

# 1 Indices for measuring network balance

Let  $G = (V, E, \sigma)$  be an undirected signed network, where  $V$  and  $E$  are the set of verices and edges, and  $\sigma$  is the sign function  $\sigma : E \rightarrow \{-1, +1\}$ . The *signed adjacency matrix* and *unsigned adjacency matrix* are defined as

$$A_{uv} = \quad (1)$$

$$|A|_{uv} = \quad (2)$$

The sign of a cycle is the product of the sign of its edges. A cycle is *balanced* if its sign is positive, and *unbalanced* if its sign is negative. The total number of positive (negative) cycles is denoted by  $O^+$  ( $O^-$ ). Similarly, the total number of positive (negative) walks is denoted by  $Q^+$  ( $Q^-$ ). Thus, the total number of cycles (walks) in the graph is  $O = O^+ + O^-$  ( $Q = Q^+ + Q^-$ ). Finally, the signed Laplacian of the graph is defined as  $L = D - A$ , where  $D_{ij} = \sum_j |a_{ij}|$  is the diagonal matrix of degrees.

There are several ways of measuring balance in signed networks (for a review on the topic, we refer the reader to [1]).

The triangle index  $T(G)$  is defined as

$$T(G) = \frac{O_3^+}{O_3} = \frac{Tr(A^3) + Tr(|A^3|)}{2Tr(|A^3|)}. \quad (3)$$

The walk-based index of balance is defined as

$$W(G) = \frac{K(G) + 1}{2}, \quad K(G) = \frac{\sum_k \frac{Q_k^+ - Q_k^-}{K!}}{\sum_k \frac{Q_k^+ + Q_k^-}{K!}} = \frac{Tr(e^A)}{Tr(e^{|A|})}. \quad (4)$$

The algebraic conflict  $\lambda(G)$  is defined as the smallest eigenvalue of the sign Laplacian matrix. This eigenvalue equals zero if and only if the networks is balanced. The normalized algebraic conflict is defined as

$$A(G) = 1 - \frac{\lambda(G)}{\bar{d}_{\max} - 1}, \quad \bar{d}_{\max} = \max_{(u,v) \in E} (d_u + d_v)/2, \quad (5)$$

and is bounded between 0 and 1 (see [2]).

## 2 Simulations

We built  $N_s = 5000$  interaction matrices with  $N_a = 10$  agents.

Let  $C_a^{(n)}$  be the complexity of the time series asociated with agent  $a$  for the simulation performed using the interaction matrix  $n$ . For each simulation, we consider as representative values the mean and maximum complexity

$$\bar{C}^{(n)} = \frac{1}{N_a} \sum_a C_a^{(n)} \quad (6)$$

$$C_{\max}^{(n)} = \max_a C_a^{(n)}. \quad (7)$$

## 3 Motives distributions

We study here the relationship between the time series complexities and motif distributions.

Starting with interaction between three agents, the following four different motives are possible:

$M_1$  = three competitive links  
 $M_2$  = two competitive links and one cooperative link  
 $M_3$  = one competitive links and two cooperative link  
 $M_4$  = three cooperative links

In Figure 1 we show the distribution of such motives for the 5000 generated interaction matrices.

