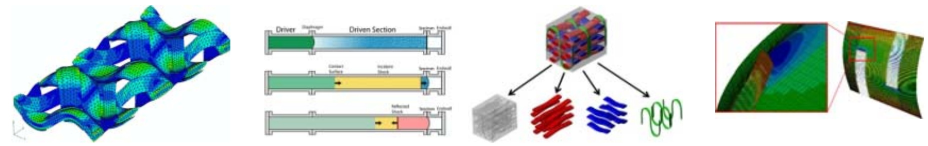

Composite Cylinder Model

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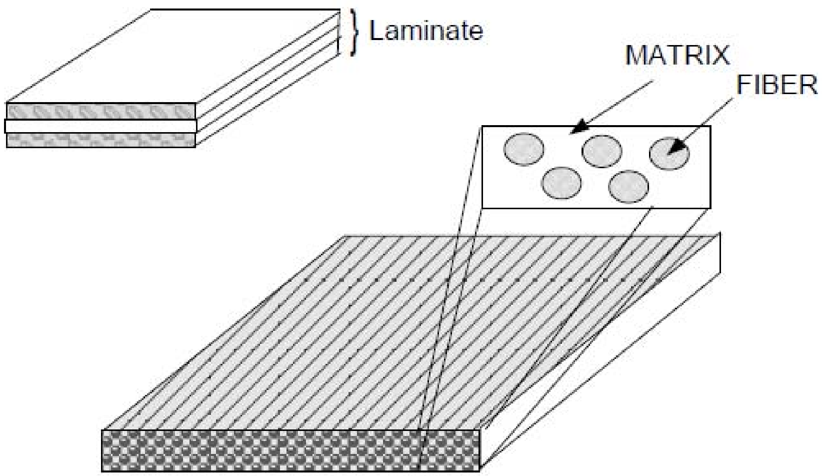
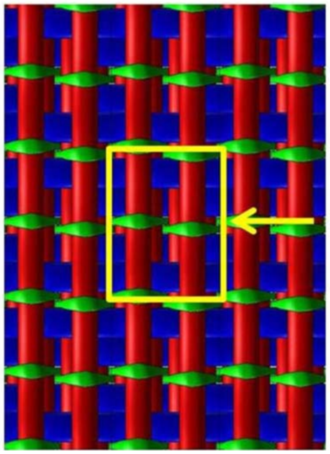
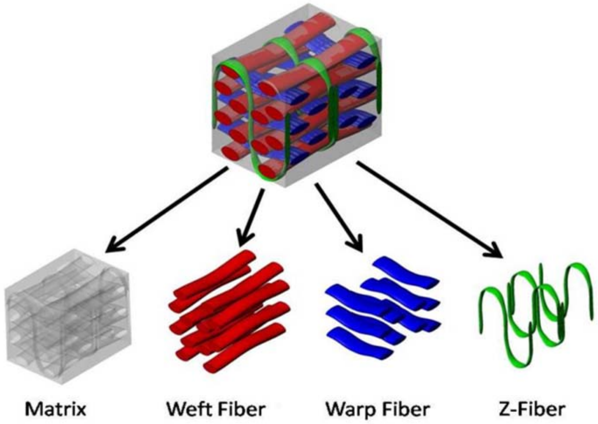
Composite Materials

