1) Find the solution $y = y(x)$ to the following **first order** differential equations:

(a) $y'(x) = \dfrac{2x - x^3}{2 + y^3}$, such that $y(0) = 2$

$(2+y^3)\, y' = 2x - x^3$

$\int (2+y^3)\, dy = \int (2x - x^3)\, dx$

$2y + \frac{1}{4} y^4 = x^2 - \frac{1}{4} x^4 + C$

$2 = y(0) \Rightarrow \quad 2 \cdot 2 + \frac{1}{4} \cdot 2^4 = C \Rightarrow C = 8$

$\frac{1}{4} y^4 + 2y = x^2 - \frac{1}{4} x^4 + 8$

or

$y^4 + 8y = 4x^2 - x^4 + 32$

(b) $x^3\, y' + 4x^2\, y = e^{-x}$, such that $y(1) = -2$, $(x > 0)$

$y' + 4x^{-1} y = e^{-x} \cdot x^{-3}$ $\quad p(x) = 4x^{-1}$

$r(x) = x^{-3} e^{-x}$

$h = \int p(x)\, dx = \int 4x^{-1}\, dx = 4 \ln x$

$y(1) = c - 2e^{-1} = -2$

$c = 2e^{-1} - 2$

$y(t) = e^{-h} \left[\int e^h r\, dx + C \right]$

$= e^{-4\ln x} \left[\int e^{4\ln x} \cdot x^{-3} e^{-x}\, dx + C \right]$

$e^{\ln x^{-4}}$ $\quad \int x^4 \cdot x^{-3} e^{-x}\, dx = \int x e^{-x}\, dx = (-1-x) e^{-x}$

$\Rightarrow \quad y(t) = x^{-4} \left[(-1-x) e^{-x} + C \right] = \dfrac{C}{x^4} - \dfrac{(1+x) e^{-x}}{x^4}$

2) A tank originally contains 120 gallons of pure water. Water containing 0.25 pounds per gallon of salt enters the tank at a rate of 2 gallons per minute and the well-stirred mixture leaves the tank at the same rate. Denote by $S(t) =$ the amount of salt in the tank at any given time.

(a) Set up an initial value problem (ODE and initial condition) that describes this flow process.

$$\begin{cases} S' = 0.25 \times 2 - \dfrac{S}{120} \times 2 \\ S(0) = 0 \end{cases}$$

(b) Find the solution the amount of salt $S(t)$ in the tank at any given time. Sketch a graph of $S(t)$.

$$S(t) = 30 - 30 e^{-\frac{1}{60}t}$$

$$\lim_{T \to \infty} S = 30$$

$$S' = 0.5 - \frac{S}{60} = \frac{1}{2} - \frac{S}{60}$$

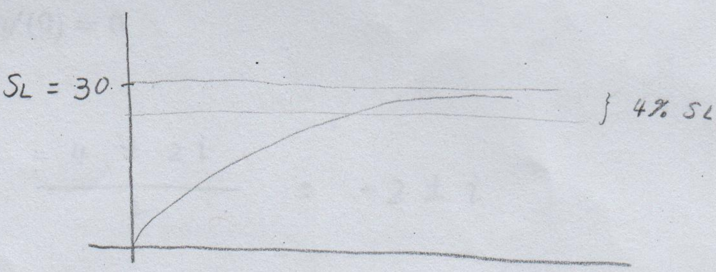
$$= -\frac{1}{60}(-30 + S)$$

$$\int \frac{1}{S-30}\, dS = \int -\frac{1}{60}\, dt$$

$$\ln|S-30| = -\frac{1}{60}t + c^*$$

$$S = C e^{-\frac{1}{60}t} + 30$$

$$0 = S(0) = 30 + C \Rightarrow C = -30$$



(c) Find the limiting amount S_L that is present after a very long time (*i.e. the equilibrium solution*). Draw it together with $S(t)$ on your graph above.

$$S_L = \lim_{t \to \infty} S(t) = 30$$

(d) Find the time T after which the salt level is <u>within</u> 4% of S_L found in (b).

$$S(T) = S_L - 4\% S_L \Rightarrow T = ?$$

$$30 - 30 e^{-\frac{1}{60}T} = 30 - 0.04 \times 30$$

$$T = -60 \ln(0.04)$$

$$\approx 193.1325495 \text{ min}$$

3) Find the solution to the following linear **second order** homogeneous equations with constant coefficients. In each case sketch its graph accurately showing the behavior of the solution.

