At this point it is worthwhile to introduce a common notation: we say that f belongs to the class C^k if f is k times continuously differentiable. Belonging to the class C^k or satisfying a Hölder condition are two possible ways to describe the *smoothness* of a function.

3 Convolutions

The notion of convolution of two functions plays a fundamental role in Fourier analysis; it appears naturally in the context of Fourier series but also serves more generally in the analysis of functions in other settings.

Given two 2π-periodic integrable functions f and g on $\mathbf{R}$, we define their **convolution** $f * g$ on $[-\pi, \pi]$ by

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,dy. \tag{2}$$

The above integral makes sense for each x, since the product of two integrable functions is again integrable. Also, since the functions are periodic, we can change variables to see that

$$(f * g)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)g(y)\,dy.$$

Loosely speaking, convolutions correspond to "weighted averages." For instance, if $g = 1$ in (2), then $f * g$ is constant and equal to $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\,dy$, which we may interpret as the average value of f on the circle. Also, the convolution $(f * g)(x)$ plays a role similar to, and in some sense replaces, the pointwise product $f(x)g(x)$ of the two functions f and g.

In the context of this chapter, our interest in convolutions originates from the fact that the partial sums of the Fourier series of f can be expressed as follows:

$$\begin{aligned}
S_N(f)(x) &= \sum_{n=-N}^{N} \hat{f}(n)e^{inx} \\
&= \sum_{n=-N}^{N}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)e^{-iny}\,dy\right)e^{inx} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\left(\sum_{n=-N}^{N} e^{in(x-y)}\right)dy \\
&= (f * D_N)(x),
\end{aligned}$$

where D_N is the N^{th} Dirichlet kernel (see Example 4) given by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}.$$

So we observe that the problem of understanding $S_N(f)$ reduces to the understanding of the convolution $f * D_N$.

We begin by gathering some of the main properties of convolutions.

Proposition 3.1 *Suppose that f, g, and h are 2π-periodic integrable functions. Then:*

(i) $f * (g + h) = (f * g) + (f * h)$.

(ii) $(cf) * g = c(f * g) = f * (cg)$ *for any* $c \in \mathbf{C}$.

(iii) $f * g = g * f$.

(iv) $(f * g) * h = f * (g * h)$.

(v) $f * g$ *is continuous.*

(vi) $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n)$.

The first four points describe the algebraic properties of convolutions: linearity, commutativity, and associativity. Property (v) exhibits an important principle: the convolution of $f * g$ is "more regular" than f or g. Here, $f * g$ is continuous while f and g are merely (Riemann) integrable. Finally, (vi) is key in the study of Fourier series. In general, the Fourier coefficients of the product fg are not the product of the Fourier coefficients of f and g. However, (vi) says that this relation holds if we replace the product of the two functions f and g by their convolution $f * g$.

Proof. Properties (i) and (ii) follow at once from the linearity of the integral.

The other properties are easily deduced if we assume also that f and g are continuous. In this case, we may freely interchange the order of integration. For instance, to establish (vi) we write

$$\begin{aligned}\widehat{f*g}(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(f*g)(x)e^{-inx}\,dx\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}f(y)g(x-y)\,dy\right)e^{-inx}\,dx\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)e^{-iny}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}g(x-y)e^{-in(x-y)}\,dx\right)dy\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)e^{-iny}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}g(x)e^{-inx}\,dx\right)dy\\ &= \hat{f}(n)\hat{g}(n).\end{aligned}$$

To prove (iii), one first notes that if F is continuous and 2π-periodic, then

$$\int_{-\pi}^{\pi}F(y)\,dy = \int_{-\pi}^{\pi}F(x-y)\,dy \quad \text{for any } x\in\mathbb{R}.$$

The verification of this identity consists of a change of variables $y\mapsto -y$, followed by a translation $y\mapsto y-x$. Then, one takes $F(y)=f(y)g(x-y)$.

Also, (iv) follows by interchanging two integral signs, and an appropriate change of variables.

Finally, we show that if f and g are continuous, then $f*g$ is continuous. First, we may write

$$(f*g)(x_1)-(f*g)(x_2) = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(y)\,[g(x_1-y)-g(x_2-y)]\,dy.$$

Since g is continuous it must be uniformly continuous on any closed and bounded interval. But g is also periodic, so it must be uniformly continuous on all of $\mathbb{R}$; given $\epsilon>0$ there exists $\delta>0$ so that $|g(s)-g(t)|<\epsilon$ whenever $|s-t|<\delta$. Then, $|x_1-x_2|<\delta$ implies $|(x_1-y)-(x_2-y)|<\delta$ for any y, hence

$$\begin{aligned}|(f*g)(x_1)-(f*g)(x_2)| &\le \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}f(y)\,[g(x_1-y)-g(x_2-y)]\,dy\right|\\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(y)|\,|g(x_1-y)-g(x_2-y)|\,dy\\ &\le \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi}|f(y)|\,dy\\ &\le \frac{\epsilon}{2\pi}2\pi B,\end{aligned}$$

where B is chosen so that $|f(x)|\le B$ for all x. As a result, we conclude that $f*g$ is continuous, and the proposition is proved, at least when f and g are continuous.

In general, when f and g are merely integrable, we may use the results established so far (when f and g are continuous), together with the following approximation lemma, whose proof may be found in the appendix.

