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Counting cycles and finite dimensional L^p norms

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Abstract

We obtain sharp bounds for the number of n-cycles in a finite graph as a function of the number of edges, and prove that the complete graph is optimal in more ways than could be imagined. We prove sharp estimates on both $\sum_{i=1}^n x_i^k$ and $\sum_{i=1}^n |x_i|^k$, subject to the constraints that $\sum_{i=1}^n x_i^2 = C$ and $\sum_{i=1}^n x_i = 0$.

Keywords: Graphs; Cycles; L^p norms; Graph spectra

Introduction

This note was inspired by the following question, which had been asked at the oral entrance exams, see [5], to the Moscow State University Mathematics Department (MekhMat) to certain applicants:

Question 1. Let G be a graph with E edges. Let T be the number of triangles of G. Show that there exists a constant C, such that $T \leqslant CE^{3/2}$ for all G.

Before proceeding any further, let us answer the question. We will assume that G is a *simple, loopless, undirected* graph; that is, there is exactly one edge

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connecting two vertices v and w of G, and there are no edges whose two endpoints are actually the same vertex.

We will need the following notation.

Definition 1. The adjacency matrix $A(G)$ is the matrix with entries

$$\bigl(A(G)_{ij}\bigr) = \begin{cases} 1, & \text{if } i\text{th and } j\text{th vertices of } G \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

We shall also need the following observations.

Observation 1. *The number of paths of length k between vertices v_i and v_j of G is equal to A^k_{ij}.*

The proof of this is immediate. It follows that the number of *closed* paths of length k in G is equal to the trace of A^k. Of course, this statement has to be formulated with some care, since the trace counts each closed path essentially $2k$ times (the 2 is for the choice of orientation, the k is for the possible starting points, the "essentially" is because this is not true for paths consisting of some path repeated l times; such one is only counted $2k/l$ times, or a path followed by retracing the steps backward—such one is counted k times, unless each half is a power of a shorter path, in which case we count it k/l times...).

Unravelling the various cases, we have:

$$\operatorname{tr} A = 0, \tag{1}$$

since G has no self-loops.

$$\operatorname{tr} A^2 = 2E(G), \tag{2}$$

where $E(G)$ is the number of edges of G.

$$\operatorname{tr} A^3 = 6T(G), \tag{3}$$

where $T(G)$ is the number of triangles of G, and

$$\operatorname{tr} A^p = 2pC_p(G), \tag{4}$$

where $C_p(G)$ is the number of cycles of length p of G and p is a prime. For general k:

$$\frac{\operatorname{tr} A^k}{2k} \leqslant \text{number of closed paths of length } k \text{ in } G \leqslant \frac{\operatorname{tr} A^k}{2}. \tag{5}$$

A much more precise general statement can be made, but this will lead us too far afield for the moment.

Since A is symmetric, the following observation is self-evident:

Observation 2.

$$\operatorname{tr} A^k = \sum_{\lambda \in \operatorname{spec} A} \lambda^k,$$

where spec A *is the spectrum of* A*—the set of all eigenvalues of* A.

To answer Question 1 we will also need the following:

Theorem 1. *Let* $\mathbf{x} = (x_1, \ldots, x_k, \ldots)$ *be a non-zero vector. Then for* $p \geqslant 2$*,*

$$\|\mathbf{x}\|_p \leqslant \|\mathbf{x}\|_2, \tag{6}$$

where

$$\|\mathbf{x}\|_p = \Big(\sum |x_i|^p\Big)^{1/p},$$

and equality case in the inequality (6) *occurs if and only if all but one of the* x_i *are equal to* 0.

Proof. It suffices to prove Theorem 1 under the assumption that $\|\mathbf{x}\|_2 = 1$; the general case follows by rescaling. This case, however, is trivial, and follows from the observation that if $0 \leqslant y \leqslant 1$, then $\alpha < \beta$ implies that $y^\alpha \geqslant y^\beta$, with equality if and only if $|y| \in \{0, 1\}$. □

Corollary 1. *Let* M *be a symmetric matrix. Then*

$$\big(\operatorname{tr} A^k\big)^2 \leqslant \big(\operatorname{tr} A^2\big)^k,$$

with equality if and only if all the eigenvalues but one of A *are* 0.

