Some more Examples and Hints for Section 7.6

Example 1. Evaluate the surface integral of $F = (3x^2, 2xy, 8)$ over the piece of the plane $z = 2x - y$ for $(x,y) \in [0,2] \times [0,2]$.

We need to compute $\iint_S F \cdot d\mathbf{S}$ where

$$S = \{(x, y, 2x-y) \,/\, (x,y) \in [0,2]\times[0,2]\}$$

This surface is a graph $z = g(x,y) = 2x - y$ over $D = [0,2]\times[0,2]$, Hence

$$\iint_S F \cdot d\mathbf{S} = \iint_D F \cdot \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right) dx\,dy$$

$$= \int_0^2 \int_0^2 (3x^2, 2xy, 8) \cdot (-2, 1, 1)\, dx\, dy$$

$$= \int_0^2 \int_0^2 -6x^2 - 2xy + 8 \, dx\, dy = \boxed{-8}.$$

Example 2: Evaluate $F = x\vec{k}$ over the unit disk in the xy plane.

Here $S = \{(x,y,0) \,/\, x^2+y^2 \leq 1\}$ and we are asked to compute $\iint_S F \cdot d\mathbf{S}$.

Since S is flat in the xy-plane its unit normal $\vec{n}$ is simply the vector $\vec{k}$. That is

$$\underbrace{d\mathbf{S}}_{\text{vector surface element}} = \underbrace{\vec{n}}_{\boxed{\text{UNIT}}} \underbrace{dS}_{\text{scaler surface element}} \overset{\downarrow}{=} \vec{k}\, dx\,dy \text{ (in this case)}$$

So

$$\iint_S F \cdot d\mathbf{S} = \iint_{Disk} \overbrace{(0,0,x)}^{F} \cdot \overbrace{(0,0,1)}^{\vec{k}}\, dx\,dy$$

(S = Disk here)

TO COMPUTE THIS WE CHANGE TO POLAR COORDINATES

$$= \iint_{Disk} x \; dx\,dy$$

$$= \underbrace{\int_0^{2\pi}\int_0^1}_{Disk} \underbrace{r\cos\theta}_{x \text{ in polar}} \; \underbrace{r\,dr\,d\theta}_{(dx\,dy)}$$

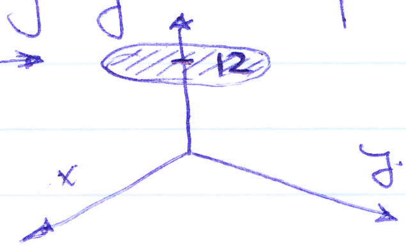
$$= \int_0^{2\pi}\int_0^1 r^2 \cos\theta \, dr\, d\theta$$

$$= \frac{1}{3}\int_0^{2\pi} \cos\theta\, d\theta = -\frac{1}{3}\sin\theta \Big|_0^{2\pi} = \boxed{0}$$

Example 3: Let $F = x\,i + y\,j + z\,k$ (position vector field) and consider the surface described by the equations: $z = 12$ $x^2+y^2 \le 25$.

Compute $\iint_S F \cdot dS$.

S = disk of radius 5 lying on the plane $z = 12$ Disk $\longrightarrow$



We can solve this in 2 ways

One way: As before S is flat so again a unit normal vector for S is $\vec{k}$. Hence

(outward unit)

$$\iint_S F \cdot dS = \iint_S \underbrace{(x,y,z)}_{F} \cdot \underbrace{(0,0,1)}_{\vec{k}}\, dx\,dy$$

since $dS = \vec{n}\, dS \stackrel{\uparrow}{=} \vec{k}\, dx\,dy$ (b/c S is horiz. flat)

($\vec{n}$: outward unit normal)

$$= \iint_{S = \text{Disk}} z \; dx\,dy = \iint_{S = \text{Disk}} 12 \; dx\,dy$$

$$= 12 \text{ Area(Disk)}$$

$$= 12\,\pi \cdot 25 = \boxed{300\pi}$$

<u>Another way</u> (via graphs) In this case

$$S = \{ (x, y, 12) \,/\, x^2+y^2 \le 25 \}$$

That is S is the graph of $z = g(x,y) = 12$ for (x,y) in $D = \{(x,y) \,/\, x^2+y^2 \le 25\}$

Hence

$F = (x,y,z)$

$$\iint_S F \cdot dS = \iint_D \overbrace{(x,y,12)}^{F \text{ on } S \text{ has } z=12} \cdot \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right) dx\,dy$$

$$= \iint_D (x,y,12) \cdot (0,0,1)\, dx\,dy$$

$$= \iint_D 12\, dx\,dy = \boxed{300\pi}$$

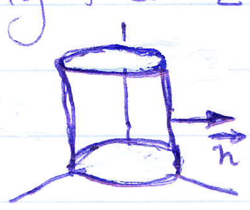
(as before).