Lemma 3.2 *Suppose f is integrable on the circle and bounded by B. Then there exists a sequence $\{f_k\}_{k=1}^{\infty}$ of continuous functions on the circle so that*

$$\sup_{x\in[-\pi,\pi]}|f_k(x)|\le B \quad \textit{for all } k=1,2,\dots,$$

and

$$\int_{-\pi}^{\pi}|f(x)-f_k(x)|\,dx\to 0 \quad \textit{as } k\to\infty.$$

Using this result, we may complete the proof of the proposition as follows. Apply Lemma 3.2 to f and g to obtain sequences $\{f_k\}$ and $\{g_k\}$ of approximating continuous functions. Then

$$f*g-f_k*g_k=(f-f_k)*g+f_k*(g-g_k).$$

By the properties of the sequence $\{f_k\}$,

$$\begin{aligned}|(f-f_k)*g(x)| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x-y)-f_k(x-y)|\,|g(y)|\,dy\\ &\le \frac{1}{2\pi}\sup_y|g(y)|\int_{-\pi}^{\pi}|f(y)-f_k(y)|\,dy\\ &\to 0 \quad \text{as } k\to\infty.\end{aligned}$$

Hence $(f-f_k)*g\to 0$ uniformly in x. Similarly, $f_k*(g-g_k)\to 0$ uniformly, and therefore f_k*g_k tends uniformly to $f*g$. Since each f_k*g_k is continuous, it follows that $f*g$ is also continuous, and we have (v).

Next, we establish (vi). For each fixed integer n we must have $\widehat{f_k*g_k}(n)\to\widehat{f*g}(n)$ as k tends to infinity since f_k*g_k converges uniformly to $f*g$. However, we found earlier that $\widehat{f_k}(n)\widehat{g_k}(n)=\widehat{f_k*g_k}(n)$ because both f_k and g_k are continuous. Hence

$$\begin{aligned}|\hat{f}(n)-\hat{f}_k(n)| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi}(f(x)-f_k(x))e^{-inx}\,dx\right|\\ &\le \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)-f_k(x)|\,dx,\end{aligned}$$

and as a result we find that $\widehat{f_k}(n) \to \hat{f}(n)$ as k goes to infinity. Similarly $\widehat{g_k}(n) \to \hat{g}(n)$, and the desired property is established once we let k tend to infinity. Finally, properties (iii) and (iv) follow from the same kind of arguments.

4 Good kernels

In the proof of Theorem 2.1 we constructed a sequence of trigonometric polynomials $\{p_k\}$ with the property that the functions p_k peaked at the origin. As a result, we could isolate the behavior of f at the origin. In this section, we return to such families of functions, but this time in a more general setting. First, we define the notion of good kernel, and discuss the characteristic properties of such functions. Then, by the use of convolutions, we show how these kernels can be used to recover a given function.

A family of kernels $\{K_n(x)\}_{n=1}^{\infty}$ on the circle is said to be a family of **good kernels** if it satisfies the following properties:

(a) For all $n \geq 1$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(x)\,dx = 1.$$

(b) There exists $M > 0$ such that for all $n \geq 1$,

$$\int_{-\pi}^{\pi} |K_n(x)|\,dx \leq M.$$

(c) For every $\delta > 0$,

$$\int_{\delta \leq |x| \leq \pi} |K_n(x)|\,dx \to 0, \quad \text{as } n \to \infty.$$

In practice we shall encounter families where $K_n(x) \geq 0$, in which case (b) is a consequence of (a). We may interpret the kernels $K_n(x)$ as weight distributions on the circle: property (a) says that K_n assigns unit mass to the whole circle $[-\pi, \pi]$, and (c) that this mass concentrates near the origin as n becomes large.[6] Figure 4 (a) illustrates the typical character of a family of good kernels.

The importance of good kernels is highlighted by their use in connection with convolutions.

[6] In the limit, a family of good kernels represents the "Dirac delta function." This terminology comes from physics.

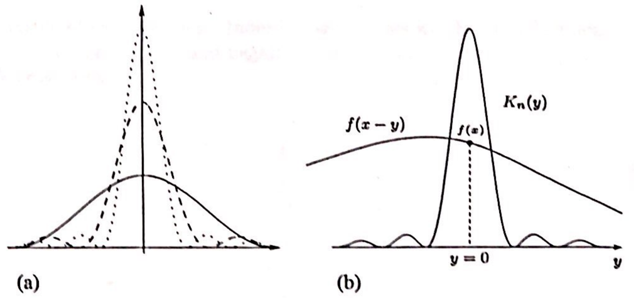

Figure 4. Good kernels

Theorem 4.1 *Let $\{K_n\}_{n=1}^{\infty}$ be a family of good kernels, and f an integrable function on the circle. Then*

$$\lim_{n\to\infty} (f * K_n)(x) = f(x)$$

whenever f is continuous at x. If f is continuous everywhere, then the above limit is uniform.

Because of this result, the family $\{K_n\}$ is sometimes referred to as an **approximation to the identity**.

We have previously interpreted convolutions as weighted averages. In this context, the convolution

$$(f * K_n)(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y)K_n(y)\,dy$$

is the average of $f(x-y)$, where the weights are given by $K_n(y)$. However, the weight distribution K_n concentrates its mass at $y = 0$ as n becomes large. Hence in the integral, the value $f(x)$ is assigned the full mass as $n \to \infty$. Figure 4 (b) illustrates this point.

