To show that the solution curves are ellipses, we need to find an "elliptical" relationship that $x(t)$ and $y(t)$ satisfy. In this case, it turns out that

$$[x(t)]^2 - 6x(t)\,y(t) + 10[y(t)]^2 = 10(k_1^2 + k_2^2).$$

In particular, the value of $x^2 - 6xy + 10y^2$ does not depend on t. It only depends on k_1 and k_2, which are, in turn, determined by the initial condition. It is an exercise in analytic geometry to show that the curves that satisfy

$$x^2 - 6xy + 10y^2 = K$$

are ellipses for any positive constant K.

You may wonder where $x^2 - 6xy + 10y^2$ comes from. See the technique for constructing Hamiltonian functions described in Section 5.3.

EXERCISES FOR SECTION 3.5

1. **(a)** The characteristic equation is

$$(-3 - \lambda)^2 = 0,$$

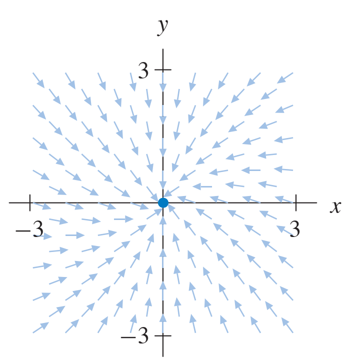
and the eigenvalue is $\lambda = -3$.

(b) To find an eigenvector, we solve the simultaneous equations

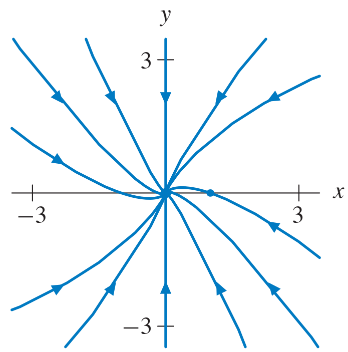
$$\begin{cases} -3x = -3x \\ x - 3y = -3y. \end{cases}$$

Then, $x = 0$, and one eigenvector is $(0, 1)$.

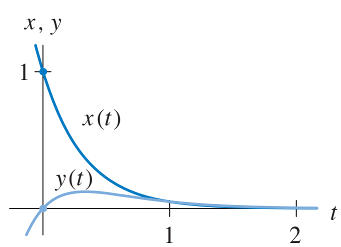
(c) Note the straight-line solutions along the y-axis.



(d) Since the eigenvalue is negative, any solution with an initial condition on the y-axis tends toward the origin as t increases. According to the direction field, every solution tends to the origin as t increases. The solutions with initial conditions in the half-plane $x > 0$ eventually approach the origin along the positive y-axis. Similarly, the solutions with initial conditions in the half-plane $x < 0$ eventually approach the origin along the negative y-axis.



(e) At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (-3, 1)$. Therefore, $x(t)$ decreases initially and $y(t)$ increases initially. The solution eventually approaches the origin tangent to the positive y-axis. Therefore, $x(t)$ monotonically decreases to zero and $y(t)$ eventually decreases toward zero. Since the solution with the initial condition $\mathbf{Y}_0$ never crosses y-axis in the phase plane, the function $x(t) > 0$ for all t.



2. **(a)** The characteristic polynomial is

$$(2 - \lambda)(4 - \lambda) + 1 = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2,$$

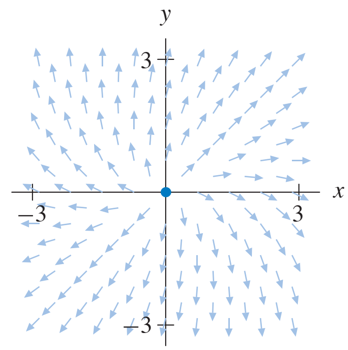
so there is only one eigenvalue, $\lambda = 3$.

(b) To find an eigenvector, we solve the equations

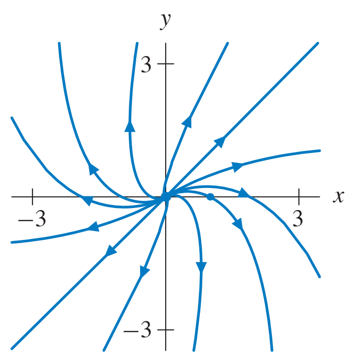
$$\begin{cases} 2x + y = 3x \\ -x + 4y = 3y. \end{cases}$$

Both equations simplify to $y = x$, so $(1, 1)$ is one eigenvector.

