ODE THEOREM USING EULER METHOD

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We want to make a proof of the fundamental theorem of Ordinary Differential Equations using Euler's integration method. We write the equation as,

$$\frac{dx}{dt} = f(x)$$

where $x \in R^n$.

Teorema Fundamental para ODEs:

Assume that there is a point x_c and a positive radius R such that $f(x)$ exists and is Lifshitz on $B_{2R}(x_c)$ (given by $|x - x_0| < 2R$). Then given any point $x_0 \in B_R(x_c)$ there is a finite time interval $[0, T$ for which an unique solution to the above system exists.

Let $F := \max_{x \in B_{2R}(x_0)} |f(x)|$, and fix $T < F/R$ so that the solution, if exist, can not leave the ball. We also define L so that $|f(x) - f(y)| < L|x - y|$ the Lifshitz constant.

We divide the integration interval $[0, T]$ into N_0 steps of size $h_0 = T/N$, and compute the finite sequence,

$$y_h^{n+1} = y_h^n + hf(x_h^n)$$

We then approximate the solution by the linear interpolation joining these points, calling it $x_h(t)$. We repeat the procedure using more points, namely $N_i = 2^i N_0$, and correspondingly $h_i = h2^{-i}$. Thus we get an infinite sequence of continuous approximations to the would be solution. Since the space of continuous functions, $C[0, T]$ is Banach (complete) under the norm *max*. If we can show that the above sequence is Cauchy we would find a unique limiting function on that space and that would be the solution we are seeking.

So we must proof now it is Cauchy in the *max* norm. It is enough to compare two consecutive elements on the sequence and show that we can make their difference as small as we want if they are far enough along it.

So we consider now the i and the $i+1$ elements of the sequences of continuous functions, and call $h_i = h$, so $h_{i+1} = h/2$. Consider any interval of the bigger sequence, say the one going from nh to $(n+1)h$, in that interval the finer sequence would consist of two straight lines, while the initial one is just a single line. The maximum distance among these lines must be at one of the three points where the lines break, nh, $(n + 1/2)h$, or $(n + 1)h$. We express them in therm of the first one, for the middle point we get,

$$\begin{aligned}
& |x_h((n + 1/2)h) - x_{h/2}(n + 1/2)h)| \\
= \quad & |x_h(nh) + h/2f(x_h(nh)) - (x_{h/2}(nh) + h/2f(x_{h/2(nh)}))| \\
= \quad & |x_h(nh) - x_{h/2}(nh) - h/2[f(x_{h/2}(nh)) - f(x_h(nh))]| \\
< \quad & |1 + hL/2||x_h(nh) - x_{h/2}(nh)|
\end{aligned}$$

For the end point we get,

$$\begin{aligned}
& |x_h((n+1)h) - x_{h/2}(n+1)h)| \\
= \; & |x_h(nh) + hf(x_h(nh)) - (x_{h/2}(nh) + h/2f(x_{h/2(nh)}) + h/2f(x_{h/2}(nh) + h/2f(x_{h/2(nh)}))| \\
< \; & |x_h(nh) - x_{h/2}(nh)| + hf(x_{h/2}(nh)) - f(x_h(nh))| + h/2|f(x_{h/2}(nh) + h/2f(x_{h/2(nh)}) - f(x_{h/2}(nh)| \\
< \; & |1 + hL/2||x_h(nh) - x_{h/2}(nh)| + h^2LF/4
\end{aligned}$$

So we can bound the approximations in the $[nh, (n+1)h]$ interval as,

$$\begin{aligned}
max_{t\in[nh,(n+1)h]}|x_h(t) - x_{h/2}(t)| &< |1 + hL/2||x_h(nh) - x_{h/2}(nh)| + h^2LF/4 \\
&< |1 + hL/2| \; max_{t\in[(n-1)h,nh]}|x_h(t) - x_{h/2}(t)| + h^2LF/4
\end{aligned}$$

Iterating this from $n = 0$ (where $x_h(0) = x_{h/2}(0)$) we get that

$$\begin{aligned}
max_{t\in[0,T]}|x_h(t) - x_{h/2}(t)| &< \sum_{j=0}^{N}(1 + hL/2)^j h^2LF/4 \\
&= \frac{(1 + hL/2)^{N+1} - 1}{(1 + hL/2) - 1} h^2LF/4 \\
&= ((1 + TL/2N)^{N+1} - 1)hF/2 \\
&\leq (e^{TL/2} - 1)hF/2
\end{aligned}$$

Since $h = 2^{-i}h_0 \to 0$, by going far enough in the sequence we can make this difference as small as we please and so conclude that the sequence is Cauchy. Thus we have a unique limiting sequence that we call $x(t)$.

We want to check now that this limiting function satisfies the integral version of our equation,

$$x(t) = x_0 + \int_0^t f(x(\tilde{t})d\tilde{t}.$$

Once we have this then we can derive it to obtain the differential equation.

Given $t \in [0, T]$ we choose n_i to be the integer part of $2^it/h_0$ then $n_ih_i \to t$ as $i \to \infty$. By continuity $x(n_ih_i) \to x(t)$, furthermore, $x_{h_i}(n_ih_i) \to x(t)$. But,

$$x_{h_i}(n_ih_i) = x_0 + \sum_{j=0}^{n_i-1} h_if(x_{h_i}(jh_i)),$$

Since f is continuos the above sum converges to the Riemann integral. This completes the existence and uniqueness proof.