4.3 The Renormalization Group Methods

Some typical results of MC simulations of a $d = 2$ Ising ferromagnet are shown in Fig. 4.9. At high T ($T = 2T_C$), there is only short-range order, the spins form small clusters. The correlation length (approximately equal to the linear size of the largest cluster) is small. Close (but above) T_C, somewhat larger patches in which most of the spins are lined up in the same direction begin to develop. When the system reaches T_C, these patches expand to infinite size, but fluctuations of smaller scale persist! At the critical temperature, spins form clusters at all lengthscales, including one infinite-size cluster (which cannot be seen on a finite system, but we know that the correlation length has to diverge at T_C). As a result, *all scales of length must be included in a theoretical description*. Another very important conclusion is reached if we substitute a block of 3×3 spins by a single spin with the orientation prescribed by the majority rule (block spin "up" if the majority of the spins in the block is "up", and vice versa). In this way the length scale of the lattice is changed by a factor $b = 3$ each time. This is a "real-space block-spin renormalization-group transformation".

Examples of such transformations are shown in Figs. 4.10-4.12. Below T_C, the system has long-range order but, because the system is close to T_C, the fluctuations form large clusters of oppositely ordered spins. However, under each transformation the size of the unit cell increases, the fluctuations become smaller and smaller and those fluctuations, which are smaller in size than the lattice spacing, disappear. Under successive transformations, thus, the system becomes more and more ordered, it effectively approaches zero temperature, see Fig. 4.13.

Above T_C (Fig. 4.11), there is no long-range order, spins form random "up" and "down" clusters. Under each transformation, the correlation length (expressed in units of the new unit cell) decreases, the clusters become smaller and smaller as if the temperature were higher, but they never disappear. At $T = T_C$ (Fig 4.12), the clusters are of *all length scales*, including one infinitely large cluster. The correlation length is infinite (exceeds the size of the lattice used in the MC simulation). Under successive transformations, it remains "infinite", there is no change in the distribution of cluster sizes, the system is invariant under scale transformations! The system remains at the critical temperature and the critical point is a *fixed point*. Other (trivial) fixed points are $T = 0$ and $T = \infty$ because the system away from T_C tends to one of these points under successive scale transformations. Decrease of the correlation length under successive transformations is an extremely useful property because it makes the fluctuations uncorrelated and we can solve the system using an approximate (e.g., mean-field) theory to calculate

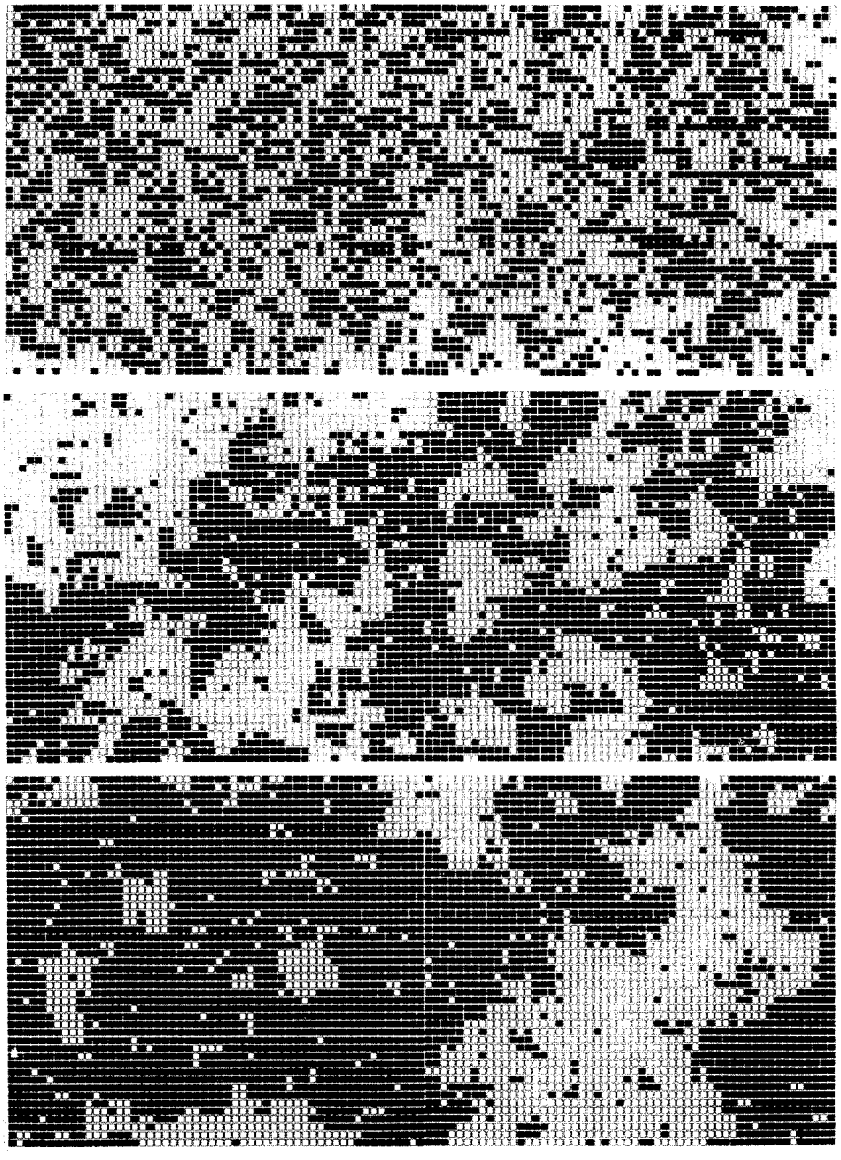

Figure 4.9: Results of MC simulations of a $d = 2$ Ising ferromagnet at different T. At high T ($T = 2T_C$; top), the correlation length is small. Close (but above) T_C (middle), somewhat larger clusters begin to develop. When the system reaches T_C (bottom), these clusters expand to infinite size, but fluctuations on smaller scale persist! (From: K.G. Wilson, Sci. Am. **241**, 140 (1979).))

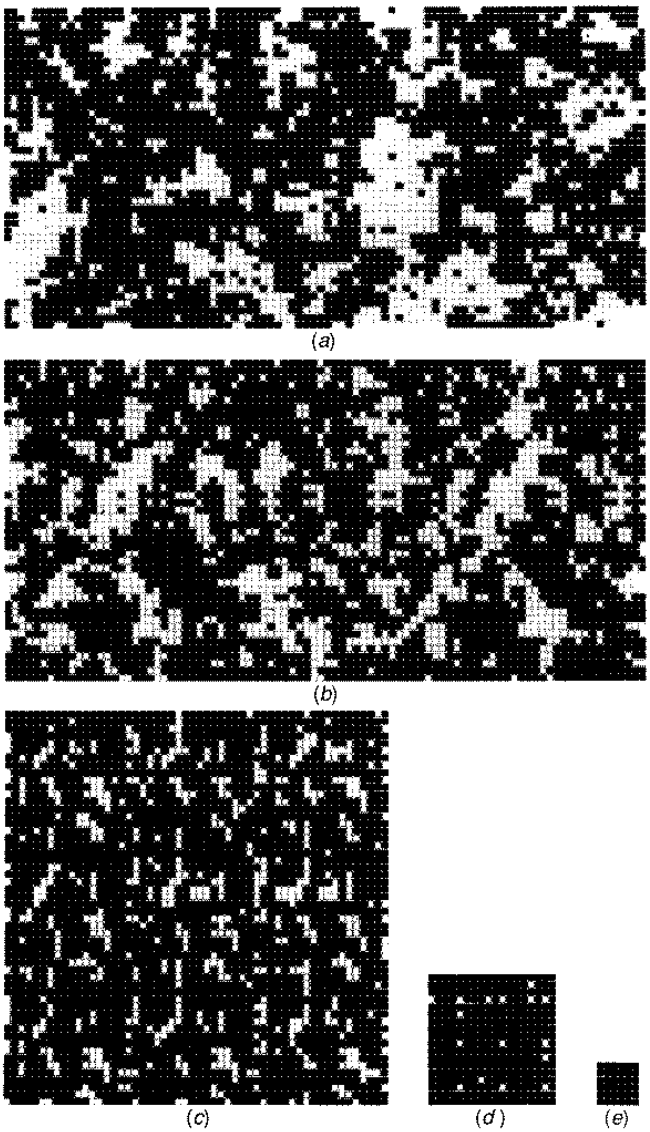

Figure 4.10: Block-spin transformation is applied to a lattice repeatedly at $T = 0.99T_C$. Each time the number of spins is reduced by a factor of 9, elucidating the behaviour of the system at a larger scale. The correlation length decreases under successive transformations. Clearly seen is the decrease of the correlation length and a suppression of the fluctuations. Under each transformation the system appears more ordered, the system flows under the renormalization transformations towards zero temperature. (After K.G. Wilson, Sci. Am. **241**, 140 (1979))

