CS-225: Discrete Structures in CS
Homework 4, Part 1

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Exercise Set 6.1

7-a

Suppositions:

(None)

Goal:

Prove $A \not\subseteq B$.

Deductions:

$4 \in A$

- Let $x \in A$.
- By definition of A, $x \in \mathbb{Z} | x = 6a + 4$.
- Then for $a = 0$, $x = 6(0) + 4 = 4$.
- Hence $4 \in A$.

$4 \notin B$.

- By defintion of B, if $4 \in B$ then $18b - 2 = 4$ for some integer b.
- By algebra, $4 = 18b - 2 \Rightarrow b = \frac{1}{3}$.
- Yet b cannot equal $\frac{1}{3}$ because b is an integer.
- Therefore, $4 \notin B$.

Since $4 \in A$ and $4 \notin B$, $A \not\subseteq B$.

Conclusion:

$A \not\subseteq B$.

7-b

Suppositions:

Let t be a particular but arbitrarily chosen element of B.

Goal:

Prove $B \subseteq A$.

Deductions:

$t \in \mathbb{Z} | t = 18b - 2$ for some integer b.

- By supposition, $t \in B$.
- Then by defintion of B, $t \in \mathbb{Z} | t = 18b - 2$ for some integer b.

$t \in A$.

- Let $a = 3b - 1$.
- By closure, a is an integer because it is the product and summation of integers.
- By definition of A, $6a + 4 \in A$
- By algebra, $t = 6a + 4$:

$$\begin{aligned} 6a + 4 &= 6(3b - 1) + 4 \\ &= 18b - 6 + 4 \\ &= 18b - 2 \\ &= t \end{aligned}$$

- By equality, $t \in A$ because $6a + 4 \in A$.
- Hence $t \in A$.

By definition of subset, $B \subseteq A$ because $t \in A \rightarrow t \in B$.

Conclusion:

Therefore, $B \subseteq A$.

7-c

By definition of set equality, $B = C$ if and only if $B \subseteq C$ and $C \subseteq B$.

Proof: $B \subseteq C$

Suppositions:

Let t be a particular but arbitrarily chosen element of B.

Goal:

Prove $B \subseteq C$.

Deductions:

$t \in \mathbb{Z} | t = 18b - 2$ for some integer b (See problem 7-b for proof).

$t \in C$.

- Let $c = b - 1$.
- By closure, c is an integer because it is the product and summation of integers.
- By definition of C, $18c + 16 \in C$
- By algebra, $t = 18c + 16$:

$$\begin{aligned} 18c + 16 &= 18(b-1) + 16) \\ &= 18b - 18 + 16 \\ &= 18b - 2 \\ &= t \end{aligned}$$

- By equality, $t \in C$ because $18c + 16 \in C$.
- Hence $t \in C$.

By definition of subset, $B \subseteq C$ because $t \in B \rightarrow t \in C$.

Conclusion:

Therefore, $B \subseteq C$.

Seeing that this first proof is complete, do not carry any established variables into the next proof. Looking ahead, the only known variable definitions should be that of B and C, and their respective formulas, from the initial prompt.

Proof: $C \subseteq B$

Suppositions:

Let t be a particular but arbitrarily chosen element of C.

Goal:

Prove $t \in B$.

Deductions:

$t \in \mathbb{Z} | t = 18c + 16$ for some integer c.

- By supposition, $t \in C$.
- Then by defintion of C, $t \in \mathbb{Z} | t = 18c + 16$ for some integer c.

$t \in B$.

- Let $b = c + 1$.
- By closure, b is an integer because it is the product and summation of integers.
- By definition of B, $18b - 2 \in B$.
- By algebra, $t = 18b - 2$:

$$\begin{aligned} 18b - 2 &= 18(c + 1) - 2 \\ &= 18c + 18 - 2 \\ &= 18c + 16 \\ &= t \end{aligned}$$

- By equality, $t \in B$ because $18b - 2 \in B$.
- Hence $t \in B$.

