Inhomogeneous Diffusion Equation.

A case study / example. We have seen how to solve the boundary value/initial value problem (Dirichlet and Neumann BC) by separation of variables & Fourier series. for the linear homogeneous diffusion equation on an interval $(0, \ell)$

- How do we proceed if the equation is Inhomogeneous instead? We illustrate this with an example. Consider the Cauchy initial value / boundary value problem with Neumann BC on $(0, \pi)$

INHOMOGENEOUS

$$(\dagger)\begin{cases} u_t(x,t) - u_{xx}(x,t) = \boxed{tx} & 0<x<\pi \quad t>0 \\ u(x,0) = 1 & 0 \leq x \leq \pi \\ u_x(0,t) = 0 = u_x(\pi,t), & t>0 \end{cases}$$

- We cannot quite do separation of variables directly. From solving the homogeneous case with Neumann BC. we know that the solution might be expressed as a cosine Fourier series. The eigenvalue problem associated to the homogeneous equation is

$$\begin{cases} V''(x) = -\lambda V(x) \\ V'(0) = V'(\pi) = 0 . \end{cases}$$

whence $\lambda_n = n^2$ and the eigenfunctions are $V_n(x) = \cos(nx)$; $n \geq 0$

We then GUESS (make the "ansatz") that a candidate for a solution to (+) might have the form

(++) $$u(x,t) = c_0(t) + \sum_{n=1}^{\infty} c_n(t)\cos(nx)$$

These functions now depend on t (since we can't directly solve an ODE in t as before).

But if such a $u(x,t)$ as in (††) solves (†)

we must have :

$$\boxed{tx} = u_t - u_{xx} = \boxed{C_0'(t) + \sum_{n=1}^{\infty} \left(C_n'(t) + \underbrace{n^2 C_n(t))\cos(nx)}_{}\right)}$$

(*)

This comes from computing u_{xx}

with (from the initial condition)

(**) $$\boxed{1} = u(x,0) = C_0(0) + \sum_{n=1}^{\infty} C_n(0)\cos(nx)$$

Our goal is to use (*) and (**) to find the coefficient functions $C_n(t)$ $n \geq 0$.

To that effect we first re express (or expand) x as a cosine (fourier) series on $[0,\pi]$

(From your previous HMWK) : $$x = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos[(2k+1)x]}{(2k+1)^2}$$

which converges uniformly in $[0,\pi]$. Hence

(***) $$tx = \frac{\pi}{2}t - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{\cos[(2k+1)x]\,t}{(2k+1)^2}$$

(NOTE EVEN TERMS ARE ZERO)

MATCH THE Cosine functions term by term

We now compare the coefficient functions in ⊛ term by term (using ⊛⊛⊛ for tx):

$$C_0'(t) = \frac{\pi}{2} t$$

$1 = 1^2$ $$C_1'(t) + C_1(t) = -\frac{4}{\pi} t$$

$4 = 4.1$ $$C_2'(t) + 4\, C_2(t) = 0$$

$9 = 3^2$ $$C_3'(t) + 9\, C_3(t) = -\frac{4}{\pi}\frac{t}{9}$$

$16 = 4.4 = 4.2^2$ $$C_4'(t) + 16\, C_4(t) = 0$$

$25 = 5^2$ $$C_5'(t) + 25\, C_5(t) = -\frac{4}{\pi}\frac{t}{25}$$

$24 = 4.6 = 4.3^2$ $$C_6'(t) + 24\, C_6(t) = 0$$

So we have: $C_0'(t) = \frac{\pi}{2} t$

$(k \geq 1)$ $C_{2k}'(t) + 4.k^2\, C_{2k}(t) = 0$

$(k \geq 0)$ $C_{2k+1}'(t) + (2k+1)^2\, C_{2k+1}(t) = -\frac{4\, t}{\pi\, (2k+1)^2}$

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On the other hand from the initial condition ⊛⊛

We also have that:

$$\text{(II)} \begin{cases} C_0(0) = 1 \\ C_{2k}(0) = 0 & (k \geq 1) \\ C_{2k+1}(0) = 0 & (k \geq 0) \end{cases}$$

Since the cosine Fourier series of 1 (Recall your HMWK)

was $1 = 1 + 0\cos x + 0\cos(2x) + \cdots$

Solving now the ODE system (I) in page 4 together with the initial conditions (II) give that

$$C_0(t) = \frac{\pi}{4}t^2 + 1$$

$$C_{2k}(t) = 0 \quad (k \geq 1)$$

$$(k \geq 0) \quad C_{2k+1}(t) = \frac{-4}{\pi(2k+1)^2}\left[t + \frac{\left(e^{-(2k+1)^2 t} - 1\right)}{(2k+1)^2}\right]$$

Hence by putting back in (††) in page 2 we find that the (formal) solution

$$u(x,t) = \frac{\pi}{4} t^2 + 1 + \sum_{k=0}^{\infty} C_{2k+1}(t) \cos\left[(2k+1)x\right]$$

where

$$C_{2k+1}(t) = -\frac{4}{\pi(2k+1)^2}\left[t + \frac{\left(e^{-(2k+1)^2 t} - 1\right)}{(2k+1)^2}\right]$$

as above.

Note: Indeed $u(x,0) = 1$ and

(You → check (formally differentiating term
☺ by term) that $u_t - u_{xx} = tx$

and that $u_x(0,t) = 0 = u_x(\pi,t)$

for x in $(0,\pi)$, $t > 0$.

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