CS-225: Discrete Structures in CS
Homework 3, Part 2

Noah Hinojos

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Exercise Set 4.7

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<u>Suppositions</u>: Suppose not. Let a be a rational number, b be a non-zero rational number, r be an irrational number, and $a + br$ be an rational number.

<u>Goal</u>:

We must arrive at a contradiction.

<u>Deductions</u>:

$r = \frac{fgd - fch}{edh}$ where c and g are integers; and d, e, f, and h are non-zero integers.:

- By definition of rational, $a = \frac{c}{d}$ where c and d are integers and d is non-zero.
- By definition of rational, $b = \frac{e}{f}$ where e and f are integers and f is non-zero.
- By eqaulity, $\frac{e}{f}$ is non-zero because b is non-zero.
- By algebra, e is non-zero because $\frac{e}{f}$ is non-zero.
- By definition of rational, $a + br = \frac{g}{h}$ where g, h are integers and $h \neq 0$.

- By substitution and algebra, r can be represented as follows:

$$\begin{aligned} a + br &= \frac{g}{h} \\ \left(\frac{c}{d}\right) + \left(\frac{e}{f}\right) r &= \frac{g}{h} \\ \frac{c}{d} + \left(\frac{e}{f}\right) r &= \frac{g}{h} \\ \left(\frac{e}{f}\right) r &= \frac{g}{h} - \frac{c}{d} \\ r &= \left(\frac{f}{e}\right)\left(\frac{g}{h} - \frac{c}{d}\right) \\ &= \frac{fg}{eh} - \frac{fc}{ed} \\ &= \frac{fgd}{ehd} - \frac{fch}{edh} \\ &= \frac{fgd - fch}{edh} \end{aligned}$$

- Hence $r = \frac{fgd - fch}{edh}$.

The product of three integers is an integer:

- Let x, y, and z be some integers.
- Then their product is xyx.
- Let $w = xy$.
- Due to closure, w is an integer because it is the product of two integers.
- Let $v = xyz$.
- $v = xyz = wz$.
- Due to closure, v is an integer because it is the product of two integers.
- Since $xyz = v$, xyz is also an integer.
- Hence, the product of three integers is an integer.

The product of three non-zero integers is a non-zero integer:

- Let i, t, and u be some non-zero integers.
- Then their product is stu.

- Let $p = st$.
- Due to closure and zero product property, p is an non-zero integer because it is the product of two non-zero integers.
- Let $q = stu$.
- $q = stu = pu$.
- Due to closure and zero product property, q is an non-zero integer because it is the product of two non-zero integers.
- Since $stu = q$, stu is also an non-zero integer.
- Hence, the product of three non-zero integers is a non-zero integer.

r is rational:

- Recall $r = \frac{fgd-fch}{edh}$.
- Let $i = fgd$.
- i is an integer because it is the product of three integers.
- Let $j = fch$.
- j is an integer because it is the product of three integers.
- Let $k = edh$.
- k is a non-zero integer because it is the product of three non-zero integers.
- $r = \frac{fgd-fch}{edh} = \frac{i-j}{k}$.
- Let $l = i - j$.
- Due to closure, l is an integer because it is the difference of two integers.
- $r = \frac{i-j}{k} = \frac{l}{k}$.
- By defintion of rational, r is a rational number because it can be expressed as $\frac{l}{k}$ where l is an integer and k is a non-zero integer.

We arrived at a contradiction because r is rational. This contradicts the supposition that r is irrational.

<u>Conclusion</u>:

Therefore, it must be true that if a and b are rational numbers, $b \neq 0$, and r is an irrational number, then $a + br$ is irraitonal.

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Suppositions:

Suppose not. Let's assume r and s are some positive real numbers and $\sqrt{r+s} = \sqrt{r}+\sqrt{s}$

Goal:

Arrive at contradiction.

Deductions:

For any positive real number a, $\sqrt{a}$ is a non-zero real number:

- Let $\sqrt{a} = b$
- By algebra, $\sqrt{a} = b \Rightarrow b^2 = a$
- By equality, b^2 is a positive real number because a is a positive real number.
- By zero product property, b is a non zero real number because b^2 is a non-zero real number.
- By equality, $\sqrt{a}$ is a non-zero real number because b is a non-zero real number.
- Hence, $\sqrt{a}$ is a non-zero real number.

The product of any three non-zero real numbers is a non-zero real number:

- Let x, y, and z be non-zero real numbers.
- Then the

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- By equality, xyz is also a non-zero real number because v is a non-zero real number.
- Hence, the product of three non-zero real numbers xyz is a non-zero real number.

Through algebra, $\sqrt{r+s} = \sqrt{r} + \sqrt{s} \Rightarrow 0 = 2\sqrt{r}\sqrt{s}$:

$$\begin{aligned}
\sqrt{r+s} &= \sqrt{r} + \sqrt{s} \\
r+s &= (\sqrt{r} + \sqrt{s})^2 \\
r+s &= r + 2\sqrt{r}\sqrt{s} + s \\
0 &= 2\sqrt{r}\sqrt{s}
\end{aligned}$$

If r and s are positive real numbers, then $2\sqrt{r}\sqrt{s} \neq 0$.

