theta \in A} \mu(\theta)\sigma(t|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t|\theta') &> \sum_{\theta \in A} \mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta' \in R} \mu(\theta')\pi_t(x_1|\theta') \\ &= \sum_{\theta \in A} \sigma(t|\theta)\mu(\theta)\pi_t(x_1|\theta) + \sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta) \\ &- \sum_{\theta' \in R} \sigma(t|\theta')\mu(\theta')\pi_t(x_1|\theta') - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta')\end{aligned}$$

We can rearrange and use the indifference condition at (x_0, t) to get

$$0 > \sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta')$$

Using the indifference condition at (x_1, t'), we can replace 0 on the LHS and get

$$\begin{aligned}&\sum_{\theta \in A} \mu(\theta)\sigma(t'|\theta)\pi_{t'}(x_1|\theta) - \sum_{\theta' \in R} \mu(\theta')\sigma(t'|\theta')\pi_{t'}(x_1|\theta') \\ &\qquad > \sum_{\theta \in A} \sigma(t'|\theta)\mu(\theta)\pi_t(x_1|\theta) - \sum_{\theta' \in R} \sigma(t'|\theta')\mu(\theta)\pi_t(x_1|\theta')\end{aligned}$$

But from the definition of the environment, for all $\theta > \theta'$,

$$\frac{\pi_t(x_1|\theta)}{\pi_t(x_1|\theta')} \geq \frac{\pi_{t'}(x_1|\theta)}{\pi_{t'}(x_1|\theta')}$$

which implies that $\mu(\theta|x_1, t) \succeq_{FOSD} \mu(\theta|x_1, t')$. Thus we get a contradiction.

A.7 Proof of Proposition 7

Suppose condition (2) holds. Suppose $(\alpha, s) \in \arg\max \min_{s'} v(\alpha, s, s')$ and $s(t_j|\theta) = 1$ for all $\theta \in A$. Take $(\tilde{\theta}_i, \tilde{\theta}_j) \in \arg\min_{\theta \in A} \pi_{t_j}(\alpha|\theta)$. Because $\pi_{t_j}(\alpha|\theta_i, \theta_j)$ is constant in θ_i, we have $(\theta_i^+, \tilde{\theta}_j) \in \arg\min_{\theta \in \Theta} \pi_{t_j}(\alpha|\theta)$ as well and from condition (2), $(\theta_i^+, \tilde{\theta}_j) \in A$. Consider the deviation to $(\tilde{\alpha}, \tilde{s})$ such that for t_i,

- $\tilde{\alpha}(\cdot, t_i)$ is set so that it has a cutoff structure and $\pi_{t_i}(\tilde{\alpha}|\theta_i^+, \tilde{\theta}_j) = \pi_{t_j}(\alpha|\theta_i^+, \tilde{\theta}_j)$ and $\tilde{\alpha}(\cdot, t_j) = \alpha(\cdot, t_j)$ otherwise.
- $\tilde{s}(t_i|\theta_i^+, \tilde{\theta}_j) = 1$ and $\tilde{s}(\cdot|\theta) = s(\cdot|\theta)$ otherwise.

Because the test t_i has the strict MLRP when restricting attention to dimension i, for all $\theta_i < \theta_i^+$, $\min_{\theta \in \Theta} \pi_{t_j}(\alpha|\theta) \geq \pi_{t_i}(\tilde{\alpha}|\theta_i^+, \theta_j) > \pi_{t_i}(\tilde{\alpha}|\theta_i, \theta_j)$. This means that mimicking $(\theta_i^+, \tilde{\theta}_j)$ is weakly dominated and $(\theta_i^+, \tilde{\theta}_j)$ has the probability of being accepted. Thus $\min_{s'} v(\alpha, s, s') \leq \min_{s'} v(\tilde{\alpha}, \tilde{s}, s')$.

Suppose condition (2) does not hold.

If condition (2) is not satisfied, then there a dimension, say 1, and $\tilde{\theta}_2 \in \Theta_2$ such that $(\theta_1^+, \tilde{\theta}_2) \in R$. By the definition of the bidimensional environment, this implies that $(\theta_1, \tilde{\theta}_2) \in R$ for all $\theta_1 \in \Theta_1$. Moreover, for all $\theta_2 < \tilde{\theta}_2$ and all $\theta_1 \in \Theta_1$, $(\theta_1, \theta_2) \in R$.

Now suppose μ is such that $\mu(\theta_1, \tilde{\theta}_2) > \sum_{\theta_2' \neq \theta_2} \mu(\theta_1, \theta_2')$ for all $\theta_1 \in \Theta_1$. And that $\mu(\theta_1, \theta_2) \approx 0$ for all $(\theta_1, \theta_2) \in R$ such that $\theta_2 > \tilde{\theta}_2$.

