Mechanics of Materials 14 (1992) 127–139
Elsevier


# Micromechanical analysis of composites by the generalized cells model

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Received 7 November 1991; revised version received 8 July 1992

In the original formulation of the micromechanical method of cells, designated for the analysis of fibrous composites with periodic structure, the repeating volume element consists of four interacting subcells. The various capabilities and reliability of the micromechanical model were verified in a recent review paper and a monograph. The present investigation offers a generalization of the method to an arbitrary number of subcells for the modeling of multiphase periodic composites. Such a generalization is particularly advantageous when dealing with elastic–plastic composites, since yielding and plastic flow of a metallic phase may take place at different locations. Effective constitutive laws that govern overall behavior of the elastic–viscoplastic composite material are established. These laws are given in terms of relationships between the average stress-rate and strain-rate of the inelastic multiphase composite. Comparisons between the response of boron/aluminum composite obtained by the present model and a finite element solution are given.

## 1. Introduction

The utilization of the material properties of the phases of a composite for the determination of its overall behavior, via a micromechanical analysis, is of a significant advantage over a macromechanical approach. This stems from the fact that no a priori assumptions are needed concerning the behavior of the composite which is, generally, anisotropic. The pronounced advantage of a micromechanical analysis is even more crucial when dealing with elastoplastic composites (e.g., metal-matrix composites) where anisotropy is induced in the metallic phase due to the plastic flow, and the resulting behavior becomes load and path dependent.

A micromechanical analysis by the method of cells model was previously presented by the second author. Its capability and reliability in providing the response of elastic, thermoelastic, viscoelastic and viscoplastic composites, as well as their yield surfaces, strength envelopes and fatigue failure curves were discussed in Aboudi (1989, 1991).

In its original formulation, the method of cells models a fibrous composite of a periodic structure. The repeating volume element consists of a fiber region together with three matrix domains. The inherent refinement involved in the model in which the matrix region is divided into three subregions, proved to be very crucial and fruitful in analyzing metal-matrix composites, because yielding and plastic flow of the metallic matrix might be different at various locations. Thus, it is advantageous, especially in elastoplastic composites, to divide the metallic phase into several subregions.

In the present paper a generalization of the method of cells for the micromechanical analysis of elastic–viscoplastic composites is presented. The proposed generalization is capable to model multiphase composites in which each phase can be considered to be an elastic–viscoplastic material. The composite is assumed to possess a peri-

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odic structure, such that a repeating rectangular representative element can be divided into any number of rectangular subcells (rather than four subcells in the original formulation). This generalization allows the modeling of composites with various types of phase arrangements and shapes. As in the original formulation of the method of cells, the determination of the overall behavior of a perfectly elastic composite is obtained in the special case when the inelastic effects are disregarded. The spatial variation of the stresses in the phase can be determined presently with high accuracy. Furthermore, it is possible to analyze various types of phase arrangements, directions of anisotropy and interphases.

The micromechanical analysis is based on the theory of continuum media in which equilibrium is ensured, and the continuity of the rates of displacements and tractions at the interfaces between subcells and between neighboring repeating cells are imposed on an average basis. The fulfillment of equilibrium in the subcell region is guaranteed by the assumption that the displacement vector is linearly expanded in terms of the local coordinates of the subcell. A smoothing operation, according to which the discrete nature of the composite is eliminated, leads to a set of continuum relations. These relations are valid at any point of the equivalent continuum that represents the multiphase viscoplastic material. The use of elastic–viscoplastic constitutive laws of the phases, or, equivalently, the use of their instantaneous stiffness tensors provide effective constitutive equations which govern the overall behavior of the composite. This behavior is presented as relationships between the average stress rates and strain rates of the multiphase composite. The derived effective constitutive laws are valid for any combination of applied loading. Finally, as a result of the present systematic formulation, the computer programming of the established constitutive laws appears to be rather simple.

The presented micromechanical analysis consists essentially of four steps. It starts by identifying a repeating volume element of the periodic multiphase composite, followed by the definition of macroscopic average stresses and strains from the microscopic ones. In the third step the continuity of traction and displacement rates are imposed at the interfaces between the constituents. These establish, in conjunction with microequilibrium, the relationship between microscopic total and plastic strains and macroscopic strains via the relevant concentration tensors. In the final step the overall macroscopic constitutive equations of the composite are determined. These four steps form the basis of micro-to-macromechanics analyses which describe the behavior of heterogeneous media (Suquet, 1985). The resulting micromechanical analysis establishes the overall elastoplastic behavior of the multiphase inelastic composite. This is expressed as a constitutive relation between the average stress, strain, and plastic strain, in conjunction with the effective elastic stiffness tensor.

The capability of the method is assessed by comparing the predicted response of a unidirectional boron/aluminum composite generated by a personal computer, with that based on a finite element solution recently obtained by Brockenbrough et al. (1991). Good agreement between the model prediction and the numerical solution is shown to exist.

For inelastic composites with periodic structures, several types of micromechanical analyses have been performed. In the periodic hexagonal array model of Dvorak and Teply (1985) a representative volume element is divided into several finite elements. Piecewise uniform fields are introduced through appropriate shape functions, and a finite element model is used to obtain estimates of instantaneous field as well as bounds on instantaneous moduli. A different approach was adopted by Accorsi and Nemat-Nasser (1986) establishing bounds of the overall instantaneous moduli of composites with periodic microstructure. Their analysis employs extremum principles in conjunction with the analytical solution of Nemat-Nasser et al. (1982) where the assumption of periodicity allows the heterogeneous field to be expanded in a Fourier series. Walker et al. (1989) employ a homogenization procedure, and integral equations are established using Fourier series or Green's function approaches in modeling the micromechanical behavior of nonlinear composite materials with periodic microstructure.

## 2. Model description

Consider a unidirectional fibrous composite in which the continuous fibers extend in the $x_1$-direction. It is assumed that the composite possesses a periodic structure such that a representative repeating volume element is of the form shown in Fig. 1. This basic element consists of $N_\beta \times N_\gamma$ subcells. Each one of these subcells is occupied by an elastic–viscoplastic material. Thus the representative volume element consists, in the most general case, of $N_\beta \times N_\gamma$ different viscoplastic materials, i.e., it represents a multiphase inelastic composite. This is of a particular importance, since even in a two-phase metal-matrix composite it is necessary to model the metallic matrix by several subcells, as yielding and plastic flow might take place at different points within the matrix. This requirement can be fulfilled by the present generalized model, since the metallic-matrix region can be represented by as many subcells as desired.

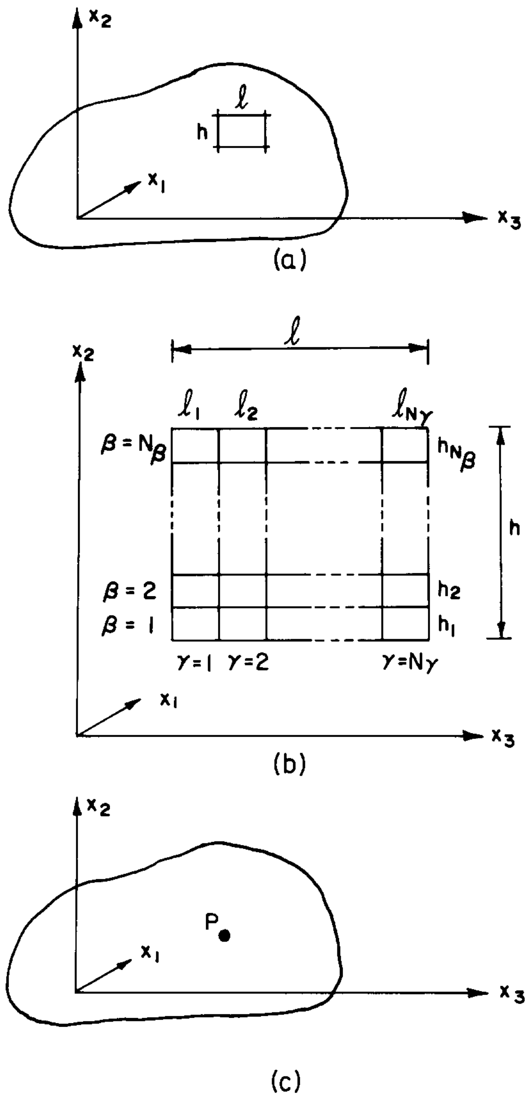


Fig. 1. (a) A repeating volume element of a multiphase unidirectional fibrous composite with a periodic structure; (b) a typical repeating volume element that consists of $N_\beta \times N_\gamma$ subcells; (c) the equivalent continuum medium in which the repeating volume element is represented by the point P.