- $\sqrt{r}$ is a non-zero real number since r is a positive real number, as shown before.
- $\sqrt{s}$ is a non-zero real number since s is a positive real number, as shown before.
- $2\sqrt{r}\sqrt{s}$ must be a non-zero real number because it is the product of three non-zero real numbers, as shown before.
- Hence $2\sqrt{r}\sqrt{s} \neq 0$.

The supposition is contradictory because the supposed equation requires $0 = 2\sqrt{r}\sqrt{s}$ to also be true. However, $2\sqrt{r}\sqrt{s} \neq 0$ because r and s are positive real integers.

Conclusion:

Therefore, for all positive real numbers r and s, $\sqrt{r+s} \neq \sqrt{r} + \sqrt{s}$ must be true.

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Suppositions:

Suppose not. Assume $a|b$, $a \nmid c$, and $a|(b+c)$.

Goal:

Arrive at a contradiction.

Deductions:

$a|c$ must be true.

- $a|b$ implies $b = ap$ where b and p are integers and a is a non-zero integer.
- By zero product property, a and p are non-zero because b is non-zero.
- $a|(b+c)$ implies $b + c = aq$ where q is some integer.
- By algebra, $c = aq - b$
- By substitution, $c = aq - ap$
- By algebra, $c = a(q - p)$
- Let $q - p = r$
- By closure, r is an integer because it is the difference of two integers.
- $c = a(q - p) = ar$
- By closure, c is an integer because it is the product of two integers.
- By defintion of divisibility, $a|c$ because c is an integer and a is a non-zero integer.

It was therefore shown that $a|c$. This contradicts the supposition that $a \nmid c$.

Conclusion:

For all integers a, b, c, if $a|b$ and $a \nmid c$, then $a \nmid (b+c)$.

4.8

18 - a

Suppositions:

Suppose a is an odd integer.

Goal:

Prove a^3 is odd.

Deductions:

The product of two odd integers is odd:

- Let the product of two odd integers be xy, where x and y are odd integers.
- By definition of odd, $x = 2s+1$ and $y = 2t+1$ where s and t are integers.
- By algebra, xy can be rewritten:

$$\begin{aligned} xy &= (2s+1)(2t+1) \\ &= 4st + 2s + 2t + 1 \\ &= 2(2st + s + t) + 1 \end{aligned}$$

- Let $n = st$.
- By closure, n is an integer because it is the product of two integers.
- Let $w = 2st$.
- By algebra, $w = 2st = 2n$.
- By closure, w is an integer because it is the product of two integers.
- Let $z = s + t$.
- By closure, z is an integer because it is the sum of two integers.
- Let $v = st + s + t$.
- By algebra, $v = st + s + t = w + z$.

- By closure, v is an integer because it is the sum of two integers.
- Let $t = xy$.
- By algebra and substitution, $t = xy = 2(st + s + t) + 1 = 2v + 1$.
- By defintion of odd, t is an odd integer.
- By equality, xy is an odd integer since t is an odd integer.
- Therefore, the product of two odd integers is an odd integer.

a^3 is an odd integer:

- Let $b = a^2$.
- Since b is the product of two odd integers, b is an odd integer, as shown before.
- Let $c = ab$.
- Since c is the product of two odd integers, c is an odd integer, as shown before.
- By algebra, $c = ab = a(a^2) = a^3$
- By equality, a^3 is an odd integer because c is an odd integer.

Thus, if a is an odd integer then a^3 is an odd integer.

Conclusion:

For every integer a, if a^3 is even then a is even.

18 - b

Suppositions:

Suppose not. Let's assume $\sqrt[3]{2} = p$ and p is a rational number.

Goal:

Arrive at a contradiction.

Deductions:

p is irrational:

- By definition of rational, $p = \frac{a}{b}$ where a is an integer and b is a non-zero integer.
- This also implies a and b also have no common factors.

- By algebra,

$$\begin{aligned} p = \sqrt[3]{2} &= \frac{a}{b} \\ \sqrt[3]{2} &= \frac{a}{b} \\ 2 &= \frac{a^3}{b^3} \\ 2b^3 &= a^3 \end{aligned}$$

- Let $n = b^3$.
- n is an integer because it is the product of three integers (See Exercise Set 4.7 #18 above).
- Let $m = 2b^3$.
- By algebra, $m = 2b^3 = 2n$.
- By definition of even, m is an even integer.
- By algebra, $a^3 = 2b^3 = m$.
- By definition of even, a^3 is an even integer. Therefore, a must also be even.
- Then $a = 2n$ where n is an integer.
- By algebra, $2b^3 = a^3 = (2n)^3 = 2 \cdot 2^2 \cdot n^3 = 2(4n^3)$
- Let $o = n^3$.
- o is an integer because it is the product of three integers (See Exercise Set 4.7 #18 above).
- Let $k = 2n^3$.
- By algebra, $k = 2n^3 = 2o$.
- By definition of even, k is an even integer.
- Let $j = 4n^3$.
- By algebra, $j = 4n^3 = 2k$.
- By definition of even, j is an even integer.
- By algebra, $2b^3 = 2(4n^3) = 2j$
- By algebra, $b^3 = j$
- By definition of even, b^3 is an even integer. Therefore, b must also be even.
- Since a and b are even, they have a common factor of 2.

- However, this contradicts the prior deduction that a and b must have no common factors.
- Hence, p must be irrational.

Since p is irrational, it contradicts the supposition that p is rational.

Conclusion:

$\sqrt[3]{2}$ is irrational.