Figure 4.11: As Fig. 5.9 but for $T = 1.22T_C$. Under each transformation, the correlation length decreases, the system flows towards infinite temperature. (After K.G. Wilson, Sci. Am. **241**, 140 (1979))

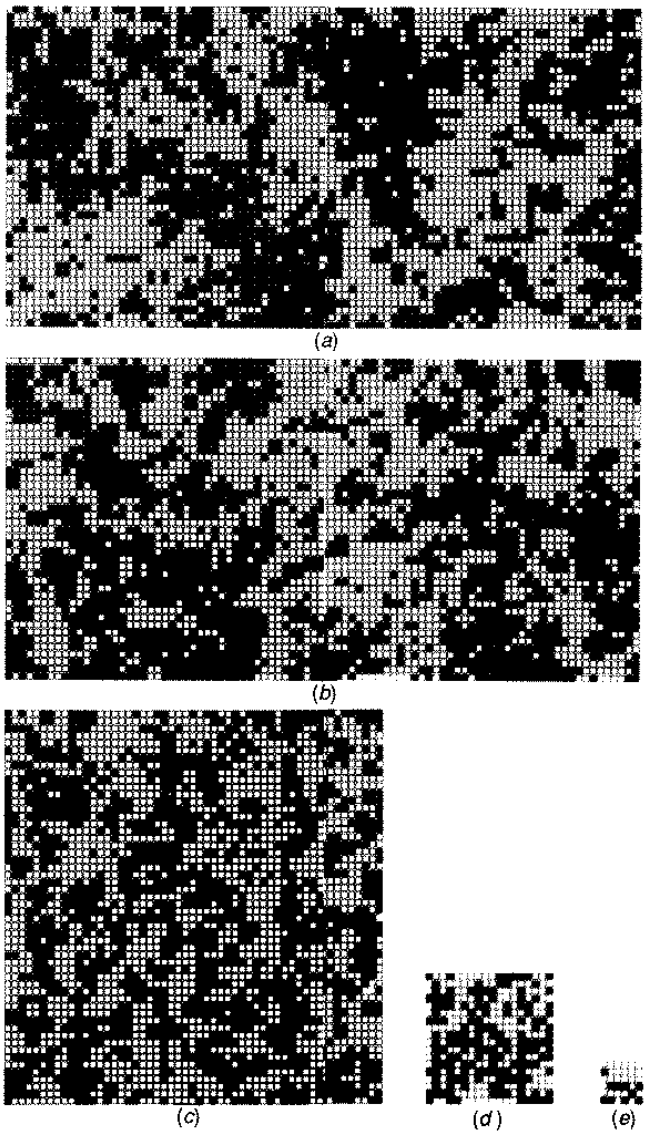

Figure 4.12: As Fig. 5.9 but for $T = T_C$: The correlation length is infinite and remains infinite under successive transformations, the system remains at the critical temperature. (After K.G. Wilson, Sci. Am. **241**, 140 (1979))

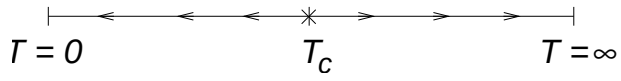

Figure 4.13: Block-spin renormalization at $H = 0$. Upon successive decimations of an initial configuration at $T < T_C$ the system becomes more and more ordered as if the temperature were lower and lower. Eventually, the system becomes completely ordered, this is the $T = 0$ fixed point. Upon decimations of a state at $T > T_C$, the system becomes more and more disordered, the correlation length decreases as if the temperature were higher, until the correlation length becomes on the order of the lattice spacing. This is the infinite–temperature fixed point. A system at T_C stays, upon successive decimations, at T_C. This is another fixed point, the critical point.

the properties in that region and then transform back to the original lattice. This is the basic idea of the renormalization-group methods and we shall exploit this idea on some simple examples.

4.3.1 A Trivial Example: The $d = 1$ Ising Model

In the renormalization-group methods the temperature changes under successive transformations, therefore it is convenient to work with the *reduced Hamiltonian*, we divide $\mathcal{H}$ by k_BT. The reduced Hamiltonian of an Ising Hamiltonian with NN exchange interaction in an external field is

$$\mathcal{H} = -K\sum_i S_iS_{i+1} - h\sum_i S_i, \qquad K = \frac{J}{k_BT} \quad h = \frac{H}{k_BT}. \tag{4.65}$$

Recursion relations. If we want to make several successive renormalization-group (RG) transformations, the structure of $\mathcal{H}$ must not change after each transformation. We are thus seeking for such a transformation from the old Hamiltonian $\mathcal{H}$ to the new Hamiltonian $\mathcal{H}'$ that keeps the structure of the Hamiltonian unchanged under successive transformations. In addition, the transformation must be such that it *does not* change the partition function. That is necessary because this ensures that the physical properties of the system are not changed. The partition function of the one-dimensional Ising model in an external field is written in

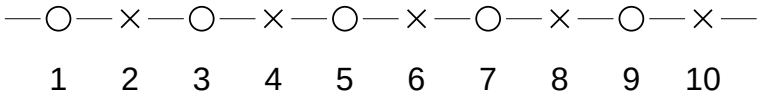

Figure 4.14: Decimation on the $d = 1$ Ising model.

the form

$$\begin{aligned} Z(N,K,h) &= \sum_{\{S\}} \mathrm{e}^{-\mathcal{H}} = \sum_{\{S\}} \mathrm{e}^{K\sum_i S_iS_{i+1}+h\sum_i S_i} \\ &= \sum_{\{S\}} \mathrm{e}^{K(S_1S_2+S_2S_3)+hS_2+\frac{h}{2}(S_1+S_3)} \\ &\quad \times \mathrm{e}^{K(S_3S_4+S_4S_5)+hS_4+\frac{h}{2}(S_3+S_5)} \times \cdots, \end{aligned} \tag{4.66}$$

We divide the lattice into the sublattices $\times$ and $\circ$, denote the spins on the $\times$ sublattice by σ_i (Fig. 4.14),

$$Z = \sum_{\{S\}} \sum_{\{\sigma\}} \mathrm{e}^{\sigma_2[K(S_1+S_3)+h]+\frac{h}{2}(S_1+S_3)} \mathrm{e}^{\sigma_4[K(S_3+S_5)+h]+\frac{h}{2}(S_3+S_5)} \cdots, \tag{4.67}$$

and carry out a partial trace, i.e., we sum over $2^{N/2}$ configurations of the $\{\sigma\}$ spins sitting on the $\times$ sublattice:

$$Z(N,K,h) = \sum_{\{S\}} \left[\mathrm{e}^{(K+\frac{h}{2})(S_1+S_3)+h} + \mathrm{e}^{(-K+\frac{h}{2})(S_1+S_3)-h} \right] \times \cdots. \tag{4.68}$$

