18/11

SEAN $f_n : E \longrightarrow \overline{\mathbb{R}}$ MEDIBLES,

$$f_n(x) \longrightarrow f(x) \quad \forall x \in E$$

¿CUÁNDO PODEMOS AFIRMAR QUE

$$\lim_{n\to\infty} \int_E f_n = \int_E f \quad ?$$

TEO: (CONVERGENCIA DOMINADA)

SUP $\exists\, \phi : E \to \overline{\mathbb{R}}$, INTEGRABLE /

$|f_n(x)| \leq \phi(x) \quad \forall x \implies$

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

EJ: $I = [0,1]$, $f : I \longrightarrow \overline{\mathbb{R}}$ INTEGRABLE

$$\implies \int_I x^n f(x)\,dx \xrightarrow{n\to\infty} 0$$

DEFINO $f_m(x) = x^m f(x)$

- $|f_m(x)| = |x^m f(x)| = \underbrace{|x^m|}_{\leq 1} |f(x)| \leq |f(x)|$

$|f|$ ES INT.

- $\lim_{m \to \infty} x^m f(x) = ?$

$\rightarrow$ SI $x=1$, $\lim_{m\to\infty} 1^m f(1) = f(1)$ NO SABEMOS SI ES 0

$\rightarrow$ SI $x<1$ Y $f(x) \in \mathbb{R}$, $\lim_{m\to\infty} \underset{\to 0}{\underset{<1}{x}^m} \cdot \underbrace{f(x)}_{\in \mathbb{R}} = 0$

$\rightarrow$ QUEDA $f(x) = \infty$, PERO f INTEGRABLE $\Rightarrow \mu(\{f(x)=\infty\}) = 0$

LUEGO $f_m(x) \xrightarrow{m\to\infty} 0$ CTP.

LUEGO POR TEO DE CONV. DOM.

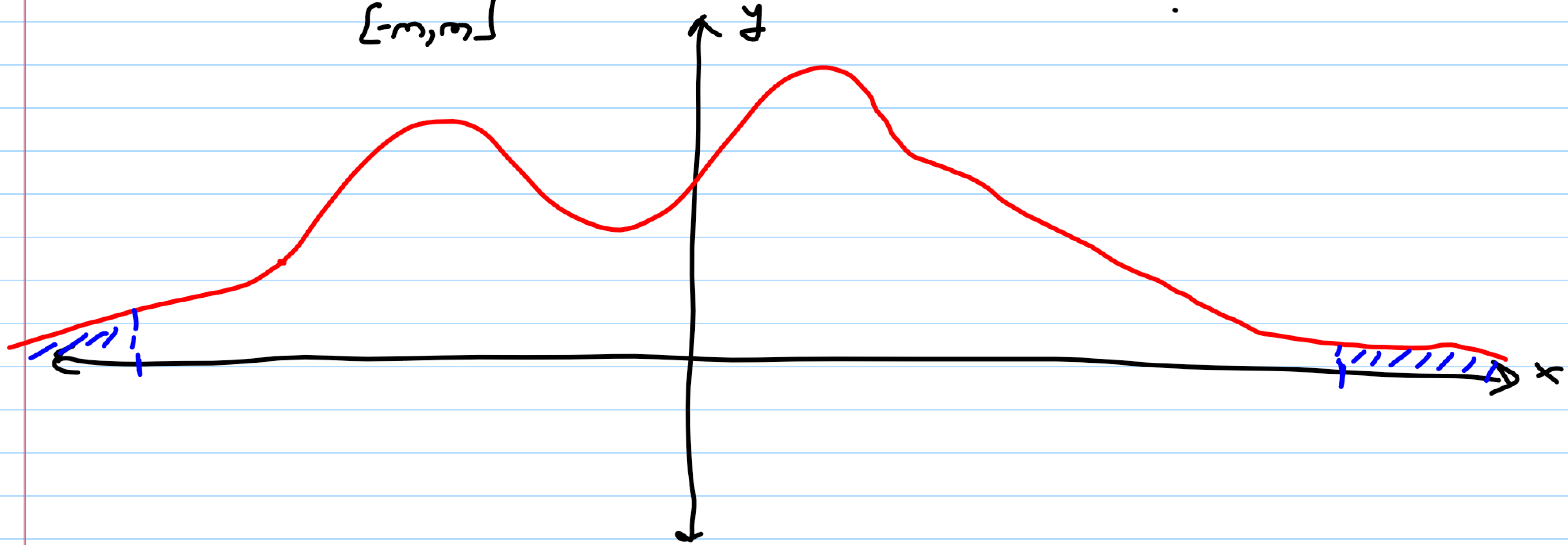
$$\lim_{m\to\infty} \int_I f_m(x)\,dx = \int_I 0 = 0$$