Proof of Theorem 4.1. If $\epsilon > 0$ and f is continuous at x, choose δ so that $|y| < \delta$ implies $|f(x-y) - f(x)| < \epsilon$. Then, by the first property of good kernels, we can write

$$\begin{aligned}(f * K_n)(x) - f(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)f(x-y)\,dy - f(x)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)]\,dy.\end{aligned}$$

Hence,

$$
\begin{aligned}
|(f * K_n)(x) - f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi} K_n(y)[f(x-y) - f(x)]\,dy\right| \\
&\le \frac{1}{2\pi}\int_{|y|<\delta} |K_n(y)|\,|f(x-y) - f(x)|\,dy \\
&\quad + \frac{1}{2\pi}\int_{\delta\le|y|\le\pi} |K_n(y)|\,|f(x-y) - f(x)|\,dy \\
&\le \frac{\epsilon}{2\pi}\int_{-\pi}^{\pi} |K_n(y)|\,dy + \frac{2B}{2\pi}\int_{\delta\le|y|\le\pi} |K_n(y)|\,dy,
\end{aligned}
$$

where B is a bound for f. The first term is bounded by $\epsilon M/2\pi$ because of the second property of good kernels. By the third property we see that for all large n, the second term will be less than ϵ. Therefore, for some constant $C > 0$ and all large n we have

$$|(f * K_n)(x) - f(x)| \le C\epsilon,$$

thereby proving the first assertion in the theorem. If f is continuous everywhere, then it is uniformly continuous, and δ can be chosen independent of x. This provides the desired conclusion that $f * K_n \to f$ uniformly.

Recall from the beginning of Section 3 that

$$S_N(f)(x) = (f * D_N)(x),$$

where $D_N(x) = \sum_{n=-N}^{N} e^{inx}$ is the Dirichlet kernel. It is natural now for us to ask whether D_N is a good kernel, since if this were true, Theorem 4.1 would imply that the Fourier series of f converges to $f(x)$ whenever f is continuous at x. Unfortunately, this is not the case. Indeed, an estimate shows that D_N violates the second property; more precisely, one has (see Problem 2)

$$\int_{-\pi}^{\pi} |D_N(x)|\,dx \ge c\log N, \quad \text{as } N \to \infty.$$

However, we should note that the formula for D_N as a sum of exponentials immediately gives

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)\,dx = 1,$$

so the first property of good kernels is actually verified. The fact that the mean value of D_N is 1, while the integral of its absolute value is large, is a result of cancellations. Indeed, Figure 5 shows that the function $D_N(x)$ takes on positive and negative values and oscillates very rapidly as N gets large.

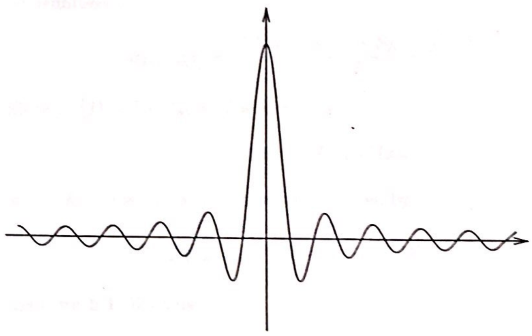

Figure 5. The Dirichlet kernel for large N

This observation suggests that the pointwise convergence of Fourier series is intricate, and may even fail at points of continuity. This is indeed the case, as we will see in the next chapter.

5 Cesàro and Abel summability: applications to Fourier series

Since a Fourier series may fail to converge at individual points, we are led to try to overcome this failure by interpreting the limit

$$\lim_{N\to\infty} S_N(f) = f$$

in a different sense.

5.1 Cesàro means and summation

We begin by taking ordinary averages of the partial sums, a technique which we now describe in more detail.

Suppose we are given a series of complex numbers

$$c_0 + c_1 + c_2 + \cdots = \sum_{k=0}^{\infty} c_k.$$

We define the n^{th} partial sum s_n by

$$s_n = \sum_{k=0}^{n} c_k,$$

and say that the series converges to s if $\lim_{n\to\infty} s_n = s$. This is the most natural and most commonly used type of "summability." Consider, however, the example of the series

$$1 - 1 + 1 - 1 + \cdots = \sum_{k=0}^{\infty} (-1)^k. \tag{3}$$

Its partial sums form the sequence $\{1, 0, 1, 0, \ldots\}$ which has no limit. Because these partial sums alternate evenly between 1 and 0, one might therefore suggest that 1/2 is the "limit" of the sequence, and hence 1/2 equals the "sum" of that particular series. We give a precise meaning to this by defining the average of the first N partial sums by

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_{N-1}}{N}.$$

The quantity σ_N is called the N^{th} **Cesàro mean**[7] of the sequence $\{s_k\}$ or the N^{th} **Cesàro sum** of the series $\sum_{k=0}^{\infty} c_k$.

If σ_N converges to a limit σ as N tends to infinity, we say that the series $\sum c_n$ is **Cesàro summable** to σ. In the case of series of functions, we shall understand the limit in the sense of either pointwise or uniform convergence, depending on the situation.

The reader will have no difficulty checking that in the above example (3), the series is Cesàro summable to 1/2. Moreover, one can show that Cesàro summation is a more inclusive process than convergence. In fact, if a series is convergent to s, then it is also Cesàro summable to the same limit s (Exercise 12).

5.2 Fejér's theorem

An interesting application of Cesàro summability appears in the context of Fourier series.

[7] Note that if the series $\sum_{k=1}^{\infty} c_k$ begins with the term $k = 1$, then it is common practice to define $\sigma_N = (s_1 + \cdots + s_N)/N$. This change of notation has little effect on what follows.

We mentioned earlier that the Dirichlet kernels fail to belong to the family of good kernels. Quite surprisingly, their averages are very well behaved functions, in the sense that they do form a family of good kernels.

To see this, we form the N^{th} Cesàro mean of the Fourier series, which by definition is

$$\sigma_N(f)(x) = \frac{S_0(f)(x) + \cdots + S_{N-1}(f)(x)}{N}.$$

Since $S_n(f) = f * D_n$, we find that

$$\sigma_N(f)(x) = (f * F_N)(x),$$

where $F_N(x)$ is the N-th **Fejér kernel** given by

$$F_N(x) = \frac{D_0(x) + \cdots + D_{N-1}(x)}{N}.$$

Lemma 5.1 *We have*

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)},$$

and the Fejér kernel is a good kernel.