The entire partition function must not change and $\mathcal{H}$ must keep its structure, therefore we have the condition

$$Z(N,K,h) = \mathrm{e}^{Ng(K,h)} Z(\frac{N}{2}, K', h') = \mathrm{e}^{Ng} \sum_{\{S\}} \mathrm{e}^{-\mathcal{H}'} \tag{4.69}$$

where

$$\mathcal{H}' = -K' \sum_{i\in\mathrm{odd}} S_iS_{i+2} - h' \sum_{i\in\mathrm{odd}} S_i \tag{4.70}$$

and g is a constant. This will ensure that the physical properties will not change under successive iterations. The two conditions (4.70) and (4.69) imply that

$$\mathrm{e}^{(K+\frac{h}{2})(S_i+S_{i+2})+h} + \mathrm{e}^{(-K+\frac{h}{2})(S_i+S_{i+2})-h}$$

$$= \mathrm{e}^{K' S_i S_{i+2} + \frac{h'}{2}(S_i + S_{i+2}) + 2g}. \tag{4.71}$$

The last equation has to hold for any $S_i, S_{i+2} = \pm 1$:

$$\begin{aligned} \mathrm{e}^{2K+2h} + \mathrm{e}^{-2K} &= \mathrm{e}^{K'+h'+2g} \\ \mathrm{e}^{2K-2h} + \mathrm{e}^{-2K} &= \mathrm{e}^{K'-h'+2g} \\ \mathrm{e}^{+h} + \mathrm{e}^{-h} &= \mathrm{e}^{-K'+2g}, \end{aligned} \tag{4.72}$$

from which we get:

$$\begin{aligned} \mathrm{e}^{4K'} &= \frac{\left(\mathrm{e}^{2K+2h} + \mathrm{e}^{-2K}\right)\left(\mathrm{e}^{2K-2h} + \mathrm{e}^{-2K}\right)}{\left(\mathrm{e}^{h} + \mathrm{e}^{-h}\right)^2} \\ \mathrm{e}^{2h'} &= \frac{\mathrm{e}^{2K+2h} + \mathrm{e}^{-2K}}{\mathrm{e}^{2K-2h} + \mathrm{e}^{-2K}} \\ \mathrm{e}^{8g} &= \left(\mathrm{e}^{2K+2h} + \mathrm{e}^{-2K}\right)\left(\mathrm{e}^{2K-2h} + \mathrm{e}^{-2K}\right)\left(\mathrm{e}^{h} + \mathrm{e}^{-h}\right)^2 \end{aligned} \tag{4.73}$$

or:

$$\begin{aligned} K' &= \frac{1}{4}\ln\frac{\cosh(2K+h)\cosh(2K-h)}{\cosh^2 h} \\ h' &= h + \frac{1}{2}\ln\frac{\cosh(2K+h)}{\cosh(2K-h)} \\ g &= \frac{1}{8}\ln\left[16\cosh(2K+h)\cosh(2K-h)\cosh^2 h\right]. \end{aligned} \tag{4.74}$$

Equations (4.73) or (4.74) are the *recursion relations* and determine the fixed points and the flow diagram of the system.

In each iteration, the number of degrees of freedom is reduced by one half, the new Hamiltonian $\mathcal{H}$ has only one half of the previous spins,

$$N' = N/b \qquad\qquad b = 2, \tag{4.75}$$

and the lattice spacing is increased,

$$a' = ba. \tag{4.76}$$

Other lengths which are measured in units of the lattice spacing are reduced by a factor b. For example, the new correlation length is

$$\xi' = \xi/b. \tag{4.77}$$

The remaining spins on the decimated lattice interact with their new nearest neighbours through the renormalized coupling constants K' and are subject to renormalized fields h'.

Fixed points. At a fixed point the parameters (the coupling constants – in our case K and h) do not change under successive decimations. This means that the system at a fixed point stays at this point in the parameter space. With the new variables

$$x = \mathrm{e}^{-4K} \qquad y = \mathrm{e}^{-2h} \qquad z = \mathrm{e}^{-8g}$$
$$0 \leq (x, y, z) \leq 1 \tag{4.78}$$

the recursion relations become

$$\begin{aligned} x' &= x\,\frac{(1+y)^2}{(x+y)(1+xy)} \\ y' &= y\,\frac{x+y}{1+xy} \\ z' &= z^2 x y^2\,\frac{1}{(x+y)(1+xy)(1+y)^2} \end{aligned} \tag{4.79}$$

The recursion relations for x and y do not depend on z. Physically, this means that the singular behaviour of the free energy does not depend on a shift in the energy scale. Therefore, we first investigate the flow diagram and fixed points in the (x, y) plane, see Fig. 4.15. From the first equation it is easy to see that $x' = x$ if $x = 1$, i.e., $x^{=}1$ is a fixed point of this system for arbitrary y . We denote the fixed points by a $*$. There is a line of fixed points at $x^* = 1$ for $0 \leq y^* \leq 1$, corresponding to an infinite-temperature sink. Along this line, the spins are completely disordered and $\xi \to 0$. If we set $x = 0$, the second equation reduces to $y' = y^2$, which has two fixed points, one at $y^* = 1$ and one at $y^* = 0$. The fixed point at $(x^*, y^*) = (0, 1)$ (this means $H = 0$ and $T = 0$) corresponds to the critical point where $\xi \to \infty$. This is correct because we know that the $d = 1$ Ising model has $T_C = 0$. This fixed point is unstable and any point in its vicinity will flow under RG transformations away from it towards the fixed points on the line at $x^* = 1$. The critical behaviour of the $d = 1$ Ising model is governed by the least stable fixed point at $(x^*, y^*) = (0, 1)$. The point at $(x^*, y^*) = (0, 0)$ is another fixed point which corresponds to $T = 0$ and $H = \infty$. At this fixed point, the spins are completely ordered.

4.3.2 RG Transformations on the $d = 2$ Ising Model

The recursion relations of the $d = 1$ Ising model could be formulated exactly. Usually, however, farther-range interactions are introduced under successive dec-

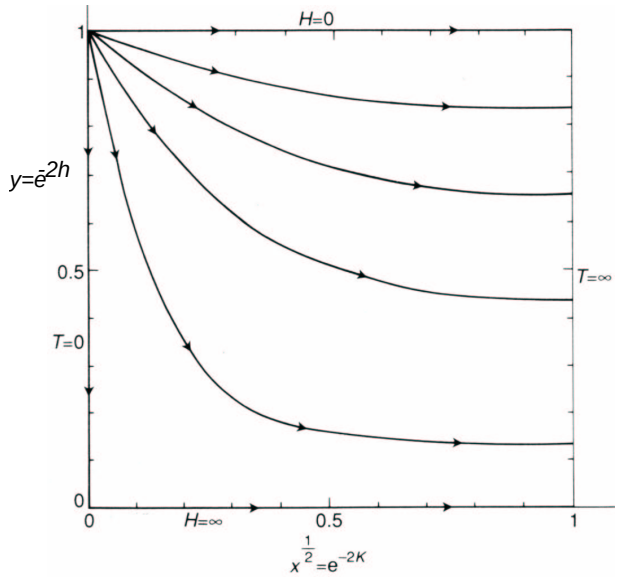

Figure 4.15: Schematic flow diagram of the $d = 1$ Ising model. (After J.M. Yeomans, Statistical Mechanics of Phase Transitions, Oxford 1992.)

imations and approximations have to be made in order to get the closed-form recursion relations.

As an example, let us apply the decimation on the $d = 2$ Ising model on a square lattice in the absence of an external field. We start with NN (nearest neighbours) exchange interactions and sum the partition function over all possible configurations of the σ spins (these are the spins that will be decimated, they are denoted by $\times$ in Fig. 4.16).

$$\begin{aligned} Z &= \sum_{\{S\}} \mathrm{e}^{K\sum_{\langle ij\rangle} S_i S_j} = \sum_{\{S\}}\sum_{\{\sigma\}} \mathrm{e}^{K\sum_i (S_{i+x}+S_{i-x}+S_{i+y}+S_{i-y})\sigma_i} \\ &= \sum_{\{S\}}\prod_i \left[\mathrm{e}^{K(S_{i+x}+S_{i-x}+S_{i+y}+S_{i-y})} + \mathrm{e}^{-K(S_{i+x}+S_{i-x}+Si+y+S_{i-y})}\right]\ldots \end{aligned} \tag{4.80}$$

$S_{\pm x}$ and $S_{\pm y}$ are the "right/left" and "upper/lower" neighbours of σ_i, respectively. The conditions are that under decimation Z must not change and that $\mathcal{H}$ should have the same structure as the original Hamiltonian. However, we soon realize that we cannot fulfill the above conditions with the renormalized form of the initial

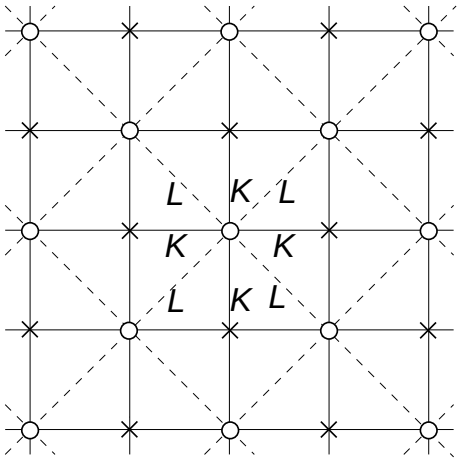

Figure 4.16: Decimation on the $d = 2$ square lattice. The spins denoted by $\times$ are decimated. K is the NN interaction and L the NNN interaction. After decimation, the lattice constant increases by $b = \sqrt{2}$, and previous NNN become NN (nearest neighbours). Solid lines represent the original lattice and dashed lines the decimated lattice.

