CS-225: Discrete Structures in CS

Homework 3, Part 1

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Exercise Set 4.2

Problem 28

Suppositions:

Suppose $n - m$ is even where n and m are integers.

Goal:

Show that $n^3 - m^3$ is even.

Deductions:

Let $n - m = r$ where r is an even integer.

The product of an even integer and some other integer is even:

- Let a be represent an even integer and b be represent some other integer.
- By defintion, $a = 2x$ where x is some integer.
- The product of a and b becomes $ab = 2xb$.
- Let $z = xb$.
- By closure, z is an integer because it is the product of two integers.
- $ab = 2xb = 2z$.
- Let $y = ab$.
- $ab = y = 2z$
- By defintion of even, y is an even integer because $y = 2z$.
- By equality, ab is an even integer because y is an even integer.

- Hence the product of an even integer and some other integer is even.

The resulting value of $n^3 - m^3$ has a factor of r:

$$\begin{aligned} n^3 - m^3 &= (n-m)(n^2+nm+m^2) \\ &= r(n^2+nm+m^2) \end{aligned}$$

$n^2 + nm + m^2 = l$ where l is some integer:

- Let $n^2 + nm + m^2 = l$.
- By closure, m^2, mn, and n^2 are all integers because each expression is the product of two integers.
- Then also by closure, l must be an integer because it is the sum of integers.

$n^3 - m^3$ is an even integer:

- $n^3 - m^3 = (n-m)(n^2+nm+m^2) = rl$.
- rl is even because it is the product of an even integer and some other integer, as shown before.
- By equality, $n^3 - m^3$ is even because rl is even.
- Hence, $n^3 - m^3$ is an even integer.

Thus, $n^3 - m^3$ was shown to be even.

Conclusion:

It was therefore shown that for all integers n and m, if $n - m$ is even then $n^3 - m^3$ is even. This was exemplified by how the even integer $n - m$ can be factored from the expression.

Problem 36

Suppositions:

Let n and m be any two consecutive integers.

Goal:

Show that $n^2 - m^2$ is odd.

Deductions:

The sum of an even integer and an odd integer is odd:

- Let a be an even integer and b be an odd integer.
- By defintion, $a = 2c$ and $b = 2d + 1$ where c and d are some integers.
- The sum of a and b becomes $a + b = 2c + 2d + 1 = 2(c + d) + 1$.
- Let $e = c + d$.
- e is an integer because it is the sum of two integers.
- So, by substitution, $a + b = 2e + 1$ where e is some integer.
- By the defintion of odd, $a + b$ is odd.

The product of two odd integers is odd:

- Let f be an odd integer and g be an odd integer.
- By defintion, $f = 2h + 1$ and $g = 2i + 1$ where h and i are some integers.
- The product of f and g becomes $fg = (2h + 1)(2i + 1) = 4hi + 2h + 2i + 1$.
- Let $j = 2hi$.
- j is an integer because it is the product of integers.
- $fg = 4hi + 2h + 2i + 1 = 2j + 2h + 2i + 1 = 2(j + h + i) + 1$
- Let $k = h + i + j$.
- k is an integer because it is the sum of integers.
- $ab = 2(h + i + j) + 1 = 2k + 1$
- By the defintion of odd, ab is odd.

For the consecutive integers n and m, one must be even and the other must be odd:

- Since the integers are consecutive, n may be greater than or less than m. n may also be even or odd yet it, even so, in either case it cannot be assumed m is even or odd. To prove that of n and m one is even and the other is odd, we must consider all cases when $n > m$ and $n < m$ as well as where n is even or odd. This creates four total cases to exhaustively check:
- In the case of <u>$n > m$ and n is even</u>:
  - Let $n = 2l$ where l is some integer.
  - Then $m = n - 1 = 2l - 1 = 2l + 1 - 2$
  - Let $o = 2l + 1$.
  - By defintion of odd, o is odd.

- Then $m = o - 2$
- m is odd because the difference between any odd integer and any even integer is odd (Thereom 4.2.1).
- Hence n is even and m is odd.