The repeating element shown in Fig. 1 is a generalization of the original method of cells which was presented by Aboudi (see Aboudi (1989, 1991) for example). In its original formulation the method models a unidirectional two-phase composites for which the representative repeating cell consists of four subcells one of which stands for the fiber material, while the other three are occupied by the matrix material, see Fig. 2. In the present generalization, on the other hand, the repeating cells contains $N_\beta \times N_\gamma$ subcells.

In Fig. 1, the rectangular fibers extend in the $x_1$-direction, and the area of the cross section of each subcell is $h_\beta l_\gamma$, with $\beta = 1, \dots, N_\beta$; $\gamma = 1, \dots, N_\gamma$. Let $N_\beta \times N_\gamma$ local coordinate systems $(x_1, \bar{x}_2^{(\beta)}, \bar{x}_3^{(\gamma)})$ be introduced whose origins are located at the center of each subcell, see Fig. 3.

Since the average behavior of the composite is sought, it turns out that a first order theory in which the displacement rates $\dot{u}_i^{(\beta\gamma)}$ (dots denote time derivatives) in the subcell are expanded linearly in terms of the distances from the center of the subcell (i.e., in terms of $\bar{x}_2^{(\beta)}$ and $\bar{x}_3^{(\gamma)}$) is sufficient for this purpose. Consequently, the following first order expansion in the subcell $(\beta\gamma)$ is considered

$$\dot{u}_i^{(\beta\gamma)} = \dot{w}_i^{(\beta\gamma)} + \bar{x}_2^{(\beta)} \dot{\phi}_i^{(\beta\gamma)} + \bar{x}_3^{(\gamma)} \dot{\psi}_i^{(\beta\gamma)}, \quad i = 1, 2, 3, \tag{1}$$

where $\dot{w}_i^{(\beta\gamma)}$ are the rate of displacement components at the center of the subcell, and $\dot{\phi}_i^{(\beta\gamma)}$, $\dot{\psi}_i^{(\beta\gamma)}$ are microvariables rates that characterize the lin-

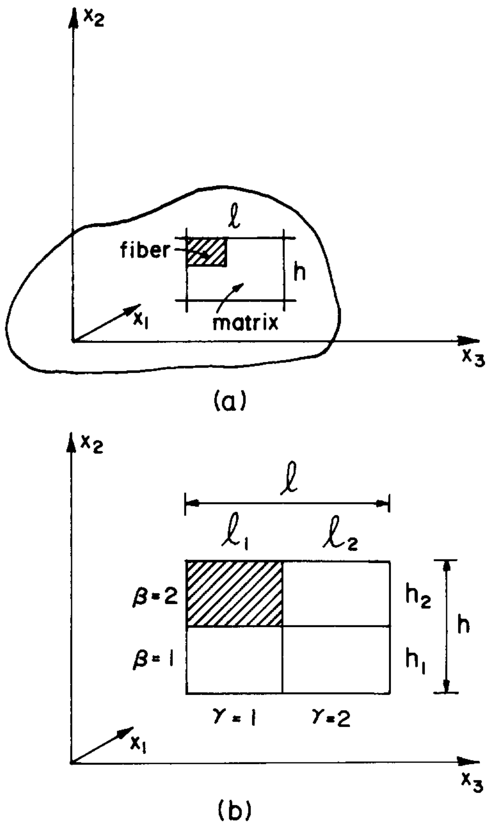


Fig. 2. (a) A repeating volume element of a two-phase unidirectional composite; (b) a typical repeating volume element that consists of 4 subcells.

ear dependence of the displacement rates on the local coordinates $\bar{x}_2^{(\beta)}$, $\bar{x}_3^{(\gamma)}$. In Eq. (1) and the sequel, repeated Greek letters do not imply summation.

The components of the small strain-rate tensor are given by

$$\eta_{ij}^{(\beta\gamma)} = \tfrac{1}{2}\left(\partial_i \dot{u}_j^{(\beta\gamma)} + \partial_j \dot{u}_i^{(\beta\gamma)}\right), \quad i, j = 1, 2, 3, \tag{2}$$

where $\partial_1 = \partial/\partial x_1$, $\partial_2 = \partial/\partial \bar{x}_2^{(\beta)}$ and $\partial_3 = \partial/\partial \bar{x}_3^{(\gamma)}$.

The average strain-rates in subcell, $\bar{\eta}_{ij}^{(\beta\gamma)}$, are given according to Eqs. (1) and (2) by

$$\begin{aligned}
&\bar{\eta}_{11}^{(\beta\gamma)} = \frac{\partial}{\partial x_1}\dot{w}_1^{(\beta\gamma)},\\
&\bar{\eta}_{22}^{(\beta\gamma)} = \dot{\phi}_2^{(\beta\gamma)},\\
&\bar{\eta}_{33}^{(\beta\gamma)} = \dot{\psi}_3^{(\beta\gamma)},\\
&2\bar{\eta}_{23}^{(\beta\gamma)} = \dot{\phi}_3^{(\beta\gamma)} + \dot{\psi}_2^{(\beta\gamma)},\\
&2\bar{\eta}_{13}^{(\beta\gamma)} = \dot{\psi}_1^{(\beta\gamma)} + \frac{\partial}{\partial x_1}\dot{w}_3^{(\beta\gamma)},\\
&2\bar{\eta}_{12}^{(\beta\gamma)} = \dot{\phi}_1^{(\beta\gamma)} + \frac{\partial}{\partial x_1}\dot{w}_2^{(\beta\gamma)}.
\end{aligned} \tag{3}$$

The constitutive law of the elastic–viscoplastic material that occupies the subcell $(\beta\gamma)$ is expressed as a relationship between the average rates of stress $\bar{\tau}_{ij}^{(\beta\gamma)}$, total strain $\bar{\eta}_{ij}^{(\beta\gamma)}$, and plastic strain $\bar{\eta}_{ij}^{\mathrm{P}(\beta\gamma)}$ in the subcell in the form

$$\bar{\tau}_{ij}^{(\beta\gamma)} = C_{ijkl}^{(\beta\gamma)}\left(\bar{\eta}_{kl}^{(\beta\gamma)} - \bar{\eta}_{kl}^{\mathrm{P}(\beta\gamma)}\right), \tag{4}$$

where $C_{ijkl}^{(\beta\gamma)}$ is the elastic stiffness tensor of material in the subcell $(\beta\gamma)$. This constitutive law of the material results from the decomposition of

