CS-225: Discrete Structures in CS

Homework 3, Part 1

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Exercise Set 4.2

Problem 28

Suppositions:

Suppose $n - m$ is even where n and m are integers. Let $r = n - m$ where r is an even integer.

Goal:

Show that $n^3 - m^3$ is even.

Deductions:

The product of an even integer and some other integer is even.

- Let a be represent an even integer and b be represent some oether integer.
- By defintion, $a = 2x$ where x is some integer.
- The product of a and b becomes $ab = 2xb$.
- Let $z = xb$.
- z is an integer because it is the product of integers.
- $ab = 2z$
- By the defintion of even, ab is even.

The resulting value of $n^3 - m^3$ has a factor r:

$$\begin{aligned} n^3 - m^3 &= (n-m)(n^2 + nm + m^2) \\ &= r(n^2 + nm + m^2) \end{aligned}$$

$n^2 + nm + m^2 = l$ where l is some integer:

- Let $n^2 + nm + m^2 = l$.
- m^2, mn, and n^2 are all integers because each expression is the product of two integers.
- Then l must be an integer because it is the sum of integers.

$n^3 - m^3$ is an even integer:

- $n^3 - m^3 = (n - m)(n^2 + nm + m^2) = rl$.
- rl is even because it is the product of an even integer and some other integer. This was demonstrated at the start of this problem.
- $n^3 - m^3$ must therefore be even.

Thus, $n^3 - m^3$ was shown to be even.

Conclusion: Therefore, for all integers n and m, if $n - m$ is even then $n^3 - m^3$ is even. This was demonstrated by the fact that an even integer could be factored out of $n^3 - m^3$.

Problem 36

Suppositions:

Let n and m be any two consecutive integers. The statement then becomes $n^2 - m^2$ is odd.

Goal:

Show that $n^2 - m^2$ is odd.

Deductions:

Before anaylzing the expression, it must first be proven that the sum of an even integer and an odd integer is odd:

- Let a be an even integer and b be an odd integer.
- By defintion, $a = 2c$ and $b = 2d + 1$ where c and d are some integers.
- The sum of a and b becomes $a + b = 2c + 2d + 1 = 2(c + d) + 1$.
- Let $e = c + d$.
- e is an integer because it is the sum of two integers.
- So, by substitution, $a + b = 2e + 1$ where e is some integer.
- By the defintion of odd, $a + b$ is odd.

Furthermore, the product of two odd integers is odd:

- Let f be an odd integer and g be an odd integer.
- By defintion, $f = 2h + 1$ and $g = 2i + 1$ where h and i are some integers.
- The product of f and g becomes $fg = (2h + 1)(2i + 1) = 4hi + 2h + 2i + 1$.
- Let $j = 2hi$.
- j is an integer because it is the product of integers.
- $fg = 4hi + 2h + 2i + 1 = 2j + 2h + 2i + 1 = 2(j + h + i) + 1$
- Let $k = h + i + j$.
- k is an integer because it is the sum of integers.
- $ab = 2(h + i + j) + 1 = 2k + 1$
- By the defintion of odd, ab is odd.

Of the consecutive integers n and m, one must be even and the other must be odd:

- For two consecutive integers, n may be greater than or less than m. Also, n may be even or odd yet it would not be known if m is even or odd. To gaurantee that of n and m, one is even and the other is odd, we must check the cases for when $n > m$ and $n < m$ as well as if n is even or odd. This creates four total cases to exhaustively check:
- In the case of $\underline{n > m \text{ and } n \text{ is even}}$:
  - Let $n = 2l$ where l is some integer.
  - Then $m = n - 1 = 2l - 1 = 2l + 1 - 2$
  - Let $o = 2l + 1$.
  - By defintion of odd, o is odd.
  - Then $m = o - 2$
  - m is odd because the difference between any odd integer and any even integer is odd (Thereom 4.2.1).
  - Hence n is even and m is odd.
- In the case of $\underline{n > m \text{ and } n \text{ is odd}}$:
  - Let $n = 2p + 1$ where p is some integer.
  - Then $m = 2n + 1 - 1 = 2n$
  - By defintion of even, m is even.
  - Hence n is odd and m is even.