- In the case of $\underline{n > m \text{ and } n \text{ is odd}}$:
  - Let $n = 2p + 1$ where p is some integer.
  - Then $m = 2n + 1 - 1 = 2n$
  - By defintion of even, m is even.
  - Hence n is odd and m is even.
- In the case of $\underline{n < m \text{ and } n \text{ is even}}$:
  - Let $n = 2r$ where r is some integer.
  - Then $m = n + 1 = 2r - 1 = 2r + 1 - 2$
  - Let $s = 2l + 1$.
  - By defintion of odd, s is odd.
  - Then $m = s - 2$
  - m is odd because it is the difference between any odd integer and any even integer is odd (Thereom 4.2.1).
  - Hence n is even and m is odd.
- In the case of $\underline{n < m \text{ and } n \text{ is odd}}$:
  - Let $n = 2t + 1$ where t is some integer.
  - Then $m = n + 1 = 2t + 2 = 2(t + 1)$
  - Let $u = t + 1$
  - u is an integer because it is the sum of two integers.
  - By substitution, $m = 2(t + 1) = 2u$
  - By defintion of even, m is even.
  - Hence n is odd and m is even.
- Therefore, as shown in call four cases, for two consecutive integers one must be even and the other must be odd.

The expression can be factored out such that $n^2 - m^2 = (n + m)(n - m)$.

$n + m = v$ where v is some odd integer:

- Let $n + m = v$.
- By closure, v is an integer because it is the sum two integers.
- v is an odd integer because it is the sum of an even integer and an odd integer is odd, as shown before.

$n - m = w$ where w is some odd integer:

- Let $n - m = w$.
- By closure, w is an integer because it is the difference two integers.
- w is odd because it is the difference between any odd integer and any even integer is odd (Thereom 4.2.1).

$n^2 - m^2$ is an odd integer:

- $n^2 - m^2 = (n + m)(n - m) = vw$.
- vw is odd because it is the product of two odd integers, as shown before.
- By equality, $n^2 - m^2$ is odd because vw is odd.

Conclusion:

Therefore, the difference of the squares of any two consecutive integers is odd. This was exemplified by how it can be rewritten as the product of two odd integers.

Exercise Set 4.3

Problem 39

This proof only holds true for $r = \frac{1}{4}$ and $s = \frac{1}{2}$. This does not mean it holds true for any r and s. It is incorrect to prove be generalizing from a single example.

Exercise Set 4.4

Problem 26

Suppositions:

Let a, b, and c be integers. Suppose $ab|c$.

Goal:

Prove $a|c$ and $b|c$.

Deductions:

$c = abk$ for some integer k:

- Let $d = ab$.
- Due to closure, d must be an integer because it is the product of two integers.
- $ab|c = d|c$.
- By definition of divisibility, d is non-zero.

- By definition of zero product property, a and b are also non-zero because $d = ab$.
- By defintion of divisibility, k is an integer such that $c = dk$.
- $c = dk = abk$.

$a|c$ must be true:

- Let $s = bk$.
- Due to closure, s must be an integer because it is the product of two integers.
- $c = abk = as$.
- By defintion of divisibility, $a|c$ because $c = sa$ where s is an integer and a is a non-zero integer.

$b|c$ must be true:

- Let $t = ak$.
- Due to closure, t must be an integer because it is the product of two integers.
- $c = abk = tb$.
- By defintion of divisibility, $b|c$ because $c = tb$ where t is an integer and b is a non-zero integer.

Thus, $a|c$ anmd $b|c$ was to be shown.

Conclusion:

For all integers a, b, and c, if $ab|c$ then $a|c$ and $b|c$ must be true.

Canvas Problems

1

Suppositions:

Let s be rational.

Goal:

Show that $9s^4 + \frac{3}{7}s - 5$ is also rational.

Deductions:

By the definition of rational, $s = \frac{a}{b}$ where a is some integer and b is some non-zero integer.

By algrebra, the original equation can be rewritten such that:

$$\begin{aligned} 9s^4 + \frac{3}{7}s - 5 &= \frac{9a^4}{b^4} + \frac{3a}{7b} - 5 \\ &= \frac{63a^4}{7b^4} + \frac{3ab^3}{7b^4} - \frac{35b^4}{7b^4} \\ &= \frac{63a^4 + 3ab^3 - 35b^4}{7b^4} \end{aligned}$$

$63a^4 = e$ where e is an integer:

- Let $c = a^2$.
- Due to closure, c is an integer because it is the product of two integers.
- Let $d = a^4$.
- $d = a^4 = c^2$.
- Due to closure, d is an integer because it is the product of two integers.
- Let $e = 63a^4$.
- $e = 63a^4 = 63d$.
- Due to closure, e is an integer because it is the product of two integers.
- Hence $63a^4 = e$ where e is an integer.

