CS-225: Discrete Structures in CS
Homework 5, Part 1

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Exercise Set 5.2

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Proof:

$P(n) \equiv \sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2$ for every integer $n \geq 0$

Basis Step:

$$
\begin{aligned}
P(0) = \sum_{i=1}^{0+1} i \cdot 2^i &= 0 \cdot 2^{0+2} + 2 \\
\sum_{i=1}^{1} i \cdot 2^i &= 2 \\
1 \cdot 2^1 &= 2 \\
2 &= 2
\end{aligned}
$$

The base case $P(0)$ holds true.

Inductive Hypothesis:

Suppose that for an aribitrary but particular integer k,

$$P(k) \equiv \sum_{i=1}^{k+1} i \cdot 2^i = k \cdot 2^{k+2} + 2 \text{ where } k \geq 0$$

is true.

Inductive Steps:

We must show that $P(k+1)$ is true. Hence we must demonstrate,

$$P(k+1) = \sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2 \text{ where } k \geq 0$$

- Let the expression $\sum_{i=1}^{k+2} i \cdot 2^i$ be referred to as the left-hand side (LHS) of the equation.
- Let the expression $(k+1) \cdot 2^{k+3} + 2$ be referred to as the right-hand side (RHS) of the equation.
- To show $P(k+1)$ is true, it must be shown that LHS is equal to RHS.

$\sum_{i=1}^{k+2} i \cdot 2^i \Rightarrow (k+1) \cdot 2^{k+3} + 2$

- Recall the LHS is $\sum_{i=1}^{k+2} i \cdot 2^i$
- By algebra and the supposition of $P(k)$, the LHS can be simplified:

$$\begin{aligned}
\sum_{i=1}^{k+2} i \cdot 2^i &= \sum_{i=1}^{k+2} i \cdot 2^i \\
&= \sum_{i=1}^{k+1} i \cdot 2^i + (k+2) \cdot 2^{k+2} \\
&= (k \cdot 2^{k+2} + 2) + (k \cdot 2^{k+2} + 2 \cdot 2^{k+2}) \\
&= 2(k \cdot 2^{k+2}) + 2^{k+3} + 2 \\
&= k(2 \cdot 2^{k+2}) + 2^{k+3} + 2 \\
&= k(2^{k+3}) + 2^{k+3} + 2 \\
&= (k+1) \cdot 2^{k+3} + 2
\end{aligned}$$

- Recall the RHS is $(k+1) \cdot 2^{k+3} + 2$
- Thus, LHS is equal to RHS

Thus, $P(k+1) = \sum_{i=1}^{k+2} i \cdot 2^i = (k+1) \cdot 2^{k+3} + 2$ where $k \geq 0$ was to be shown.

<u>Conclusion</u>:

Since both the basis step and the inductive step have been proved, the original expression,

$$P(n) \equiv \sum_{i=1}^{n+1} i \cdot 2^i = n \cdot 2^{n+2} + 2 \text{ for every integer } n \geq 0$$

must be true.

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Proof:

$P(n) = \prod_{i=2}^{n} \left(1 - \frac{1}{i}\right) = \frac{1}{n}$ for every integer $n \geq 2$

Basis Step:

$$\begin{aligned} P(2) = \prod_{i=2}^{2} \left(1 - \frac{1}{i}\right) &= \frac{1}{2} \\ \left(1 - \frac{1}{2}\right) &= \frac{1}{2} \\ \frac{1}{2} &= \frac{1}{2} \end{aligned}$$

The base case $P(2)$ holds true.

Inductive Hypothesis:

Suppose that for an aribitrary but particular integer k,

$$P(k) \equiv \prod_{i=2}^{k} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{k} \text{ where } k \geq 2$$

is true.

Inductive Steps:

We must show that $P(k+1)$ is true. Hence we must demonstrate,

$$P(k+1) = \prod_{i=2}^{k+1} \left(1 - \tfrac{1}{i}\right) = \tfrac{1}{k+1} \text{ where } k \geq 2$$

- Let the expression $\prod_{i=2}^{k+1} \left(1 - \frac{1}{i}\right)$ be referred to as the left-hand side (LHS) of the equation.
- Let the expression $\frac{1}{k+1}$ be referred to as the right-hand side (RHS) of the equation.
- To show $P(k+1)$ is true, it must be shown that LHS is equal to RHS.

