Monte Carlo Simulation of areas

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Monte Carlo Integration for $\int_1^b e^{-x^3}\,dx$

Step 1: Approximate the Integral for $b = 6$ Using Monte Carlo Integration

1. **Define the Function**: The function we want to integrate is

$$f(x) = e^{-x^3}$$

   over the interval $[1, b]$ with $b = 6$.

2. **Generate Uniformly Distributed Random Variables**: We generate a set of random variables x_i uniformly distributed between 1 and 6. We choose a large number of samples, say $N = 10,000$, to get a good approximation.

3. **Calculate the Monte Carlo Estimate**:

   Using Monte Carlo integration, we can approximate the integral as follows:

$$\int_1^6 e^{-x^3}\,dx \approx \frac{b-a}{N} \sum_{i=1}^{N} f(x_i)$$

   where $a = 1$ and $b = 6$, and x_i are our uniformly distributed samples over $[1, 6]$.

```r
# Define parameters
  N <- 10000
  a <- 1
  b <- 6

  # Generate random samples and evaluate the function at these points
  x <- runif(N, min = a, max = b)
  f_x <- exp(-x^3)

  # Monte Carlo estimate of the integral
  integral_approx <- (b - a) * mean(f_x)
  integral_approx
```

```
## [1] 0.08862736
```

4. **Comparison Using Exact Integration:** To verify the accuracy of our Monte Carlo approximation, we compare it to the exact integral using R's integrate function:

```r
# Compute the exact integral
exact_result <- integrate(function(x) exp(-x^3), lower = 1, upper = 6)
exact_result$value
```

```
## [1] 0.08546833
```

Step 2: Approximate the Integral for $b = \infty$ Using Monte Carlo Integration

1. **Define the Function**: The function we want to integrate is:

$$f(x) = e^{-x^3}$$

   over the interval $[1, \infty)$.

2. **Choose a Suitable Density**: Since our integration range is $[1, \infty)$, we choose an exponential distribution with support $[0, \infty)$ and shift it by 1. Let $X = 1 + Y$ where $Y \sim \text{Exponential}(\lambda = 1)$. Then we approximate the integral as:

$$\int_1^\infty e^{-x^3}\, dx \approx \frac{1}{N} \sum_{i=1}^{N} \frac{e^{-(1+y_i)^3}}{\lambda e^{-\lambda y_i}}.$$

```r
# Define parameters
N <- 10000
lambda <- 1

# Generate random samples from the exponential distribution
y <- rexp(N, rate = lambda)

# Calculate the Monte Carlo estimate
integral_approx_infinity <- mean(exp(-(1 + y)^3) / (lambda * exp(-lambda * y)))
integral_approx_infinity
```

```
## [1] 0.08583759
```

Comparison Using Exact Integration: To verify the accuracy of our Monte Carlo approximation, we use R's integrate function to compute the integral with an upper limit of infinity:

```r
# Define the function
f <- function(x) exp(-x^3)

# Compute the exact integral from 1 to infinity
exact_result_infinity <- integrate(f, lower = 1, upper = Inf)
exact_result_infinity$value
```

```
## [1] 0.08546833
```

step 3: Why Monte Carlo integration works better in step 2 that step 1?

In step 1, the Monte Carlo integration used a **uniform distribution** over $[1, 6]$ for the integral

$$\int_1^6 e^{-x^3}\, dx.$$

Since e^{-x^3} decreases rapidly as x increases, most of the integral's value comes from x near 1. The uniform distribution samples points evenly across the entire interval, including regions where the function is very small, leading to a less accurate estimate.

In step 2, we used an **exponential distribution** shifted to $[1, \infty)$ for

$$\int_1^\infty e^{-x^3}\, dx.$$

This distribution places more samples near $x = 1$, where e^{-x^3} is largest, reducing variance and improving accuracy. This alignment between the sampling density and the function's decay results in a closer match with the `integrate` function's result.