Proof of Corollary 1. Since the eigenvalues of a symmetric matrix are real, and by Observation 2, $\operatorname{tr} A^l = \sum_{\lambda \in \operatorname{spec} A} \lambda^l$, this follows immediately from Theorem 1. □

Applying the corollary in the case $k = 3$, together with Eqs. (2), (3), we get:

$$(2E)^{3/2} \geqslant 6T, \tag{7}$$

and so

$$T \leqslant \frac{2^{1/2}}{3} E^{3/2}. \tag{8}$$

We have answered Question 1, but we have done more: we found an explicit value for the constant C ($\sqrt{2}/3$), and the method works without change to show that

$$C_p \leqslant \frac{2^{p/2-1}}{p} E^{p/2}, \tag{9}$$

for prime p, while

$$C_k \leqslant 2^{k/2-1} E^{k/2} \tag{10}$$

in general.

1. Sharper estimates for odd *n*

Something not quite satisfactory remains about the above argument (aside from the weak bound for general k): it is clear that the equality case in the estimates (8) and (9) is never attained. This is so, because we know that the equality would correspond to the spectrum of A consisting of all 0s and one non-zero eigenvalue, but this contradicts Eq. (1). So, potentially we could get a tighter bound by taking (1) into account. No easier said than done. We now have the following optimization problem (for the number of triangles):

Maximize

$$\sum_{i=1}^{n} \lambda_i^3$$

subject to

$$\sum_{i=1}^{n} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i^2 = 2E.$$

This is a typical constrained optimization problem, best approached with Lagrange multipliers. To avoid (or increase) future confusion, we note that by scale invariance the optimization problem is equivalent to:

Maximize

$$\sum_{i=1}^{n} x_i^3$$

subject to

$$\sum_{i=1}^{n} x_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} x_i^2 = 1.$$

We know that to find the maximum we need to solve the following gradient constraint:

$$\nabla\left(\sum_{i=1}^{n} x_i^3\right) = \lambda_1 \nabla\left(\sum_{i=1}^{n} x_i\right) + \lambda_2 \nabla\left(\sum_{i=1}^{n} x_i^2\right).$$

In coordinates, we have a system of n equations, with the ith being

$$E_i: \quad x_i^2 = \lambda_1 + \lambda_2 x_i.$$

This already tells us that whatever λ_1 and λ_2 might be, there are only two possible values of x_i (independently of i)—the two roots of the quadratic equation.

Summing all the equations, we obtain

$$1 = n\lambda_1,$$

so $\lambda_1 = 1/n$. On the other hand, multiplying E_i by x_i and summing, we see that

$$\sum_{i=1}^{n} x_i^3 = \lambda_2. \tag{11}$$

The left-hand side of Eq. (11) is just the function we are trying to maximize! It remains, thus, to find a good λ_2.

Rewriting the equation E_i as

$$x_i^2 - \lambda_2 x_i - 1/n = 0$$

gives us

$$x_i = \frac{1}{2}\left[\lambda_2 \pm \sqrt{\lambda_2^2 + \frac{4}{n}}\right]. \tag{12}$$

Let us assume that the number of i for which we take the plus sign in the quadratic formula (12) exceeds the number of i for which we take the minus sign by k. Summing all of the x_i we derive

$$0 = \sum_{i=1}^{n} x_i = n\lambda_2 + k\sqrt{\lambda_2^2 + \frac{4}{n}}.$$

(This implies already that $k < 0$.) This translates to the following equation for λ_2:

$$\lambda_2^2 = \frac{4k^2}{n(n^2 - k^2)}.$$

Since we want to make λ_2 as large as possible (by Eq. (11)), we want to make k^2 as large as possible on the right-hand side. Since at least one of the x_i has to be negative and at least one positive, $-k$ cannot exceed $n - 2$. Thus, the biggest possible value for λ_2 is

$$\lambda_2 = \frac{n-2}{\sqrt{n(n-1)}},$$

so, after all this work, we have improved our estimate (8) to

$$T \leqslant \frac{V-2}{\sqrt{V(V-1)}}\frac{2^{1/2}}{3}E^{3/2} \tag{13}$$

(V being the number of vertices of our graph G). This is somehow disappointing: as E (and thus V) goes to infinity, the improvement disappears, and we have the same constant as before. All the work was not done for nothing, however;