(a) $y'' - y' + \dfrac{1}{4}y = 0; \qquad y(2) = 0,\ y'(2) = 1$

$\lambda^2 - \lambda + \frac{1}{4} = 0$

$(\lambda - \frac{1}{2})^2 = 0$

$\lambda_{1,2} = \frac{1}{2}$

$y(t) = (C_1 + C_2 t)\, e^{\frac{1}{2}t}$

$y'(t) = \frac{1}{2}(C_1 + C_2 t)\, e^{\frac{1}{2}t} + C_2 e^{\frac{1}{2}t}$

$0 = y(2) = [C_1 + 2C_2]\, e$

$1 = y'(2) = \frac{1}{2}[C_1 + 2C_2]\, e + C_2 e = [\frac{1}{2}C_1 + 2C_2]\, e$

$\Rightarrow \quad C_1 = -2e^{-1}, \quad C_2 = e^{-1}$

$y(t) = [-2e^{-1} + e^{-1}t]\, e^{\frac{1}{2}t}$

$= [-2 + t]\, e^{\frac{1}{2}t - 1}$

(b) $y'' + 4y' + 5y = 0; \qquad y(0) = 1,\ y'(0) = 0$

$\lambda^2 + 4\lambda + 5 = 0$

$\lambda = \dfrac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = \dfrac{-4 \pm 2i}{2} = -2 \pm i$

$y(t) = e^{-2t}[A\cos t + B\sin t]$

$y'(t) = -2e^{-2t}[A\cos t + B\sin t] + e^{-2t}[-A\sin t + B\cos t]$

$1 = y(0) = A$

$0 = y'(0) = -2A + B$

$\Rightarrow \quad A = 1, \quad B = 2$

$y(t) = e^{-2t}[\cos t + 2\sin t]$

4) a) Consider the second order linear equation $y'' + 5y' + 4y = 0$.

(i) Find the general solution $y = y(x)$

$\lambda^2 + 5\lambda + 4 = 0$

$(\lambda+4)(\lambda+1) = 0$

$\lambda = -4 \quad \lambda = -1$

$y(x) = c_1 e^{-x} + c_2 e^{-4x}$

(ii) Suppose $y_1(x)$ and $y_2(x)$ are two solutions satisfying respectively the initial conditions:

$$\begin{cases} y_1(0) = 3, & y_1'(0) = 0 & \text{for } y_1 \\ y_2(0) = 0, & y_2'(0) = 3 & \text{for } y_2 \end{cases}$$

Set up and compute the Wronskian $W(y_1, y_2)(0)$ and use your answer to determine whether $y_1(x)$ and $y_2(x)$ are linearly independent or not.

$W(y_1, y_2) = y_1 y_2' - y_2 y_1'$

linearly independent

$W(y_1, y_2)(0) = 3 \cdot 3 - 0 \cdot 0 = 9 \neq 0$

(b) Find the **general** solution $y = y(x)$ to the in-homogeneaous equation

$$y'' + 5y' + 4y = 20e^{-3x}.$$

Hint. Find a particular solution $y_p(x)$ and use the part a)i) giving $y_h(x)$

$y_p = c e^{-3x}$

$y_p' = -3c e^{-3x}$

$y(x) = c_1 e^{-x} + c_2 e^{-4x} - 10 e^{-3x}$

$y_p'' = 9c e^{-3x}$

$y_p'' + 5y_p' + 4y_p = 20e^{-3x}$

$9C - 15C + 4C = 20$

$\Rightarrow \quad C = -10$

$y_p = -10 e^{-3x}$

5) A body weighing 8 pounds stretches a spring 6 inches. Suppose the body is pushed upwards (negative direction) contracting the spring a distance of 2 inch and then set into motion with a downward (positive) velocity of 1 foot per second. Assume there is no damping. Use pounds, <u>feet</u> and seconds as units.