$\prod_{i=2}^{k+1} \left(1 - \frac{1}{i}\right) \Rightarrow \frac{1}{k+1}$

- Recall the LHS is $\prod_{i=2}^{k+1} \left(1 - \frac{1}{i}\right)$

- By algebra and the supposition of $P(k)$, the LHS can be simplified:

$$\begin{aligned}
\prod_{i=2}^{k+1}\left(1-\frac{1}{i}\right) &= \prod_{i=2}^{k}\left(1-\frac{1}{i}\right)\cdot\left(1-\frac{1}{k+1}\right)\\
&= \left(\frac{1}{k}\right)\cdot\left(1-\frac{1}{k+1}\right)\\
&= \frac{1}{k}-\frac{1}{k}\cdot\frac{1}{k+1}\\
&= \frac{k+1}{k+1}\cdot\frac{1}{k} \quad - \quad \frac{1}{k}\cdot\frac{1}{k+1}\\
&= \frac{k+1}{k^2+k}-\frac{1}{k^2+k}\\
&= \frac{k}{k^2+k}\\
&= \frac{1}{k+1}
\end{aligned}$$

- Recall the RHS is $\frac{1}{k+1}$
- Thus, LHS is equal to RHS.

Thus, $P(k+1) = \prod_{i=2}^{k+1}\left(1-\frac{1}{i}\right) = \frac{1}{k+1}$ where $k \geq 2$ was to be shown.

Conclusion:

Since both the basis and inductive step have been proved, the original expression,

$$P(n) = \prod_{i=2}^{n}\left(1-\frac{1}{i}\right) = \frac{1}{n} \text{ for every integer } n \geq 2$$

must be true.

Exercise Set 5.3

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Proof:

$P(n) \equiv 7^n - 2^n$ is divisible by 5, where $n \geq 0$

Basis Step:

$$\begin{aligned} P(0) &= 7^0 - 2^0 \text{ is divisible by } 5 \\ &= 1 - 1 \text{ is divisible by } 5 \\ &= 0 \text{ is divisible by } 5 \end{aligned}$$

The base case for $P(0)$ holds true.

Inductive Hypothesis:

Suppose that for an arbitrary but particular integer k, such that $k \geq 0$,

$$P(k) \equiv 7^k - 2^k \text{ is divisible by } 5$$

is true.

By definition of divisibility, this means that,

$$P(k) \equiv 7^k - 2^k = 5r \text{ for some integer } r$$

Inductive Steps:

We must show that $P(k+1)$ is true. Hence we must demonstrate,

$$P(k+1) = 7^{k+1} - 2^{k+1} \text{ is divisible by } 5$$

$P(k+1)$ is indeed divisible by 5.

- The expression for $P(k+1)$ can be simplified using algebra and the supposition of $P(k)$:

$$\begin{aligned} P(k+1) &= 7^{k+1} - 2^{k+1} \\ &= 7 \cdot 7^k - 2 \cdot 2^k \\ &= (5+2) \cdot 7^k - 2 \cdot 2^k \\ &= 5 \cdot 7^k - 2(7^k - 2^k) \\ &= 5 \cdot 7^k - 2(5r) \\ &= 5(7^k - 2r) \end{aligned}$$

- Hence, $P(k+1) = 5(7^k - 2r)$
- By closure, the internal expression $7^k - 2r$ is an integer because it is the product and summation of integers.
- Then by definition of divisibility, $5(7^k - 2r)$ is divisible by 5.
- By equallity, $P(k+1)$ must also be divisible by 5.

- Hence, $P(k+1)$ is divisible by 5.

Thus, it was shown that $P(k+1) = 7^{k+1} - 2^{k+1}$ is divisible by 5.

Conclusion:

Since both the basis and inductive step have been proved, the original expression,

$$P(n) \equiv 7^n - 2^n \text{ is divisible by 5, where } n \geq 0$$

must be true.

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Proof:

$P(n) \equiv n(n^2+5)$ is divisible by 6, for each integer $n \geq 0$

Basis Step:

$$\begin{aligned} P(0) &= 0 \cdot (0^2+5) \text{ is divisible by 6} \\ &= 0 \text{ is divisible by 6} \end{aligned}$$

The base case for $P(0)$ holds true.