consider the complete graph on n vertices K_n. $E(K_n) = n(n-1)/2$, while $T(K_n) = n(n-1)(n-2)/6$ (since any pair of vertices defines an edge, while any triple defines a triangle). A simple computation shows that

$$T(K_n) = \frac{n-2}{\sqrt{n(n-1)}} \frac{2^{1/2}}{3} E(K_n)^{3/2}; \tag{14}$$

so the inequality (13) is actually an equality in this case. Hence the estimate (13) is sharp (since it becomes an equality for an infinite family of graphs), and therefore the constant $2^{1/2}/3$ is also sharp.

A few remarks are in order (as usual).

Firstly, we have inadvertently computed the spectrum of the complete graph.

The estimate (13) and the identity (14) together show that the complete graph K_n is actually maximal (in terms of the number of triangles) of all the graphs with the same number of vertices and edges as it. This sounds wonderful, until we realize that it is the *only* graph with n vertices and $n(n-1)/2$ edges. The identity (14) together with (13) do seem to *suggest* that the complete graph is maximal (for the number of triangles) of all the graphs with the same number of edges. We state this as

Question 2. Show that the complete graph K_n is the graph containing the most triangles of the graphs with $(n-1)n/2$ edges.

This question turns out to be not too difficult. The answer is the subject of the following theorem.

Theorem 2. *In a graph G with no more than $n(n-1)/2$ edges, each edge is contained, on the average, in no more than $n-2$ triangles. Equality holds only for the complete graph K_n.*

Proof. We will prove the theorem by induction. Let v be a vertex in G of maximal degree d. Such a vertex is contained in, at most, $T_v = \min(d(d-1)/2, E(G)-d)$ triangles. This is because there is at most one triangle per edge connecting two vertices adjacent to v, and removing v together with the edges incident to it leaves a graph G' with $T(G) - T_v$ triangles, $E(G) - d$ edges, and $V(G) - 1$ vertices.

Note, first of all, that if the two endpoints of an edge e in G have valences d_1 and d_2, then, if $m = \min(d_1, d_2)$, e is contained in at most $m-1$ triangles. So if the degree of v (assumed to be maximal) was smaller than $n-1$, *no* edge of G was contained in as many as $n-2$ triangles, so we are done.

If $d > n-1$, then G' has $n(n-1)/2 - d$ edges, and so each edge incident to v is contained, on the average, in at most $[n(n-1) - 2d]/d$ triangles. Now,

$$n(n-1) - 2d - d(n-2) = n(n-1) - dn = n(n-1-d) < 0;$$

so the edges incident to d are contained, on the average, in fewer than $n-2$ triangles. The number of edges of G' is smaller than $(n-1)(n-2)/2$ (by a simple calculation); so each of them is contained, on the average, in at most $n-3$ triangles. Since, at best, each of them was contained in one more triangle containing v, this tells us that the average was smaller than $n-2$.

If $d = n-1$, repeating the argument as above shows us that for the equality to hold G' has to be a complete graph on $n-1$ vertices, and so G is a complete graph on n vertices. □

Since most numbers are not triangular (triangular numbers being those of the form $n(n-1)/2$), one can naturally ask the following questions.

Question 3. Is there a simple characterization of graphs with k edges which are "triangle maximal" (for all k)?

Question 4. Consider all graphs with E edges and V vertices. Is there a way to characterize the one with the most triangles.

2. Estimates on power sums

Moving away from graphs as such, the reader will have noted, perhaps, that our way to maximize $\sum_{i=1}^n x_i^p$, subject to the constraints $\|\mathbf{x}\| = 1$ and $\sum_{i=1}^n x_i = 0$, does not work so well for $p \neq 3$, which brings up the questions:

Question 5. Which point $\mathbf{x}$ on the unit sphere $\mathbb{S}^{n-1} \in \mathbb{R}^n$ satisfying $\sum_{i=1}^n x_i = 0$ has the biggest value $\sum_{i=1}^n x_i^p$? Which has the biggest L^p norm (this question is the same for even integer p, but quite different for odd p). For non-integer p, the first question does not make that much sense...