The proof of the formula for F_N (a simple application of trigonometric identities) is outlined in Exercise 15. To prove the rest of the lemma, note that F_N is positive and $\frac{1}{2\pi}\int_{-\pi}^{\pi} F_N(x)\,dx = 1$, in view of the fact that a similar identity holds for the Dirichlet kernels D_n. However, $\sin^2(x/2) \geq c_\delta > 0$, if $\delta \leq |x| \leq \pi$, hence $F_N(x) \leq 1/(Nc_\delta)$, from which it follows that

$$\int_{\delta \leq |x| \leq \pi} |F_N(x)|\,dx \to 0 \quad \text{as } N \to \infty.$$

Applying Theorem 4.1 to this new family of good kernels yields the following important result.

Theorem 5.2 *If f is integrable on the circle, then the Fourier series of f is Cesàro summable to f at every point of continuity of f.*

Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Cesàro summable to f.

We may now state two corollaries. The first is a result that we have already established. The second is new, and of fundamental importance.

Corollary 5.3 *If f is integrable on the circle and $\hat{f}(n) = 0$ for all n, then $f = 0$ at all points of continuity of f.*

The proof is immediate since all the partial sums are 0, hence all the Cesàro means are 0.

Corollary 5.4 *Continuous functions on the circle can be uniformly approximated by trigonometric polynomials.*

This means that if f is continuous on $[-\pi, \pi]$ with $f(-\pi) = f(\pi)$ and $\epsilon > 0$, then there exists a trigonometric polynomial P such that

$$|f(x) - P(x)| < \epsilon \quad \text{for all } -\pi \le x \le \pi.$$

This follows immediately from the theorem since the partial sums, hence the Cesàro means, are trigonometric polynomials. Corollary 5.4 is the periodic analogue of the Weierstrass approximation theorem for polynomials which can be found in Exercise 16.

5.3 Abel means and summation

Another method of summation was first considered by Abel and actually predates the Cesàro method.

A series of complex numbers $\sum_{k=0}^{\infty} c_k$ is said to be **Abel summable** to s if for every $0 \le r < 1$, the series

$$A(r) = \sum_{k=0}^{\infty} c_k r^k$$

converges, and

$$\lim_{r \to 1} A(r) = s.$$

The quantities $A(r)$ are called the **Abel means** of the series. One can prove that if the series converges to s, then it is Abel summable to s. Moreover, the method of Abel summability is even more powerful than the Cesàro method: when the series is Cesàro summable, it is always Abel summable to the same sum. However, if we consider the series

$$1 - 2 + 3 - 4 + 5 - \cdots = \sum_{k=0}^{\infty} (-1)^k (k+1),$$

then one can show that it is Abel summable to 1/4 since

$$A(r) = \sum_{k=0}^{\infty} (-1)^k (k+1) r^k = \frac{1}{(1+r)^2},$$

but this series is not Cesàro summable; see Exercise 13.

5.4 The Poisson kernel and Dirichlet's problem in the unit disc

To adapt Abel summability to the context of Fourier series, we define the Abel means of the function $f(\theta) \sim \sum_{n=-\infty}^{\infty} a_n e^{in\theta}$ by

$$A_r(f)(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta}.$$

Since the index n takes positive and negative values, it is natural to write $c_0 = a_0$, and $c_n = a_n e^{in\theta} + a_{-n} e^{-in\theta}$ for $n > 0$, so that the Abel means of the Fourier series correspond to the definition given in the previous section for numerical series.

We note that since f is integrable, $|a_n|$ is uniformly bounded in n, so that $A_r(f)$ converges absolutely and uniformly for each $0 \le r < 1$. Just as in the case of Cesàro means, the key fact is that these Abel means can be written as convolutions

$$A_r(f)(\theta) = (f * P_r)(\theta),$$

where $P_r(\theta)$ is the **Poisson kernel** given by

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}. \tag{4}$$

In fact,

$$\begin{aligned} A_r(f)(\theta) &= \sum_{n=-\infty}^{\infty} r^{|n|} a_n e^{in\theta} \\ &= \sum_{n=-\infty}^{\infty} r^{|n|} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) e^{-in\varphi}\, d\varphi \right) e^{in\theta} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\varphi) \left(\sum_{n=-\infty}^{\infty} r^{|n|} e^{-in(\varphi-\theta)} \right) d\varphi, \end{aligned}$$

where the interchange of the integral and infinite sum is justified by the uniform convergence of the series.

Lemma 5.5 *If $0 \le r < 1$, then*

$$P_r(\theta) = \frac{1 - r^2}{1 - 2r\cos\theta + r^2}.$$

The Poisson kernel is a good kernel,[8] *as r tends to 1 from below.*

Proof. The identity $P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta+r^2}$ has already been derived in Section 1.1. Note that

$$1 - 2r\cos\theta + r^2 = (1-r)^2 + 2r(1-\cos\theta).$$

Hence if $1/2 \leq r \leq 1$ and $\delta \leq |\theta| \leq \pi$, then

$$1 - 2r\cos\theta + r^2 \geq c_\delta > 0.$$

Thus $P_r(\theta) \leq (1-r^2)/c_\delta$ when $\delta \leq |\theta| \leq \pi$, and the third property of good kernels is verified. Clearly $P_r(\theta) \geq 0$, and integrating the expression (4) term by term (which is justified by the absolute convergence of the series) yields

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta)\, d\theta = 1,$$

thereby concluding the proof that P_r is a good kernel.

Combining this lemma with Theorem 4.1, we obtain our next result.

