CS-225: Discrete Structures in CS

Homework 6

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Given:

$e_k = 5e_{k-1} + 3$ for all integers $k \geq 2$
$e_1 = 2$

Iterations:

$$
\begin{aligned}
e_1 &= 2 \\
e_2 &= 5e_1 + 3 \\
e_3 &= 5e_2 + 3 \\
&= 5(5e_1 + 3) + 3 \\
e_4 &= 5e_3 + 3 \\
&= 5(5(5e_1 + 3) + 3) + 3 \\
&= 5^3 \cdot e_1 + 5^2 \cdot 3 + 5 \cdot 3 + 3 \\
&= 5^3 \cdot 2 + 3 \sum_{i=0}^{3} 5^{i-1} \\
&= 5^{4-1} \cdot 2 + 3 \sum_{i=0}^{4-2} 5^i \\
e_n &= 5^{n-1} \cdot 2 + 3 \sum_{i=0}^{n-2} 5^i
\end{aligned}
$$

Guess:

$e_n = 5^{n-1} \cdot 2 + 3 \sum_{i=0}^{n-2} 5^i$ for every integer $n \geq 1$

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Proof:

Given the sequence $e_1, e_2, \ldots, e_n$

$e_1 = 2$
$e_k = 5e_{k-1} + 3$ for all integers $k \geq 2$

Let $P(n) \equiv e_n = 5^{n-1} \cdot 2 + 3\sum_{i=0}^{n-2} 5^i$ for every integer $n \geq 1$

Basis Step:

$$\begin{aligned} P(1) = e_1 &= 5^{1-1} \cdot 2 + 3\sum_{i=0}^{1-2} 5^i \\ &= 5^0 \cdot 2 + 3\sum_{i=0}^{-1} 5^i \\ &= 2 + 3 \cdot 0 \\ &= 2 \end{aligned}$$

The base case for $P(1)$ holds true.

Inductive Hypothesis:

Suppose that for an arbitrary but particular integer j,

$$P(j) \equiv e_j = 5^{j-1} \cdot 2 + 3\sum_{i=0}^{j-2} 5^i \text{ for every integer } j \geq 1$$

is true.

Inductive Steps:

We must show that $P(j+1)$ is true. Hence we must demonstrate that,

$$P(j+1) \equiv e_{j+1} = 5^j \cdot 2 + 3\sum_{i=0}^{j-1} 5^i$$

By algebra, $j+1$ can be plugged into the original sequence defintion:

$$
\begin{aligned}
P(j+1) = e_{j+1} &= 5e_j + 3 \\
&= 5\left(5^{j-1} \cdot 2 + 3\sum_{i=0}^{j-2} 5^i\right) + 3 \qquad \text{(Subsitution of Inductive Hypothesis)} \\
&= 5 \cdot 5^{j-1} \cdot 2 + 5 \cdot 3\sum_{i=0}^{j-2} 5^i + 3\sum_{i=0}^{0} 5^i \\
&= 5^j \cdot 2 + 3\sum_{i=0}^{j-2} 5^{i+1} + 3\sum_{i=0}^{0} 5^i \\
&= 5^j \cdot 2 + 3\sum_{i=1}^{j-1} 5^i + 3\sum_{i=0}^{0} 5^i \\
&= 5^j \cdot 2 + 3\left(\sum_{i=1}^{j-1} 5^i + \sum_{i=0}^{0} 5^i\right) \\
&= 5^j \cdot 2 + 3\sum_{i=0}^{j-1} 5^i
\end{aligned}
$$

Thus, with was shown that $P(j+1) \equiv e_{j+1} = 5^j \cdot 2 + 3\sum_{i=0}^{j-1} 5^i$ holds true.

<u>Conclusion</u>:

Since both the basis and inductive step have been proved, the original expression,

$$P(n) \equiv e_n = 5^{n-1} \cdot 2 + 3\sum_{i=0}^{n-2} 5^i \text{ for every integer } n \geq 1$$

is true.

3

Given:

$t_k = t_{k-1} + 7k^3 + 4k + 5$ for all integers $k \geq 1$
$t_0 = 2$

Iterations:

$$
\begin{aligned}
t_0 &= 2 \\
t_1 &= t_0 + 7 \cdot 1^3 + 4 \cdot 1 + 5 \\
&= 2 + 7 \cdot 1^3 + 4 \cdot 1 + 5 \\
t_2 &= t_1 + 7 \cdot 2^3 + 4 \cdot 2 + 5 \\
&= (2 + 7 \cdot 1^3 + 4 \cdot 1 + 5) + 7 \cdot 2^3 + 4 \cdot 2 + 5 \\
&= 2 + 7(1^3 + 2^3) + 4(1 + 2) + 2(5) \\
t_3 &= t_2 + 7 \cdot 3^3 + 4 \cdot 2 + 5 \\
&= ((2 + 7 \cdot 1^3 + 4 \cdot 1 + 5) + 7 \cdot 2^3 + 4 \cdot 2 + 5) + 7 \cdot 3^3 + 4 \cdot 2 + 5 \\
&= 2 + 7(1^3 + 2^3 + 3^3) + 4(1 + 2 + 3) + 3(5) \\
t_n &= 7 \sum_{i=1}^{n} n^3 + 4 \sum_{i=1}^{n} n + 5n + 2
\end{aligned}
$$

Guess:

$t_n = 7 \sum_{i=1}^{n} n^3 + 4 \sum_{i=1}^{n} n + 5n + 2$ for every integer $n \geq 0$

4

I - Base: The strings ab and ba are in set S.

II - Recursion: New strings are formed according to the following rules:

a. If u is any string in S, or if u is either the individual character a or b; then

$$aub \text{ is a string in } S$$

where aub is the concatenation of a, u and b; aub is obtained by appending a on the left of u, and b on the right of u.

b. If u is any string in S, or if u is either the individual character a or b; then

$$bua \text{ is a string in } S$$

where bua is the concatenation of b, u and a; bua is obtained by appending b on the left of u, and a on the right of u.

III - Restriction: Every string in S is obtainable from the base and the recursion.

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I - Base: The individual string characters a and b are in set S.

II - Recursion: New strings are formed according to the following rules:

a. If u is any string in S, then

$$aua \text{ is a string in } S$$

where aua is the concatenation of a, u and b; aua is obtained by appending a on both the left and right side of u.

b. If u is any string in S, then

$$bub \text{ is a string in } S$$

where bub is the concatenation of a, u and b; bub is obtained by appending b on both the left and right side of u.

c. If u is any string in S, then

$$aub \text{ is a string in } S$$

where aub is the concatenation of a, u and b; aub is obtained by appending a on the left of u, and b on the right of u.

d. If u is any string in S, then

$$bua \text{ is a string in } S$$

where bua is the concatenation of a, u and b; bua is obtained by appending b on the left of u, and a on the right of u.

III - Restriction: Every string in S is obtainable from the base and the recursion.