3. Odd *p*

It turns out that it is easiest to minimize the sum of pth powers for p odd. The maximum in this case is attained a the point satisfying the constraints of largest L^∞ norm. For arbitrary p, the argument is a little more subtle—see the proof of Theorem 6.

Theorem 3. *The maximal value of* $\sum_{i=1}^n x_i^{2p+1}$, *subject to the constraints* $\sum_{i=1}^n x_i^2 = 1$ *and* $\sum_{i=1}^n x_i = 0$, *is attained at the point where*

$$x_1 = \sqrt{\frac{n-1}{n}} \quad \textit{and} \quad x_j = -\sqrt{\frac{1}{(n-1)n}}, \quad j = 2, \ldots, n.$$

The value of this maximum is $M_{n,2p+1}$, *where*

$$M_{n,k} = \left(\frac{n-1}{n}\right)^{k/2} + (-1)^k(n-1)\left(\frac{1}{(n-1)n}\right)^{k/2} \tag{15}$$
$$= n^{-k/2}\left[(n-1)^{k/2} + (-1)^k(n-1)^{1-k/2}\right]. \tag{16}$$

Proof. As before, we set up the Lagrange multiplier problem, which has n equations of the form

$$E_i\colon \quad x_i^{2p} = \lambda_1 + \lambda_2 x_i. \tag{17}$$

Summing up all the equations, we find that

$$n\lambda_1 = \sum_{i=1}^{n} x_i^{2p}, \tag{18}$$

while multiplying E_i by x_i and summing the results we obtain

$$\lambda_2 = \sum_{i=1}^{n} x_i^{2p+1}; \tag{19}$$

so that that sought-after sum is equal to λ_2, as before.

Further, note that the derivative of $x^{2p} - \lambda_2 x - \lambda_1$ is equal to $(2p-1) \times x^{2p-1} - \lambda_2$, which has exactly one real zero (whatever the value of λ_2). Therefore, the equation $x^{2p} - \lambda_2 x - \lambda_1 = 0$ has at most two real roots. The specifics of our problem is such that we know that there are exactly two roots, one positive, an other negative. Denote the positive root α_1, the negative root α_2, and suppose that n_1 of x_i are equal to α_1, while $n_2 = n - n_1$ of x_i are equal to α_2. It follows that

$$\alpha_1 = -\frac{n_2}{n_1}\alpha_2. \tag{20}$$

By Eqs. (18) and (19) it follows that

$$\lambda_1 = \frac{1}{n}\left(n_1\alpha_1^{2p} + n_2\alpha_2^{2p}\right), \qquad \lambda_2 = n_1\alpha_1^{2p+1} + n_2\alpha_2^{2p+1}.$$

From Eq. (17), we have the following equation for α_2 (where we have substituted α_2 from the Eq. (20):

$$\alpha_2^{2p} = \frac{1}{n}\left(n_1\left(-\frac{n_2}{n_1}\alpha_2\right)^{2p} + n_2\alpha_2^{2p}\right) + \left(n_1\left(-\frac{n_2}{n_1}\alpha_2\right)^{2p+1} + n_2\alpha_2^{2p+1}\right)\alpha_2. \tag{21}$$

Dividing by α_2^{2p}

$$1 = \frac{1}{n}\left[\frac{n_2^{2p}}{n_1^{2p-1}} + n_2\right] + \alpha_2^2\left[-\frac{n_2^{2p+1}}{n_1^{2p}} + n_2\right];$$

from where, rearranging terms, and replacing n by $n_1 + n_2$, we derive

$$\alpha_2^2 = \frac{\frac{1}{n_2} - \frac{1}{n_1+n_2}[(n_2/n_1)^{2p-1} + 1]}{1 - (n_2/n_1)^{2p}} = \frac{n_1}{n_2(n_1+n_2)},$$

since, amazingly, everything cancels after clearing denominators.

So, finally, we see that

$$\alpha_2^2 = \frac{n_1}{n_2(n_1+n_2)}, \quad \text{while} \quad \alpha_1^2 = \frac{n_2}{n_1(n_1+n_2)},$$

thus showing the first part of the theorem.