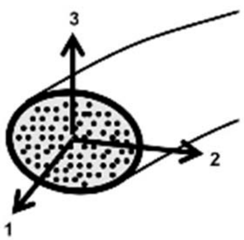
Laminated Composites

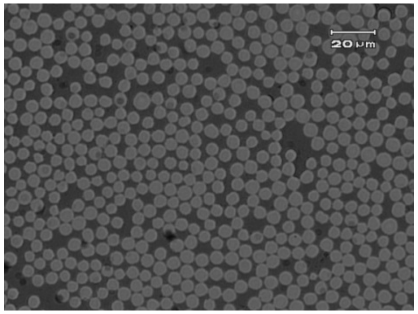


Textile Composites





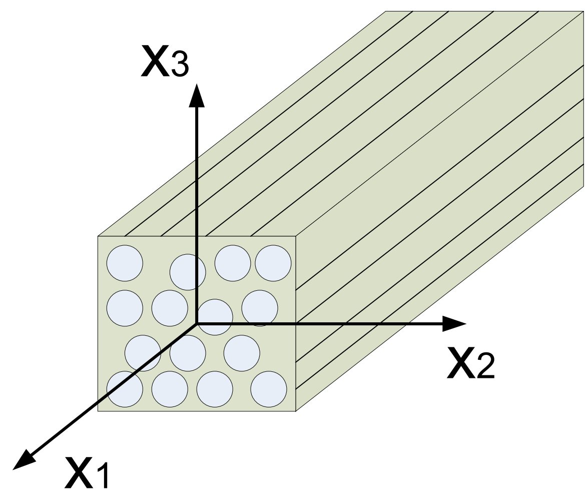




Material System

- Transversely isotropic material

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{C_{22}-C_{23}}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix}$$



Five independent properties!

$$E_1 = C_{11} - \frac{2C_{12}^2}{C_{22}+C_{23}} \qquad \upsilon_{12} = \frac{C_{12}}{C_{22}+C_{23}} \qquad G_{12} = C_{66}$$

plane-strain bulk modulus

$$K_{23} = \frac{1}{2}\left(C_{22}+C_{23}\right) \qquad G_{23} = \frac{1}{2}\left(C_{22}-C_{23}\right)$$

Transversely Isotropic Material

- The selection of the five independent constants is not unique. The elastic moduli, E_1, v_{12}, K_{23}, G_{12}, G_{23}, are chosen to be solved using the composite cylinder model (CCM)
- Subsequently, other engineering properties of interest can be calculated as,

$$E_2 = \frac{4G_{23}k_{23}}{k_{23} + \left(1 + \frac{4k_{23}\nu_{12}^2}{E_1}\right)G_{23}} \qquad \nu_{23} = \frac{E_2}{2G_{23}} - 1$$

- The stiffness matrix can be rewritten as,

$$\mathbf{C} = \begin{bmatrix} E_1 + 4\nu_{12}^2 K_{23} & 2\nu_{12}K_{23} & 2\nu_{12}K_{23} & 0 & 0 & 0 \\ 2\nu_{12}K_{23} & K_{23}+G_{23} & K_{23}-G_{23} & 0 & 0 & 0 \\ 2\nu_{12}K_{23} & K_{23}-G_{23} & K_{23}+G_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12} \end{bmatrix}$$