Hamiltonian. After decimation we get a more complicated Hamiltonian:

$$\begin{aligned}\mathcal{H}' = \; & - \sum_i [K'(S_{i+x}S_{i+y} + S_{i+y}S_{i-x} + S_{i-x}S_{i-y} + S_{i-y}S_{i+x}) \\ & + \; L'(S_{i+x}S_{i-x} + S_{i+y}S_{i-y}) + Q'S_{i+x}S_{i+y}S_{i-x}S_{i-y} + C'] \,,\end{aligned} \tag{4.81}$$

where K' is the new NN interaction, L' the new NNN (next nearest neighbours) interaction, Q' the four-spin interaction, and C' a constant. Only with the Hamiltonian of this form, it is possible to satisfy the condition $Z' = Z$ for arbitrary configurations of $S_{\pm x,y}$. Although we started with only NN interactions, decimation generated NNN interaction and even four-spin interactions. So, we have to start from the beginning again with a more complicated Hamiltonian which includes all these interactions. In the following we shall neglect Q' and assume that L is small. In this case we obtain the following approximate recursion relations:

$$\begin{aligned} K' &\approx L + 2K^2 \\ L' &\approx K^2 \end{aligned} \tag{4.82}$$

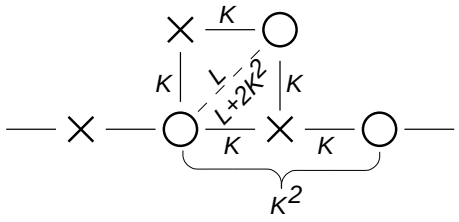

Figure 4.17: The renormalized interaction after a decimation is $L + 2K^2$.

In fact, these relations could be guessed immediately, see Fig. 4.17. The new nearest neighbours are the previous NNN, therefore the term L in the first equation. Besides, they were connected by two pairs of consecutive bonds, this brings $2K^2$. In the second equation, the new next nearest neighbours (NNN) are connected by two consecutive NN bonds on the old lattice - each of them contributing a factor K.

The recursion relations 4.82 have two trivial fixed points, $(K^*, L^*) = (0, 0)$ and $(K^*, L^*) = (\infty, \infty)$ and one non-trivial fixed point at $(1/3, 1/9)$. The first one is the infinite-temperature, paramagnetic fixed point, the second one is the zero-temperature, ferromagnetic fixed point. The non-trivial fixed point corresponds to the critical point of the system.

4.3.3 General Properties of the RG Transformations

So far, we have seen that the renormalization-group transformations work by changing the length scale and by reducing the number of the degrees of freedom. Only at criticality the system remains unaltered under the change in scale and the critical behaviour is determined by a fixed point of the transformation.

We start with an effective Hamiltonian, which is characterized by a set of coupling constants and fields $K,\ L,\ \cdots$. Each set of these constants defines a point in the "Hamiltonian space," the space of coupling constants. The renormalization-group transformations work in this space,

$$\boxed{\mathcal{H}' = \mathcal{R}\,\mathcal{H}.} \tag{4.83}$$

The renormalization-group operator $\mathcal{R}$ reduces the number of degrees of freedom

from N to N'. This defines the scale factor b:

$$b^d = \frac{N}{N'}. \tag{4.84}$$

In the previous examples, the reduction was made in real space either by block-spin transformation or by decimation; later we shall do the reduction also by integrating out large momenta (short wavelengths) in momentum space.

The essential condition to be satisfied by any RG transformation is that the partition function must not change,

$$\boxed{Z_{N'}(\mathcal{H}') = Z_N(\mathcal{H}).} \tag{4.85}$$

Therefore the total free energy does not change but the free energy per unit cell (spin) increases as

$$f(\mathcal{H}') = b^d f(\mathcal{H}). \tag{4.86}$$

All lengths, measured in units of the new lattice spacing, are reduced by the factor b. Thus, the correlation length scales as

$$\xi' = \frac{1}{b}\,\xi. \tag{4.87}$$

Fixed points. The iteration of $\mathcal{R}$ traces a trajectory in the Hamiltonian space, this is called the "Hamiltonian flow." The trajectories end in fixed points $\mathcal{H}^*$, defined by

$$\mathcal{H}^* = \mathcal{R}\mathcal{H}^* \qquad \Longleftrightarrow \qquad \mathcal{H}' = \mathcal{H} \equiv \mathcal{H}^*. \tag{4.88}$$

At the fixed points, the system is invariant under subsequent scale changes and thus the correlation length does not change

$$\xi' = \xi \equiv \xi^*. \tag{4.89}$$

Let us now take a closer look at the Hamiltonian flow near the fixed points. We introduce a reduced Hamiltonian which we write in the general form

$$\mathcal{H} = \vec{\mu} \cdot \vec{\phi} \tag{4.90}$$

and investigate the flow in the *parameter space* spanned by the generalized fields $\vec{\mu}$, conjugate to the operator $\vec{\phi}$ which contains product of operators (like spins).

- *Example:* In the case of the $d = 2$ Ising model on a square lattice, the components of $\vec{\phi}$ include the terms linear in S (the conjugate filed is h), quadratic in S (coming from interactions with NN, NNN neighbours), four-spin terms, etc. The corresponding conjugate fields are the magnetic field h and the coupling constants K, L, etc. In the beginning, many components of $\vec{\mu}$ are zero, but under successive transformations, more and more complex interactions emerge and an increasing number of components of $\vec{\mu}$ becomes $\neq 0$.

Under successive RG transformations, the Hamiltonian evolves and the system moves through the parameter space:

$$\vec{\mu}' = \mathcal{R}\vec{\mu} \tag{4.91}$$

until it reaches a fixed point when:

$$\vec{\mu}' = \vec{\mu} \equiv \vec{\mu}^*. \tag{4.92}$$

Every particular type of criticality corresponds to a particular fixed point and deviations from the critical point are related to deviations from the corresponding fixed point. To get insight into the critical properties we must investigate the flow in the Hamiltonian space in the *vicinity* of the fixed points.

Near a fixed point we linearize the operator $\mathcal{R}$. We write

$$\begin{aligned} \vec{\mu} &= \vec{\mu}^* + \delta\vec{\mu} \\ \vec{\mu}' &= \vec{\mu}^* + \delta\vec{\mu}' \end{aligned} \tag{4.93}$$

where the small deviations from $\vec{\mu}^*$ are related by

$$\vec{\mu}' = \mathcal{R}(\vec{\mu}^*) + \left.\frac{\partial\mathcal{R}}{\partial\vec{\mu}}\right|_{\vec{\mu}=\vec{\mu}^*} \delta\vec{\mu} = \vec{\mu}^* + \mathcal{L}(\vec{\mu}^*)\delta\vec{\mu} \tag{4.94}$$

$$\delta\vec{\mu}' = \mathcal{L}(\vec{\mu}^*)\delta\vec{\mu}. \tag{4.95}$$

The linear matrix $\mathcal{L}$ is evaluated at the fixed point and is thus a constant. The eigenvalues λ_i and the eigenvectors $\vec{\nu}_i$ of $\mathcal{L}$,

$$\mathcal{L}\vec{\nu}_i = \lambda_i\vec{\nu}_i, \tag{4.96}$$

determine the critical properties of the Hamiltonian.