I am going to show that $\{t_2\}$ is optimal when t_1 fully reveals dimension 1. Because this test can replicate the strategies of any t_1, it is enough to prove our claim.

Suppose there is an optimal menu $\{t_1, t_2\}$. From our assumptions on μ, the DM follows a cutoff strategy after t_2. That's because his payoff is monotone along that dimension, ignoring

$(\theta_1, \theta_2) \in R$ such that $\theta_2 > \tilde{\theta}_2$ whose prior probability is close to zero. So it does not upset the cutoff structure of the best-response. This implies that $\pi_{t_2}(\alpha|\theta_1, \theta_2) > \pi_{t_2}(\alpha|\theta_1, \tilde{\theta}_2)$ for all $\theta_2 > \tilde{\theta}_2$ because the likelihood ratio is strictly increasing.

Suppose that some $(\theta_1, \tilde{\theta}_2)$ chooses t_1 with probability 1 in equilibrium. Because $\mu(\theta_1, \tilde{\theta}_2) > \sum_{\theta_2' \neq \theta_2} \mu(\theta_1, \theta_2')$ for all $\theta_1 \in \Theta_1$, it must be that the best-response is $\alpha(x = \theta_1, t_1) = 0$ (recall that t_1 fully reveals θ_1). Thus $\pi_{t_2}(\alpha|\theta_1, \theta_2) = 0$ for all $\theta_2 \in \Theta_2$, otherwise there is a profitable deviation. Either this contradicts the fact that the DM best replies or in equilibrium the DM rejects after all signals in every test. But then he is weakly better off only offering t_2.

Thus to have $\{t_1, t_2\}$ strictly better, it must be that all $(\theta_1, \tilde{\theta}_2)$ choosing t_1 mix in equilibrium. This means that $\pi_{t_1}(\alpha|\theta_1, \tilde{\theta}_2) = \pi_{t_1}(\alpha|\theta_1, \tilde{\theta}_2)$. But by the cutoff structure of $\alpha(\cdot, t_2)$, we have $\pi_{t_2}(\alpha|\theta_1, \theta_2) \geq \pi_{t_2}(\alpha|\theta_1, \tilde{\theta}_2)$ for all $\theta_2 > \tilde{\theta}_2$ and $\pi_{t_2}(\alpha|\theta_1, \theta_2) \leq \pi_{t_2}(\alpha|\theta_1, \tilde{\theta}_2)$ for all $\theta_2 < \tilde{\theta}_2$. Thus t_1 is weakly dominated in the auxiliary game for all $(\theta_1, \theta_2) \in A$. Thus choosing only $\{t_2\}$ is an optimal menu.

A.8 Proof of Proposition 8

Proof. I will first prove the following lemma. This result already exists in the literature and I provide a proof for completeness.

Lemma 8. *For any $t \succeq t'$ and $\alpha(\cdot, t')$, there is $\alpha(\cdot, t)$ such that*

$$\sum_x \alpha(x, t)\pi_t(x|\theta) \geq \sum_x \alpha(x, t')\pi_{t'}(x|\theta)$$

$$\textit{for all } \theta' \in R, \quad \sum_x \alpha(x, t)\pi_t(x|\theta') \leq \sum_x \alpha(x, t')\pi_{t'}(x|\theta')$$

Proof. We can prove this lemma by using a theorem of the alternative (see e.g., Rockafellar (2015) Section 22). Only one of the following statement is true:

- There exists $\alpha(\cdot, t)$ such that

$$\sum_x \alpha(x,t)\pi_t(x|\theta) \geq \sum_x \alpha(x,t')\pi_{t'}(x|\theta)$$

$$\text{for all } \theta' \in R, \quad \sum_x \alpha(x,t)\pi_t(x|\theta') \leq \sum_x \alpha(x,t')\pi_{t'}(x|\theta')$$

$$\text{for all } x \in X, \quad \alpha(x,t) \leq 1$$

$$\text{for all } x \in X, \quad \alpha(x,t) \geq 0$$

- There exists $z, y \geq 0$ such that

$$\text{for all } x \in X, \quad -z_\theta \pi_t(x|\theta) + \sum_{\theta' \in R} z_{\theta'} \pi_t(x|\theta') + y_x \geq 0 \tag{4}$$

$$-z_\theta \sum_{x'} \alpha(x',t')\pi_{t'}(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_{x'} \alpha(x',t')\pi_{t'}(x'|\theta') + \sum_{x'} y_{x'} < 0 \tag{5}$$