Transversely Isotropic Material

- Compliance matrix

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{Bmatrix}$$

$$S_{11} = \frac{1}{E_{11}} \qquad S_{22} = \frac{1}{E_{22}}$$

$$S_{12} = -\frac{\nu_{12}}{E_{11}} \qquad S_{23} = -\frac{\nu_{23}}{E_{22}}$$

$$S_{44} = \frac{1}{G_{23}} = \frac{2(1+\nu_{23})}{E_{22}}$$

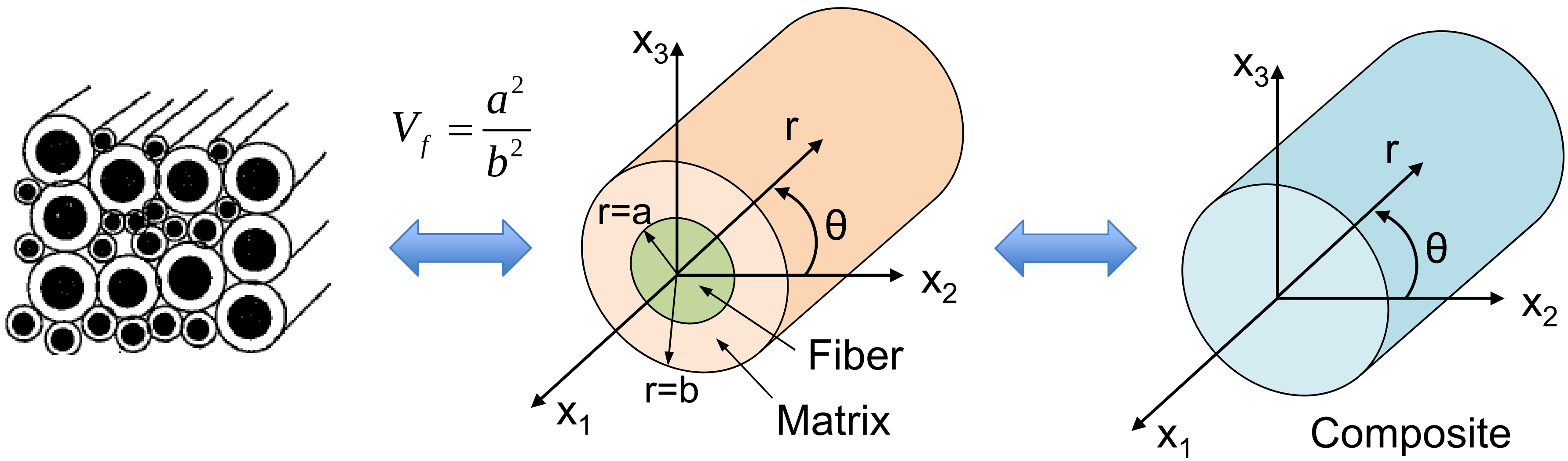
$$S_{66} = \frac{1}{G_{12}}$$

- Relation between **[S]** and **[C]**

$$[\mathbf{C}] = [\mathbf{S}]^{-1}$$

Composite Cylinder Model

- The composite cylinder model was proposed by Hashin and Rosen (1964).



- The key is to impose stress or deformation conditions on the concentrically assembled cylinders to find the elastic properties of the equivalent homogenous single-material cylinder .
- The composite properties can be expressed in terms of the fiber and matrix properties. The strain fields of the fiber and matrix can be computed by knowing the prescribed strains on the composite.

Fiber Direction Tension

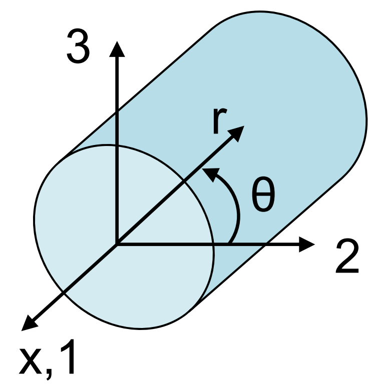
- Equilibrium eqn: $\dfrac{d\sigma_r}{dr}+\dfrac{\sigma_r-\sigma_\theta}{r}=0$

- Stress-strain relations for fiber and matrix:

$$\begin{Bmatrix}\sigma_x^f\\ \sigma_\theta^f\\ \sigma_r^f\end{Bmatrix}=\begin{bmatrix}C_{11}^f & C_{12}^f & C_{12}^f\\ C_{12}^f & C_{22}^f & C_{23}^f\\ C_{12}^f & C_{23}^f & C_{22}^f\end{bmatrix}\begin{Bmatrix}\varepsilon_x^f\\ \varepsilon_\theta^f\\ \varepsilon_r^f\end{Bmatrix}\qquad \begin{Bmatrix}\sigma_x^m\\ \sigma_\theta^m\\ \sigma_r^m\end{Bmatrix}=\begin{bmatrix}C_{11}^m & C_{12}^m & C_{12}^m\\ C_{12}^m & C_{11}^m & C_{12}^m\\ C_{12}^m & C_{12}^m & C_{11}^m\end{bmatrix}\begin{Bmatrix}\varepsilon_x^m\\ \varepsilon_\theta^m\\ \varepsilon_r^m\end{Bmatrix}$$

- Strain-displacement relations: $\varepsilon_x=\dfrac{\partial u_x}{\partial x}\qquad \varepsilon_\theta=\dfrac{u_r}{r}\qquad \varepsilon_r=\dfrac{du_r}{dr}$
- Displacement equilibrium eqn:

$$\frac{d^2u_r}{dr^2}+\frac{1}{r}\frac{du_r}{dr}-\frac{u_r}{r^2}=0 \quad\Longrightarrow\quad u_r=Ar+\frac{B}{r}$$



- Displacement fields for fiber and matrix:

$$u_r^f(r)=A^f r+\cancel{\frac{B^f}{r}}\qquad (0\le r\le a)\qquad\qquad u_x^f(r)=\varepsilon_1^f x\qquad (0\le r\le a)$$

$$u_r^m(r)=A^m r+\frac{B^m}{r}\qquad (a\le r\le b)\qquad\qquad u_x^m(r)=\varepsilon_1^m x\qquad (a\le r\le b)$$