Theorem 5.6 *The Fourier series of an integrable function on the circle is Abel summable to f at every point of continuity. Moreover, if f is continuous on the circle, then the Fourier series of f is uniformly Abel summable to f.*

We now return to a problem discussed in Chapter 1, where we sketched the solution of the steady-state heat equation $\Delta u = 0$ in the unit disc with boundary condition $u = f$ on the circle. We expressed the Laplacian in terms of polar coordinates, separated variables, and expected that a solution was given by

$$u(r,\theta) = \sum_{m=-\infty}^{\infty} a_m r^{|m|} e^{im\theta}, \tag{5}$$

where a_m was the m^{th} Fourier coefficient of f. In other words, we were led to take

$$u(r,\theta) = A_r(f)(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi) P_r(\theta-\varphi)\, d\varphi.$$

We are now in a position to show that this is indeed the case.

[8] In this case, the family of kernels is indexed by a continuous parameter $0 \leq r < 1$, rather than the discrete n considered previously. In the definition of good kernels, we simply replace n by r and take the limit in property (c) appropriately, for example $r \to 1$ in this case.

Theorem 5.7 *Let f be an integrable function defined on the unit circle. Then the function u defined in the unit disc by the Poisson integral*

$$u(r,\theta) = (f * P_r)(\theta) \tag{6}$$

has the following properties:

(i) *u has two continuous derivatives in the unit disc and satisfies $\Delta u = 0$.*

(ii) *If θ is any point of continuity of f, then*

$$\lim_{r\to 1} u(r,\theta) = f(\theta).$$

If f is continuous everywhere, then this limit is uniform.

(iii) *If f is continuous, then $u(r,\theta)$ is the unique solution to the steady-state heat equation in the disc which satisfies conditions* (i) *and* (ii).

Proof. For (i), we recall that the function u is given by the series (5). Fix $\rho < 1$; inside each disc of radius $r < \rho < 1$ centered at the origin, the series for u can be differentiated term by term, and the differentiated series is uniformly and absolutely convergent. Thus u can be differentiated twice (in fact infinitely many times), and since this holds for all $\rho < 1$, we conclude that u is twice differentiable inside the unit disc. Moreover, in polar coordinates,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2},$$

so term by term differentiation shows that $\Delta u = 0$.

The proof of (ii) is a simple application of the previous theorem. To prove (iii) we argue as follows. Suppose v solves the steady-state heat equation in the disc and converges to f uniformly as r tends to 1 from below. For each fixed r with $0 < r < 1$, the function $v(r,\theta)$ has a Fourier series

$$\sum_{n=-\infty}^{\infty} a_n(r) e^{in\theta} \quad \text{where} \quad a_n(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} v(r,\theta) e^{-in\theta}\, d\theta.$$

Taking into account that $v(r,\theta)$ solves the equation

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0, \tag{7}$$

we find that

$$a_n''(r) + \frac{1}{r}a_n'(r) - \frac{n^2}{r^2}a_n(r) = 0. \tag{8}$$

Indeed, we may first multiply (7) by $e^{-in\theta}$ and integrate in θ. Then, since v is periodic, two integrations by parts give

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\partial^2 v}{\partial \theta^2}(r,\theta)e^{-in\theta}\,d\theta = -n^2 a_n(r).$$

Finally, we may interchange the order of differentiation and integration, which is permissible since v has two continuous derivatives; this yields (8).

Therefore, we must have $a_n(r) = A_n r^n + B_n r^{-n}$ for some constants A_n and B_n, when $n \neq 0$ (see Exercise 11 in Chapter 1). To evaluate the constants, we first observe that each term $a_n(r)$ is bounded because v is bounded, therefore $B_n = 0$. To find A_n we let $r \to 1$. Since v converges uniformly to f as $r \to 1$ we find that

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)e^{-in\theta}\,d\theta.$$

By a similar argument, this formula also holds when $n = 0$. Our conclusion is that for each $0 < r < 1$, the Fourier series of v is given by the series of $u(r,\theta)$, so by the uniqueness of Fourier series for continuous functions, we must have $u = v$.

Remark. By part (iii) of the theorem, we may conclude that if u solves $\Delta u = 0$ in the disc, and converges to 0 uniformly as $r \to 1$, then u must be identically 0. However, if uniform convergence is replaced by pointwise convergence, this conclusion may fail; see Exercise 18.

6 Exercises

1. Suppose f is 2π-periodic and integrable on any finite interval. Prove that if $a, b \in \mathbf{R}$, then

$$\int_a^b f(x)\,dx = \int_{a+2\pi}^{b+2\pi} f(x)\,dx = \int_{a-2\pi}^{b-2\pi} f(x)\,dx.$$

Also prove that

$$\int_{-\pi}^{\pi} f(x+a)\,dx = \int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi+a}^{\pi+a} f(x)\,dx.$$

2. In this exercise we show how the symmetries of a function imply certain properties of its Fourier coefficients. Let f be a 2π-periodic Riemann integrable function defined on $\mathbf{R}$.

(a) Show that the Fourier series of the function f can be written as

$$f(\theta) \sim \hat{f}(0) + \sum_{n\geq 1}[\hat{f}(n) + \hat{f}(-n)]\cos n\theta + i[\hat{f}(n) - \hat{f}(-n)]\sin n\theta.$$

(b) Prove that if f is even, then $\hat{f}(n) = \hat{f}(-n)$, and we get a cosine series.

(c) Prove that if f is odd, then $\hat{f}(n) = -\hat{f}(-n)$, and we get a sine series.

(d) Suppose that $f(\theta + \pi) = f(\theta)$ for all $\theta \in \mathbf{R}$. Show that $\hat{f}(n) = 0$ for all odd n.

(e) Show that f is real-valued if and only if $\overline{\hat{f}(n)} = \hat{f}(-n)$ for all n.

3. We return to the problem of the plucked string discussed in Chapter 1. Show that the initial condition f is *equal* to its Fourier sine series

$$f(x) = \sum_{m=1}^{\infty} A_m \sin mx \quad \text{with} \quad A_m = \frac{2h}{m^2}\frac{\sin mp}{p(\pi - p)}.$$

[Hint: Note that $|A_m| \leq C/m^2$.]