- In the case of $\underline{n < m \text{ and } n \text{ is even}}$:
  - Let $n = 2r$ where r is some integer.
  - Then $m = n + 1 = 2r - 1 = 2r + 1 - 2$
  - Let $s = 2l + 1$.
  - By defintion of odd, s is odd.
  - Then $m = s - 2$
  - m is odd because the difference between any odd integer and any even integer is odd (Thereom 4.2.1).
  - Hence n is even and m is odd.
- In the case of $\underline{n < m \text{ and } n \text{ is odd}}$:
  - Let $n = 2t + 1$ where t is some integer.
  - Then $m = n + 1 = 2t + 2 = 2(t + 1)$
  - Let $u = t + 1$
  - u is an integer because it is the sum of two integers.
  - By substitution, $m = 2(t + 1) = 2u$
  - By defintion of even, m is even.
  - Hence n is odd and m is even.
- Therefore, regardless of whether $n > m$ and that n is even or odd, for the consecutive integers n and m one must be even and the other must be odd.

The expression can be factored out such that $n^2 - m^2 = (n + m)(n - m)$.

$n + m = v$ where v is some odd integer:

- Let $n + m = v$.
- v is the sum of an even integers and an odd integer.
- The sum of an even integer and an odd integer is odd, as shown in the first proof.
- Thus, v must be an odd integer.

$n - m = w$ where w is some odd integer:

- Let $n - m = w$.
- w is the difference of an even integers and an odd integer.
- The difference of an even integer and an odd integer is odd (Theorem 4.2.1).
- Thus, w must be an odd integer.

$n^2 - m^2$ is an odd integer:

- $n^2 - m^2 = (n + m)(n - m) = vw$.

- kl is odd because it is the product of two odd integers, as shown in my second proof.
- $n^2 - m^2$ is odd.

Conclusion:

Therefore, the difference of the squares of any two consecu- tive integers is odd. This was demonstrated by how it can be rewritten as the product of two odd integers.

Exercise Set 4.3

Problem 39

This proof only holds true for $r = \frac{1}{4}$ and $s = \frac{1}{2}$. This does not mean it holds true for any r and s. It is incorrect to generalize from a single example like this.

Exercise Set 4.4

Problem 26

Suppositions:

Let a, b, and c be integers. Suppose $ab|c$.

Goal:

Prove $a|c$ and $b|c$.

Deductions:

$c = abk$ for some integer k:

- Let $d = ab$.
- Due to closure, d must be an integer because it is the product of two integers.
- $ab|c = d|c$.
- By defintion of divisibility, $\exists k \in \mathbb{Z}$ such that $c = dk$.
- $c = dk = abk$.

$a|c$ must be true:

- Let $s = bk$.
- Due to closure, s must be an integer because it is the product of two integers.
- $c = abk = as$.

- Since $\exists s \in \mathbb{Z}$ such that $c = as$, the $a|c$ is also true.

$b|c$ must be true:

- Let $t = ak$.
- Due to closure, t must be an integer because it is the product of two integers.
- $c = abk = tb$.
- Since $\exists t \in \mathbb{Z}$ such that $c = tb$, the $b|c$ is also true.

Conclusion:

Canvas Problems

1

Suppositions:

Let s be rational.

Goal:

Show that $9s^4 + \frac{3}{7}s - 5$ is also rational.

Deductions:

By the definition of rational, $s = \frac{a}{b}$ where a is some integer and b is some non-zero integer.

$$\begin{aligned} 9s^4 + \frac{3}{7}s - 5 &= \frac{9a^4}{b^4} + \frac{3a}{7b} - 5 \\ &= \frac{63a^4}{7b^4} + \frac{3ab^3}{7b^4} - \frac{35b^4}{7b^4} \\ &= \frac{63a^4 + 3ab^3 - 35b^4}{7b^4} \end{aligned}$$

$63a^4 = e$ where e is an integer:

- Let $c = a^2$.
- Due to closure, c is an integer because it is the product of two integers.

- Let $d = a^4$.
- $d = a^4 = c^2$.
- Due to closure, d is an integer because it is the product of two integers.
- Let $e = 63a^4$.
- $e = 63a^4 = 63d$.
- Due to closure, e is an integer because it is the product of two integers.
- Hence $63a^4 = e$ where e is an integer.