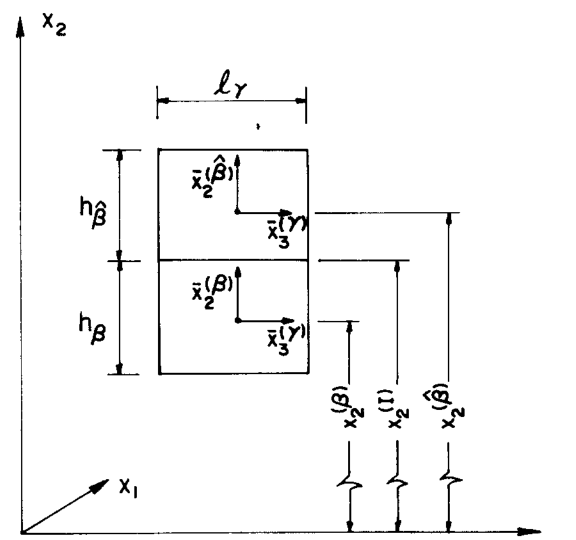


Fig. 3. The local coordinates and locations of interface and centerlines in typical two neighboring subcells.

the total strain-rates into elastic (reversible) and plastic (irreversible) strain-rates in conjunction with the rate of Hooke's law of the anisotropic material.

Alternatively, the constitutive law of the material in subcell $(\beta\gamma)$ can be represented in the form

$$\bar{\tau}_{ij}^{(\beta\gamma)} = C_{ijkl}^{\mathrm{VP}(\beta\gamma)}\bar{\eta}_{kl}^{(\beta\gamma)}, \tag{5}$$

where $C_{ijkl}^{\mathrm{VP}(\beta\gamma)}$ are the components of the instantaneous stiffness tensor of the material. This tensor depends on the deformation history, loading path and applied loading rate. Its explicit form can be found in Paley and Aboudi (1991). In the special case of a perfectly elastic material this fourth order tensor reduces to the standard elastic stiffness tensor which characterizes the behavior of the material in the subcell $(\beta\gamma)$.

It will be shown in the following that by employing the rate of displacements and tractions continuity conditions at the interfaces between subcells and between neighboring repeating cells, it is possible to eliminate the microvariables and obtain, via a smoothing operation procedure, a set of continuum equations that model the overall behavior of the multiphase viscoplastic composite.

## 3. Interfacial continuity of displacement-rates

### 3.1. Derivation of a set of continuum relations in terms of microvariables

At any instant, the components of the displacement-rates should be continuous at the interfaces between the subcells of the repeating cell, and between neighboring repeating cells. This implies that for $\beta = 1,\ldots,N_\beta$; $\gamma = 1,\ldots,N_\gamma$, the following relations hold

$$\dot{u}_i^{(\beta\gamma)}\Big|_{\bar{x}_2^{(\beta)}=h_\beta/2} = \dot{u}_i^{(\hat{\beta}\gamma)}\Big|_{\bar{x}_2^{(\hat{\beta})}=-h_{\hat{\beta}}/2}, \tag{6}$$

$$\dot{u}_i^{(\beta\gamma)}\Big|_{\bar{x}_3^{(\gamma)}=l_\gamma/2} = \dot{u}_i^{(\beta\hat{\gamma})}\Big|_{\bar{x}_3^{(\hat{\gamma})}=-l_{\hat{\gamma}}/2}, \tag{7}$$

where $\hat{\beta}$ and $\hat{\gamma}$ are defined by

$$\hat{\beta} = \begin{cases} \beta+1, & \beta < N_\beta, \\ 1, & \beta = N_\beta \end{cases} \tag{8}$$

and

$$\hat{\gamma} = \begin{cases} \gamma+1, & \gamma < N_\gamma, \\ 1, & \gamma = N_\gamma. \end{cases} \tag{9}$$

Note that since $\hat{\beta} = 1$ for $\beta = N_\beta$, and $\hat{\gamma} = 1$ for $\gamma = N_\gamma$, continuity of the displacement rates at the interfaces between neighboring repeating cells is ensured.

Continuity conditions (6) are imposed at the interfaces on an average basis, i.e.,

$$\int_{-l_\gamma/2}^{l_\gamma/2} \dot{u}_i^{(\beta\gamma)}\Big|_{\bar{x}_2^{(\beta)}=h_\beta/2} \mathrm{d}\bar{x}_3^{(\gamma)} = \int_{-l_\gamma/2}^{l_\gamma/2} \dot{u}_i^{(\hat{\beta}\gamma)}\Big|_{\bar{x}_2^{(\hat{\beta})}=-h_{\hat{\beta}}/2} \mathrm{d}\bar{x}_3^{(\gamma)}. \tag{10}$$

Using Eq. (1), it follows that

$$\dot{w}_i^{(\beta\gamma)} + \tfrac{1}{2}h_\beta\dot{\phi}_i^{(\beta\gamma)} = \dot{w}_i^{(\hat{\beta}\gamma)} - \tfrac{1}{2}h_{\hat{\beta}}\dot{\phi}_i^{(\hat{\beta}\gamma)}. \tag{11}$$

A similar relation is obtained from Eq. (7) where the integration is with respect to $\bar{x}_2^{(\beta)}$ from $-h_\beta/2$ to $h_\beta/2$ yielding

$$\dot{w}_i^{(\beta\gamma)} + \tfrac{1}{2}l_\gamma\dot{\psi}_i^{(\beta\gamma)} = \dot{w}_i^{(\beta\hat{\gamma})} - \tfrac{1}{2}l_{\hat{\gamma}}\dot{\psi}_i^{(\beta\hat{\gamma})}. \tag{12}$$

All field quantities in Eq. (11) are evaluated at the centerline $x_2^{(\beta)}$ of the subcell $(\beta\gamma)$, and the centerline $x_2^{(\hat{\beta})}$ of the subcell $(\hat{\beta}\gamma)$, see Fig. 3. Let us denote by $x_2^{(\mathrm{I})}$ the location of the interface between subcell $(\beta\gamma)$ and the following one $(\hat{\beta}\gamma)$. It follows that

$$x_2^{(\beta)} = x_2^{(\mathrm{I})} - \tfrac{1}{2}h_\beta. \tag{13}$$

In addition, the location of the centerline $x_2^{(\hat{\beta})}$ of the following subcell $(\hat{\beta}\gamma)$ with respect to $x_2^{(\mathrm{I})}$ is given by

$$x_2^{(\hat{\beta})} = x_2^{(\mathrm{I})} + \tfrac{1}{2}h_{\hat{\beta}}. \tag{14}$$

By employing a Taylor expansion of the field variables in Eq. (11) and omitting second and higher order terms we have

$$\dot{w}_i^{(\beta\gamma)} - \tfrac{1}{2}h_\beta\left(\frac{\partial}{\partial x_2}\dot{w}_i^{(\beta\gamma)} - \dot{\phi}_i^{(\beta\gamma)}\right) = \dot{w}_i^{(\hat{\beta}\gamma)} + \tfrac{1}{2}h_{\hat{\beta}}\left(\frac{\partial}{\partial x_2}\dot{w}_i^{(\hat{\beta}\gamma)} - \dot{\phi}_i^{(\hat{\beta}\gamma)}\right), \tag{15}$$

where all field variables are evaluated at the interface $x_2^{(1)}$. The discrete structure of the composite is eliminated by mapping the repeating volume element into a point P within the equivalent homogeneous medium that effectively represents the multiphase composite. The transition to the continuum model relies on the assumption that Eq. (15) are simultaneously valid at point P for all $\beta = 1, \dots, N_\beta$ and $\gamma = 1, \dots, N_\gamma$. For a composite subjected to homogeneous boundary conditions, the behavior of all repeating cells is identical, and a uniform field exists at the equivalent homogeneous medium.