Take inequality (4) from the second alternative and multiply by $\beta(x, x')$ as described in Definition 6 and sum over $x \in X$:

$$-z_\theta \sum_x \beta(x,x')\pi_t(x|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_x \beta(x,x')\pi_t(x|\theta') + \sum_x \beta(x,x')y_x \geq 0$$

Because $t \succeq_\theta t'$, we get for all $x' \in X$,

$$-z_\theta \pi_{t'}(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \pi_{t'}(x'|\theta') + \sum_x \beta(x,x')y_x \geq 0$$

We can then multiply by $\alpha(x', t')$ and sum over $x' \in X$:

$$-z_\theta \sum_{x'} \alpha(x', t')\pi_t(x'|\theta) + \sum_{\theta' \in R} z_{\theta'} \sum_{x'} \alpha(x', t')\pi_{t'}(x'|\theta') + \sum_{x,x'} \alpha(x', t')\beta(x, x')y_x \geq 0 \quad (6)$$

Because $\sum_{x'} \beta(x, x') \leq 1$ and $\alpha(x', t') \leq 1$ for all $x' \in X$, we have $\sum_{x,x'} \alpha(x', t')\beta(x, x')y_x \leq \sum_x y_x$. Therefore, the inequality (5) cannot hold and the first alternative holds. $\square$

With this result in hand, we can now prove our result. Suppose that t is not part of the optimal menu. Thus we can find an equilibrium of the zero-sum game of Theorem 1, (α, s, s') with $s(t|\theta) = 0$ for all $\theta \in A$. Take a test t' used in equilibrium by some $\theta \in A$. Then from Lemma 8, we can construct a $\tilde{\alpha}$ such that

$$\pi_t(\tilde{\alpha}|\theta) \geq \pi_{t'}(\alpha|\theta)$$

$$\text{for all } \theta' \in R, \quad \pi_t(\tilde{\alpha}|\theta') \leq \pi_{t'}(\alpha|\theta')$$

If the first inequality is strict or the second such that $s'(\theta'|\theta) > 0$ is strict then we have a strict profitable deviation. Otherwise, we have constructed a new equilibrium of the zero-sum game.

$\square$

A.9 Proof of Proposition 9

Proof. ($\Leftarrow$) For each $\theta \in A$, let t_θ such that

$$\operatorname{supp} \pi_t(\cdot|\theta) \cap \left(\cup_{\theta' \in R} \operatorname{supp} \pi_t(\cdot|\theta') \right) = \emptyset$$

Then posting a menu $(t_\theta)_{\theta\in A}$ is optimal (eliminating duplicates if there are some). Each $\theta \in A$ chooses t_θ. For any strategy of $\theta' \in R$, the DM accepts after any $(x,t) \in \cup_{\theta:\sigma(t|\theta)=1} \operatorname{supp} \pi_t(\cdot|\theta)$ and rejects otherwise. This gives the DM and the A-types maximal payoffs and the R-types get rejected for any strategy they follow.

($\Rightarrow$) Suppose the DM's payoffs are maximal and there is $\theta \in A$ and for all $t \in T$ there is $\theta' \in R$ and $x \in X$ such that $\pi_t(x|\theta), \pi_t(x|\theta') > 0$. Then when θ chooses t out of the menu of tests, if θ' chooses t as well, at x, either the DM accepts θ' or rejects θ. Therefore, payoffs cannot be maximal. $\square$

A.10 Proof of Theorem 2

The only thing we need prove is that it is optimal to have a different message for each type $\theta \in A$, the rest follows from Theorem 1. Suppose it is not the case and take a saddle-point (α, s, s') of the zero-sum game.

There is $\theta_1, \theta_2 \in A$ and $(t,m) \in T \times M$ such that $s(t,m|\theta_1) = s(t,m|\theta_2) = 1$ (if they use a different test then we can also change the message and nothing is changed). Then consider the alternative strategy α' where, for some unused (t,m') in the original mechanism, $\alpha'(t,m',x) = \alpha(t,m,x)$ for all $x \in X$ and $\alpha'(t'',m'',x) = \alpha'(t'',m'',x)$ for all other $(t'',m'') \in T \times M$ and all $x \in X$ otherwise. The new strategy α' is thus the same as α but makes sure that if the pair (t,m') is chosen, it uses the same actions as (t,m). Now consider the following strategy $\tilde{s}(\cdot|\theta)$ for $\theta \in A$ in the auxiliary game, $\tilde{s}(\cdot|\theta) = s(\cdot|\theta)$ for $\theta \neq \theta_1$ and $\tilde{s}(t,m'|\theta_1) = 1$. In the zero-sum game under the strategy α', the payoffs are the same than under (α, s, s') for all types. Moreover, any deviations under α' gives the same payoff than under α. Therefore, $(\alpha', \tilde{s}, s')$ is an equilibrium of the zero-sum game and $v(\alpha, s, s') = v(\alpha', \tilde{s}, s')$. Either α' is a best response to $(\tilde{s}, s')$, $(\alpha', \tilde{s}, s')$ is saddle-point of