4. Consider the 2π-periodic odd function defined on $[0,\pi]$ by $f(\theta) = \theta(\pi - \theta)$.

(a) Draw the graph of f.

(b) Compute the Fourier coefficients of f, and show that

$$f(\theta) = \frac{8}{\pi}\sum_{k \text{ odd } \geq 1} \frac{\sin k\theta}{k^3}.$$

5. On the interval $[-\pi,\pi]$ consider the function

$$f(\theta) = \begin{cases} 0 & \text{if } |\theta| > \delta, \\ 1 - |\theta|/\delta & \text{if } |\theta| \leq \delta. \end{cases}$$

Thus the graph of f has the shape of a triangular tent. Show that

$$f(\theta) = \frac{\delta}{2\pi} + 2\sum_{n=1}^{\infty}\frac{1 - \cos n\delta}{n^2\pi\delta}\cos n\theta.$$

6. Let f be the function defined on $[-\pi,\pi]$ by $f(\theta) = |\theta|$.

(a) Draw the graph of f.

(b) Calculate the Fourier coefficients of f, and show that

$$\hat{f}(n) = \begin{cases} \dfrac{\pi}{2} & \text{if } n = 0, \\ \dfrac{-1+(-1)^n}{\pi n^2} & \text{if } n \neq 0. \end{cases}$$

(c) What is the Fourier series of f in terms of sines and cosines?

(d) Taking $\theta = 0$, prove that

$$\sum_{n \text{ odd } \geq 1} \frac{1}{n^2} = \frac{\pi^2}{8} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

See also Example 2 in Section 1.1.

7. Suppose $\{a_n\}_{n=1}^N$ and $\{b_n\}_{n=1}^N$ are two finite sequences of complex numbers. Let $B_k = \sum_{n=1}^k b_n$ denote the partial sums of the series $\sum b_n$ with the convention $B_0 = 0$.

(a) Prove the **summation by parts** formula

$$\sum_{n=M}^{N} a_n b_n = a_N B_N - a_M B_{M-1} - \sum_{n=M}^{N-1} (a_{n+1} - a_n) B_n.$$

(b) Deduce from this formula Dirichlet's test for convergence of a series: if the partial sums of the series $\sum b_n$ are bounded, and $\{a_n\}$ is a sequence of real numbers that decreases monotonically to 0, then $\sum a_n b_n$ converges.

NOTE

8. Verify that $\dfrac{1}{2i} \sum_{n \neq 0} \dfrac{e^{inx}}{n}$ is the Fourier series of the 2π-periodic **sawtooth** function illustrated in Figure 6, defined by $f(0) = 0$, and

$$f(x) = \begin{cases} -\dfrac{\pi}{2} - \dfrac{x}{2} & \text{if } -\pi < x < 0, \\ \dfrac{\pi}{2} - \dfrac{x}{2} & \text{if } 0 < x < \pi. \end{cases}$$

Note that this function is not continuous. Show that nevertheless, the series converges for every x (by which we mean, as usual, that the symmetric partial sums of the series converge). In particular, the value of the series at the origin, namely 0, is the average of the values of $f(x)$ as x approaches the origin from the left and the right.

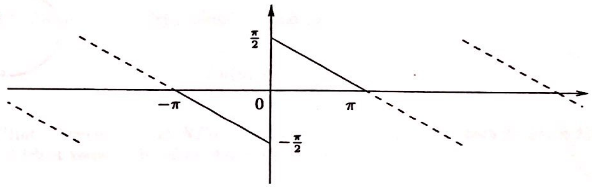

Figure 6. The sawtooth function

[Hint: Use Dirichlet's test for convergence of a series $\sum a_n b_n$.]

9. Let $f(x) = \chi_{[a,b]}(x)$ be the characteristic function of the interval $[a, b] \subset [-\pi, \pi]$, that is,

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a, b], \\ 0 & \text{otherwise.} \end{cases}$$

NICE PROBLEM

(a) Show that the Fourier series of f is given by

$$f(x) \sim \frac{b-a}{2\pi} + \sum_{n \neq 0} \frac{e^{-ina} - e^{-inb}}{2\pi i n} e^{inx}.$$

The sum extends over all positive and negative integers excluding 0.

(b) Show that if $a \neq -\pi$ or $b \neq \pi$ and $a \neq b$, then the Fourier series does not converge absolutely for any x. [Hint: It suffices to prove that for many values of n one has $|\sin n\theta_0| \geq c > 0$ where $\theta_0 = (b-a)/2$.]

(c) However, prove that the Fourier series converges at every point x. What happens if $a = -\pi$ and $b = \pi$?

10. Suppose f is a periodic function of period 2π which belongs to the class C^k. Show that

$$\hat{f}(n) = O(1/|n|^k) \quad \text{as } |n| \to \infty.$$

This notation means that there exists a constant C such $|\hat{f}(n)| \leq C/|n|^k$. We could also write this as $|n|^k \hat{f}(n) = O(1)$, where $O(1)$ means bounded.
[Hint: Integrate by parts.]

NOTE

11. Suppose that $\{f_k\}_{k=1}^{\infty}$ is a sequence of Riemann integrable functions on the interval $[0, 1]$ such that

$$\int_0^1 |f_k(x) - f(x)|\, dx \to 0 \quad \text{as } k \to \infty.$$

Show that $\hat{f}_k(n) \to \hat{f}(n)$ uniformly in n as $k \to \infty$.

12. Prove that if a series of complex numbers $\sum c_n$ converges to s, then $\sum c_n$ is Cesàro summable to s.

[Hint: Assume $s_n \to 0$ as $n \to \infty$.]