(SIMILAR TO <u>HMWK 1</u>)

<u>Example 4</u>: Calculate the flux of $F = x\vec{i} + y\vec{j} - y\vec{k}$ over the surface defined by $x^2+y^2 = 1$, $0 \le z \le 1$ with normal pointing <u>out</u> of the surface

The surface is a cylinder (the boundary of a piece of a cylinder) of <u>radius 1</u> standing vertically along the z-axis between 0 and 1

S.



<u>NOTE</u>: since $x^2+y^2=1$ the surface DOES <u>NOT</u> include the top and bottom disks !!

To find $\iint_S F \cdot dS$ we parametrize S by cylindrical coordinates. Consider

$$\Phi : \underset{\theta}{[0, 2\pi]} \times \underset{z}{[0, 1]} \longrightarrow \mathbb{R}^3$$

$\boxed{r = 1}$

$$\Phi(\theta, z) = (\underbrace{\cos\theta}_{x}, \underbrace{\sin\theta}_{y}, \underbrace{z}_{z})$$

(S = image of Φ). Then

$$\vec{n} = T_\theta \times T_z = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} =$$

outward (check) normal (not unit)

$$= \cos\theta\, \vec{i} - (-\sin\theta)\vec{j} + 0\, \vec{k}$$

$$= \cos\theta\, \vec{i} + \sin\theta\, \vec{j} + 0\, \vec{k}$$

$$= (\cos\theta, \sin\theta, 0) = (x, y, 0)$$

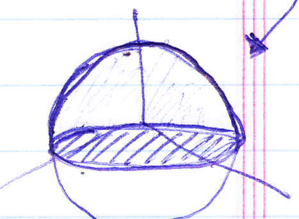
Then $$\iint_S F \cdot dS = \int_0^{2\pi}\int_0^1 F \cdot (T_\theta \times T_z)\, d\theta\, dz$$

Recall $F = (x, y, -y)$

$$= \int_0^{2\pi}\int_0^1 (\cos\theta, \sin\theta, -\sin\theta) \cdot (\cos\theta, \sin\theta, 0)\, d\theta\, dz$$

$$= \int_0^{2\pi}\int_0^1 \underbrace{(\cos^2\theta + \sin^2\theta)}_{= 1 \text{ by Pythagoras}}\, d\theta\, dz = \boxed{2\pi}$$

(SIMILAR TO HMWK 3
ALSO in HMWK1
have to find
$F = -\nabla T$
as here)
(closed surface)



Example 5: Let T = temperature at (x, y, z) in $S \subseteq \mathbb{R}^3$. Suppose
$T(x, y, z) = 3x^2 + 3y^2 + 3z^2$ and that
S = hemisphere $x^2 + y^2 + z^2 = 1$, $z \geq 0$ and its base $x^2 + y^2 \leq 1$, $z = 0$

Compute the heat flux across S if $k = 1$.

We need to find $\iint_S F \cdot dS$ where

$$F = -k \nabla T \underset{k=1}{=} -\nabla T = (-6x, -6y, -6z)$$

and $S = H \cup D$ where

$$H = \{(x, y, z) \,/\, x^2 + y^2 + z^2 = 1, z \geq 0\}$$

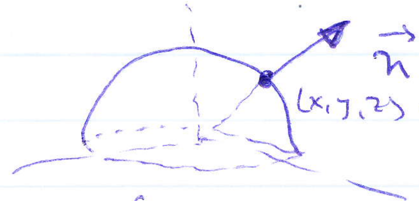
$$D = \{(x, y, 0) \,/\, x^2 + y^2 \leq 1\}$$

Then

$$\iint_S F \cdot dS = \iint_H F \cdot dS + \iint_D F \cdot dS$$

We compute each of this separately:

For the upper UNIT hemisphere H we know that the outward UNIT normal



is $\vec{n} = x\,i + y\,j + z\,k$

$= (x, y, z) =$ position (NOTE $\|\vec{n}\| = 1$ b/c. $(x,y,z) \in H$ $\Rightarrow x^2+y^2+z^2 = 1$)

