CS-225: Discrete Structures in CS
Homework 5, Part 2

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Exercise Set 5.4

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<u>Proof</u>:

Given the sequence $c_0, c_1, c_2...$ defined below:

$$c_0 = 2, c_1 = 2, c_2 = 6,$$
$$c_k = 3c_{k-3} \text{ for every integer } k \geq 3.$$

Let $P(n) \equiv c_n$ is even for each integer $n \geq 0$.

<u>Basis Step</u>:

For the case $P(0)$:

$c_0 = 2$

Since $c_0 = 2$ is even, $P(0)$ holds true.

For the case $P(1)$:

$c_1 = 2$

Since $c_1 = 2$ is even, $P(1)$ holds true.

For the case $P(2)$:

$c_2 = 6$

Since $c_2 = 6$ is even, $P(2)$ holds true.

<u>Inductive Hypothesis</u>:

Suppose that for an arbitrary but particular integer k, such that $k \geq 0$,

$$P(i) \equiv c_i \text{ is even for each integer } k \geq i \geq 0$$

Inductive Steps:

We must show that $P(k+1)$ is true. Hence we must demonstrate,

$$P(k+1) \equiv c_{k+1} \text{ is even for each integer } k+1 \geq 0$$

$P(k+1)$ holds true.

- For $P(k+1)$, $c_{k+1} = 3c_{k-2}$.
- By inductive hypotheses, c_{k-2} is even.
- c_{k+1} must also be even because it is the product of 3 and an even number c_{k-2}.
- Since c_{k+1} is even, $P(k+1)$ holds true.
- Hence, $P(k+1)$ holds true.

Thus, it was shown to be true that $P(k+1) \equiv c_{k+1}$ is even for each integer $k+1 \geq 0$.

Conclusion:

Since both the basis and inductive step have been proved, the original expression,

$$P(n) \equiv c_n \text{ is even for each integer } n \geq 0.$$

is true.

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Proof:

Given the sequence $h_0, h_1, h_2 ...$ defined below:

$$h_0 = 1, h_1 = 2, h_2 = 3,$$
$$h_k = h_{k-1} + h_{k-2} + h_{k-3} \text{ for every integer } k \geq 3.$$

Let $P(n) \equiv h_n \leq 3^n$ for every integer $n \geq 0$.

Basis Step:

For the case $P(0)$:

$h_0 = 1$

Since $1 \leq 3^0$, $P(0)$ holds true.

For the case $P(1)$:

$h_1 = 2$

Since $2 \leq 3^1$, $P(1)$ holds true.

For the case $P(2)$:

$h_2 = 3$

Since $3 \leq 3^2$, $P(2)$ holds true.

Inductive Hypothesis:

Suppose that for an arbitrary but particular integer k, such that $k \geq 0$,

$$P(i) \equiv h_i \leq 3^i \text{ for each integer } k \geq i \geq 0$$

Inductive Steps:

We must show that $P(k+1)$ is true. Hence we must demonstrate,

$$P(k+1) \equiv h_{k+1} \leq 3^{k+1} \text{ for each integer } k+1 \geq 0$$

$P(k+1)$ holds true.

- For $P(k+1)$:

$$\begin{aligned} h_{k+1} &= h_k + h_{k-1} + h_{k-2} \\ &\leq 3^k + 3^{k-1} + 3^{k-2} \\ &\leq 3^k + 3^k + 3^k \\ &\leq 3^{k+1} \end{aligned}$$

- Hence, $P(k+1)$ holds true.

Thus, it was shown to be true that $P(k+1) \equiv h_{k+1} \leq 3^{k+1}$ for each integer $k+1 \geq 0$.

Conclusion:

Since both the basis and inductive step have been proved, the original expression,

$$P(n) \equiv h_n \leq 3^n \text{ for every integer } n \geq 0.$$

is true.

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Proof:

Given the sequence $a_0, a_1, a_2 ...$ defined below:

$a_1 = 1, a_2 = 3$
$a_k = a_{k-1} + a_{k-2}$ for every integer $k \geq 3$.

Let $P(n) \equiv a_n \leq \left(\frac{7}{4}\right)^n$ for every integer $n \geq 1$.

Basis Step:

For the case $P(1)$:

$a_1 = 1$

Since $1 \leq \frac{7}{4}$, $P(1)$ holds true.

For the case $P(2)$:

$a_2 = 3$

Since $3 \leq \frac{49}{16}$, $P(2)$ holds true.

Inductive Hypothesis:

Suppose that for an arbitrary but particular integer k, such that $k \geq 1$,

$$P(i) \equiv a_i \leq \left(\tfrac{7}{4}\right)^i \text{ for each integer } k \geq i \geq 1$$

Inductive Steps:

We must show that $P(k+1)$ is true. Hence we must demonstrate,

$$P(k+1) \equiv a_{k+1} \leq \left(\tfrac{7}{4}\right)^{k+1} \text{ for each integer } k+1 \geq 1$$

$P(k+1)$ holds true.