$$\|$$

$$\lim_{m\to\infty} \int_I x^m f(x)\,dx$$

EJ: f INTEGRABLE EN $\mathbb{R} \Rightarrow$

$$\lim_{n \to \infty} \int_{[-n,n]^c} f = 0$$



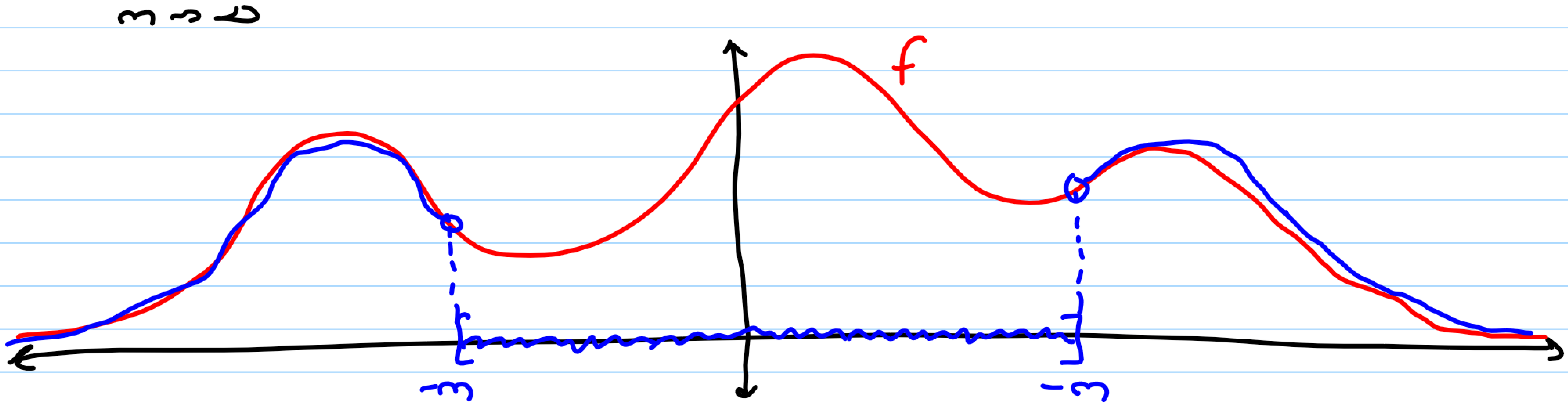
QUIERO DEFINIR f_n PARA QUE $\int_{\mathbb{R}} f_n = \int_{[-n,n]^c} f$

Y APLICAR CONV. DOM.

NOTEMOS QUE $\int_{[-n,n]^c} f = \int_{\mathbb{R}} f \cdot \chi_{[-n,n]^c}$

INSPIRADO POR ESTO, DEFINO $f_n(x) = f(x) \cdot \chi_{[-n,n]^c}$

- $|f_n(x)| = |f(x)| |\chi_{[-n,n]^c}| \leq |f(x)|$ INTEGRABLE
- $\lim_{n \to \infty} f_n(x) = 0$



(*) PARA VER ESTO, FIJO $x \in \mathbb{R}$ Y NOTO QUE
SI $m > |x| \Rightarrow f_m(x) = 0$

POR TCD,

$$\lim_{m \to \infty} \int_{\mathbb{R}} f_m(x)\,dx = \int_{\mathbb{R}} 0 = 0$$

$$\parallel$$

$$\lim_{m \to \infty} \int_{[-m,m]^c} f(x)\,dx$$

EJ: DERIVACIÓN BAJO EL SIGNO INTEGRAL

$f : I \times J \longrightarrow \mathbb{R}$

- $\forall x, \quad f(x, \_) : J \longrightarrow \mathbb{R}$ ES INTEGRABLE
$$y \longmapsto f(x,y)$$

- $\forall x, y \quad \exists \frac{\partial f}{\partial x}(x,y)$ Y ES ACOT, ie $\exists M > 0 /$
$\forall x, y \quad \left|\frac{\partial f}{\partial x}(x,y)\right| \leq M$

ENTONCES SI DEF: $\frac{\partial f}{\partial x}(x,\_) : I \longrightarrow \mathbb{R}$

$$y \longmapsto \frac{\partial f}{\partial x}(x,y)$$

- $\forall x,\ \frac{\partial f}{\partial x}(x,\_)$ ES MEDIBLE
- $\forall x,\ \frac{\partial}{\partial x}\left[\int_I f(x,y)\,dy\right] = \int_I \frac{\partial f}{\partial x}(x,y)\,dy$

RESOLUC:

RECORDEMOS. $\frac{\partial f}{\partial x}(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$

FIJO $x\in I$. TOMAMOS ALGUNA SUC. $(h_n)_n \,/\, h_n \xrightarrow{n} 0$

$$\frac{\partial f}{\partial x}(x,\_) = \lim_{n\to\infty} \underbrace{\frac{f(x+h_n,\_)-f(x,\_)}{h_n}}_{g_n(y)}$$

$g_n(y)$ ES MEDIBLE $\forall n. \Rightarrow \frac{\partial f}{\partial x}(x,\_)$ ES MEDIBLE

POR SER LÍMITE DE MEDIBLES.

$$\frac{\partial f}{\partial x}(x,y) = \lim_{n\to\infty} g_n(y)$$

ξ ESTÁ ENTRE x y $x+h_n$

$$|g_n(y)| = \left|\frac{f(x+h_n,y)-f(x,y)}{h_n}\right| \overset{TVM}{=} \left|\frac{\frac{\partial f}{\partial x}(\xi,y)\cdot \cancel{h_n}}{\cancel{h_n}}\right|$$

$\leq M \to$ ES INTEGRABLE EN I

LUEGO POR EL TCD

$$\int_I \frac{\partial f}{\partial x}(x,y)\,dy = \int_I \lim_{n\to\infty} g_n(y)\,dy \overset{TCD}{=}$$

$$\lim_{n\to\infty}\int_I g_n(y)\,dy = \lim_{n\to\infty}\int_I \frac{f(x+h_n,y)-f(x,y)}{h_n}\,dy$$

$$\overbrace{\lim_{n\to\infty}\frac{\int_I f(x+h_n,y)\,dy - \int_I f(x,y)\,dy}{h_n}}^{\circledast} =$$

VALE PORQUE $\circledast$ VALE $\forall (h_n)_n$ TAL QUE $h_n \xrightarrow{n} 0$

$$\lim_{h\to 0}\frac{\int_I f(x+h,y)\,dy - \int_I f(x,y)\,dy}{h} = \frac{\partial}{\partial x}\int_I f(x,y)\,dy$$