Then $\displaystyle\iint_H F \cdot d\mathbf{S} = \iint_H F \cdot \vec{n}\, \underbrace{dS}_{\text{scalar surface element}}$

since $\underbrace{d\mathbf{S}}_{\text{vector surface element}} = \underbrace{\vec{n}}_{\text{UNIT OUTWARD NORMAL}}\, dS$

$$= \iint_H (-6x, -6y, -6z) \cdot (x, y, z)\, \underbrace{dS}_{\text{scaler surface element}}$$

$$= \iint_H -6x^2 - 6y^2 - 6z^2 \, dS$$

$(x^2+y^2+z^2 = 1)$ $\displaystyle = -6 \underbrace{\iint_H dS}_{= \frac{1}{2}(\text{surface area of unit sphere})} = -6 \cdot \frac{1}{2} \text{Area (Unit sphere)}$

$= -6 \cdot \frac{1}{2} \cdot 4\pi$

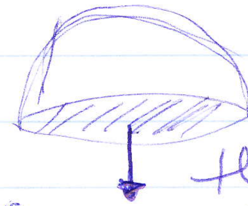
$= \boxed{-12\pi}$

For the base of $\mathcal{S}$ we need to compute

$$\iint_D F \cdot d\mathbf{S}$$

D = <u>unit</u> <u>disk</u> sitting on $z=0$.

Now on D the outward unit normal to $\mathcal{S}$ is $-\vec{k}$ (b/c D is horiz. flat)



$(0,0,-1)$

NOTE! the outward unit normal to $\mathcal{S} = H \cup D$ Points DOWNWARD on D

Hence on D $d\mathbf{S} = -\vec{k}\, dS$

$$\iint_D F \cdot d\mathbf{S} = \iint (-6x,-6y,-6z)\cdot(0,0,-1)\, dS$$

$$D = \{(x,y,z) \,/\, x^2+y^2 \leq 1 \text{ and } z=0\}$$

$$= \iint_D 6z \, dS \underset{\substack{z=0 \\ \text{on } D}}{=} \iint_D 0 \, dS = \boxed{0}$$

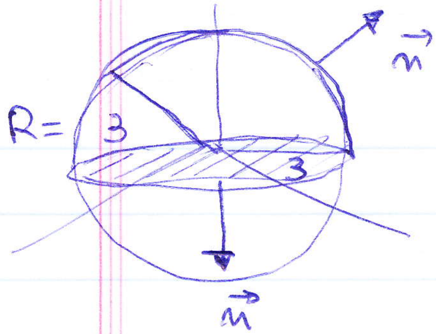
<u>Hence</u>

$$\iint_{\mathcal{S}} F \cdot d\mathbf{S} = -12\pi + 0 = \boxed{-12\pi}$$

(SIMILAR TO HMWK 7)

<u>Example 6</u> : Let $\mathcal{S}$ be the surface of the half ball $x^2+y^2+z^2 \leq 9$ $z \geq 0$

(This is the closed surface which is the union of the upper hemisphere of the sphere of radius 3 $x^2+y^2+z^2=9$, $z>0$ UNION the base which is Disk $x^2+y^2 \leq 9$, $z=0$)



$$S = H \cup D$$

$$H := \{(x,y,z) \,/\, x^2+y^2+z^2 = 9,\ z > 0\}$$

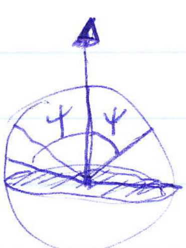
$$D := \{(x,y,0) \,/\, x^2+y^2 \leq 9\}.$$

Compute $\iint_S F \cdot dS$ for $F(x,y,z) = (2x - y^3,\ 2y + 7xz,\ 2z + 3xy)$

As Before we compute $\iint_H F \cdot dS + \iint_D F \cdot dS$

- Because our F is more complicated we use spherical coordinates to parametrize H

In spherical coordinates H is the image of



$$\Phi : \underset{\psi}{\left[0, \tfrac{\pi}{2}\right]} \times \underset{\theta}{[0, 2\pi]} \longrightarrow \mathbb{R}^3$$

(upper hemisphere)

ψ = angle of point in sphere with the positive z-axis

$R = 3$

$$\Phi(\psi, \theta) = (\overbrace{3\sin\psi\cos\theta}^{x}, \overbrace{3\sin\psi\sin\theta}^{y}, \overbrace{3\cos\psi}^{z})$$

$$dS = (T_\psi \times T_\theta)\, d\psi\, d\theta$$

outward normal (not unit)

$$= 3\sin\psi\,(3\sin\psi\cos\theta,\ 3\sin\psi\sin\theta,\ 3\cos\psi)\, d\psi\, d\theta$$

Now first rewrite $F = (2x - y^3, 2y + 7xz, 2z + 3xy)$

in spherical coordinates. Get.