- For $P(k+1)$:

$$\begin{aligned} a_{k+1} &= a_k + a_{k-1} \\ &\leq \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^{k-1} \\ &\leq \left(\frac{7}{4}\right)^k + \left(\frac{7}{4}\right)^k \\ &\leq \left(\frac{7}{4}\right)^{k+1} \end{aligned}$$

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Thus, it was shown to be true that $P(k+1) \equiv a_{k+1} \leq \left(\frac{7}{4}\right)^{k+1}$ for each integer $k+1 \geq 1$.
Conclusion:

Since both the basis and inductive step have been proved, the original expression,

$$P(n) \equiv a_n \leq \left(\tfrac{7}{4}\right)^n \text{ for every integer } n \geq 1.$$

is true.

Canvas Problem

Proof:

Let $P(n) \equiv$ a postage of n cents can be formed using just 5-cent and 7-cent stamps, where n is any integer ≥ 24.

Basis Step:

The case $P(24)$ where the postage is 24 cents can be formed using two 5-cent and two 7-cent stamps.

Inductive Hypothesis:

Suppose that for an arbitrary but particular integer k, such that $k \geq 24$,

A postage of k cents can be formed using just 5-cent and 7-cent stamps.

Inductive Steps:

We must show that $P(k+1)$ is true. Hence we must demonstrate,

A postage of $k+1$ cents can be formed using just 5-cent and 7-cent stamps.

There are six total cases for $P(k)$ of how a postage can be represented:

- Case 1: The postage is formed using no 5-cent stamps.
- Case 2: The postage is formed using one 5-cent stamp.
- Case 3: The postage is formed using two 5-cent stamps.
- Case 4: The postage is formed using three 5-cent stamps.
- Case 5: The postage is formed using four 5-cent stamps.

- Case 6: The postage is formed using five or more 5-cent stamps.

We will exhaustively demonstrate how each of the six cases of $P(k)$ can be easily transformed for $P(k+1)$.

Case 1: The postage is formed using no 5-cent stamps.

- In Case 1 of $P(k)$, the postage is formed using no 5-cent stamps.
- Hence, the postage is made up of at least four 7-cent stamps. These four stamps summate to 28 cents.
- To convert to $P(k+1)$, replace these four stamps with three 5-cent stamps and two 7-cent stamps. This new replacement summates to 29 cents.

Case 2: The postage is formed using one 5-cent stamp.

- In Case 2 of $P(k)$, the postage is formed using one 5-cent stamp.
- Hence, the postage is made up of at least three 7-cent stamps. These four stamps summate to 26 cents.
- To convert to $P(k+1)$, replace the these four with one 7-cent stamp and four 5-cent stamps. This new replacement summates to 27 cents.

Case 3: The postage is formed using two 5-cent stamps.

- In Case 3 of $P(k)$, the postage is formed using two 5-cent stamps.
- Hence, the postage is made up of at least two 7-cent stamps. These four stamps summate to 24 cents.
- To convert to $P(k+1)$, replace the these four with five 5-cent stamps. This new replacement summates to 25 cents.

Case 4: The postage is formed using three 5-cent stamps.

- In Case 4 of $P(k)$, the postage is formed using three 5-cent stamps.
- Hence, the postage is made up of at least two 7-cent stamp. These four stamps summate to 29 cents.
- To convert to $P(k+1)$, replace the these four with six 5-cent stamps. This new replacement summates to 30 cents.

Case 5: The postage is formed using four 5-cent stamps.

- In Case 5 of $P(k)$, the postage is formed using four 5-cent stamps.
- Hence, the postage is made up of at least one 7-cent stamps. These four stamps summate to 27 cents.
- To convert to $P(k+1)$, replace the these four with four 7-cent stamps. This new replacement summates to 28 cents.

Case 6: The postage is formed using five or more 5-cent stamps.

- In Case 6 of $P(k)$, the postage is formed using five or more 5-cent stamps. Let's consider the five 5-cent stamps, which summate to 25 cents.
- To convert to $P(k+1)$, replace the these five stamps stamps with three 7-cent stamps and one 5-cent stamps. This new replacement summates to 26 cents.

Thus, it was shown to be true that $P(k+1) \equiv$ a postage of n cents can be formed using just 5-cent and 7-cent stamps, where n is any integer ≥ 24.

<u>Conclusion</u>:

Since both the basis and inductive step have been proved, the original expression,

$P(n) \equiv$ a postage of n cents can be formed using just 5-cent and 7-cent stamps, where n is any integer ≥ 24.

is true.