Let us define

$$F_i^{(\beta)} = \dot{w}_i^{(\beta\gamma)} + f_i^{(\beta)} - \dot{w}_i^{(\hat{\beta}\gamma)} + f_i^{(\hat{\beta})}, \tag{16}$$

where

$$f_i^{(\beta)} = -\tfrac{1}{2} h_\beta \left( \frac{\partial}{\partial x_2} \dot{w}_i^{(\beta\gamma)} - \dot{\phi}_i^{(\beta\gamma)} \right). \tag{17}$$

It follows that Eqs. (15) can be expressed in the compact form

$$F_i^{(\beta)} = 0, \quad \beta = 1, \dots, N_\beta. \tag{18}$$

Similarly, Eqs. (12) would provide the continuum relations

$$\dot{w}_i^{(\beta\gamma)} - \tfrac{1}{2} l_\gamma \left( \frac{\partial}{\partial x_3} \dot{w}_i^{(\beta\gamma)} - \dot{\psi}_i^{(\beta\gamma)} \right) = \dot{w}_i^{(\beta\hat{\gamma})} + \tfrac{1}{2} l_{\hat{\gamma}} \left( \frac{\partial}{\partial x_3} \dot{w}_i^{(\beta\hat{\gamma})} - \dot{\psi}_i^{(\beta\hat{\gamma})} \right). \tag{19}$$

These relations can be written in the form

$$G_i^{(\gamma)} = 0, \quad \gamma = 1, \dots, N_\gamma, \tag{20}$$

where the following definitions have been employed

$$G_i^{(\gamma)} = \dot{w}_i^{(\beta\gamma)} + g_i^{(\gamma)} - \dot{w}_i^{(\beta\hat{\gamma})} + g_i^{(\hat{\gamma})}, \tag{21}$$

$$g_i^{(\gamma)} = -\tfrac{1}{2} l_\gamma \left( \frac{\partial}{\partial x_3} \dot{w}_i^{(\beta\gamma)} - \dot{\psi}_i^{(\beta\gamma)} \right). \tag{22}$$

Like Eqs. (18), Eqs. (20) are valid simultaneously at point P.

From (18) and (20) we obtain, respectively, that $\sum_{\beta=1}^{N_\beta} F_i^{(\beta)} = 0$ and $\sum_{\gamma=1}^{N_\gamma} G_i^{(\gamma)} = 0$. It can be easily verified that these last two summations imply, respectively, that

$$\sum_{\beta=1}^{N_\beta} f_i^{(\beta)} = 0 \tag{23}$$

and

$$\sum_{\gamma=1}^{N_\gamma} g_i^{(\gamma)} = 0. \tag{24}$$

Since

$$\frac{\partial}{\partial x_2} f_i^{(\beta)} = 0 \tag{25}$$

and

$$\frac{\partial}{\partial x_3} g_i^{(\gamma)} = 0, \tag{26}$$

for all $\beta = 1, \dots, N_\beta$ and $\gamma = 1, \dots, N_\gamma$, it follows, by differentiation of Eqs. (18) and (20) with respect to $x_2$ and $x_3$, respectively, that the following relations result for all $\beta$ and $\gamma$:

$$\frac{\partial}{\partial x_2} \dot{w}_i^{(\beta\gamma)} = \frac{\partial}{\partial x_2} \dot{w}_i^{(\hat{\beta}\gamma)} \tag{27}$$

and

$$\frac{\partial}{\partial x_3} \dot{w}_i^{(\beta\gamma)} = \frac{\partial}{\partial x_3} \dot{w}_i^{(\beta\hat{\gamma})}. \tag{28}$$

Equations (27) and (28) are satisfied by assuming that common velocity functions, $\dot{w}_i$, exist such that

$$\dot{w}_i^{(\beta\gamma)} = \dot{w}_i, \tag{29}$$

for all $\beta$ and $\gamma$. This assumption follows from the fact that the entire repeating volume element of the periodic composite is mapped by a smoothing operation into a single point within the equivalent continuum medium that effectively represents the multiphase composite. The particle velocity at this point is given by $\dot{w}_i$. It will be shown that the governing constitutive laws of this equivalent continuum can be established by the generalized method of cells analysis.

The following $N_\beta + N_\gamma$ continuum relations result from Eqs. (23) and (24), respectively:

$$\sum_{\beta=1}^{N_\beta} h_\beta \dot{\phi}_i^{(\beta\gamma)} = h \frac{\partial}{\partial x_2} \dot{w}_i, \quad \gamma = 1, \ldots, N_\gamma, \tag{30}$$

$$\sum_{\gamma=1}^{N_\gamma} l_\gamma \dot{\psi}_i^{(\beta\gamma)} = l \frac{\partial}{\partial x_3} \dot{w}_i, \quad \beta = 1, \ldots, N_\beta, \tag{31}$$

where $h = \sum_{\beta=1}^{N_\beta} h_\beta$ and $l = \sum_{\gamma=1}^{N_\gamma} l_\gamma$. These continuum equations are valid at point P of the equivalent medium, and have been derived from the continuity conditions of the rate of displacements at the interfaces of the discrete model.

### *3.2. Derivation of the continuum relations in terms of average subcell strain-rates*

The previous set of continuum equations (i.e, Eqs. (30) and (31) are given in terms of the microvariables $\dot{\phi}_i$, $\dot{\psi}_i$. It is possible to derive an equivalent system of equations expressed in terms of the average subcell strain-rates $\bar{\eta}_{ij}^{(\beta\gamma)}$. To this end the standard definition of the composite's average strain-rate $\bar{\eta}_{ij}$ is employed in the form

$$\bar{\eta}_{ij} = \frac{1}{hl} \sum_{\beta=1}^{N_\beta} \sum_{\gamma=1}^{N_\gamma} h_\beta l_\gamma \bar{\eta}_{ij}^{(\beta\gamma)}. \tag{32}$$

Let us first show that

$$\bar{\eta}_{ij} = \frac{1}{2}\left( \frac{\partial \dot{w}_i}{\partial x_j} + \frac{\partial \dot{w}_j}{\partial x_i} \right). \tag{33}$$

For $i = j = 1$, this result follows immediately from (29) and (32) and the first relation in (3), i.e., $\bar{\eta}_{11} = \partial \dot{w}_i / \partial x_1$.

For $i = j = 2$, let us multiply Eq. (30) by $l_\gamma$ and perform a summation over $\gamma$ from 1 to $N_\gamma$. This leads to

$$\sum_{\beta=1}^{N_\beta} \sum_{\gamma=1}^{N_\gamma} h_\beta l_\gamma \dot{\phi}_2^{(\beta\gamma)} = hl \frac{\partial \dot{w}_2}{\partial x_2}. \tag{34}$$

Comparison with Eq. (32), and using the second relation in (3) gives that $\bar{\eta}_{22} = \partial \dot{w}_2 / \partial x_2$.

For $i = 1$, $j = 2$ the corresponding result follows by multiplying Eq. (30) by $l_\gamma$ and summing over $\gamma$. The use of the sixth relation in Eq. (3) readily yields that

$$\bar{\eta}_{12} = \frac{1}{2}\left( \frac{\partial \dot{w}_2}{\partial x_1} + \frac{\partial \dot{w}_1}{\partial x_2} \right).$$

The other three relations in eq. (33) can be similarly established.