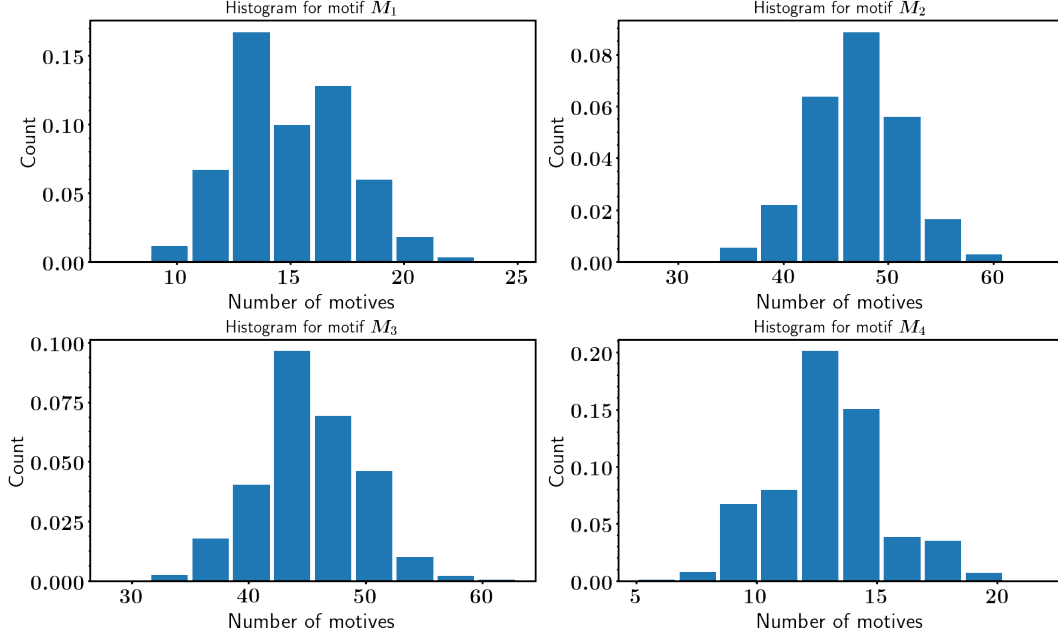


Figure 1: Histogram for the four different motives associated with 3-agent interactions.

For a full interaction matrix with  $N_a$  agents and  $E^+$  ( $E^-$ ) cooperative (competitive) interaction, the following relationships between the four 3-agent motives hold:

$$\sum_{i=1}^4 M_i = \binom{N}{3} = \frac{N(N-1)(N-2)}{6}$$

$$3M_1 + M_2 - M_3 - M_4 = 8(E^- - E^+)$$

## 4 Correlation between balance and complexity

We try to determine if there is a dependence of the time series complexities on the balance of their corresponding interaction matrices.

In Figure 2 (left), we show the histogram for the number of matrices with a given value of the triangle index  $T$ . [TODO: prove that the histogram can be expressed in terms of the Binomial distribution with parameters  $q = 0.5$  and  $N = \text{number of cycles}$ ]. For each value of  $T$ , we plot in the center and right panel of the figure, the maximum and the mean complexity, averaged over all the matrices having the same  $T$ . It can be observed that there is not clear dependence of the complexity on this balance index.

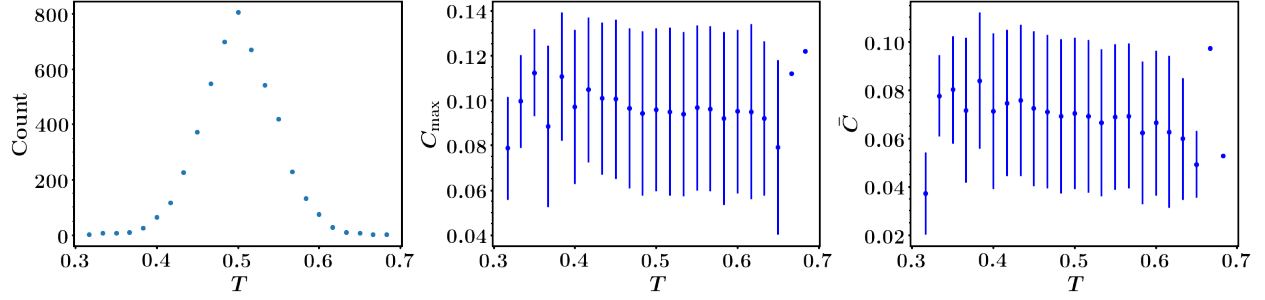


Figure 2: (Left) Histogram for the triangle index distribution in the 5000 interaction matrices generated. Values for the max (center) and mean (right) complexities as a function of the triangle index. Dots and bars represent the mean and standard deviation computed over all the matrices with the same value of  $T$

In Figure 3 we show the maximum and mean complexity plotted against the normalized algebraic conflict. No clear correlation can be seen for this balance measure.