$$F(\Phi(\varphi,\theta)) = \Big(6\sin\varphi\cos\theta - 27\sin^3\varphi\sin^3\theta,$$

$$6\sin\varphi\sin\theta + 7(3\sin\varphi\cos\theta)(3\cos\varphi),$$

$$2(3\cos\varphi) + 3(3\sin\varphi\cos\theta)(3\sin\varphi\sin\theta)\Big)$$

Then compute

$$F(\Phi(\varphi,\theta)) \cdot \Big[3\sin\varphi\,(3\sin\varphi\cos\theta, 3\sin\varphi\sin\theta, 3\cos\varphi)\Big]$$

After some algebra and simplifications

$$= \Big[18 + 10\cdot 27\sin^2\varphi\cos\varphi\cos\theta\sin\theta -$$

$$- 81\sin^4\varphi\cos\theta\sin^3\theta\Big]\, 3\sin\varphi$$

$$\therefore \int_0^{\pi/2}\int_0^{2\pi} F\cdot(T_\varphi \times T_\theta)\, d\varphi\, d\theta = \text{integrate } \theta \text{ first!}$$

$$\int_0^{\pi/2}\int_0^{2\pi}\Big(54\sin\varphi + 810\sin^3\varphi\cos\varphi\cos\theta\sin\theta -$$

$$- 243\sin^5\varphi\cos\theta\sin^3\theta\Big)\, d\theta\, d\varphi$$

$$= \int_0^{\pi/2}(108\pi)\sin\varphi\, d\varphi = \boxed{108\pi}$$ (check!)

For $\iint_D F \cdot dS$ we proceed as in the previous example since the outward unit normal to S along D is the downward unit vector $-\vec{k} = (0,0,-1)$

Hence $\iint_D (2x-y^3, 2y+7xz, 2z+3xy) \cdot (0,0,-1)\, dx\,dy$

since $dS = -\vec{k}\, \underbrace{dS} = -\vec{k}\, dx\,dy$

(for flat horizontal surface $dS = dx\,dy$)

$= \iint_D -3xy \, dx\,dy$ to compute change to polar coordinates

$D = \{(x,y,0) \,/\, x^2+y^2 \le 9\}$

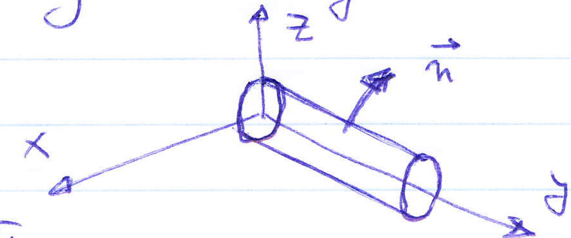
$x = r\cos\theta$
$y = r\sin\theta$

$= \int_0^{2\pi} \int_0^3 -3(r\cos\theta)(r\sin\theta)\, r\,dr\,d\theta$

$= -3 \int_0^{2\pi} \int_0^3 r^3 \cos\theta \sin\theta \, dr\,d\theta$

$= \boxed{0}$ $\therefore \iint_S F \cdot dS = 108\pi + 0 = \boxed{108\pi}$

Some **Hints:** **Problem 1** is similar to Example 4 above except that the cylinder has radius $\sqrt{2}$ and is horizontally sitting on the y-axis.
See also Example 5 to find the heat v.f. $F = -k\nabla T$

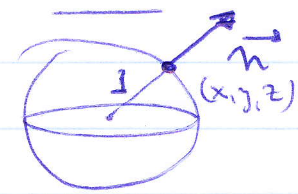


Problem 2: don't parametrize the unit sphere. Just use that

$$d\mathbf{S} = \underbrace{\vec{n}}_{\text{outward unit normal}}\, dS \quad \text{where } \vec{n} = x\vec{i} + y\vec{j} + z\vec{k}$$

(= position) for the UNIT SPHERE

Compute $F = -k\nabla T$ with $k = 1$.

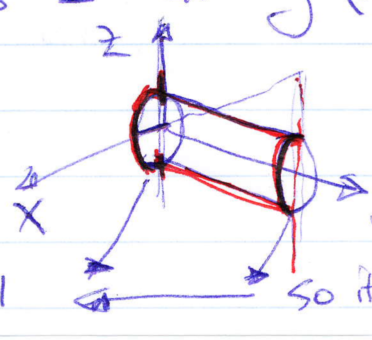


Problem 3 is like Example 5 above

Problem 4: Compute the flux across S

$$\iint_S F \cdot d\mathbf{S} \quad \text{where } F = \sqrt{y}\, i = (\sqrt{y}, 0, 0)$$

and S is half the surface of a cylinder of radius 1 sitting (horizontally) along the y-axis



$0 \le y \le 1$
but
$0 \le x \le 1$ only

$x^2 + z^2 = 1$ so it's half of circle

Problem 7 is like Example 6 above

Problem 9: you do ☺ !!