It is now possible to express continuum Eqs. (30) and (31) in terms of $\bar{\eta}_{ij}^{(\beta\gamma)}$ and $\bar{\eta}_{ij}$. Setting $i = 2$ in Eq. (30) and $i = 3$ in Eq. (31) yields

$$\sum_{\beta=1}^{N_\beta} h_\beta \bar{\eta}_{22}^{(\beta\gamma)} = h \bar{\eta}_{22}, \quad \gamma = 1, \ldots, N_\gamma, \tag{35}$$

$$\sum_{\gamma=1}^{N_\gamma} l_\gamma \bar{\eta}_{33}^{(\beta\gamma)} = l \bar{\eta}_{33}, \quad \beta = 1, \ldots, N_\beta. \tag{36}$$

Addition of $h\, \partial \dot{w}_2 / \partial x_1$ to Eq. (30), and $l\, \partial \dot{w}_3 / \partial x_1$ to Eq. (31), both equations with $i = 1$, yield, respectively,

$$\sum_{\beta=1}^{N_\beta} h_\beta \left( \dot{\phi}_1^{(\beta\gamma)} + \frac{\partial \dot{w}_2}{\partial x_1} \right) = 2h \bar{\eta}_{12}, \quad \gamma = 1, \ldots, N, \tag{37}$$

and

$$\sum_{\gamma=1}^{N_\gamma} l_\gamma \left( \dot{\psi}_1^{(\beta\gamma)} + \frac{\partial \dot{w}_3}{\partial x_1} \right) = 2l \bar{\eta}_{13}, \quad \beta = 1, \ldots, N_\beta, \tag{38}$$

or, using Eq. (3),

$$\sum_{\beta=1}^{N_\beta} h_\beta \bar{\eta}_{12}^{(\beta\gamma)} = h \bar{\eta}_{12}, \quad \gamma = 1, \ldots, N_\gamma, \tag{39}$$

$$\sum_{\gamma=1}^{N_\gamma} l_\gamma \bar{\eta}_{13}^{(\beta\gamma)} = l \bar{\eta}_{13}, \quad \beta = 1, \ldots, N_\beta. \tag{40}$$

Equations (35), (36), (39) and (40) together with $\bar{\eta}_{11}^{(\beta\gamma)} = \bar{\eta}_{11}$, and Eq. (32) with $i = 2$, $j = 3$ form a system of $2(N_\beta + N_\gamma) + N_\beta N_\gamma + 1$ continuum relations. These relations can be written in a matrix form as follows

$$\boldsymbol{A}_{\mathrm{G}}\boldsymbol{\eta}_{\mathrm{s}}=\boldsymbol{J}\bar{\boldsymbol{\eta}}, \tag{41}$$

where the 6-order average strain-rate vector is of the form

$$\bar{\boldsymbol{\eta}}=\left(\bar{\eta}_{11}, \bar{\eta}_{22}, \bar{\eta}_{33}, 2\bar{\eta}_{23}, 2\bar{\eta}_{13}, 2\bar{\eta}_{12}\right) \tag{42}$$

and the $6N_\beta N_\gamma$ order subcells strain-rate vector is defined by

$$\boldsymbol{\eta}_{\mathrm{s}}=\left(\bar{\boldsymbol{\eta}}^{(11)}, \bar{\boldsymbol{\eta}}^{(12)},\ldots,\bar{\boldsymbol{\eta}}^{(N_\beta N_\gamma)}\right), \tag{43}$$

where the 6 components of the vector $\bar{\boldsymbol{\eta}}^{(\beta\gamma)}$ are arranged as in Eq. (42). The matrix $\boldsymbol{A}_{\mathrm{G}}$ is $2(N_\beta+N_\gamma)+N_\beta N_\gamma+1$ by $6N_\beta N_\gamma$, while $\boldsymbol{J}$ is a $2(N_\beta+N_\gamma)+N_\beta N_\gamma+1$ by 6 matrix. It should be noted that the matrix $\boldsymbol{A}_{\mathrm{G}}$ involves the geometrical properties of the repeating cell of Fig. 1.

## 4. Interfacial continuity of traction-rates

The continuity of the rates of tractions must be imposed at the interfaces between the subcells, and between neighboring repeating cells. Those conditions are imposed on an average basis leading to

$$\bar{\tau}_{2j}^{(\beta\gamma)}=\bar{\tau}_{2j}^{(\hat{\beta}\gamma)} \tag{44}$$

and

$$\bar{\tau}_{3j}^{(\beta\gamma)}=\bar{\tau}_{3j}^{(\beta\hat{\gamma})}, \tag{45}$$

where $j=1, 2, 3$; $\beta=1,\ldots,N_\beta$ and $\gamma=1,\ldots,N_\gamma$. It can be easily verified that there are only $5N_\beta N_\gamma-2(N_\beta+N_\gamma)-1$ independent conditions in Eqs. (44) and (45). These independent interfacial relations are

$$\bar{\tau}_{22}^{(\beta\gamma)}=\bar{\tau}_{22}^{(\hat{\beta}\gamma)},\quad \beta=1,\ldots,N_\beta-1,\quad \gamma=1,\ldots,N_\gamma, \tag{46}$$

$$\bar{\tau}_{33}^{(\beta\gamma)}=\bar{\tau}_{33}^{(\beta\hat{\gamma})},\quad \beta=1,\ldots,N_\beta,\quad \gamma=1,\ldots,N_\gamma-1, \tag{47}$$

$$\bar{\tau}_{23}^{(\beta\gamma)}=\bar{\tau}_{23}^{(\hat{\beta}\gamma)},\quad \beta=1,\ldots,N_\beta-1,\quad \gamma=1,\ldots,N_\gamma, \tag{48}$$

$$\bar{\tau}_{32}^{(\beta\gamma)}=\bar{\tau}_{32}^{(\beta\hat{\gamma})},\quad \beta=N_\beta,\quad \gamma=1,\ldots,N_\gamma-1, \tag{49}$$

$$\bar{\tau}_{21}^{(\beta\gamma)}=\bar{\tau}_{21}^{(\hat{\beta}\gamma)},\quad \beta=1,\ldots,N_\beta-1,\quad \gamma=1,\ldots,N_\gamma, \tag{50}$$

$$\bar{\tau}_{31}^{(\beta\gamma)}=\bar{\tau}_{31}^{(\beta\hat{\gamma})},\quad \beta=1,\ldots,N_\beta,\quad \gamma=1,\ldots,N_\gamma-1. \tag{51}$$

The continuum model requires the use of these relations altogether at point P of the equivalent medium that effectively models the models the composite behavior.

By expressing the average stress rates $\bar{\tau}_{ij}^{(\beta\gamma)}$ in the subcells in terms of the average strain rates $\bar{\eta}_{ij}^{(\beta\gamma)}$ via the instantaneous viscoplastic stiffness tensor $C_{ijkl}^{\mathrm{VP}(\beta\gamma)}$ (see Eq. (5)), one obtains from (46)–(51) a system of continuum equations which can be written in the following matrix form

$$\boldsymbol{A}_{\mathrm{M}}^{\mathrm{VP}}\boldsymbol{\eta}_{\mathrm{s}}=\boldsymbol{0}, \tag{52}$$

where the $5N_\beta N_\gamma-2(N_\beta+N_\gamma)-1$ by $6N_\beta N_\gamma$ matrix $\boldsymbol{A}_{\mathrm{M}}^{\mathrm{VP}}$ involves the instantaneous properties of the viscoplastic materials in the various subcells.