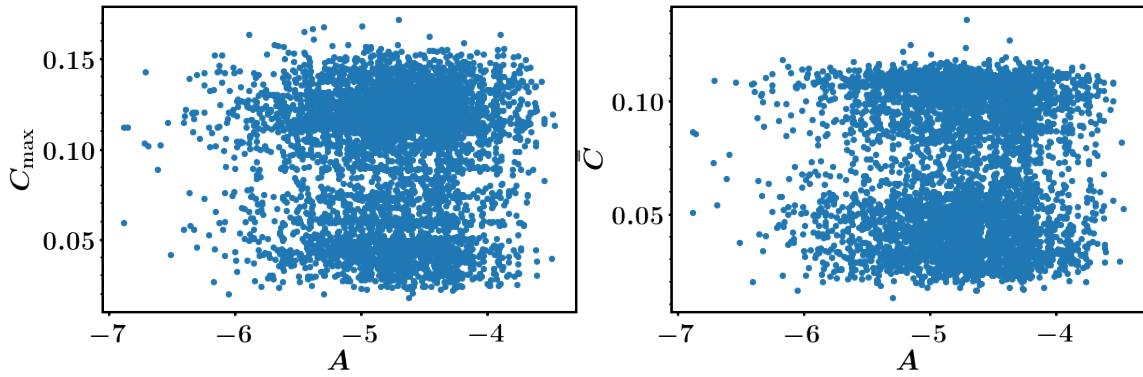


Figure 3: Max (left) and mean (right) complexities of each realization plotted against the normalized algebraic conflict.

In Figure 4 we show the maximum and mean complexity plotted against the walk-based index of balance. Again, there is not a good correlation using this measure.

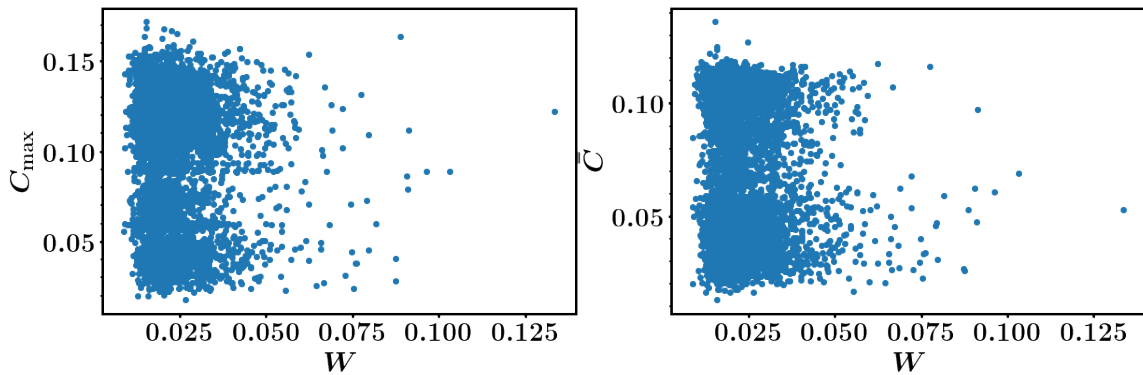


Figure 4: Max (left) and mean (right) complexities of each realization plotted against the walk-based measure of balance.

## References

- [1] S. Aref and M. C. Wilson, “Measuring partial balance in signed networks,” *Journal of Complex Networks*, vol. 6, pp. 566–595, Sept. 2017.
- [2] F. Belardo, “Balancedness and the least eigenvalue of laplacian of signed graphs,” *Linear Algebra and its Applications*, vol. 446, pp. 133–147, Apr. 2014.