The eigenvalues λ_i are functions of b. Two successive transformations with scale change b make a total scale change b^2, they are thus equivalent to a single transformation with the scale change b^2.

$$\lambda_i(b)\lambda_i(b) = \lambda_i(b^2) \tag{4.97}$$

and the eigenvalues must be of the form

$$\lambda_i(b) = b^{x_i}. \tag{4.98}$$

x_i are the critical exponents and, as we shall see, they are closely related to the usual critical exponents α, β, $\cdots$. So, if we know λ_i, we can obtain the critical exponents x_i from

$$x_i = \frac{\ln \lambda_i}{\ln b}. \tag{4.99}$$

- *Example 1:* For the $d = 1$ Ising model, $\mu_1 = x$ and $\mu_2 = y$. After linearization of the recursion relations (4.73)around the fixed point at $\vec{\mu}^* = (0, 1)$ we get:

  $$\delta x' = 4\delta x \qquad\qquad \delta y' = 2\delta y. \tag{4.100}$$

  Hence, $\mathcal{L}$ is diagonal in this representation. The eigenvalues and corresponding eigenvectors are:

  $$\begin{aligned} \lambda_1 &= 4 \qquad & \nu_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \lambda_2 &= 2 \qquad & \nu_1 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{4.101}$$

  It follows that the two critical exponents are both positive:

  $$x_1 = 2 \qquad\qquad x_2 = 1. \tag{4.102}$$

- *Example 2:* For the $d = 2$ Ising model, $\mu_1 = K$ and $\mu_2 = L$, and after linearization Eqs. (4.82) become

  $$\begin{aligned} \delta K' &= 4K^*\delta K + \delta L \\ \delta L' &= 2K^*\delta K. \end{aligned} \tag{4.103}$$

For the non-trivial fixed point at $(K^*, L^*) = (\frac{1}{3}, \frac{1}{9})$, the RG transformation matrix is:

$$\mathcal{L} = \begin{vmatrix} 4K^* & 1 \\ 2K^* & 0 \end{vmatrix} = \begin{vmatrix} 4/3 & 1 \\ 2/3 & 0 \end{vmatrix} \tag{4.104}$$

The eigenvalues of $\mathcal{L}$ are:

$$\lambda_1 = \frac{2+\sqrt{10}}{3} \approx 1.72, \qquad \lambda_2 = \frac{2-\sqrt{10}}{3} \approx -0.387 \tag{4.105}$$

and the eigenvectors:

$$\begin{aligned} \vec{\nu}_1 &\propto \begin{pmatrix} 1+\sqrt{10}/2 \\ 1 \end{pmatrix} \approx \begin{pmatrix} 2.58 \\ 1 \end{pmatrix} \\ \vec{\nu}_2 &\propto \begin{pmatrix} 1-\sqrt{10}/2 \\ 1 \end{pmatrix} \approx \begin{pmatrix} -0.58 \\ 1 \end{pmatrix} \end{aligned} \tag{4.106}$$

(Question: what does negative λ_2 mean physically, how does the system behave under successive RG transformations if λ is negative?) The corresponding critical exponents are:

$$\begin{aligned} x_1 &= 2\frac{\ln[(2+\sqrt{10})/3]}{\ln 2} \approx 1.57 \\ x_2 &= 2\frac{\ln|(2-\sqrt{10})/3|}{\ln 2} \approx -2.74, \end{aligned} \tag{4.107}$$

one is positive and the other one is negative.

Flow in the parameter space. We first express $\delta\vec{\mu}$ near the fixed point at $\vec{\mu}^*$ in terms of the eigenvectors $\vec{\nu}_i$ of $\mathcal{L}$. In case of the $d = 1$ Ising model this means that

$$\begin{aligned} \delta x' &= \lambda_1 \delta x = 4\delta x \\ \delta y' &= \lambda_2 \delta y = 2\delta y. \end{aligned} \tag{4.108}$$

In general we write $\delta\vec{\mu}$ as:

$$\delta\vec{\mu} = \sum_i h_i \vec{\nu}_i. \tag{4.109}$$

The coefficients h_i are the *linear scaling fields*. They are the components of the generalized field in the coordinate system of the eigenvectors. Under a renormalization

$$\delta\vec{\mu}' = \mathcal{L}\delta\vec{\mu} = \sum_i h_i \mathcal{L}\vec{\nu}_i = \sum_i h_i \lambda_i \vec{\nu}_i, \tag{4.110}$$

where $\lambda_i = b^{x_i}$. On the other hand:

$$\delta\vec{\mu}' = \sum_i h_i'\vec{\nu}_i, \tag{4.111}$$

therefore

$$h_i' = b^{x_i}h_i \qquad\qquad (b > 1). \tag{4.112}$$

It follows that the flow of the Hamiltonian in the parameter space close to the fixed point depends crucially on the exponents x_i. For positive x_i ($\lambda_i > 1$), the scaling filed h_i increases under repeated RG transformations and drives the system away from the fixed point. In this case, h_i is a *relevant scaling field*. If the exponent x_i is negative, the corresponding scaling field h_i decreases under subsequent RG transformations, thus driving the system towards the fixed point. Such fields are called *irrelevant scaling fields*.

Thus the stability of the fixed point and the topology of the flow lines around it depend on the number of relevant and irrelevant scaling fields associated with it. To illustrate this, Fig. 4.18 shows the the trajectories in the parameter space with two irrelevant and one relevant scaling field.

In general, any system which has initially the component of the relevant field different from zero, will after repeated RG transformations eventually be driven away from the fixed point (dotted lines in Fig. 4.18). Physically this corresponds to the system moving away from criticality.

The points in which the relevant field vanishes, define the *critical surface*. All the points on the critical surface flow *into* the fixed point. The points on the critical surface represent systems with an infinite correlation length. Under RG transformations, the systems stays on the critical surface (ξ remains infinite), but it moves towards the fixed point.

- *Example:* The $d = 1$ Ising model has $h_1 = x$ and $h_2 = y$. For the non-trivial fixed point at $x^* = 0$ and $y^* = 1$ the critical exponents are $x_1 = 2 > 0$ and $x_2 = 1 > 0$, that means that both, temperature and the external magnetic field, are relevant.

  In the case of the $d = 2$ Ising model on the square lattice, $x_1 > 0$ and $x_2 < 0$, as we have seen. It has one relevant scaling field (h_1) and one irrelevant field (h_2). The relevant scaling field is related to reduced temperature whereas the irrelevant field is related to L, the coupling with NNN. Notice that here we considered the Ising model without external field. If the field

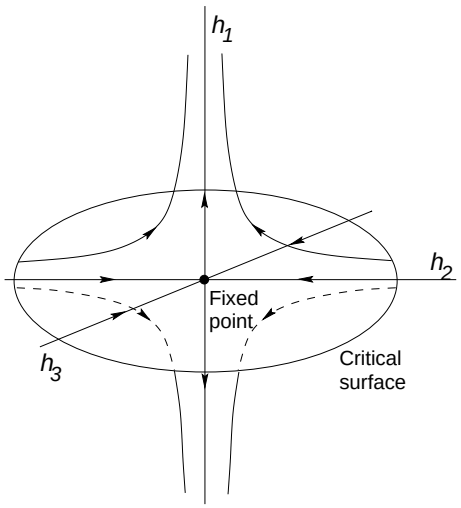

Figure 4.18: A fixed point with one relevant (h_1) and two irrelevant (h_2 and h_3) scaling fields. All the points on the two-dimensional critical surface flow to the fixed point (solid trajectories). The points slightly above (or below) the critical surface first flow towards the fixed point but eventually they are driven away from it as the relevant scaling field increases (dotted trajectories).

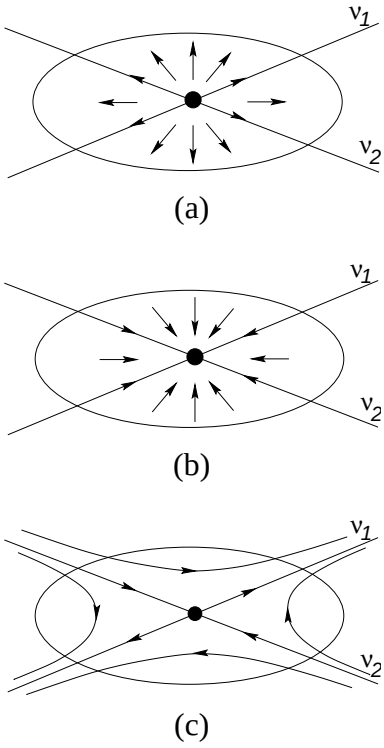

Figure 4.19: Topology of the RG flow near fixed points with: (a) two relevant fields, (b) two irrelevant fields, and (c) one relevant and one irrelevant field.

is included, then we have two relevant scaling fields and one irrelevant (the magnetic field is always relevant).