Now, the sum S which we seek is given by

$$S = n_1\alpha_1^{2p+1} + n_2\alpha_2^{2p+1} = \frac{1}{n^{p+1/2}}\left[\frac{n_2^{p+1/2}}{n_1^{p-1/2}} - \frac{n_1^{p+1/2}}{n_2^{p-1/2}}\right].$$

It is obviously maximal when n_2 is as large as possible, to wit $n-1$, from which the second part of the theorem follows immediately. □

Notice that since the values of x_i are independent of p, it follows from Theorem 3 that we have proved the following theorem.

Theorem 4. *Let p be an odd prime. A graph G with V vertices and E edges has at most*

$$C_{V,E} = \frac{(V-1)^{p-1} - 1}{V^{(p+1)/2}(V-1)^{(p-1)/2-1}} \frac{2^{p/2-1}}{p} E^{p/2}$$

p-cycles, where equality holds if and only if G is the complete graph $K_{|V|}$.

4. General p

The remainder of the paper will be devoted to the proof of the following theorem.

Theorem 5. *Let $p > 2$. Then the maximum of the sum*

$$S_{n,p} = \sum_{i=1}^{n} x^p,$$

subject to the constraints

$$\sum_{i=1}^{n} x_i = 0 \quad \textit{and} \quad \sum_{i=1}^{n} x_i^2 = 1,$$

is achieved at the point

$$x_1 = (1 - 1/n)^{1/2}, \qquad x_2 = \cdots = x_n = \left[n(n-1)\right]^{-1/2}.$$

The maximal value of $S_{n,p}$ then equals

$$S^*_{n,p} = \frac{(n-1)^{k-1} + 1}{n^{k/2}(n-1)^{k/2-1}}.$$

Theorem 5 was already shown above for the case of p odd. The proof for even p will proceed as follows. First, we show Theorem 6, which works with all but a finite number of exceptions. Then, in Sections 5 and 6 we will consider the exceptions. It should be noted that the proof of Theorem 6 does not rely on the integrality of p in any essential way, and can be viewed as a result on general L^p norms on finite-dimensional vector spaces. The proof leaves a white spot for small dimensions and degree p; but it should be noted that Sections 5 and 6 are devoted just to the integer version of the theorem as stated above and that our result for arbitrary L^p norms is quite non-complete.

Theorem 6. *Let $p > 2$ be even and such that the pair (n, p) is not in the set*

$$E = \left\{(3,4), (4,4), (5,4), (6,4), (7,4), (3,6), (4,6), (3,8), (3,10), (3,12)\right\}.$$

Then the maximum of the sum

$$S_{n,p} = \sum_{i=1}^{n} x^p,$$

subject to the constraints

$$\sum_{i=1}^{n} x_i = 0 \quad \textit{and} \quad \sum_{i=1}^{n} x_i^2 = 1,$$

is achieved at the point

$$x_1 = (1 - 1/n)^{1/2}, \qquad x_2 = \cdots = x_n = \left[n(n-1)\right]^{-1/2}.$$

The maximal value of $S_{n,p}$ then equals

$$S^*_{n,p} = \frac{(n-1)^{p-1} + 1}{n^{p/2}(n-1)^{p/2-1}}.$$

First, we need a lemma.

Lemma 1. *Let $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ be a maximizer for our optimization problem. Then $x_1^{p-2} \geqslant M_{n,p}$, where $M_{n,k}$ is defined in the statement of Theorem* 3.

Proof.

$$M_{n,p} \leqslant S^*_{n,p} = \sum_{i=1}^{n} x_i^p = \sum_{i=1}^{n} x_i^{p-2} x_i^2 \leqslant x_1^{p-2} \sum_{i=1}^{n} x_i^2 = x_1^{p-2},$$

where the first equality uses the fact that $(x_1, \ldots, x_p)$ is a maximizer. □

Notation. We will denote $M_{n,p}^{1/(p-2)}$ by $N_{n,p}$.