(a) Formulate the initial value problem that governs the motion of the body (an ODE and initial conditions.

no damping $\quad k = \dfrac{w \text{ lb}}{6 \text{ in}} = \dfrac{8 \text{ lb}}{6 \times \frac{1}{12} \text{ ft}} = 16 \text{ lb/ft}$

$m = \dfrac{w}{g} = \dfrac{8 \text{ lb}}{32 \text{ ft/s}^2} = \dfrac{1}{4} \dfrac{\text{lb}\cdot\text{s}^2}{\text{ft}}$

$\dfrac{k}{m} = 16 \times 4 = 64 \qquad w_0 = \sqrt{\dfrac{k}{m}} = 8$

$$\begin{cases} u'' + 64u = 0 \\ u(0) = -2 \text{ in} \times \frac{1}{12} \text{ ft/in} = -\frac{1}{6} \text{ ft} \\ u'(0) = 1 \text{ ft/s} \end{cases}$$

(b) Find the displacement $y(t)$ at any time t.

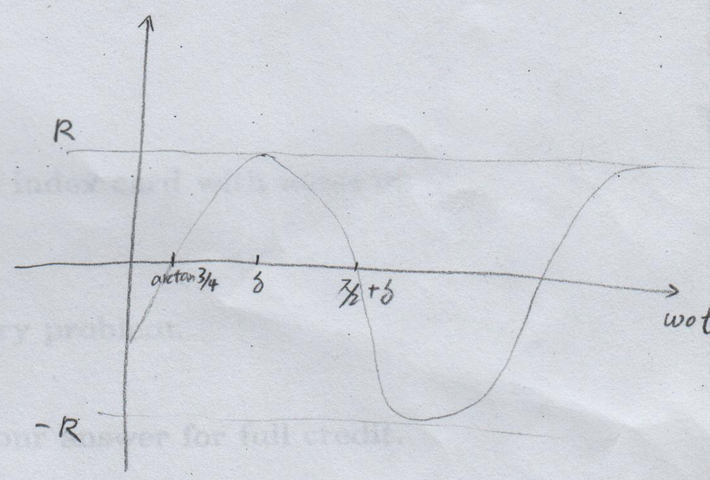
$u = A \cos w_0 t + B \sin w_0 t$

$u(0) = A = -\frac{1}{6}$

$u' = -w_0 A \sin w_0 t + w_0 B \cos w_0 t$

$u'(0) = w_0 B = 1 \Rightarrow B = \frac{1}{w_0} = \frac{1}{8}$

$u = -\frac{1}{6} \cos 8t + \frac{1}{8} \sin 8t$



(c) Find the frequency ω_0, period T, phase δ and amplitude R of the motion. Re-express the solution $y(t)$ as $R\cos(\omega_0 t - \delta)$ and sketch its graph in the $(\omega_0 t,\ y)$ axes, indicating the amplitude lines $y = \pm R$.

$w_0 = 8 \qquad T = \dfrac{2\pi}{w_0} = \dfrac{2\pi}{8} = \dfrac{1}{4}\pi$

$R = \sqrt{A^2 + B^2} = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{8}\right)^2} = \sqrt{\frac{1}{36} + \frac{1}{64}} = \sqrt{\frac{1}{4\cdot 9} + \frac{1}{4\cdot 16}}$

$= \sqrt{\dfrac{9+16}{4\cdot 9\cdot 16}} = \sqrt{\dfrac{25}{4\cdot 9\cdot 16}} = \dfrac{5}{2\cdot 3\cdot 4} = \dfrac{5}{24}$

$\tan\delta = \dfrac{B}{A} = \frac{1}{8}\cdot(-6) = -\frac{3}{4}$ $\qquad \delta = \left(\arctan\frac{3}{4}\right) + \frac{\pi}{2}$

$$\left.\begin{aligned} A &= R\cos\delta = -\tfrac{1}{6} < 0 \Rightarrow \cos\delta < 0 \\ B &= R\sin\delta = \tfrac{1}{8} > 0 \Rightarrow \sin\delta > 0 \end{aligned}\right\} \Rightarrow \text{2nd quadrant}$$