13. The purpose of this exercise is to prove that Abel summability is stronger than the standard or Cesàro methods of summation.

(a) Show that if the series $\sum_{n=1}^{\infty} c_n$ of complex numbers converges to a finite limit s, then the series is Abel summable to s. [Hint: Why is it enough to prove the theorem when $s = 0$? Assuming $s = 0$, show that if $s_N = c_1 + \cdots + c_N$, then $\sum_{n=1}^{N} c_n r^n = (1-r)\sum_{n=1}^{N} s_n r^n + s_N r^{N+1}$. Let $N \to \infty$ to show that

$$\sum c_n r^n = (1-r)\sum s_n r^n.$$

Finally, prove that the right-hand side converges to 0 as $r \to 1$.]

(b) However, show that there exist series which are Abel summable, but that do not converge. [Hint: Try $c_n = (-1)^n$. What is the Abel limit of $\sum c_n$?]

(c) Argue similarly to prove that if a series $\sum_{n=1}^{\infty} c_n$ is Cesàro summable to σ, then it is Abel summable to σ. [Hint: Note that

$$\sum_{n=1}^{\infty} c_n r^n = (1-r)^2 \sum_{n=1}^{\infty} n\sigma_n r^n,$$

and assume $\sigma = 0$.]

(d) Give an example of a series that is Abel summable but not Cesàro summable. [Hint: Try $c_n = (-1)^{n-1}n$. Note that if $\sum c_n$ is Cesàro summable, then c_n/n tends to 0.]

IMPORTANT

The results above can be summarized by the following implications about series:

$$\text{convergent} \Longrightarrow \text{Cesàro summable} \Longrightarrow \text{Abel summable},$$

and the fact that none of the arrows can be reversed.

14. This exercise deals with a theorem of Tauber which says that under an additional condition on the coefficients c_n, the above arrows can be reversed.

(a) If $\sum c_n$ is Cesàro summable to σ and $c_n = o(1/n)$ (that is, $nc_n \to 0$), then $\sum c_n$ converges to σ. [Hint: $s_n - \sigma_n = [(n-1)c_n + \cdots + c_2]/n$.]

(b) The above statement holds if we replace Cesàro summable by Abel summable. [Hint: Estimate the difference between $\sum_{n=1}^{N} c_n$ and $\sum_{n=1}^{N} c_n r^n$ where $r = 1 - 1/N$.]

15. Prove that the Fejér kernel is given by

NOTE

$$F_N(x) = \frac{1}{N}\frac{\sin^2(Nx/2)}{\sin^2(x/2)}.$$

[Hint: Remember that $NF_N(x) = D_0(x) + \cdots + D_{N-1}(x)$ where $D_n(x)$ is the Dirichlet kernel. Therefore, if $\omega = e^{ix}$ we have

$$NF_N(x) = \sum_{n=0}^{N-1} \frac{\omega^{-n} - \omega^{n+1}}{1-\omega}.]$$

16. The Weierstrass approximation theorem states: Let f be a continuous function on the closed and bounded interval $[a,b] \subset \mathbb{R}$. Then, for any $\epsilon > 0$, there exists a polynomial P such that

$$\sup_{x\in[a,b]} |f(x) - P(x)| < \epsilon.$$

Prove this by applying Corollary 5.4 of Fejér's theorem and using the fact that the exponential function e^{ix} can be approximated by polynomials uniformly on any interval.

17. In Section 5.4 we proved that the Abel means of f converge to f at all points of continuity, that is,

IMPORTANT

$$\lim_{r\to 1} A_r(f)(\theta) = \lim_{r\to 1}(P_r * f)(\theta) = f(\theta), \qquad \text{with } 0 < r < 1,$$

whenever f is continuous at θ. In this exercise, we will study the behavior of $A_r(f)(\theta)$ at certain points of discontinuity.

An integrable function is said to have a **jump discontinuity** at θ if the two limits

$$\lim_{\substack{h\to 0\\ h>0}} f(\theta+h) = f(\theta^+) \quad \text{and} \quad \lim_{\substack{h\to 0\\ h>0}} f(\theta-h) = f(\theta^-)$$

exist.

(a) Prove that if f has a jump discontinuity at θ, then

$$\lim_{r\to 1} A_r(f)(\theta) = \frac{f(\theta^+) + f(\theta^-)}{2}, \qquad \text{with } 0 \le r < 1.$$

[Hint: Explain why $\frac{1}{2\pi}\int_{-\pi}^{0} P_r(\theta)\,d\theta = \frac{1}{2\pi}\int_{0}^{\pi} P_r(\theta)\,d\theta = \frac{1}{2}$, then modify the proof given in the text.]

(b) Using a similar argument, show that if f has a jump discontinuity at θ, the Fourier series of f at θ is Cesàro summable to $\frac{f(\theta^+)+f(\theta^-)}{2}$.

18. If $P_r(\theta)$ denotes the Poisson kernel, show that the function

$$u(r,\theta) = \frac{\partial P_r}{\partial \theta},$$

defined for $0 \le r < 1$ and $\theta \in \mathbb{R}$, satisfies:

(i) $\Delta u = 0$ in the disc.

(ii) $\lim_{r \to 1} u(r,\theta) = 0$ for each θ.

However, u is not identically zero.

19. Solve Laplace's equation $\Delta u = 0$ in the semi infinite strip

$$S = \{(x,y) : 0 < x < 1,\ 0 < y\},$$

subject to the following boundary conditions

$$\begin{cases} u(0,y) = 0 & \text{when } 0 \le y, \\ u(1,y) = 0 & \text{when } 0 \le y, \\ u(x,0) = f(x) & \text{when } 0 \le x \le 1 \end{cases}$$

where f is a given function, with of course $f(0) = f(1) = 0$. Write

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

and expand the general solution in terms of the special solutions given by

$$u_n(x,y) = e^{-n\pi y} \sin(n\pi x).$$

Express u as an integral involving f, analogous to the Poisson integral formula (6).