## 5. Effective composite constitutive relations

### 5.1. Overall instantaneous stiffness tensor

The micromechanically based constitutive law that governs the overall behavior of the multiphase viscoplastic composite is established by combining the rate of displacements and tractions continuum equations, Eqs. (41) and (52). This yields a system of $6N_\beta N_\gamma$ continuum relations which can be written in the form

$$\tilde{\boldsymbol{A}}^{\mathrm{VP}}\boldsymbol{\eta}_{\mathrm{s}}=\boldsymbol{K}\bar{\boldsymbol{\eta}}, \tag{53}$$

where the $6N_\beta N_\gamma$ order square matrix $\tilde{\boldsymbol{A}}^{\mathrm{VP}}$ is combined of

$$\tilde{\boldsymbol{A}}^{\mathrm{VP}}=\begin{bmatrix}\boldsymbol{A}_{\mathrm{M}}^{\mathrm{VP}}\\ \boldsymbol{A}_{\mathrm{G}}\end{bmatrix} \tag{54}$$

and the $6N_\beta N_\gamma$ by 6 matrix $\boldsymbol{K}$ is

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{J} \end{bmatrix} \tag{55}$$

Solving Eq. (53) for the subcells strain rates $\boldsymbol{\eta}_s$ provides

$$\boldsymbol{\eta}_s = \boldsymbol{A}^{VP}\bar{\boldsymbol{\eta}}, \tag{56}$$

where

$$\boldsymbol{A}^{VP} = [\tilde{\boldsymbol{A}}^{VP}]^{-1}\boldsymbol{K}, \tag{57}$$

which is the instantaneous strain-rate concentration tensor that relates the average strain-rates in the subcells to the average total strain-rates.

The concentration matrix $\boldsymbol{A}^{VP}$ can be partitioned into $N_\beta N_\gamma$ 6-order square submatrices as follows:

$$\boldsymbol{A}^{VP} = \begin{bmatrix} \boldsymbol{A}^{VP(11)} \\ \vdots \\ \boldsymbol{A}^{VP(N_\beta N_\gamma)} \end{bmatrix}. \tag{58}$$

Each submatrix $\boldsymbol{A}^{VP(\beta\gamma)}$ relates the average strain-rates in the subcell $(\beta\gamma)$ to the average total strain rate, namely,

$$\bar{\boldsymbol{\eta}}^{(\beta\gamma)} = \boldsymbol{A}^{VP(\beta\gamma)}\bar{\boldsymbol{\eta}}. \tag{59}$$

Employing Eq. (5) in (59) yields

$$\bar{\boldsymbol{\tau}}^{(\beta\gamma)} = \boldsymbol{C}^{VP(\beta\gamma)}\boldsymbol{A}^{VP(\beta\gamma)}\bar{\boldsymbol{\eta}}. \tag{60}$$

The average stress-rate in the composite is defined by

$$\bar{\boldsymbol{\tau}} = \frac{1}{hl}\sum_{\beta=1}^{N_\beta}\sum_{\gamma=1}^{N_\gamma} h_\beta l_\gamma \bar{\boldsymbol{\tau}}^{(\beta\gamma)}. \tag{61}$$

Consequently the following composite effective constitutive law for the average field quantities has been established

$$\bar{\boldsymbol{\tau}} = \boldsymbol{B}^{*VP}\bar{\boldsymbol{\eta}}, \tag{62}$$

where the effective instantaneous viscoplastic tensor of the multiphase composite is given by

$$\boldsymbol{B}^{*VP} = \frac{1}{hl}\sum_{\beta=1}^{N_\beta}\sum_{\gamma=1}^{N_\gamma} h_\beta l_\gamma \boldsymbol{C}^{VP(\beta\gamma)}\boldsymbol{A}^{VP(\beta\gamma)}. \tag{63}$$

*5.2. Overall elastic–plastic behavior*

The constitutive law, Eqs. (62), of the composite is given in terms of the effective instantaneous stiffness tensor $\boldsymbol{B}^{*VP}$. Alternatively, it is possible to establish the overall elastoplastic constitutive behavior of the composite in terms of the effective elastic stiffness tensor and the average plastic strain-rate tensor. To this end, we start from Eq. (4) (rather than (5) in the previous case). Following the procedure presented in Sections 3–4, we arrive to the following relation (which replaces Eq. (52)):

$$\boldsymbol{A}_M(\boldsymbol{\eta}_s - \boldsymbol{\eta}_s^P) = \boldsymbol{0}, \tag{64}$$

where the $5N_\beta N_\gamma - 2(N_\beta + N_\gamma) - 1$ by $6N_\beta N_\gamma$ matrix $\boldsymbol{A}_M$ involves presently the elastic properties $\boldsymbol{C}^{(\beta\gamma)}$ of the material in the subcell, and $\boldsymbol{\eta}_s^P$ is the vector that represents the plastic strain-rates of the subcells, i.e., $\boldsymbol{\eta}_s^P = (\bar{\boldsymbol{\eta}}^{P(11)}, \bar{\boldsymbol{\eta}}^{P(12)}, \ldots, \bar{\boldsymbol{\eta}}^{P(N_\beta, N_\gamma)})$. The combination of Eq. (41) and (64) leads to

$$\tilde{\boldsymbol{A}}\boldsymbol{\eta}_s - \tilde{\boldsymbol{A}}^P\boldsymbol{\eta}_s^P = \boldsymbol{K}\bar{\boldsymbol{\eta}}, \tag{65}$$

where

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A}_M \\ \boldsymbol{A}_G \end{bmatrix}$$

and

$$\tilde{\boldsymbol{A}}^P = \begin{bmatrix} \boldsymbol{A}_M \\ \boldsymbol{0} \end{bmatrix}.$$

From Eq. (65) we obtain (like Eq. (56)) an expression for the subcells strain-rates $\boldsymbol{\eta}_s$ in terms of the composite average strain rates

$$\boldsymbol{\eta}_s = \boldsymbol{A}\bar{\boldsymbol{\eta}} + \boldsymbol{A}^P\boldsymbol{\eta}_s^P, \tag{66}$$

where

$$\boldsymbol{A} = \tilde{\boldsymbol{A}}^{-1}\boldsymbol{K}$$

and

$$\boldsymbol{A}^P = \tilde{\boldsymbol{A}}^{-1}\tilde{\boldsymbol{A}}^P.$$

The concentration matrix $A$ can be partitioned into $N_\beta N_\gamma$ 6-order square submatrices in the form:

$$A = \begin{bmatrix} A^{(11)} \\ \vdots \\ A^{(N_\beta N_\gamma)} \end{bmatrix}. \tag{67}$$

Similarly, let

$$A^{\mathrm{P}} = \begin{bmatrix} A^{\mathrm{P}(11)} \\ \vdots \\ A^{\mathrm{P}(N_\beta N_\gamma)} \end{bmatrix}. \tag{68}$$

Using (67) and (68), the following relation between the average strain-rates in the subcell $(\beta\gamma)$ to the average strain-rate can be established

$$\bar{\boldsymbol{\eta}}^{(\beta\gamma)} = \boldsymbol{A}^{(\beta\gamma)}\bar{\boldsymbol{\eta}} + \boldsymbol{A}^{\mathrm{P}(\beta\gamma)}\boldsymbol{\eta}_{\mathrm{s}}^{\mathrm{P}}. \tag{69}$$

Employing Eq. (4) in (69) yields

$$\bar{\boldsymbol{\tau}}^{(\beta\gamma)} = \boldsymbol{C}^{(\beta\gamma)}\left(\boldsymbol{A}^{(\beta\gamma)}\bar{\boldsymbol{\eta}} + \boldsymbol{A}^{\mathrm{P}(\beta\gamma)}\boldsymbol{\eta}_{\mathrm{s}}^{\mathrm{P}} - \bar{\boldsymbol{\eta}}^{\mathrm{P}(\beta\gamma)}\right). \tag{70}$$