Corollary 2. *Let $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ be a maximizer for the optimization problem. Then*

$$x_2^2 \leqslant 1 - N_{n,p}^2.$$

Proof. Immediate from the constraints. □

Proof of Theorem 6. Setting up the Lagrange multiplier problem as before, we see that

$$x_i^{p-1} = \lambda_1 + \lambda_2 x_i \tag{22}$$

must hold at the maximum. As in the proof for odd p, if we let $f_p(x) = x^{p-1} - \lambda_1 - \lambda_2 x$, we note that $f_p'(x) = (p-1)x^{p-2} - \lambda_2$. Since $f_p'(x)$ has exactly two real zeros $z_\pm = \pm(\lambda_2/(p-1))^{1/(p-2)}$, we will write $z = |z_\pm|$. $f_p(x)$ has at most 3 real zeros $t_1 \leqslant t_2 \leqslant t_3$, where $t_1 \leqslant z_-$ and $t_3 \geqslant z_+$. If we succeed in showing that $z > \sqrt{1 - N_{n,p}^2}$, then it will follow that there are at most two distinct values of x_i, and the argument for odd p will lead us to the desired conclusion. To do that, we note that, by multiplying Eqs. (22) by x_i and adding them over i, we see that $\lambda_2 = S^*_{n,p} \geqslant M_{n,p}$, and therefore

$$z > \bigl(M_{n,p}/(p-1)\bigr)^{1/(p-2)}.$$

Thus, our conclusion will follow if we show that

$$\left(\frac{M_{n,p}}{p-1}\right)^{2/(p-2)} > 1 - N_{n,p}^2 = 1 - M_{n,p}^{2/(p-2)},$$

or equivalently:

$$M_{n,p}^{2/(p-2)} > \frac{1}{1 + (p-1)^{-2/(p-2)}}.$$

Since it is clear that

$$M_{n,p} \geqslant \left(\frac{n-1}{n}\right)^{p/2},$$

it would suffice to show that

$$1 - \frac{1}{n} > \frac{1}{(1 + (p-1)^{-2/(p-2)})^{(p-2)/p}}. \tag{23}$$

Let us denote the right-hand side in the desired inequality (23) by $g(p)$.

Lemma 2. *The function $g(p)$ is monotonically decreasing for $p > 2$ and* $\lim_{p\to\infty} = 1/2$.

Proof of Lemma 2. Note that $g(p) = 1/h(p)$, where

$$h(p) = \left(1 + (p-1)^{-2/(p-2)}\right)^{(p-2)/p}.$$

The fact that $\lim_{p\to\infty} h(p) = 2$ is obvious. Since $(p-2)/p$ is an increasing function of p, it is enough to show that $k(p) = (p-1)^{-2/(p-2)}$ is an increasing function of p. Write

$$l(p) = \log k(p) = -\frac{2}{p-2}\log(p-1).$$

Now

$$\begin{aligned}\frac{\mathrm{d}l}{\mathrm{d}p} &= -\frac{2}{(p-2)(p-1)} + \frac{2}{(p-2)^2}\log(p-1)\\ &= \frac{2}{p-2}\left(-\frac{1}{p-1} + \frac{\log(p-1)}{p-2}\right).\end{aligned}$$

Since $\log(p-1) > (p-2)/(p-1)$, it follows that $\mathrm{d}l/\mathrm{d}p > 0$ whenever $p > 2$, and the assertion of the lemma follows. □

The proof of the theorem now follows easily: by Lemma 2, the statement of the theorem is true for any pair (n, p) such that $n > n(p)$, where $1 - 1/n(p) \geqslant g(p)$, and $n(p)$ is chosen to be minimal with this property. The explicit form of the exceptional set follows by a simple machine computation. □

5. Power sums and symmetric functions and some optima

5.1. A brief introduction to symmetric functions

Let us first introduce the *elementary symmetric functions* $e_k(x_1,\dots,x_n)$. These are defined simply as

$$e_k(x_1,\dots,x_n) = (-1)^k,$$
$$\text{times the coefficient of } x^{n-k} \text{ in } (x-x_1)\cdots(x-x_n),$$

while $e_k(x_1,\dots,x_n) = 0$ for $k > n$. The *symmetric function theorem* (see, e.g., [4]) tells us that any symmetric polynomial of $x_1,\dots,x_n$ can be written as a polynomial in $e_1(x_1,\dots,x_n),\dots,e_n(x_1,\dots,x_n)$. Recall that a polynomial f is *symmetric* if $f(x_1,\dots,x_n) = f(x_{\sigma(1)},\dots,x_{\sigma(n)})$, where σ is an arbitrary permutation of n letters. An example of a symmetric polynomial is the kth *power sum* $t_k(x_1,\dots,x_n) = x_1^k + \cdots + x_n^k$. In this case, the algorithm to express t_k in terms of e_k was found by Isaac Newton, and can be summarized as follows:

(a) When $n > k$, then

$$t_k(x_1, \ldots, x_{n-1}) = f\left(e_1(x_1, \ldots, x_{n-1}), \ldots, e_n(x_1, \ldots, x_{n-1})\right) \tag{24}$$

implies that

$$t_k(x_1, \ldots, x_n) = f\left(e_1(x_1, \ldots, x_n), \ldots, e_n(x_1, \ldots, x_n)\right). \tag{25}$$

(b) When $n \leqslant k$, then

$$t_k(x_1, \ldots, x_n) + \sum_{i=1}^{n} (-1)^i e_i(x_1, \ldots, x_n) t_{k-i}(x_1, \ldots, x_n). \tag{26}$$

Both parts (a) and (b) are easily shown: part (a) by noting that the difference between the right- and the left-hand sides of Eq. (25) vanishes when $x_n = 0$, and so, by symmetry, that difference must be divisible by $x_1 \cdots x_n$, and hence is identically 0 (since the degree is smaller than n); part (b) by considering a matrix A with eigenvalues $x_1, \ldots, x_n$, remarking that A satisfies its characteristic polynomial, then taking traces.

5.2. $n = 3$

First, note that our constraints $t_1(x_1, \ldots, x_n) = 0$ and $t_2(x_1, \ldots, x_n) = 1$ imply that $e_1(x_1, \ldots, x_n) = t_1(x_1, \ldots, x_n) = 0$, while, since $t_1^2 - 2e_2 = t_2$, it follows that $e_2(x_1, \ldots, x_n) = -1/2$.

Specializing to $n = 3$, we see from Eq. (26) that

$$t_k(x_1, x_2, x_3) = \frac{1}{2} t_{k-2}(x_1, x_2, x_3) + t_{k-3}(x_1, x_2, x_3) e_3(x_1, x_2, x_3),$$

which implies firstly that

- $t_4(x_1, x_2, x_3) = 1/2$,
- $t_k(x_1, x_2, x_3)$ is a polynomial in $e_3(x_1, x_2, x_3)$ *with positive coefficients*.

This means that the maximum of $t_k(x_1, x_2, x_3)$ is achieved for those values of x_1, x_2, x_3 which maximize the value of $x_1 x_2 x_3$ (subject to our constraints). But since we know that for k large (or k odd) this happens at $x_1 = \sqrt{2/3}$, $x_2 = x_3 = -\sqrt{1/6}$, this finishes the proof of $(n = 3)$.

5.3. $n = 4$

A routine computation using Eqs. (25) and (26) leads to the following:

$$t_3(x_1, x_2, x_3, x_4) = 3e_3(x_1, x_2, x_3, x_4), \tag{27}$$

$$t_4(x_1, x_2, x_3, x_4) = \frac{1}{2} - 4e_4(x_1, x_2, x_3, x_4), \tag{28}$$

$$t_6(x_1, x_2, x_3, x_4) = \frac{1}{4} + 3e_3^2(x_1, x_2, x_3, x_4) - 3e_4(x_1, x_2, x_3, x_4). \tag{29}$$

Since we know that $t_3(x_1, x_2, x_3, x_4)$ is maximized at the point $x_1 = \sqrt{3/4}$, $x_2 = x_3 = x_4 = -\sqrt{1/12}$, we know that $e_3(x_1, x_2, x_3, x_4)$ is maximized at that point, and thus, to finish the case $n = 4$ we need to show that $t_4(x_1, x_2, x_3, x_4)$ is *minimized* at the same point. In general, to minimize $e_n(x_1, \ldots, x_n)$, subject to $t_1(x_1, \ldots, x_n) = 0$ and $t_2(x_1, \ldots, x_n) = 1$, we set up the usual Lagrange multiplier problem, and have the Lagrange equations for the critical points:

$$\frac{e_n(x_1, \ldots, x_n)}{x_i} = \lambda_1 + \lambda_2 x_i \quad \text{or} \quad e_n(x_1, \ldots, x_n) = \lambda_1 x_i + \lambda_2 x_i^2.$$

