CHAPTER 5 (5.1, 5.2, 5.3, 5.4) FOURIER SERIES

I) Basic Facts:

First recall the following trigonometric identities

- $\cos a \cdot \cos b = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$

- $\sin a \cdot \cos b = \frac{1}{2}\left[\sin(a+b) + \sin(a-b)\right]$

DEFINITION: (PERIODIC FUNCTIONS ON $\mathbb{R}$): A function $\phi : \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) is said to be PERIODIC if there exists $P > 0$ such that

$$\boxed{\phi(x+P) = \phi(x) \quad \forall x \in \mathbb{R}} \quad (*)$$

REMARK: If ϕ is periodic, the smallest such $P > 0$ for which $(*)$ holds is called the period of ϕ.

Examples: i) $\sin x$, $x \in \mathbb{R}$ is periodic with period 2π

② ii) $\cos 2x$, $x \in \mathbb{R}$ is periodic with period π

iii) $\sin \frac{x}{2}$, $x \in \mathbb{R}$ is periodic with period 4π

** iv) $\tan x$, $x \in \mathbb{R} \setminus \{\text{odd multiples of } \frac{\pi}{2}\}$ is periodic with period $\pi/2$. (note ** we need to adjust domain).

REMARKS: 1) If ϕ has period p, then

$$\phi(x + np) = \phi(x) \quad \forall n \in \mathbb{Z}, x \in \mathbb{R}$$

2) Given 2 periodic functions f of period $P_1 > 0$ and g with period $P_2 > 0$ we have:

ⓐ If $P_1 = m\pi$ and $P_2 = n\pi$ $(m, n \in \mathbb{N})$ then $f \pm g$ is periodic with period $\max\{P_1, P_2\}$.

ⓑ Otherwise, $f \pm g$ is periodic **ONLY IF** $P_1/P_2 \in \mathbb{Q}$ (rational)

③ in that case, the period p is the lowest common multiple of P_1 and P_2.

<u>Example</u>: $\sin x$ has period 2π; $\tan x$ has period π $\Rightarrow$ $\sin x + \tan x$ is periodic with period 2π.

3) If $\phi : \mathbb{R} \to \mathbb{R}$ is periodic of period $P > 0$ then $\int_a^{a+P} \phi(x)\,dx$ is independent of a $(a \in \mathbb{R})$.

In particular, $\int_a^{a+P} \phi(x)\,dx = \int_0^P \phi(x)\,dx \quad \forall a \in \mathbb{R}$

(assuming ϕ is integrable)

4) A <u>periodic function</u> of period $P > 0$ is fully specified if we give its values on $(0, P)$ or on any other interval I of length P $\left(\left(-\frac{P}{2}, \frac{P}{2}\right) \text{ or } \left(\sqrt{2}, \sqrt{2}+P\right), \text{etc} \right)$.

(4) Suppose now that we are given a function on an interval I of length L. Then such function can be extended to $\mathbb{R}$ as a periodic function with the period L in only one way.

More precisely suppose that $\phi : (-\ell, \ell) \to \mathbb{R}$ (think $2\ell = L$) is given. Then its periodic extension $\phi_{per} : \mathbb{R} \to \mathbb{R}$ is defined by:

(Periodization of ϕ with period 2ℓ)
$$\begin{cases} \phi_{per}(x) := \phi(x - 2\ell m) \text{ for} \\ -\ell + 2\ell m < x < \ell + 2\ell m \quad \forall m \in \mathbb{Z} \end{cases}$$

REMARK: Note that this definition of $\phi_{per}(x)$, $x \in \mathbb{R}$ does NOT specify the values of ϕ_{per} at the

⑤ endpoints $x = -\ell + 2\ell m$ and $x = \ell + 2\ell m$, $m \in \mathbb{Z}$

In fact, ϕ_{per} may be discontinuous at these endpoints $\boxed{\text{UNLESS}}$ $\boxed{\phi(\ell^-) = \phi(-\ell^+)}$.

In other words $\boxed{\text{UNLESS}}$

$$\boxed{\lim_{\substack{x \to \ell^- \\ (x<\ell)}} \phi(x) = \lim_{\substack{x \to -\ell^+ \\ (x>-\ell)}} \phi(x)}$$

FINALLY RECALL THE FOLLOWING (PROVE THEM!):

1) $$\int_0^\ell \sin\frac{n\pi x}{\ell} \cdot \sin\frac{m\pi}{\ell}x \, dx = \begin{cases} 0 & n \neq m \\ \frac{\ell}{2} & n = m \; (\neq 0) \end{cases}$$

2) $$\int_0^\ell \cos\frac{n\pi}{\ell}x \cdot \cos\frac{m\pi}{\ell}x \, dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{\ell}{2} & \text{if } n = m \; (\neq 0) \end{cases}$$

⑥ In case 2) if $n=m=0$ we get $\int_0^{\ell} 1\,dx = \ell$.

(in case 1) if $n=m=0$ we get 0).

Finally also recall from calculus the following:

3) $\displaystyle\int_{-\ell}^{\ell} \cos\frac{n\pi}{\ell}x \ \sin\frac{m\pi}{\ell}x \ dx = 0 \quad \forall n,m\in\mathbb{Z}$

4) $\displaystyle\int_{-\ell}^{\ell} \cos\frac{n\pi x}{\ell}\cdot\cos\frac{m\pi}{\ell}x\ dx = 0 \quad \text{for } n\neq m$

5) $\displaystyle\int_{-\ell}^{\ell} \sin\frac{n\pi}{\ell}x\cdot\sin\frac{m\pi}{\ell}x\ dx = 0 \quad \text{for } n\neq m$

6) $\displaystyle\int_{-\ell}^{\ell} 1\cdot\cos\frac{n\pi}{\ell}x\ dx = 0 = \int_{-\ell}^{\ell} 1\cdot\sin\frac{n\pi}{\ell}x\ dx \quad \forall n\in\mathbb{Z}$

7) $\displaystyle\int_{-\ell}^{\ell} \cos^2\frac{n\pi}{\ell}x\ dx = \ell = \int_{-\ell}^{\ell} \sin^2\frac{n\pi}{\ell}x\ dx \quad \forall n\in\mathbb{Z}$