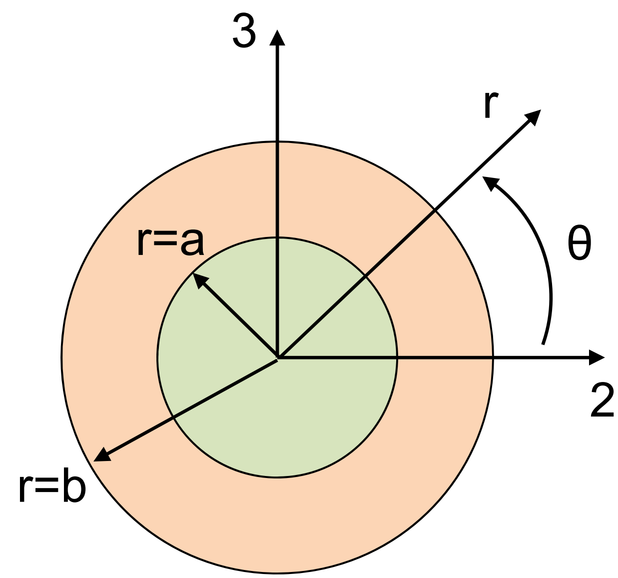
Fiber Direction Tension

- Strains in the fiber and matrix:

$$\varepsilon_x^f = \varepsilon_1^f = \varepsilon_1 \qquad \varepsilon_x^m = \varepsilon_1^m = \varepsilon_1$$

$$\varepsilon_\theta^f = A^f \qquad \varepsilon_\theta^m = A^m + \frac{B^m}{r^2}$$

$$\varepsilon_r^f = A^f \qquad \varepsilon_r^m = A^m - \frac{B^m}{r^2}$$



- Boundary conditions due to continuity:

$$u_r^f(a) = u_r^m(a)$$

$$\sigma_r^f(a) = \sigma_r^m(a)$$

- Boundary condition at $r = b$:

$$\text{if } \sigma_r^m(b) = 0 \Longrightarrow \text{Compute } E_{11}$$

$$\text{if } u_r^m(b) = 0 \Longrightarrow \text{Compute } C_{11}$$

- Hence, A^f, A^m, and B^m can be solved in terms of ε_x

The Effective Axial Modulus, E_1

- Boundary conditions at *r* = *b*: $\sigma_r^m(b) = 0$

- Apply axial load *P* on the composite cylinder:

$$\sigma_x = \frac{P}{A} = \frac{P}{\pi b^2} \qquad \varepsilon_x = \frac{\sigma_x}{E_1} = \frac{P}{\pi b^2 E_1}$$

- The integrated effect of the axial stress in the fiber and matrix equals the applied axial load *P*:

$$\int_A \sigma_x dA = \int_0^{2\pi} \int_0^r \sigma_x r dr d\theta = 2\pi \left\{ \int_0^a \sigma_x^f r dr + \int_a^b \sigma_x^m r dr \right\} = P$$

- Hence,

$$\boxed{E_1 = \frac{2\pi \left\{ \int_0^a \sigma_x^f r dr + \int_a^b \sigma_x^m r dr \right\}}{\pi b^2 \varepsilon_x}}$$

Effective Axial Poisson's Ratio, ν_{12}

- Boundary conditions at $r = b$: $\sigma_r^m(b) = 0$

- Apply axial load P on the composite cylinder:

$$\sigma_x = \frac{P}{A} = \frac{P}{\pi b^2} \qquad \varepsilon_x = \frac{\sigma_x}{E_1} = \frac{P}{\pi b^2 E_1}$$

- Define the lateral strain of the cylinder as the change in radius per unit length associated with an axial strain (ε_x) due to the applied stress (σ_x)

$$\nu_{12} = -\frac{u_r^m(b)/b}{\varepsilon_x}$$

- Hence,

$$\boxed{\nu_{12} = -\frac{1}{\varepsilon_x}\left(A_m + \frac{B_m}{b^2}\right)}$$

Effective Plane-Strain Bulk Modulus, K_{23}

- K_{23} – plane-strain bulk modulus
- Apply a radial stress σ at the outer radius and restrain the axial strain to be zero, hence for the composite:

$$\sigma_\theta = \sigma_r = \sigma \qquad \varepsilon_\theta = \varepsilon_r = \frac{\sigma}{C_{22} + C_{23}}$$

- Boundary conditions at $r=b$: $\sigma_r^m(b) = \sigma$
- Volumetric change:

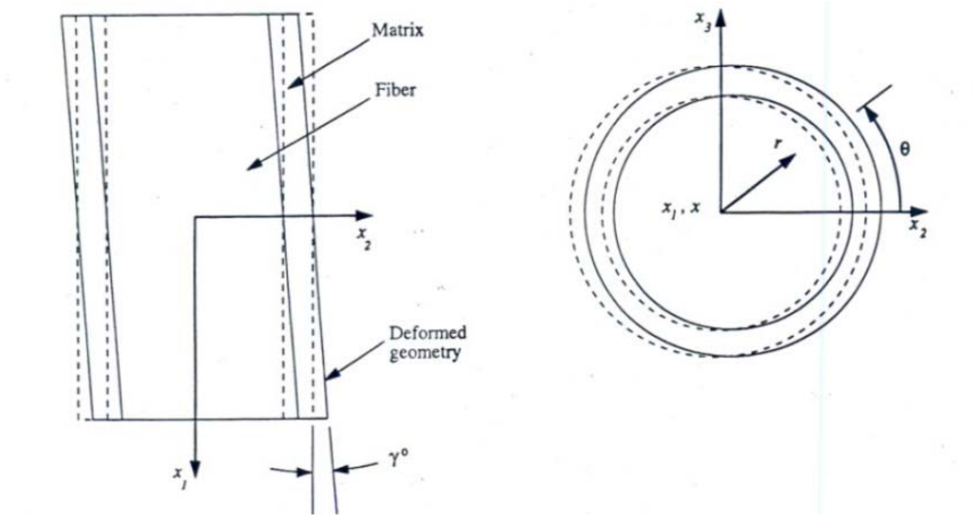
$$\frac{\Delta V}{V} = \frac{\pi\left(b + u_r^m(b)\right)^2 - \pi b^2}{\pi b^2} = \frac{2u_r^m(b)}{b} = 2\left(A^m + \frac{B^m}{b^2}\right)$$

- Hence,

$$\boxed{K_{23} = \frac{\sigma}{\frac{\Delta V}{V}} = \frac{\sigma}{2\left(A^m + \frac{B^m}{b^2}\right)}}$$

Effective Axial Shear Modulus, G_{12}

- Deformed shape



For composites:

$$\gamma_{12} = \gamma^0$$

$$u_x = 0$$

$$u_\theta = -\gamma^0 x \sin\theta$$

$$u_r = \gamma^0 x \cos\theta$$

- Displacements for fiber and matrix:

$$u_x^f = \left(A^f r + \tfrac{B^f}{r}\right)\cos\theta \qquad u_x^m = \left(A^m r + \tfrac{B^m}{r}\right)\cos\theta$$

$$u_\theta^f = -C^f x \sin\theta \qquad u_\theta^m = -C^m x \sin\theta$$

$$u_r^f = C^f x \cos\theta \qquad u_r^m = C^m x \cos\theta$$

Effective Axial Shear Modulus, G_{12}

- Nonzero stresses in the fiber and matrix:

$$\sigma_{xr}^{f} = G_{12}^{f}\left(A^{f} + C^{f} - \tfrac{B^{f}}{r}\right)\cos\theta \qquad \sigma_{xr}^{m} = G^{m}\left(A^{m} + C^{m} - \tfrac{B^{m}}{r}\right)\cos\theta$$

$$\sigma_{x\theta}^{f} = -G_{12}^{f}\left(A^{f} + C^{f} + \tfrac{B^{f}}{r}\right)\sin\theta \qquad \sigma_{x\theta}^{m} = -G^{m}\left(A^{m} + C^{m} + \tfrac{B^{m}}{r}\right)\sin\theta$$

- Boundary conditions:

$B^{f} = 0 \longleftarrow$ Finite displacement at $r = 0$

Continuity at $r = a$
$$\begin{cases} u_x^{f}(a) = u_x^{m}(a) \quad u_\theta^{f}(a) = u_\theta^{m}(a) \quad u_r^{f}(a) = u_r^{m}(a) \\ \sigma_{xr}^{f}(a) = \sigma_{xr}^{m}(a) \end{cases}$$