Figure 4.19 shows possible scenarios in the case of two scaling fields. Ferromagnets have two relevant fields, external field and temperature (or NN exchange interaction) and the fixed point is of the type shown in Fig. 4.19 (a).

Universality. The critical behaviour is completely determined by the behaviour of the system close to the fixed point, where the transformation matrix can be linearized. The transformation matrix depends on the values of the scaling fields at the critical point and not on the initial values of the irrelevant scaling fields. Thus, *all systems which flow close to the fixed point, independent of where in the parameter space they start, will exhibit the same critical exponents determined by the eigenvalues of the linear transformation matrix at that fixed point.* The critical

exponents are universal and do not depend on the initial point in the parameter space (like the coupling strength).

Critical exponent ν. To illustrate how the critical exponent ν is obtained from x_i, let us consider ferromagnetic systems. We know that there the critical point is specified by H and T. These are the relevant fields for magnetic systems. Therefore we identify h_1 by the reduced temperature, $h_1 = t$ and h_2 by the reduced magnetic field, $h_2 = h$. Under RG transformations ξ and t scale as

$$\xi' = \frac{\xi}{b} \qquad\qquad t' = b^{x_t} t \tag{4.113}$$

The two equations tell us that $\xi^{x_t} t$ is constant, it is invariant under RG transformations. Therefore

$$\xi(t) \propto t^{-1/x_t} \tag{4.114}$$

Since, by definition of ν, $\xi \propto t^{-\nu}$, we get

$$\boxed{\nu = \frac{1}{x_t}} \tag{4.115}$$

The last equation can be obtained also in a different way. Unless the system is initially at the critical point, the correlation length will decrease under successive block-spin RG transformations. The system will eventually reach a point where the correlation length ξ' will become equal to the size of the renormalized unit cell, so that the nearest-neighbour spins become uncorrelated. This point corresponds to a certain temperature t_0 which cannot depend on the initial temperature. Suppose, the system is initially at the temperature t, and arrives at t_0 after n iterations. Then,

$$t_0 = b^{n x_t} t \qquad and \qquad \xi' \equiv 1 = \xi / b^n \Rightarrow \xi = b^n \tag{4.116}$$

From these equations we get Eq. 4.115 again.

- We return again to the $d = 2$ Ising model. Above, we considered the model at $h = 0$. Thus, the exponent x_t corresponds to the relevant exponent $x_1 = 1.57$ from Eq. (4.107) and therefore $\nu = 1/x_1 = 0.64$. This is better than the mean-field value of $1/2$, but still far from the exact value $\nu = 1$. The discrepancy has its origin in the approximations we have made to obtain the RG recursion relations.

Other critical exponents. The original and renormalized Hamiltonians $\mathcal{H}$ and $\mathcal{H}'$ correspond to positions in the parameter space, $\vec{\mu}$ and $\vec{\mu}'$. We apply relation (4.86) to the *singular part* of the reduced free energy per spin:

$$g_s(\vec{\mu}') = b^d g_s(\vec{\mu}). \tag{4.117}$$

Near a fixed point $\vec{\mu}$ and $\vec{\mu}'$ can be expressed in terms of the scaling fields h_i and h_i' which are related by (4.112). Hence eq. (4.117) becomes:

$$g_s(h_1, h_2, ...) = b^{-d} g_s(b^{x_1} h_1, b^{x_2} h_2, ...), \tag{4.118}$$

where g_s is a homogeneous function of $\vec{\mu}$. This is the *scaling form* of the singular part of the free energy. This equation holds for arbitrary b because we can repeat transformations many times. In fact, b can be any real number > 1! Again we identify $h_1 = t$, $h_2 = h$ and assume that other scaling fields are irrelevant. Then, eq. (4.118) becomes

$$g_s(t, h) = b^{-d} g_s(b^{x_1} t, b^{x_2} h). \tag{4.119}$$

Now, to come to the specific-heat critical exponent α, we remind that

$$c_H = \left(\frac{\partial^2 g_s}{\partial t^2} \right)_{h=0} \equiv g_{tt}(h = 0) \propto |t|^{-\alpha}. \tag{4.120}$$

Differentiating Eq. (4.119) twice with respect to temperature and putting the scaling field h equal to zero gives:

$$g_{tt}(t, h = 0) \propto b^{-d} g_{\tau\tau}(\tau, 0) \frac{d\tau}{dt} = b^{-d+2x_1} g_{\tau\tau}, \tag{4.121}$$

where we have used $\tau = b^{x_1} t$. Initially, t is small whereas $b > 1$ so that the product $b^{x_1} t$ increases with each iteration. We repeat the iterations until

$$b^{x_1} |t| = 1. \tag{4.122}$$

In this way, the arguments of $g_{\tau\tau}$ and therefore also the function $g_{\tau\tau}$ are constant, all the temperature dependence of the specific heat is in the prefactor b^{-d+2x_1}:

$$g_{tt}(t, h = 0) \propto |t|^{(d-2x_1)/x_1} g_{tt}(\pm 1, 0) \tag{4.123}$$

and

$$\alpha = 2 - \frac{d}{x_1}. \tag{4.124}$$

Similarly we find

$$\begin{aligned} \beta &= \frac{d - x_2}{x_1} \\ \gamma &= \frac{2x_2 - d}{x_1} \\ \delta &= \frac{x_2}{d - x_2} \end{aligned} \tag{4.125}$$

We have expressed five critical exponents in terms of two independent exponents x_1 and x_2. That means that there must be relations between the exponents $\alpha \cdots \nu$. One can easily verify the scaling laws:

$$\begin{aligned} \alpha + 2\beta + \gamma &= 2 \\ \gamma &= \beta(\delta - 1) \\ 2 - \alpha &= d\nu. \end{aligned} \tag{4.126}$$

4.4 Momentum-Space Renormalization Group

RG methods in momentum space are much more powerful and more widely used than their counterparts in real space. The problem is only that they are much more complicated than the above real-space prototype examples. We shall define the RG transformation for the Landau model, that means that we shall now use continuous instead of discrete spin variables. The main reason is that *discrete spins become essentially continuous after (block-spin) transformations*.

We start with the free energy of the form:

$$\begin{aligned} \mathcal{H}(h, m(r)) = \int_V \mathrm{d}^d\vec{r} \, \Big[&\frac{1}{2} g |\nabla m(r)|^2 - m(r)h \\ &+ \frac{1}{2} r m^2(r) + u m^4(r) \Big] \end{aligned} \tag{4.127}$$

This is also called the Landau-Ginzburg-Wilson (LGW) Hamiltonian. The main procedure of all the RG transformations is "coarse graining"(i.e., transformation to larger unit cells) followed by rescaling (to make the system look like the original one). In real-space RG this was made either by introducing block spins or by decimation. Now we shall make coarse graining in *momentum space*. Instead

of *expanding* the size of the unit cell in real space, we shall *shrink* the size of the Brillouin zone in the momentum space. In the following the method of the momentum-space renormalization group will be demonstrated on the Gaussian model.

4.4.1 The Gaussian Model

The Hamiltonian of the Gaussian model in the momentum space is:

$$\mathcal{H} = \frac{1}{2} \sum_{|\vec{q}|<\Lambda} (r + gq^2)|m(q)|^2 - hm(0). \tag{4.128}$$

We assume a homogeneous magnetic field and we are only interested in the stability of the $m = 0$ solution ($r > 0$), therefore we take $u = 0$; this will simplify the summation over $\vec{q}$). Λ is the cutoff momentum (Brillouin zone boundary). Taking $u = 0$ means that the modes with different momenta become uncoupled.

The corresponding partition function is equal to the functional integral over al functions $m(r)$. In momentum space, the functional integral becomes:

$$\int \mathrm{D}m(r) \to \prod_{|q|<\Lambda} \int_0^\infty \mathrm{d}m(q) \tag{4.129}$$

and the partition function:

$$Y(t,h) = \prod_{|q|<\Lambda} \int_0^\infty \mathrm{d}m(q)\, \mathrm{e}^{-\frac{1}{2}\sum_{|q|<\Lambda}(r+gq^2)|m(q)|^2 - hm(0)}. \tag{4.130}$$

The RG transformation consists of "integrating out" the wavevectors that are outside the sphere of the radius Λ/b and then rescaling. One RG cycle consists of three steps:

1. **Integration.** After "integrating out" the momenta within a thin shell between the radii Λ/b and Λ, the partition function (which should not change) becomes:

$$Y = \mathrm{e}^{\Omega} \prod_{|q|<\Lambda/b} \int \mathrm{d}m(q)\, \mathrm{e}^{-\mathcal{H}'}, \tag{4.131}$$

   where the first term, $\exp(\Omega)$, is a function of Λ, b, and coupling constants. It comes from the integration over momenta in the shell between Λ/b and Λ.