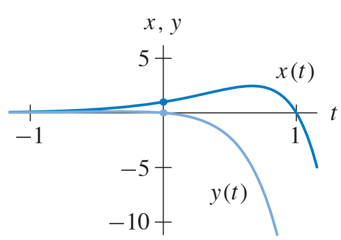
(c) Note the straight-line solutions along the line $y = x$.



(d) Since the sole eigenvalue is positive, all solutions except the equilibrium solution are unbounded as t increases. As $t \to -\infty$, the solutions with initial conditions in the half-plane $y > x$ tend to the origin tangent to the half-line $y = x$ with $y < 0$. Similarly, solutions with initial conditions in the half-plane $y < x$ tend to the origin tangent to the half-line $y = x$ with $y > 0$. Note the solution curve that goes through the initial condition $(1, 0)$.



(e) At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (2, -1)$. Hence, $x(t)$ is initially increasing, and $y(t)$ is initially decreasing.



3. **(a)** The characteristic equation is

$$(-2 - \lambda)(-4 - \lambda) + 1 = (\lambda + 3)^2 = 0,$$

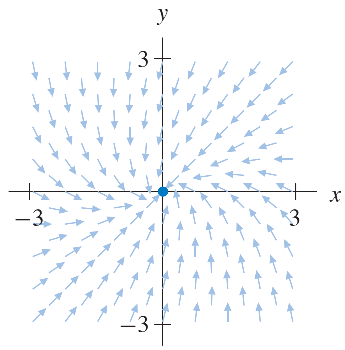
and the eigenvalue is $\lambda = -3$.

(b) To find an eigenvector, we solve the simultaneous equations

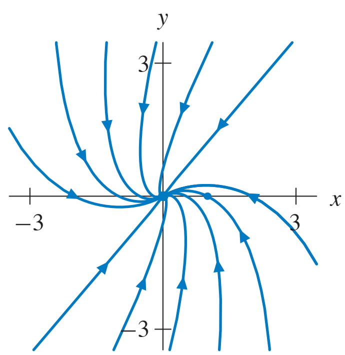
$$\begin{cases} -2x - y = -3x \\ x - 4y = -3y. \end{cases}$$

Then, $y = x$, and one eigenvector is $(1, 1)$.

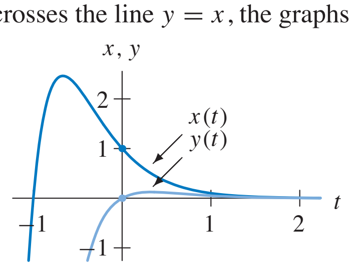
(c) Note the straight-line solutions along the line $y = x$.



(d) Since the eigenvalue is negative, any solution on the line $y = x$ tends toward the origin along $y = x$ as t increases. According to the direction field, every solution tends to the origin as t increases. The solutions with initial conditions that lie in the half-plane $y > x$ eventually approach the origin tangent to the half-line $y = x$ with $y < 0$. Similarly, the solutions with initial conditions that lie in the half-plane $y < x$ eventually approach the origin tangent to the line $y = x$ with $y > 0$.



(e) At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (-2, 1)$. Therefore, $x(t)$ initially decreases and $y(t)$ initially increases. The solution eventually approaches the origin tangent to the line $y = x$. Since the solution curve never crosses the line $y = x$, the graphs of $x(t)$ and $y(t)$ do not cross.



4. **(a)** The characteristic polynomial is

$$(-\lambda)(-2-\lambda)+1=\lambda^2+2\lambda+1=(\lambda+1)^2,$$

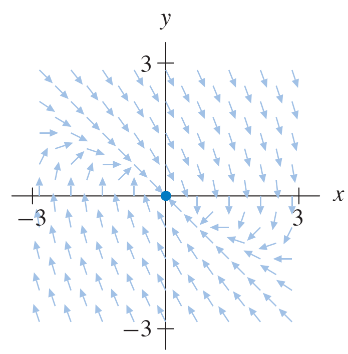
so there is only one eigenvalue, $\lambda=-1$.

