CS-225: Discrete Structures in CS

Initial- Post Wk 3 & 4 Problem Sets #7 and #11

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Question

Prove the following statement (using either direct or indirect proof method):

For every real number x, if $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 \geq 0$, then $x \geq 0$.

Answer

[PROOF BY CONTRAPOSITIVE]

Contrapositive of the given statement:

For any real number x, if $x < 0$ then $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$.

Suppositions:

x is a real number and $x < 0$.

Goal:

Prove that $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$.

Deductions:

(i) $r^p > 0$ where r is some non-zero real number and p is some even integer.

- Let r be some non-zero real number.
- Let p be an even integer.
- Since p is even, it can be represented as $p = 2n$ where n is some integer.
- $r^p = r^{2n} = (r^n)^2$.

- $(r^n)^2 > 0$ because the square of any non-zero number is positive.
- By equality, $r^p > 0$ because $(r^n)^2 > 0$.
- Hence, $r^p > 0$ where r is some real number and p is some even integer.

(ii) $s^q < 0$ where s is some negative real number and q is some odd integer.

- Let s be some negative real number.
- Let q be some odd integer.
- Since q is odd, it can be represented as $q = 2m + 1$ where m is some integer.
- $s^q = s^{2m+1} = (s^m)^2$.
- $(s^m)^2 > 0$ because the square of a non-zero real number is positive.
- $s(s^m)^2 < 0$ because it is the product of some positive real number $(s^m)^2$ and some negative real number s.
- By equality, $s^q < 0$ because $s(s^m)^2 < 0$.

(iii) $3x < 0$

- The product of a positive real number and a negative real number is a negative real number.

(iv) $-x^2 < 0$

- $x^2 > 0$ because the exponent is an even integer. See Deduction (i).
- Then $-x^2 < 0$ because it is the opposite sign of a positive number.

(v) $3x^3 < 0$

- $x^3 < 0$ because the exponent is a negative integer. See Deduction (ii).
- Then $3x^3 < 0$ because it is the product of a positive and negative number.

(vi) $-4x^4 < 0$

- $x^4 > 0$ because the exponent is an even integer. See Deduction (i).
- Then $-4x^4 < 0$ because it is the product of a positive and negative number.

(vii) $x^5 < 0$

- $x^5 < 0$ because the exponent is a negative integer. See Deduction (ii).

By Deductions (iii) through (vii), every individual expression in $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4$ is less than 0. Then by closure, $x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$.

Conclusion:

$x^5 - 4x^4 + 3x^3 - x^2 + 3x - 4 < 0$.

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Question

a) Find $\bigcup_{i=1}^{4} A_i$ and $\bigcap_{i=1}^{4} A_i$ if for every positivie integer i.

I. $A_i = \{i, i+1, i+2, ...\}$. Are A_1, A_2, A_3, A_4 mutually disjoint? Justify your answer.

II. $A_i = \{0, i\}$. Are A_1, A_2, A_3, A_4 mutually disjoint? Justify your answer.

III. $A_i = (0, i)$, that is, the set of real numbers x with $0 < x < i$. Are A_1, A_2, A_3, A_4 mutually disjoint? Justify your answer.

IV. $A_i = (0, \infty)$, that is, the set of real numbers x with $x > i$. Are A_1, A_2, A_3, A_4 mutually disjoint? Justify your answer.

b) Let R, S, and T, be defined as follows:

$R = \{x \in Z | x \text{ is divisible by } 2\}$

$S = \{x \in Z | x \text{ is divisible by } 6\}$

$T = \{x \in Z | x \text{ is divisible by } 5\}$

Prove or disprove each of the following statements.

I. $R \subseteq T$

II. $T \subseteq R$

III. $T \subseteq S$

Answer

a - I)

$$\begin{aligned}\bigcup_{i=1}^{4} A_i &= \bigcup_{i=1}^{4}\{i, i+1, i+2, ...\}\\ &= \{1, 1+1, 1+2, ...\} \cup \{2, 2+1, 2+2, ...\} \cup \{3, 3+1, 3+2, ...\} \cup \{4, 4+1, 4+2, ...\}\\ &= \{1, 2, 3, ...\} \cup \{2, 3, 4, ...\} \cup \{3, 4, 5, ...\} \cup \{4, 5, 6, ...\}\\ &= \{1, 2, 3, 4, 5, 6, ...\}\end{aligned}$$

$$\begin{aligned}\bigcap_{i=1}^{4} A_i &= \bigcap_{i=1}^{4}\{i, i+1, i+2, ...\}\\ &= \{1, 1+1, 1+2, ...\} \cap \{2, 2+1, 2+2, ...\} \cap \{3, 3+1, 3+2, ...\} \cap \{4, 4+1, 4+2, ...\}\\ &= \{1, 2, 3, ...\} \cap \{2, 3, 4, ...\} \cap \{3, 4, 5, ...\} \cap \{4, 5, 6, ...\}\\ &= \{4, 5, 6, ...\}\end{aligned}$$