The second term is the partition function of the states with momenta $|q| < \Lambda/b$. Notice that this separation into two contributions was only possible because we set $u = 0$, the partition function is then a product of independent Gaussian integrals. The new Hamiltonian,

$$\mathcal{H}'(m(q)) = \sum_{|q|<\Lambda/b} \frac{1}{2}(r + gq^2)|m(q)|^2 - hm(0), \tag{4.132}$$

depends only on $m(q)$ with $|q| < \Lambda/b$ and has the same structure as the original one.

2. **Rescaling.** The second step restores the old cutoff by blowing up the radius of integration to the original value Λ in such a way that we change the variable of integration to

$$\vec{q'} = b\vec{q}. \tag{4.133}$$

In this way the cutoff momentum is restored back to Λ. This step corresponds to putting $a' = ba$ in real-space RG. The Hamiltonian now reads:

$$\mathcal{H}'(m) = b^{-d} \sum_{|q'|<\Lambda} \frac{1}{2}(r + g\frac{q'^2}{b^2})|m(\frac{q'}{b})|^2 - hm(0). \tag{4.134}$$

3. **Normalization.** The step 3 restores the standard normalization of the order parameter. This step corresponds to putting the new spin equal to 1 after block-spin or decimation in real-space RG. Renormalization is restored with $g' = g$ if we set:

$$m'(q') = \sqrt{\frac{1}{b^{d+2}}} m(\frac{q'}{b}). \tag{4.135}$$

The transformed Hamiltonian and the partition function (which must not change) now look like

$$\mathcal{H}'(m) = \sum_{|q'|<\Lambda} \frac{1}{2}(r' + gq'^2)|m'(q')|^2 - h'm'(0). \tag{4.136}$$

and

$$Y = \mathrm{e}^{\Omega} \prod_{|q'|<\Lambda} \int \mathrm{d}m(q')\, \mathrm{e}^{-\beta\mathcal{H}'}. \tag{4.137}$$

In this way we integrated out large momenta and transformed $\mathcal{H}$ to its original form. The recursion relations are:

$$\begin{aligned} r' &= b^2 r \\ h' &= b^{(d+2)/2}\, h. \end{aligned} \tag{4.138}$$

Since $b \approx 1$ we can write $b = 1 + \mathrm{d}\tau$, thus,

$$b^n \approx 1 + n \ln b \equiv 1 + n\mathrm{d}\tau. \tag{4.139}$$

We can say that τ counts the number of RG cycles. Then, r and h obey the following differential equations:

$$\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}\tau} &= 2r \\ \frac{\mathrm{d}h}{\mathrm{d}\tau} &= \frac{d+2}{2} h \end{aligned} \tag{4.140}$$

with the solution

$$\begin{aligned} r &= r_0 \mathrm{e}^{x_t \tau} \qquad & x_t &= 2 \\ h &= h_0 \mathrm{e}^{x_h \tau} \qquad & x_h &= \frac{d+2}{2}. \end{aligned} \tag{4.141}$$

Since q is continuous (for macroscopic systems), the RG transformations was carried out in infinitesimal steps ($b \approx 1$) and the problem was reduced to solving a set of differential equations.

The Gaussian model has two trivial fixed points at $h^* = 0$, one at $r^* = \infty$ (very high T, $m = 0$) and one at $r^* = -\infty$ (very low T, where $m \to \infty$ because we set $u = 0$!), and one nontrivial fixed point (the critical point) at $h^* = r^* = 0$. At the nontrivial fixed point the critical exponents are both positive and r and h are both relevant scaling fields. From x_i, we shall calculate other critical exponents in the Section 4.4.3.

As we shall see in the next Section, for $d < 4$ the Gaussian model is unstable with respect to the parameter u coming from the four-spin interaction in the Hamiltonian.

4.4.2 The Landau-Wilson Model

In the Gaussian model, we have neglected the fourth and higher-order terms in the Landau Hamiltonian. The next step is to include also the m^4 term. This will lead to non-MFA exponents, as we shall see. Here we shall bring only an outline of the RG procedure and discuss the results. We start with the LGW Hamiltonian in real space,

$$\mathcal{H}(m) = \int \mathrm{d}^d\vec{r}\ \left[\frac{1}{2}g|\nabla m(\vec{r})|^2 + \frac{1}{2}rm^2(\vec{r}) + um^4(\vec{r})\right] \tag{4.142}$$

where $u > 0$. We first integrate out an infinitesimally thin shell of the thickness δq in momentum space by taking $b \approx 1$. Then,

$$\delta q/\Lambda = \delta\Lambda/\Lambda = \delta \ln \Lambda = b - 1 \approx \ln b \tag{4.143}$$

and we write

$$m(\vec{r}) = \bar{m}(\vec{r}) + \delta m(\vec{r}) \tag{4.144}$$

where $\delta m(\vec{r})$ contains the Fourier components to be integrated out and $\bar{m}$ the rest. After some approximations we get the following renormalization-group (differential) equations for r and u [Huang, 1987]:

$$\begin{aligned} \frac{\mathrm{d}r(\tau)}{\mathrm{d}\tau} &= 2r + 12\Lambda^{d-2}u\left(1 - \frac{r}{\Lambda^2}\right) \\ \frac{\mathrm{d}u(\tau)}{\mathrm{d}\tau} &= (4-d)u - 36\Lambda^{d-4}u^2. \end{aligned} \tag{4.145}$$

To analyse these equations, it is convenient to introduce the dimensionless coupling constants:

$$x = \frac{r}{\Lambda^2}, \qquad y = \frac{u}{\Lambda^{4-d}} \tag{4.146}$$

then, the RG equations become:

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}\tau} &= 2x + 12y(1-x) \\ \frac{\mathrm{d}y}{\mathrm{d}\tau} &= \epsilon y - 36y^2 \end{aligned} \tag{4.147}$$

where

$$\epsilon \equiv 4 - d \tag{4.148}$$

We shall assume that ϵ is small. The system has two fixed points at finite x and y, fixed by $dx/d\tau = dy/d\tau = 0$:

$$\left.\begin{aligned} x^* &= 0 \\ y^* &= 0 \end{aligned}\right\} \quad \text{Gaussian fixed point}$$
$$\left.\begin{aligned} x^* &= -\epsilon/6 \\ y^* &= \epsilon/36 \end{aligned}\right\} \quad \text{“Nontrivial” fixed point.} \tag{4.149}$$

The Gaussian fixed point is at $x = y = 0$ whereas the “nontrivial” (also called the “Wilson–Fisher”) fixed point lies in the upper $(y > 0)$ half of the (x, y) plane only if $\epsilon > 0$, $d < 4$. In the lower half of the (x, y) plane, u is negative and the system is unstable. The nontrivial fixed point approaches the Gaussian fixed point as $\epsilon \to 0$.