v and characterises an optimal menu. Or, α' is not a best-response and there is $\tilde{\alpha}$ such that $v(\tilde{\alpha}, \tilde{s}, s') > v(\alpha', \tilde{s}, s') = v(\alpha, s, s')$. This would contradict that (α, s, s') is a saddle point of v.

A.11 Proof of Proposition 10

The way this proof proceed is by first arguing that an optimal mechanism $\tilde{\sigma} : \Theta \to \Delta(T \times M)$ does weakly better than an optimal GR-mechanism, τ. Then I will show that the outcome of the optimal mechanism $\tilde{\sigma}$ can be implemented by a GR-game.

To see the first part, note that a GR-mechanism can be rewritten as a mechanism $\tilde{\tau} : M \to \Delta(T)$ and a DM-strategy $\tilde{\alpha} : M \times T \times X \to [0,1]$. Then we can implement any equilibrium outcome of $(\tilde{\tau}, \tilde{\alpha}, \delta)$, where δ is the agent's strategy by a mechanism and strategy of the DM, $(\tilde{\sigma}, \alpha)$ by setting $\tilde{\sigma} = \tilde{\tau} \circ \delta$, the composition of the GR-mechanism and the agent's strategy and $\alpha = \tilde{\alpha}$. This does not change the outcome so all the agent's incentives are preserved.

I will now show that the outcome of the menu game with communication can be implemented in a GR-game.

Remember that we have established that in the zero-sum game, all the A-types play a pure strategy and send a different message (Theorem 2). This implies that it is without loss of optimality to decompose the A-types' strategy s in choosing a message $m \in M$, call it $\phi : A \to M$ and a test for each message, call it $\rho : M \to T$.

Abusing notation define

$$\pi_t(\alpha\theta, m) = \sum_x \alpha(t, x, m)\pi_t(x|\theta)$$

$$v(\alpha, \phi, \rho, s') = \sum_m \mathbb{1}\left[(t,\theta) : t = \rho(m), m = \phi(\theta)\right]\left[\mu(\theta)\pi_t(\alpha|\theta, m) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_t(\alpha|\theta', m)\right]$$

To understand the new version of v, we sum over all messages and for each message, we select the the test associated with it and the A-type choosing that message.

We get,

$$\min_{s'} \max_{\alpha,s} v(\alpha, s, s') = \max_{\alpha,s} \min_{s'} v(\alpha, s, s') = \max_{\alpha,\phi,\rho} \min_{s'} v(\alpha, \phi, \rho, s') = \min_{s'} \max_{\alpha,\phi,\rho} v(\alpha, \phi, \rho, s')$$

But now observe that we could equivalently interpret ρ as being chosen by the DM as it maximises his payoffs. We are left to check that ϕ and s' generate equilibrium strategies.

As before the R-types select an A-type's strategy. Because they are playing a pure strategy, this is equivalent to choosing an on-path m taking into account that the test will be $t = \phi(m)$ to maximise $\pi_{t=\phi(m)}(\alpha|\theta', m)$. The A-types choose m if

$$\mu(\theta)\pi_{\phi(m)}(\alpha|\theta, m) - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_{\phi(m)}(\alpha|\theta', m) \geq \mu(\theta)\pi_{\phi(m')}(\alpha|\theta, m') - \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\pi_{\phi(m')}(\alpha|\theta'$$

$$\Leftrightarrow \mu(\theta)\big[\pi_{\phi(m)}(\alpha|\theta, m) - \pi_{\phi(m')}(\alpha|\theta, m')\big] \geq \sum_{\theta' \in R} \mu(\theta')s'(\theta|\theta')\big(\pi_{\phi(m)}(\alpha|\theta', m) - \pi_{\phi(m')}(\alpha|\theta', m')\big) \geq 0$$

where the last line uses the equilibrium behaviour of R-types to get that $s'(\theta|\theta')$ implies $\pi_{\phi(m)}(\alpha|\theta', m) - \pi_{\phi(m')}(\alpha|\theta', m') \geq 0$.