No, the sets A_1, A_2, A_3, and A_4 are not mutually disjoint. This is because they all share the elements all integers that are greater than or equal to 4.

a - II)

$$\begin{aligned}\bigcup_{i=1}^{4} A_i &= \bigcup_{i=1}^{4}\{0, i\}\\ &= \{0, 1\} \cup \{0, 2\} \cup \{0, 3\} \cup \{0, 4\}\\ &= \{0, 1, 2, 3, 4\}\end{aligned}$$

$$\begin{aligned}\bigcap_{i=1}^{4} A_i &= \bigcap_{i=1}^{4}\{0, i\}\\ &= \{0, 1\} \cap \{0, 2\} \cap \{0, 3\} \cap \{0, 4\}\\ &= \{0\}\end{aligned}$$

No, the sets A_1, A_2, A_3, and A_4 are not mutually disjoint. This is because they all share the element 0.

a - III)

$$\begin{aligned}\bigcup_{i=1}^{4} A_i &= \bigcup_{i=1}^{4}(0, i)\\ &= (0, 1) \cup (0, 2) \cup (0, 3) \cup (0, 4)\\ &= (0, 4)\end{aligned}$$

$$\begin{aligned}\bigcap_{i=1}^{4} A_i &= \bigcap_{i=1}^{4}(0, i)\\ &= (0, 1) \cap (0, 2) \cap (0, 3) \cap (0, 4)\\ &= (0, 1)\end{aligned}$$

No, the sets A_1, A_2, A_3, and A_4 are not mutually disjoint. This is because they all share the set of numbers within the range $(0, 1)$.

a - IV)

$$\begin{aligned}\bigcup_{i=1}^{4} A_i &= \bigcup_{i=1}^{4} (i, \infty) \\ &= (1, \infty) \cup (2, \infty) \cup (3, \infty) \cup (4, \infty) \\ &= (1, \infty)\end{aligned}$$

$$\begin{aligned}\bigcap_{i=1}^{4} A_i &= \bigcap_{i=1}^{4} (i, \infty) \\ &= (1, \infty) \cap (2, \infty) \cap (3, \infty) \cap (4, \infty) \\ &= (4, \infty)\end{aligned}$$

No, the sets A_1, A_2, A_3, and A_4 are not mutually disjoint. This is because they all share the set of numbers within the range $(4, \infty)$.

b - I)

[DISPROOF BY COUNTEREXAMPLE]

Suppositions:

Suppose not. Let's assume $R \subseteq T$.

Goal:

We must arrive at a contradiction.

Deductions:

$4 \in R$ but $4 \notin T$

- $4 \in R$ because 4 is divisible by 2.
- $4 \notin T$ because 4 is not divisible by 8.
- Hence, $4 \in R$ but $4 \notin T$.

By definition of subset, $R \not\subseteq T$ since there exists an element in R that is not in T.

Conclusion:

$R \not\subseteq T$

b - II)

[ELEMENTAL PROOF]

Suppositions:

Let x be a particular but arbitrarily chosen element of T.

Goal:

Prove $T \subseteq R$.

Deductions:

$x \in R$

- By definition of divisibility, $x = 8n$ where n is some integer. This is because x is divisible by 8.
- Let $m = 4n$.
- By closure, m is some integer because it is the product of two integers.
- Then through substitution, $x = 8n = 2(4n) = 2m$.
- By definition of divisibility, x is divisible by 2 because $x = 2m$.
- Therefore, $x \in R$.

By definition of subset, $T \subseteq R$ because the arbitrarily but particular element x of T is in R.

Conclusion:

$T \subseteq R$

b - III)

[ELEMENTAL PROOF]

Suppositions:

Suppose not. Let's assume $S \subseteq T$.

Goal:

We must arrive at a contradiction.

<u>Deductions</u>:

$12 \in S$ but $12 \notin T$

- $12 \in S$ because 12 is divisible by 6.
- $12 \notin T$ because 12 is not divisible by 8.
- Hence, $12 \in S$ but $12 \notin T$.

By definition of subset, $S \nsubseteq T$ since there exists an element in S that is not in T.

<u>Conclusion</u>:

$S \nsubseteq T$