(b) To find an eigenvalue we solve

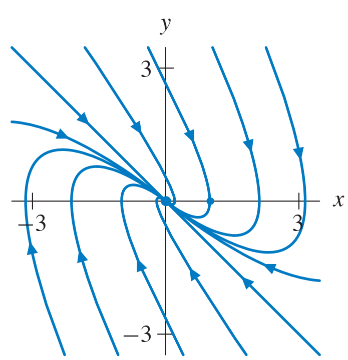
$$\begin{cases} y=-x \\ -x-2y=-y. \end{cases}$$

These equations both simplify to $y=-x$, so $(1,-1)$ is one eigenvector.

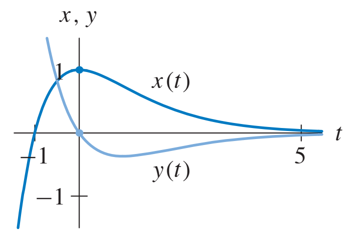
(c) Note the straight-line solutions along the line $y=-x$.



(d) Since the eigenvalue is negative, all solutions approach the origin as t increases. Solutions with initial conditions on the line $y=-x$ approach the origin along $y=-x$. Solutions with initial conditions that lie in the half-plane $y>-x$ approach the origin tangent to the half-line $y=-x$ with $y<0$. Solutions with initial conditions that lie in the half-plane $y<-x$ approach the origin tangent to the half-line $y=-x$ with $y>0$.



(e) At the point $\mathbf{Y}_0 = (1, 0)$, $d\mathbf{Y}/dt = (0, -1)$. Therefore, $x(t)$ assumes a maximum at $t = 0$ and then decreases toward 0. Also, $y(t)$ becomes negative. Then, it assumes a (negative) minimum, and finally it is asymptotic to 0 without crossing $y = 0$.



5. **(a)** According to Exercise 1, there is one eigenvalue, -3, with eigenvectors of the form $(0, y_0)$, where $y_0 \neq 0$.

To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\begin{aligned}\mathbf{V}_1 &= \left[\begin{pmatrix} -3 & 0 \\ 1 & -3 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{V}_0 \\ &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ x_0 \end{pmatrix}.\end{aligned}$$

We obtain the general solution

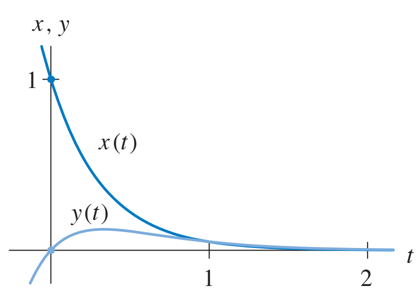
$$\mathbf{Y}(t) = e^{-3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-3t}\begin{pmatrix} 0 \\ x_0 \end{pmatrix}.$$

(b) The solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$ is

$$\mathbf{Y}(t) = e^{-3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Hence, $x(t) = e^{-3t}$ and $y(t) = te^{-3t}$.

(c) Compare the graphs of $x(t) = e^{-3t}$ and $y(t) = te^{-3t}$ with the sketches obtained in part (e) of Exercise 1.



6. **(a)** From Exercise 2, we know that there is only one eigenvalue, $\lambda = 3$, and the eigenvectors (x_0, y_0) satisfy the equation $y_0 = x_0$.

To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\begin{aligned}\mathbf{V}_1 &= \left[\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} - 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{V}_0 \\ &= \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &= \begin{pmatrix} y_0 - x_0 \\ y_0 - x_0 \end{pmatrix}.\end{aligned}$$

We obtain the general solution

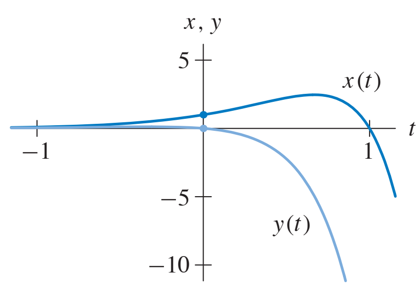
$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{3t}\begin{pmatrix} y_0 - x_0 \\ y_0 - x_0 \end{pmatrix}.$$

(b) The solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$ is

$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Hence, $x(t) = e^{3t}(1 - t)$ and $y(t) = -te^{3t}$.

(c) Compare the graphs of $x(t) = e^{3t}(1-t)$ and $y(t) = -te^{3t}$ with the sketches obtained in part (e) of Exercise 2.