$3ab^3 = h$ where h is an integer:

- Let $f = b^2$.
- Due to zero product property and closure, f is a non-zero integer because it is the product of two non-zero integers.
- Let $g = b^3 = fb$.
- Due to closure, g is an integer because it is the product of two integers.
- Let $h = 3ab^3 = 3g$.
- Due to closure, h is an integer because it is the product of two integers.
- Hence $3ab^3 = h$ where h is an integer.

$35b^4 = j$ where j is an integer:

- Let $i = h^2$.
- Due to closure, i is an integer because it is the product of two integers.
- Let $j = 35b^4 = 35i$.
- Due to closure, j is an integer because it is the product of two integers.
- Hence $35b^4 = j$ where j is an integer.

$63a^4 + 3ab^3 - 35b^4 = l$ where l is an integer:

- $63a^4 + 3ab^3 - 35b^4 = e + f - j$.
- Let $k = e + f$.
- Due to closure, k is an integer because it is the sum of two integers.
- $63a^4 + 3ab^3 - 35b^4 = e + f - j = k - j$
- Let $l = k - j$.
- Due to closure, l is an integer because it is the difference of two integers.
- Hence $63a^4 + 3ab^3 - 35b^4 = l$ where l is an integer.

$7b^4 = n$ where n is a non-zero integer:

- $7b^4 = 7f^2$.
- Let $m = f^2$.
- Due to zero product property and closure, m is a non-zero integer because it is the product of two non-zero integers.
- $7b^4 = 7f^2 = 7m$.
- Let $n = 7m$.
- Due to zero product property and closure, n is a non-zero integer because it is the product of two non-zero integers.
- Hence $7b^4 = n$ where n is a noon-zero integer.

$9s^4 + \frac{3}{7}s - 5$ is rational:

- $\frac{63a^4+3ab^3-35b^4}{7b^4} = \frac{l}{n}$.
- By definition of rational, $\frac{63a^4+3ab^3-35b^4}{7b^4}$ is rational because it can be expressed as $\frac{l}{n}$ where l is an integer and n is a non-zero integer.
- Recall $9s^4 + \frac{3}{7}s - 5 = \frac{63a^4+3ab^3-35b^4}{7b^4}$.
- By equality, $9s^4 + \frac{3}{7}s - 5$ is rational because $\frac{63a^4+3ab^3-35b^4}{7b^4}$ is rational.
- Hence $9s^4 + \frac{3}{7}s - 5$ is rational.

Thus $9s^4 + \frac{3}{7}s - 5$ was shown to be rational.

Conclusion:

It must must be true that $9s^4 + \frac{3}{7}s - 5$ is rational. This was exmplified by how, through algebra, it can be represeted as an integer divided by a non-zero integer.

2

Suppositions:

Suppose $a|b$ and $a|(b^2 - c)$ where a, b, and c are integers.

Goal:

Prove $a|c$ must be true.

Deductions:

$\frac{c}{a} = am^2 - n$ where m and n are non-zero integers:

- By definition of divisibility, a is a non-zero integer because $a|b$.
- By defintion of divisibility, $a|b$ implies $b = am$ for some non-zero integer m.
- By defintion of divisibility, $a|(b^2 - c)$ states $b^2 - c = an$ for some non-zero integer n.
- By substitution, $b^2 - c = an$ can be rewritten:

$$\begin{aligned} b^2 - c &= an \\ (am)^2 - c &= an \\ a^2m^2 - c &= an \\ c &= a^2m^2 - an \\ c &= a(am^2 - n) \\ \frac{c}{a} &= am^2 - n \end{aligned}$$

$\frac{c}{a} = q$ where q is an integer:

- Let $o = m^2$.
- Due to closure, o is an integer because it is the product of two integers.
- $\frac{c}{a} = am^2 - n = ao - n$
- Let $p = ao$.
- Due to closure, p is an integer because it is the product of two integers.
- $\frac{c}{a} = ao - n = p - n$
- Let $q = p - n$.
- Due to closure, q is an integer because it is the difference of two integers.
- $\frac{c}{a} = p - n = q$
- Hence $\frac{c}{a} = q$ where q is an integer.

$a|c$ must be true.

- Recall $\frac{c}{a} = q$ where c and q are integers, and a is a non-zero integer.
- $\frac{c}{a} = q \Rightarrow c = aq$
- By defintion of divisibility, $a|c$ because $c = aq$ where q is an integer and a is a non-zero integer.

<u>Conclusion</u>:

It is therefore true that $a|c$. This was exemplified by how a is definitively a non zero integer, and by how c is an integer through algrebra.