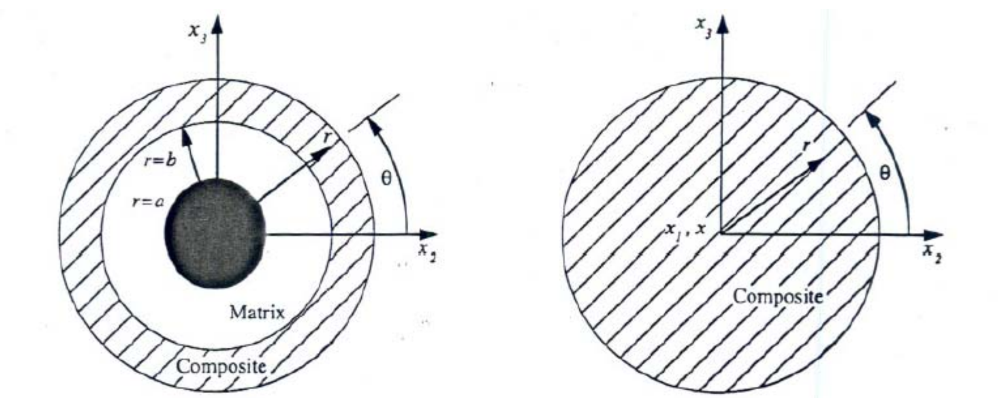
at $r = b$
$$\begin{cases} u_x^{m}(b) = 0 \quad u_\theta^{m}(b) = -\gamma^{0} x \sin\theta \quad u_r^{m}(b) = \gamma^{0} x \cos\theta \\ \sigma_{xr}^{m}(r = b, \theta = 0) = G^{m}\left(A^{m} + C^{m} - \tfrac{B^{m}}{b}\right) \equiv \sigma_{12} \end{cases}$$

- Hence,

$$\boxed{G_{12} = \frac{\sigma_{12}}{\gamma^{0}} = G^{m}\frac{G^{f}(1+V_f) + G^{m}(1-V_f)}{G^{f}(1-V_f) + G^{m}(1+V_f)}}$$

Effective Transverse Shear Modulus, G_{23}

- It is impossible to determine the *effective transverse shear modulus*, G_{23}, using the previous 2-phase composite cylinder model.
- Christensen and Lo (1979) obtained a close-form expression for G_{23} by using a self-consistent method. This model is based upon a three-phase cylinder in which the fiber and matrix are embedded in the equivalent composite material.



- Hashin's lower bound (1965) has been widely accepted

$$G_{23} = G^m \left[1 + \frac{V_f}{\dfrac{G^m}{G_{23}^f - G^m} + \dfrac{\left(K_{23}^m + 2G^m\right)\left(1 - V_f\right)}{2\left(K_{23}^m + G^m\right)}} \right]$$

Composite Elastic Properties

$$E_1 = E_1^f V_f + E^m(1-V_f) + \frac{4V_f(1-V_f)(\nu_{12}^f - \nu^m)^2 G^m}{\frac{(1-V_f)G^m}{K_{23}^f} + \frac{V_f G^m}{K_{23}^m} + 1}$$

$$\nu_{12} = \nu_{12}^f V_f + \nu^m(1-V_f) + \frac{V_f(1-V_f)(\nu_{12}^f - \nu^m)\left[\frac{G^m}{K_{23}^m} - \frac{G^m}{K_{23}^f}\right]}{\frac{(1-V_f)G^m}{K_{23}^f} + \frac{V_f G^m}{K_{23}^m} + 1}$$

$$K_{23} = K_{23}^m + \frac{V_f}{\frac{1}{K_{23}^f - K_{23}^m} + \frac{1-V_f}{K_{23}^m + G_m}}$$

$$G_{12} = G^m \frac{G_{12}^f(1+V_f) + G^m(1-V_f)}{G_{12}^f(1-V_f) + G^m(1+V_f)}$$

$$G_{23} = G^m \left[1 + \frac{V_f}{\frac{G^m}{G_{23}^f - G^m} + \frac{\left(K_{23}^m + 2G^m\right)\left(1-V_f\right)}{2\left(K_{23}^m + G^m\right)}}\right]$$

- By knowing these five constants

$$E_2 = \frac{4G_{23}K_{23}}{K_{23} + \left(1 + \frac{4K_{23}\nu_{12}^2}{E_1}\right)G_{23}}$$

$$\nu_{23} = \frac{E_2}{2G_{23}} - 1$$