Since the right-hand side is a quadratic, it follows immediately that there are exactly two different values of the coordinates, and the rest of the argument (at least when $n = 4$) is routine and shows that $e_4(x_1, \ldots, x_4)$ is minimized exactly when $e_3(x_1, \ldots, x_4)$ is maximized, which does it for $t_4(x_1, x_2, x_3, x_4)$ and $t_6(x_1, x_2, x_3, x_4)$.

6. $p = 4$

In this section we eliminate the exceptional cases of the form $(n, 4)$, for all n. Consider, then, the vector $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^n x_i = 0$, $\sum_{i=1}^n x_i^2 = 1$, and $\sum_{i=1}^n x_i^4$ is maximal. Such a vector must satisfy the Lagrange multiplier equations:

$$x_i^3 = \lambda x_i + \mu. \tag{30}$$

If there are only two different values of the coordinates, then we are done. It is easy to see directly that there are at most *three* distinct values, as follows: Suppose $x_1 \neq x_j$. Then, subtracting the Lagrange equation (30) for x_1 from that of x_j, we obtain

$$x_1^3 - x_j^3 = \lambda(x_1 - x_j).$$

Dividing by $x_1 - x_j$ we obtain

$$x_j^2 + x_j x_1 + x_1^2 - \lambda = 0. \tag{31}$$

This is a quadratic equation for x_j, and we see that x_j could be either one of the two roots. So, if there are more than two distinct values of the coordinates, there are exactly three, call them $\alpha = x_1, \beta, \gamma$. Since β, γ are the two roots of the quadratic equation (31), we see that

$$\beta + \gamma = -\alpha. \tag{32}$$

Let us assume that $\alpha = \max(\alpha, \beta, \gamma)$. Furthermore, let us assume that $n \geqslant 4$ (since the case $n = 3$ was dealt with above). That being the case, it is clear that

$$1 = \sum_{i=1}^{n} x_i^2 \geqslant \alpha^2 + 2\beta^2 + \gamma^2. \tag{33}$$

Let α be fixed. Then, the minimum of $2\beta^2 + \gamma^2$, subject to the relation (32), is achieved for $\beta = -(\frac{2}{3}\alpha)$ and $\gamma = -\frac{1}{3}\alpha$ (this is easily shown using Lagrange multipliers), and so

$$\alpha^2 + 2\beta^2 + \gamma^2 \geqslant \alpha^2\left(1 + 2\frac{4}{9} + \frac{1}{9}\right) = 2\alpha^2.$$

This, together with the inequality (33), implies that

$$\alpha^2 \leqslant \frac{1}{2}.$$

On the other hand, Lemma 1 tells us that

$$\alpha^2 \geqslant \left(\frac{n-1}{n}\right)^2$$

if our vector is a maximizer. The last two inequalities imply that

$$\left(\frac{n-1}{n}\right)^2 \leqslant \frac{1}{2},$$

which is satisfied just for $n = 1, 2, 3$. The argument is now complete. $\square$

Notes on the bibliography

It is hoped that this paper is reasonably self-contained, however, I would be remiss not to give some references to related literature. The literature on graph eigenvalues is vast. For some entry points, the reader is advised to look at the books of Biggs [1] and Cvetkovic et al. [3] for a general introduction to graph theory, Bollobas' book [2] is good, among many others.

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References

[1] N. Biggs, Algebraic Graph Theory, 2nd edition, in: Cambridge Math. Library, Cambridge University Press, Cambridge, 1993.
[2] B. Bollobas, Modern Graph Theory, in: Graduate Texts in Math., Vol. 184, Springer-Verlag, New York, 1998.
[3] D. Cvetkovic, M. Doob, H. Sachs, Spectra of Graphs, Theory and Applications, 3rd edition, Barth, Heidelberg, 1995.
[4] S. Lang, Algebra, 2nd edition, Addison–Wesley, 1984.
[5] I. Vardi, Mekh-mat entrance examination questions, IHES Preprint, 1999.