Consequently, in conjunction with Eq. (61), the following effective elastic–plastic constitutive law of the composite can be established

$$\bar{\boldsymbol{\tau}} = \boldsymbol{B}^*(\bar{\boldsymbol{\eta}} - \bar{\boldsymbol{\eta}}^{\mathrm{P}}), \tag{71}$$

where

$$\boldsymbol{B}^* = \frac{1}{hl}\sum_{\beta=1}^{N_\beta}\sum_{\gamma=1}^{N_\gamma} h_\beta l_\gamma \boldsymbol{C}^{(\beta\gamma)}\boldsymbol{A}^{(\beta\gamma)}, \tag{72}$$

which is the initial value of $\boldsymbol{B}^{*\mathrm{VP}}$ defined by Eq. (63), and

$$\bar{\boldsymbol{\eta}}^{\mathrm{P}} = -\boldsymbol{B}^{*-1}\sum_{\beta=1}^{N_\beta}\sum_{\gamma=1}^{N_\gamma} h_\beta l_\gamma \boldsymbol{C}^{(\beta\gamma)} \times\left(\boldsymbol{A}^{\mathrm{P}(\beta\gamma)}\boldsymbol{\eta}_{\mathrm{s}}^{\mathrm{P}} - \bar{\boldsymbol{\eta}}^{\mathrm{P}(\beta\gamma)}\right)/(hl). \tag{73}$$

According to representation (71), the overall behavior of the elastic–viscoplastic composite is given in terms of the effective elastic stiffness tensor $\boldsymbol{B}^*$ and the composite plastic strain-rate $\bar{\boldsymbol{\eta}}^{\mathrm{P}}$. Since $\boldsymbol{B}^*$ is a time-independent tensor, one can integrate Eq. (71) with respect to time to obtain the average composite stresses in terms of the average strains and plastic strains.

## 6. Applications

The capability of the generalized cells model (GCM) is assessed herein by comparison with the elastoplastic response of a unidirectional boron/aluminum composite recently obtained by Brockenbrough et al. (1991) by employing a finite element analysis using a supercomputer. The unidirectional boron fibers are elastic and isotropic, while the aluminum matrix (6061-0) is considered as elastoplastic work-hardening material. The inelastic response of the aluminum can be described by Bodner and Partom (1975) unified viscoplastic theory. This theory does not assume the existence of a yield condition, which eliminates the need to specify loading or unloading criteria, and the same equations can be directly used in all stages of loading and unloading. According to these equations plastic deformation always exists, but it is negligibly small when the material behavior should be essentially elastic. The material inelastic behavior is characterized in the framework of this theory by five parameters: $D_0$, $Z_0$, $Z_1$, $m$ and $n$. The parameter $D_0$ is the limiting strain rate, $Z_0$ is related to the "yield stress" of a uniaxial stress–strain curve, and $Z_1$ is proportional to the ultimate stress. The material constant $m$ determines the rate of work hardening, and the rate sensitivity is controlled by the parameter $n$. For $n = 10$ the material is essentially rate insensitive. In Table 1, the material parameters of the fiber and matrix phases are given.

Let us first compare the effective elastic moduli of the composite as predicted by the general-

Table 1
Material parameters of the boron (perfectly elastic and isotropic) and 6061-0 aluminum (elastic–viscoplastic isotropic hardening)

| | $E$ (GPa) | $\nu$ | $D_0$ ($s^{-1}$) | $Z_0$ (MPa) | $Z_1$ (MPa) | $m$ | $n$ |
|---|---|---|---|---|---|---|---|
| boron | 410 | 0.2 | | | | | |
| aluminum | 69 | 0.33 | 10000 | 52 | 135 | 31 | 10 |

$E$ and $\nu$ denote the elastic Young's modulus and Poisson's ratio. The other parameters characterize the material in the plastic region in accordance with Bodner–Partom unified theory.

Table 2
Comparison between the elastic effective moduli as predicted by the model and finite element solution (FE) for a boron/aluminum composite ($v_f = 0.46$)

| | $E_L^*$ (GPa) | $\nu_L^*$ | $E_T^*$ (GPa) | $G_T^*$ (GPa) |
|---|---|---|---|---|
| square edge-packing | | | | |
| model | 226.1 | 0.26 | 146 | 56.2 |
| FE | 228 | 0.26 | 152 | 57 |
| square diagonal-packing | | | | |
| model | 225.9 | 0.27 | 121 | 43 |
| FE | 227 | 0.26 | 127 | 45 |
| triangular-packing | | | | |
| model | 226 | 0.26 | 136 | 50 |
| FE | 227 | 0.26 | 137 | 50 |

The subscripts L and T denote longitudinal and transverse directions, respectively.

ized cells model and the finite element solution for three types of the fiber phase arrangement namely, square edge-packing, square diagonal-packing, and triangle-packing. Such a comparison is given in Table 2 that shows that the correspondence between the two methods is satisfactory.

In Fig. 4 the average transverse stress–strain response of the composite is shown for a square edge packing (denoted in the figure in a schematic form) with a fiber volume fraction $v_f = 0.46$. The figure exhibits the predicted behavior by GCM for a repeating volume element with 4 subcells ($N_\beta = N_\gamma = 2$) and 16 subcells ($N_\beta = N_\gamma = 4$). Also shown in the figure is the finite element solution of Brockenbrough et al. (1991). The capability of the GCM is very well demonstrated by this figure if one takes into account that the response can be generated with a personal computer with small amounts of running time and memory. As it is expected, the increase of the number of matrix subcells exhibits an improvement of the predicted response.

The composite average transverse response is shown in Fig. 5 for $v_f = 0.2$. The agreement between the GCM prediction and finite element solution is very good.

It is also possible to compare the GCM prediction with the finite element solution of Brockenbrough et al. (1991) for a composite with a square diagonal-packing. Figures 6 and 7 exhibit such comparisons for $v_f = 0.46$ and 0.2, respectively.

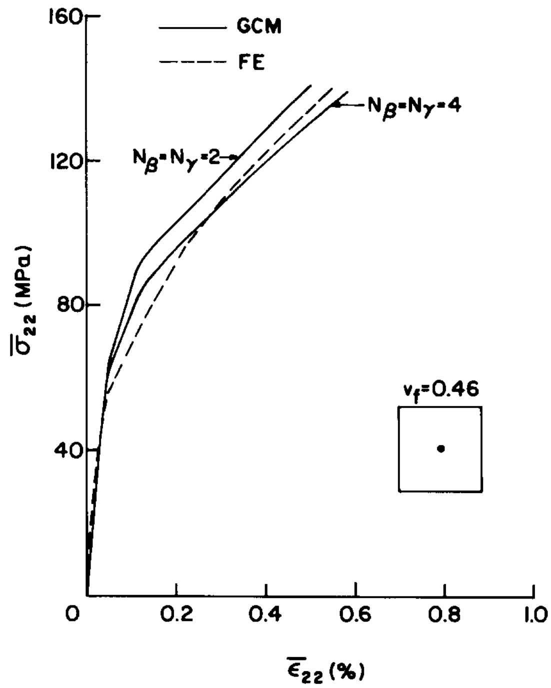


Fig. 4. The average transverse stress–strain response of a unidirectional boron/aluminum composite as predicted by GCM and the finite element method for a square edge-packing with $v_f = 0.46$.