$3ab^3 = h$ where h is an integer:

- Let $f = b^2$.
- Due to zero product property and closure, f is a non-zero integer because it is the product of two non-zero integers.
- Let $g = b^3 = fb$.
- Due to closure, g is an integer because it is the product of two integers.
- Let $h = 3ab^3 = 3g$.
- Due to closure, h is an integer because it is the product of two integers.
- Hence $3ab^3 = h$ where h is an integer.

$35b^4 = j$ where j is an integer:

- Let $i = h^2$.
- Due to closure, i is an integer because it is the product of two integers.
- Let $j = 35b^4 = 35i$.
- Due to closure, j is an integer because it is the product of two integers.
- Hence $35b^4 = j$ where j is an integer.

$63a^4 + 3ab^3 - 35b^4 = l$ where l is an integer:

- $63a^4 + 3ab^3 - 35b^4 = e + f - j$.
- Let $k = e + f$.
- Due to closure, k is an integer because it is the sum of two integers.

- $63a^4 + 3ab^3 - 35b^4 = e + f - j = k - j$
- Let $l = k - j$.
- Due to closure, l is an integer because it is the sum of two integers.
- Hence $63a^4 + 3ab^3 - 35b^4 = l$ where l is an integer.

$7b^4 = n$ where n is a non-zero integer:

- $7b^4 = 7f^2$.
- Let $m = f^2$.
- Due to zero product property and closure, m is a non-zero integer because it is the product of two non-zero integers.
- $7b^4 = 7f^2 = 7m$.
- Let $n = 7m$.
- Due to zero product property and closure, n is a non-zero integer because it is the product of two non-zero integers.
- Hence $7b^4 = n$ where n is an integer.

$\frac{63a^4+3ab^3-35b^4}{7b^4}$ is rational:

- $\frac{63a^4+3ab^3-35b^4}{7b^4} = \frac{l}{n}$.
- Since $\exists$ integers l and n such that $\frac{63a^4+3ab^3-35b^4}{7b^4} = \frac{l}{n}$ and $n \neq 0$, $\frac{63a^4+3ab^3-35b^4}{7b^4}$ is therefore rational.

Thus $9s^4 + \frac{3}{7}s - 5$ was shown to be rational.

Conclusion:

Therefore, it must be true that $9s^4 + \frac{3}{7}s - 5$ is rational.

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Suppositions:

Suppose $a|b$ and $a|(b^2 - c)$ where b and c are some integers, By definition of divisibility, a is a non-zero integer.

Goal:

Prove $a|c$ must be true.

<u>Deductions</u>:

$b = am$ where m is a non-zero integer:

- By defintion of divisibility, $a|b$ states $b = am$ for some non-zero integer m.

$b^2 - c = an$ where n is a non-zero integer:

- By defintion of divisibility, $a|(b^2 - c)$ states $b^2 - c = an$ for some non-zero integer n.

$\frac{c}{a} = am^2 - n$:

- By substitution, $b^2 - c = an$ can be rewritten:

$$\begin{aligned} b^2 - c &= an \\ (am)^2 - c &= an \\ a^2m^2 - c &= an \\ c &= a^2m^2 - an \\ c &= a(am^2 - n) \\ \frac{c}{a} &= am^2 - n \end{aligned}$$

$\frac{c}{a} = q$ where q is an integer:

- Recall $\frac{c}{a} = am^2 - n$.
- Let $o = m^2$.
- Due to closure, o is an integer because it is the product of two integers.
- $\frac{c}{a} = am^2 - n = ao - n$
- Let $p = ao$.
- Due to closure, p is an integer because it is the product of two integers.
- $\frac{c}{a} = ao - n = p - n$
- Let $q = p - n$.
- Due to closure, q is an integer because it is the difference of two integers.

- $\frac{c}{a} = p - n = q$
- Hence $\frac{c}{a} = q$ where q is an integer.

$a|c$ must be true.

- Recall $\frac{c}{a} = q$ where c and q are integers, and a is an non-zero integer.
- $\frac{c}{a} = q \Rightarrow c = aq$
- By defintion of divisibility, this implies $a|c$.

Conclusion:

Therefore, it is true that $a|c$.