20. Consider the Dirichlet problem in the annulus defined by $\{(r,\theta) : \rho < r < 1\}$, where $0 < \rho < 1$ is the inner radius. The problem is to solve

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

subject to the boundary conditions

$$\begin{cases} u(1,\theta) = f(\theta), \\ u(\rho,\theta) = g(\theta), \end{cases}$$

where f and g are given continuous functions.

Arguing as we have previously for the Dirichlet problem in the disc, we can hope to write

$$u(r,\theta) = \sum c_n(r) e^{in\theta}$$

with $c_n(r) = A_n r^n + B_n r^{-n}$, $n \neq 0$. Set

$$f(\theta) \sim \sum a_n e^{in\theta} \quad \text{and} \quad g(\theta) \sim \sum b_n e^{in\theta}.$$

We want $c_n(1) = a_n$ and $c_n(\rho) = b_n$. This leads to the solution

$$u(r,\theta) = \sum_{n \neq 0} \left(\frac{1}{\rho^n - \rho^{-n}}\right) \left[((\rho/r)^n - (r/\rho)^n)\, a_n + (r^n - r^{-n}) b_n \right] e^{in\theta}$$
$$+ a_0 + (b_0 - a_0) \frac{\log r}{\log \rho}.$$

Show that as a result we have

$$u(r,\theta) - (P_r * f)(\theta) \to 0 \qquad \text{as } r \to 1 \text{ uniformly in } \theta,$$

and

$$u(r,\theta) - (P_{\rho/r} * g)(\theta) \to 0 \qquad \text{as } r \to \rho \text{ uniformly in } \theta.$$

7 Problems

1. One can construct Riemann integrable functions on $[0,1]$ that have a dense set of discontinuities as follows.

(a) Let $f(x) = 0$ when $x < 0$, and $f(x) = 1$ if $x \ge 0$. Choose a countable dense sequence $\{r_n\}$ in $[0,1]$. Then, show that the function

$$F(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} f(x - r_n)$$

is integrable and has discontinuities at all points of the sequence $\{r_n\}$. [Hint: F is monotonic and bounded.]

(b) Consider next

$$F(x) = \sum_{n=1}^{\infty} 3^{-n} g(x - r_n),$$

where $g(x) = \sin 1/x$ when $x \neq 0$, and $g(0) = 0$. Then F is integrable, discontinuous at each $x = r_n$, and fails to be monotonic in any subinterval of $[0,1]$. [Hint: Use the fact that $3^{-k} > \sum_{n>k} 3^{-n}$.]

(c) The original example of Riemann is the function

$$F(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2},$$

where $(x) = x$ for $x \in (-1/2, 1/2]$ and (x) is continued to $\mathbb{R}$ by periodicity, that is, $(x+1) = (x)$. It can be shown that F is discontinuous whenever $x = m/2n$, where $m, n \in \mathbb{Z}$ with m odd and $n \neq 0$.

2. Let D_N denote the Dirichlet kernel IMPORTANT

$$D_N(\theta) = \sum_{k=-N}^{N} e^{ik\theta} = \frac{\sin((N+1/2)\theta)}{\sin(\theta/2)},$$

and define

$$L_N = \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(\theta)| \, d\theta.$$

(a) Prove that

$$L_N \geq c \log N$$

for some constant $c > 0$. [Hint: Show that $|D_N(\theta)| \geq c\frac{\sin((N+1/2)\theta)}{|\theta|}$, change variables, and prove that

$$L_N \geq c \int_{\pi}^{N\pi} \frac{|\sin\theta|}{|\theta|} \, d\theta + O(1).$$

Write the integral as a sum $\sum_{k=1}^{N-1} \int_{k\pi}^{(k+1)\pi}$. To conclude, use the fact that $\sum_{k=1}^{n} 1/k \geq c \log n$.] A more careful estimate gives

$$L_N = \frac{4}{\pi^2} \log N + O(1).$$

(b) Prove the following as a consequence: for each $n \geq 1$, there exists a continuous function f_n such that $|f_n| \leq 1$ and $|S_n(f_n)(0)| \geq c' \log n$. [Hint: The function g_n which is equal to 1 when D_n is positive and -1 when D_n is negative has the desired property but is not continuous. Approximate g_n in the integral norm (in the sense of Lemma 3.2) by continuous functions h_k satisfying $|h_k| \leq 1$.]

3.* Littlewood provided a refinement of Tauber's theorem:

(a) If $\sum c_n$ is Abel summable to s and $c_n = O(1/n)$, then $\sum c_n$ converges to s.

(b) As a consequence, if $\sum c_n$ is Cesàro summable to s and $c_n = O(1/n)$, then $\sum c_n$ converges to s.

These results may be applied to Fourier series. By Exercise 17, they imply that if f is an integrable function that satisfies $\hat{f}(\nu) = O(1/|\nu|)$, then:

MUCH MORE THAN CONTINUOUS! (SEE PAGE 43 ALSO.

(i) If f is continuous at θ, then

$$S_N(f)(\theta) \to f(\theta) \quad \text{as } N \to \infty.$$

(ii) If f has a jump discontinuity at θ, then

$$S_N(f)(\theta) \to \frac{f(\theta^+) + f(\theta^-)}{2} \quad \text{as } N \to \infty.$$

(iii) If f is continuous on $[-\pi, \pi]$, then $S_N(f) \to f$ uniformly.

For the simpler assertion (b), hence a proof of (i), (ii), and (iii), see Problem 5 in Chapter 4.