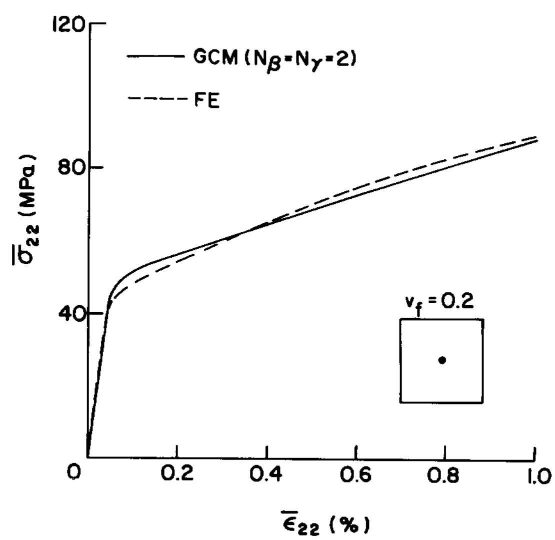


Fig. 5. The average transverse stress–strain response of the composite as predicted by GCM and the finite element method for a square edge-packing with $v_f = 0.2$.

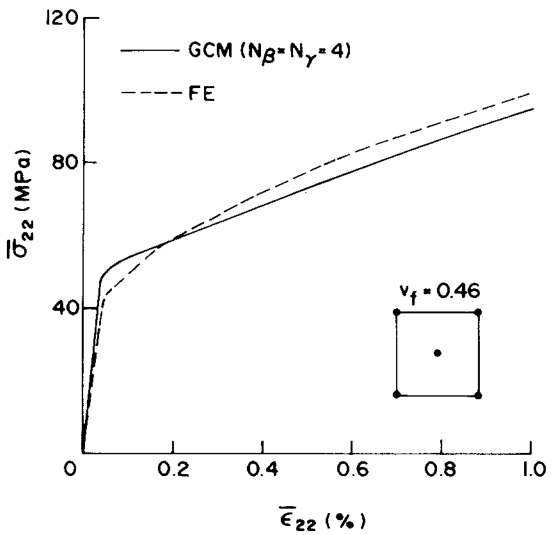


Fig. 6. The average transverse stress–strain response of a unidirectional boron/aluminum composite as predicted by GCM and the finite element method for a square diagonal packing with $v_f = 0.46$.

The agreement between the two methods of analyses is pronounced.

Figure 8 presents a comparison between GCM prediction and the finite element results for a composite with triangle-packing arrangement. It is clearly seen that fair agreement between the two approaches exists.

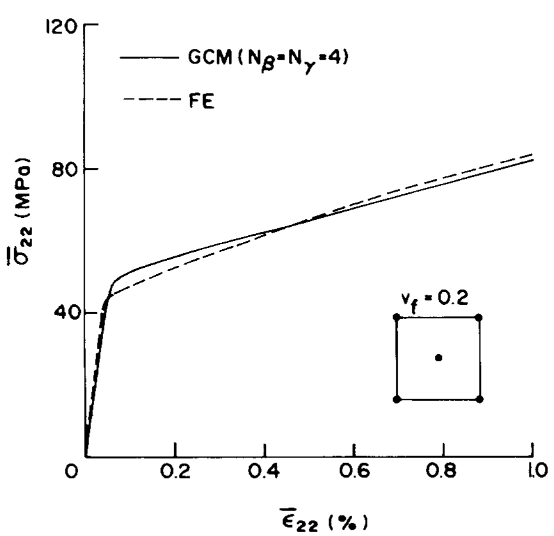


Fig. 7. The average transverse stress–strain response of the composite as predicted by GCM and the finite element method for a square diagonal packing with $v_f = 0.2$.

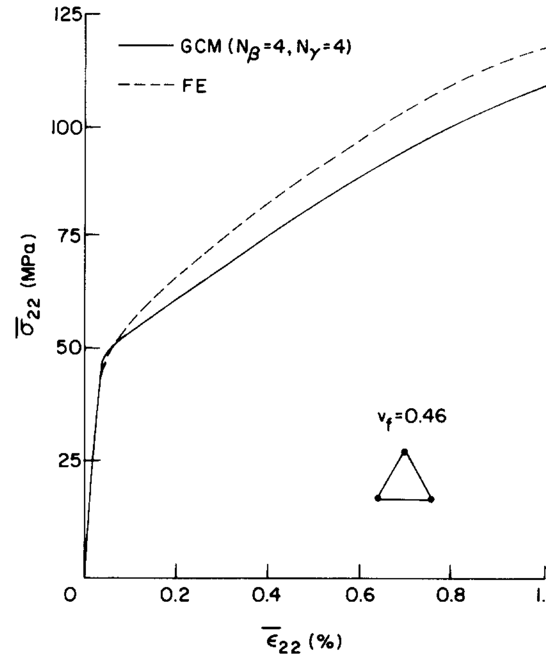


Fig. 8. The average transverse stress–strain response of the composite as predicted by GCM and the finite element method for a triangle packing with $v_f = 0.46$.

## 7. Discussion

A generalization of the micromechanical method of cells for the modeling of periodic multiphase fibrous composites has been presented. The analysis is applicable to inelastic composites in which the various phases behave, in general, as elastic–viscoplastic materials. The micromechanical analysis imposes the continuity of the displacement and traction rates at the interfaces on average bases. A homogenization procedure leads to a set of equations which are valid at any point of an equivalent continuum medium that effectively represents the multiphase composite. Further, effective viscoplastic constitutive laws are established which govern the overall behavior of the composite. The derived effective constitutive relations are given in the form of relationships between the average stress and strain rates of the composite. These rates are connected via the effective instantaneous viscoplastic stiffness tensor (Eq. (62)), or by the effective elastic stiffness tensor and the composite plastic strain-rate (Eq. (71)). The established overall constitutive laws readily provide the multiphase composite response to any prescribed combined loading conditions.

It is sometimes noticed that the method of cells is confused with the finite element method. The purpose of the following discussion is to emphasize that essential differences between the two entirely different approaches to the micromechanical analysis of composites exist. Firstly, it should be emphasized, that when a repeating cell of a periodic composite is analyzed by a finite element procedure, different strategies must be employed when imposing the boundary conditions that are associated with a specific loading. For example, the boundary conditions associated with a normal loading transverse to the fiber direction are completely different from those associated with pure shear type of loading (see Brockenbrough et al. (1991), for example). Furthermore, the appropriate symmetry conditions, which arise from the concrete applied loading, have to be utilized in formulating the boundary conditions on the unit cell. The established continuum relations by the method of cells, on the other hand, are applicable for any type of loading irrelevant whether symmetry applies or not. This is not surprising since one needs to distinguish between a continuum model (such as the method of cells and its generalization) and a numerical method for solving the exact elasticity equations (such as the finite element or finite difference methods).

The use of a standard finite element method for the analysis of a repeating cell of a composite may require several hundred elements to obtain meaningful results (e.g., over 800 elements in Brockenbrough et al. (1991)). This necessitates, due to the large number of equations involved, the use of a supercomputer when inelastic composites are considered. The continuum model derived in this paper involves small number of equations and thus a personal computer can be used for this purpose.

The original method of cells with four subcells was previously shown to yield good and reliable results. It is expected that the proposed generalization will improve the accuracy of solution by a small increase of the number of subcells. This has been demonstrated in Fig. 4 in the specific example of a boron/aluminum composite.

Finally, it should be emphasized that in cells method and its present generalization, the average fields in the subcells are used in the analysis of the metal-matrix composite. This is of importance since when the average field in the entire matrix constituent is used (as it is the case in the self-consistent and Mori–Tanaka methods, for example) inaccurate results are obtained (Aboudi and Pindera, 1990; Dvorak, 1991).

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