By definition of subset, $C \subseteq B$ because $t \in C \rightarrow t \in B$.

Conclusion:

Therefore, $C \subseteq B$.

By definition of set equality, $B = C$ because $B \subseteq C$ and $C \subseteq B$.

26 - a

$$\begin{aligned}
\bigcup_{i=1}^{4} R_i &= \bigcup_{i=1}^{4} \left[1, 1 + \frac{1}{i}\right] \\
&= \left[1, 1 + \frac{1}{1}\right] \cup \left[1, 1 + \frac{1}{2}\right] \cup \left[1, 1 + \frac{1}{3}\right] \cup \left[1, 1 + \frac{1}{4}\right] \\
&= [1, 2] \cup \left[1, \frac{3}{2}\right] \cup \left[1, \frac{4}{3}\right] \cup \left[1, \frac{5}{4}\right] \\
&= [1, 2]
\end{aligned}$$

26 - b

$$\begin{aligned}
\bigcap_{i=1}^{4} R_i &= \bigcap_{i=1}^{4} \left[1, 1 + \frac{1}{i}\right] \\
&= \left[1, 1 + \frac{1}{1}\right] \cap \left[1, 1 + \frac{1}{2}\right] \cap \left[1, 1 + \frac{1}{3}\right] \cap \left[1, 1 + \frac{1}{4}\right] \\
&= [1, 2] \cap \left[1, \frac{3}{2}\right] \cap \left[1, \frac{4}{3}\right] \cap \left[1, \frac{5}{4}\right] \\
&= \left[1, \frac{5}{4}\right]
\end{aligned}$$

26 - c

No, the sets R_1, R_2, R_3,... are not mutually disjoint. This is because all sets share the element, 1. In fact, for $i \neq j$, no set R_i is mutually disjoint from R_j.

29

Yes. This is because all real numbers are either positive, negative, or zero. Also, there are no real numbers that are both positive and negative, and zero is neither of the two.

30

Yes. All integers can be represented by the expressions $4k$, $4k+1$,$4k+2$, and $4k+3$ where k is an integer, and this through to the quotient-remainder theorem. Here, the range of the 'remainder' is $[0,3]$, i.e., all possible remainder values when dividing by 4. Since any value can be respresented by 4 times any permissible remainder value of 4, then all integers can be derived. The sets are also mutually disjoint through similar reasoning, none of the values of any expression could possibly overlap with values from any other expression.

33 - b

$$\begin{aligned}\mathscr{P}(\mathscr{P}(\varnothing)) &= \mathscr{P}(\{\varnothing\}) \\ &= \{\varnothing, \{\varnothing\}\}\end{aligned}$$

33 - c

$$\begin{aligned}\mathscr{P}(\mathscr{P}(\mathscr{P}(\varnothing))) &= \mathscr{P}(\mathscr{P}(\{\varnothing\})) \\ &= \mathscr{P}(\{\varnothing, \{\varnothing\}\}) \\ &= \{\varnothing, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}\end{aligned}$$

34 - b

$$\begin{aligned}(A_1 \cup A_2) \times A_3 &= (\{1\} \cup \{u, v\}) \times \{m, n\} \\ &= \{1, u, v\} \times \{m, n\} \\ &= \{(1, m), (1, n), (u, m), (u, n), (v, m), (v, n)\}\end{aligned}$$

35 - d

$$\begin{aligned}(A \times B) \cap (A \times C) &= (\{a, b\} \times \{1, 2\}) \cap (\{a, b\} \times \{2, 3\}) \\ &= \{\{a, 1\}, \{a, 2\}, \{b, 1\}, \{b, 2\}\} \cap \{\{a, 2\}, \{a, 3\}, \{b, 2\}, \{b, 3\}\} \\ &= \{\{a, 2\}, \{b, 2\}\}\end{aligned}$$