In the neighbourhood of the fixed points we linearize the RG equations

$$x = x^* + \delta x, \qquad y = y^* + \delta y \tag{4.150}$$

and get the following linearized recursion relations:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\delta x &= (2 - 12y^*)\delta x + 12(1 - x^*)\delta y \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\delta y &= (\epsilon - 72y^*)\delta y \end{aligned} \tag{4.151}$$

which are written in the general form as:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix} = \mathcal{L}\begin{pmatrix}\delta x \\ \delta y\end{pmatrix} \tag{4.152}$$

At the Gaussian fixed point the linear matrix $\mathcal{L}$ is:

$$\mathcal{L} = \begin{vmatrix} 2 & 12 \\ 0 & \epsilon \end{vmatrix} \tag{4.153}$$

The matrix $\mathcal{L}$ has the following eigenvalues and eigenvectors:

$$\lambda_1 = 2 \qquad \vec{\nu}_1 = \begin{pmatrix}1 \\ 0\end{pmatrix}$$

$$\lambda_2 \;=\; \epsilon \qquad\qquad \vec{\nu_2} \propto \begin{pmatrix} 1 \\ -(2-\epsilon)/12 \end{pmatrix} \tag{4.154}$$

The corresponding scaling fields are h_1 and h_2. Under the (infinitesimal) RG transformations the scaling fields change as $dh_i/d\tau = \lambda_i h_i$,

$$h_i(\tau) = h_{i0}\mathrm{e}^{\lambda_i \tau} = h_{i0} b^{\lambda_i} \tag{4.155}$$

Thus, the critical exponents are equal to the eigenvalues λ_i: $x_t = \lambda_1 = 2$ and $x_2 = \lambda_2 = \epsilon$

In the neighbourhood of the nontrivial fixed point the linear transformation matrix $\mathcal{L}$ is:

$$\mathcal{L} = \begin{vmatrix} 2(1-6y^*) & 12(1-x^*) \\ 0 & \epsilon - 72y^* \end{vmatrix} = \begin{vmatrix} 2(1-\epsilon/6) & 12(1+\epsilon/6) \\ 0 & -\epsilon \end{vmatrix} \tag{4.156}$$

It has the following eigenvalues and eigenvectors:

$$\begin{aligned} \lambda_1 \;&=\; 2-\epsilon/3 \qquad & \nu_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \lambda_2 \;&=\; -\epsilon \qquad & \nu_2 &= \begin{pmatrix} -\frac{6+\epsilon}{1+\epsilon/3} \\ 1 \end{pmatrix} \end{aligned} \tag{4.157}$$

Now we have the following picture. For $d < 4$ ($\epsilon > 0$) the nontrivial fixed point lies in the upper half of the (x, y) plane whereas the Gaussian fixed point is (as before) at $x = y = 0$, see Fig. 4.20. For the nontrivial fixed point, h_2 is irrelevant (it does not affect the critical exponents) and $h_1 = r \propto t$ the relevant scaling field whereas for the Gaussian fixed point, both fields are relevant. Thus, the line ν_2 is the critical line. For $d > 4$ ($\epsilon < 0$)the nontrivial fixed point lies in the lower half of the (x, y) plane, i.e., in the unphysical region ($u < 0$), see Fig. 4.21, and the critical behaviour is governed by the Gaussian fixed point. $d = 4$ is the special case when both fixed points coincide. The linearized recursion relations give $x_2 = 0$, h_2 is marginal, and x_2 cannot tell us the direction of RG flow. In this case we must include higher–order terms which were neglected when we linearized the transformation matrix. These terms tell us that the scaling field h_2 is irrelevant for $u > 0$ at $d = 4$.

4.4.3 Critical Exponents

We have seen that the critical exponents of the nontrivial fixed point are

$$x_t \;=\; 2-\epsilon/3$$

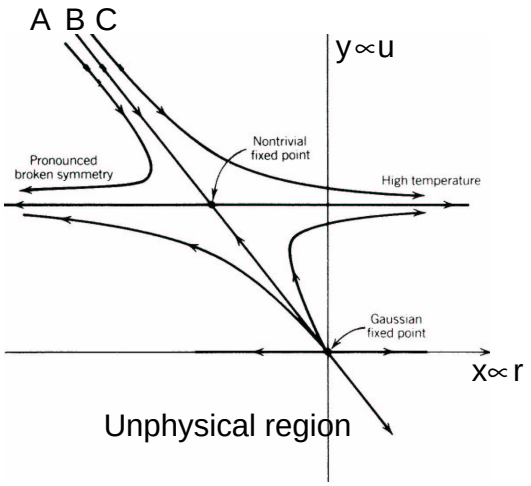

Figure 4.20: Fixed points and flow diagram of the Landau-Ginzburg-Wilson model for $d < 4$. The Gaussian fixed point at $(x^*, y^*) = (0, 0)$ has the mean-field exponents. The non-trivial fixed point at $(x^*, y^*) = (-\epsilon/6, \epsilon/36)$ is unstable against the relevant perturbation $\delta x \sim t$ and stable against the irrelevant perturbation δy. The system at A has $t < 0$ and it flows towards the low-temperature fixed point, r becomes increasingly negative. At the point C the system has $t > 0$ and it flows to the high-temperature fixed point, r becomes increasingly positive. The point B is on the critical line and the system flows towards the non-trivial fixed point. (After K. Huang, Statistical Mechanics)

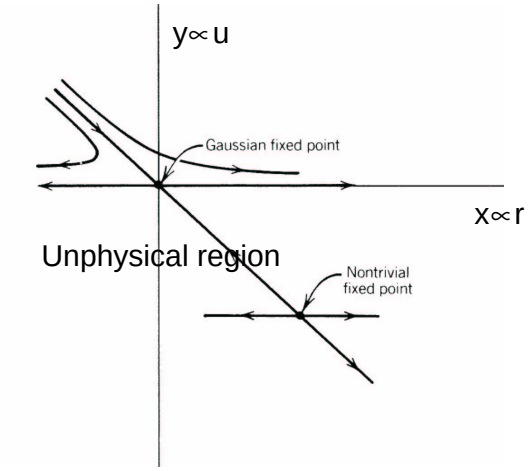

Figure 4.21: Fixed points and flow diagram of the Landau-Ginzburg-Wilson model for $d > 4$. The nontrivial fixed point is in the unphysical region. To stabilize the system in the lower half of the (x, y) plane, one needs an m^6 term. (After K. Huang, Statistical Mechanics)

$$\begin{aligned} x_u &= -\epsilon \\ x_h &= 1 + d/2. \end{aligned} \tag{4.158}$$

In fact, we calculated x_h only for the Gaussian fixed point in the Eq. (4.138). However, the term with m^4 does not affect the value of x_h so that x_h of the nontrivial fixed point is the same. With these values of x_i we get from Eqs. (4.115, 4.124, 4.125) to the first order in ϵ for $d < 4$:

$$\begin{aligned} \alpha &= \frac{\epsilon}{6} \\ \beta &= \frac{1}{2} - \frac{\epsilon}{6} \\ \gamma &= 1 + \frac{\epsilon}{6} \\ \delta &= 3 + \epsilon \\ \nu &= \frac{1}{2} + \frac{\epsilon}{12} \\ \eta &= 0. \end{aligned} \tag{4.159}$$

For $d \geq 4$ ($\epsilon \leq 0$) we only have the Gaussian fixed point in the physical region ($u > 0$) with the exponents

$$\begin{aligned} x_t &= 2 \\ x_u &= \epsilon \\ x_h &= 1 + d/2. \end{aligned} \tag{4.160}$$

With Eqs. (4.115, 4.124, 4.125) we find:

$$\begin{aligned} \alpha &= 2 - \frac{d}{2} \\ (\beta &= \frac{1}{2} - \frac{\epsilon}{4}) \\ \gamma &= 1 \\ (\delta &= 3 + \epsilon) \\ \nu &= \frac{1}{2} \\ \eta &= 0. \end{aligned} \tag{4.161}$$

The above expressions for β and δ are wrong, because we have neglected the m^4 term in considering the Gaussian fixed point. Below T_C, r is negative and for m to be finite, the term um^4 with $u > 0$ must be included in the Hamiltonian. Although u is an irrelevant field, it helps to stabilize m at a finite value and in this way influences the critical behaviour. By treating the um^4 term in the Gaussian approximation (independent modes) we get the usual mean-field values for the exponents β and η:

$$\begin{aligned} \beta &= \frac{1}{2} \\ \delta &= 3 \end{aligned} \tag{4.162}$$

for all $d \geq 4$. These results are correct and also agree with the results of the Landau (MFA) model in Sec. 4.1.4.

The Gaussian model gives the same critical exponents as the mean-field theory except for the specific-heat exponent α. Therefore the Gaussian fixed point is identified as the mean-field fixed point. In the Gaussian model the specific heat

diverges ($\alpha > 0$) for $d < 4$. This divergence is caused by the long-wavelength fluctuations which were neglected in the calculation of the free energy in the MF approximation.

In $d = 3$ ($\epsilon = 1$) we find $\alpha \approx 0.17$ and $\nu \approx 0.58$. The results of this paragraph are based on an expansion of d around $d = 4$ and are valid for small $\epsilon > 0$ or for $d \geq 4$. Nevertheless, we put $\epsilon = 1$ when $d = 3$. That means that ϵ is not small and higher-order terms in ϵ must be included. This is then the $\epsilon-$expansion, which gives good values for the critical exponents.