Inductive Hypothesis:

Suppose that for an arbitrary but particular integer k, such that $k \geq 0$,

$$P(k) \equiv n(n^2+5) \text{ is divisible by 6}$$

is true.

By definition of divisibility, this means that,

$$P(k) \equiv n(n^2+5) = 6r \text{ for some integer } r$$

Inductive Steps:

We must show that $P(k+1)$ is true. Hence we must demonstrate,

$$P(k+1) = (n+1) \cdot ((n+1)^2+5) \text{ is divisible by 6}$$

$p(p+1)$ can be expressed as $2q$ where p and q are some integers.

- Let p and q be integers.
- Let $t = p(p+1)$.
- By closure, t is an integer because it is the product and summation of integers.
- By the Parity Property (Theorem 4.5.2), p and $p+1$ have opposite parity; one expression is even and the other is odd.
- By definition of even, t is even because it is the product of an even and an odd integer.
- Hence, t can be expressed as $2q$.
- By equality, $p(p+1) = t = 2q$.
- Therefore, $p(p+1)$ can be expressed as $2q$ where p and q are some integers.

$P(k+1)$ is indeed divisible by 6.

- The expression for $P(k+1)$ can be simplified using algebra and the supposition of $P(k)$:

$$\begin{aligned}
P(k+1) &= (n+1) \cdot ((n+1)^2 + 5) \\
&= (n+1) \cdot (n^2 + 5 + n + 1) \\
&= (n+1) \cdot ((n^2 + 5) + (2n+1)) \\
&= n(n^2 + 5) + (n^2 + 5) + (n+1)(2n+1) \\
&= 6r + 3n^2 + 3n + 6 \\
&= 6(r+1) + 3n^2 + 3n \\
&= 6(r+1) + 3(n(n+1))
\end{aligned}$$

- Let m be some integer.
- The expression $n(n+1) = 2m$ as demonstrated by the first deduction.
- This can be subsituted into our new expression for $P(k+1)$. Contiuing the algebra from before:

$$\begin{aligned}
P(k+1) &= 6(r+1) + 3(n(n+1)) \\
&= 6(r+1) + 3(2m) \\
&= 6(r+1) + 6m \\
&= 6(r+1+m)
\end{aligned}$$

- Hence, $P(k+1) = 6(r+1+m)$

- By closure, the internal expression $r+1+m$ is an integer because it is the summation of integers.
- Then by definition of divisibility, $6(r+1+m)$ is divisible by 6.
- By equality, $P(k+1)$ must also be divisible by 6.
- Hence, $P(k+1)$ is divisible by 6.

Thus, it was shown that $P(k+1)$ is divisible by 6.

Conclusion:

Since both the basis and inductive step have been proved, the original expression,

$$P(n) \equiv n(n^2+5) \text{ is divisible by 6, for each integer } n \geq 0$$

must be true.

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Proof:

$P(n) \equiv n! > n^2$, for each integer $n \geq 4$

Basis Step:

$$\begin{aligned} P(4) &= 4! > 4^2 \\ &= 24 > 16 \end{aligned}$$

The base case for $P(4)$ holds true.

Inductive Hypothesis:

Suppose that for an arbitrary but particular integer k, such that $k \geq 4$,

$$P(k) \equiv k! > k^2$$

is true.

Inductive Steps:

We must show that $P(k+1)$ is true. Hence we must demonstrate,

$$P(k+1) = (k+1)! > (k+1)^2$$

- Let $(k+1)!$ be referred to as the left-hand side (LHS) of the equation.
- Let $(k+1)^2$ be referred to as the right-hand side (RHS) of the equation.
- To show $P(k+1)$ is true, it must be shown that LHS is greater than RHS.

$P(k+1)$ is indeed true.

- The expression for $P(k+1)$ can be simplified using algebra:

$$\begin{aligned} P(k+1) = (k+1)! &> (k+1)^2 \\ (k+1)k! &> (k+1)^2 \\ k! &> (k+1) \end{aligned}$$

- By Proof#2 in the provided Additional Proofs, it is true that $k! > (k+1)$ for every integer $k \geq 4$.
- Hence, $P(k+1)$ is indeed true.

Thus, it was shown that $P(k+1)$ is true.

<u>Conclusion</u>:

Since both the basis and inductive step have been proved, the original expression,

$$P(n) \equiv n! > n^2, \text{ for each integer } n \geq 4$$

must be true.