7. **(a)** From Exercise 3, we know that there is only one eigenvalue, $\lambda = -3$, and the eigenvectors (x_0, y_0) satisfy the equation $y_0 = x_0$.

To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\begin{aligned}\mathbf{V}_1 &= \left[\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{V}_0 \\ &= \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}.$$

We obtain the general solution

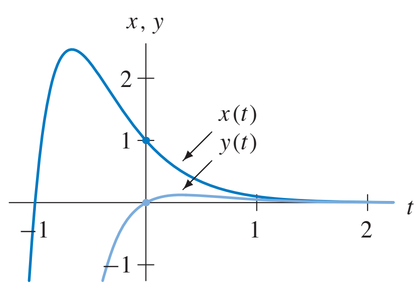
$$\mathbf{Y}(t) = e^{-3t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-3t} \begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}.$$

(b) The solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$ is

$$\mathbf{Y}(t) = e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, $x(t) = e^{-3t}(t + 1)$ and $y(t) = te^{-3t}$.

(c) Compare the graphs of $x(t) = e^{-3t}(t + 1)$ and $y(t) = te^{-3t}$ with the sketches obtained in part (e) of Exercise 3.



8. **(a)** From Exercise 4, we know that there is only one eigenvalue, $\lambda = -1$, and the eigenvectors (x_0, y_0) satisfy the equation $y_0 = -x_0$.

To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\mathbf{V}_1 = \left[\begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix} + 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \mathbf{V}_0$$

$$= \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= \begin{pmatrix} x_0 + y_0 \\ -x_0 - y_0 \end{pmatrix}.$$

We obtain the general solution

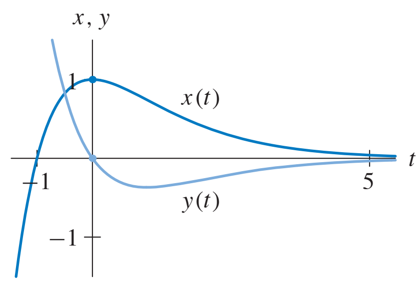
$$\mathbf{Y}(t) = e^{-t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-t} \begin{pmatrix} x_0 + y_0 \\ -x_0 - y_0 \end{pmatrix}.$$

(b) The solution that satisfies the initial condition $(x_0, y_0) = (1, 0)$ is

$$\mathbf{Y}(t) = e^{-t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Hence, $x(t) = e^{-t}(t+1)$ and $y(t) = -te^{-t}$.

(c) Compare the graphs of $x(t) = e^{-t}(t+1)$ and $y(t) = -te^{-t}$ with those obtained in part (e) of Exercise 4.



9. **(a)** By solving the quadratic equation, we obtain

$$\lambda = \frac{-\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}.$$

Therefore, for the quadratic to have a double root, we must have

$$\alpha^2 - 4\beta = 0.$$

(b) If zero is a root, we set $\lambda = 0$ in $\lambda^2 + \alpha\lambda + \beta = 0$, and we obtain $\beta = 0$.

10. **(a)** To compute the limit of $te^{\lambda t}$ as $t \to \infty$ if $\lambda > 0$, we note that both t and $e^{\lambda t}$ go to infinity as t goes to infinity. So $te^{\lambda t}$ blows up as t tends to infinity, and the limit does not exist.

(b) To compute the limit of $te^{\lambda t}$ as $t \to \infty$ if $\lambda < 0$, we write

$$\lim_{t\to\infty} te^{\lambda t} = \lim_{t\to\infty} \frac{t}{e^{-\lambda t}} = \lim_{t\to\infty} \frac{1}{-\lambda e^{-\lambda t}}$$

where the last equality follows from L'Hôpital's Rule. Because $e^{-\lambda t}$ tends to infinity as $t \to \infty$ $(-\lambda > 0)$, the fraction tends to 0.

11. The characteristic equation is

$$-\lambda(-p-\lambda) + q = \lambda^2 + p\lambda + q = 0.$$

Solving the quadratic equation, one obtains

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

(a) Therefore, in order for $\mathbf{A}$ to have two real eigenvalues, p and q must satisfy $p^2 - 4q > 0$.

(b) In order for $\mathbf{A}$ to have complex eigenvalues, p and q must satisfy $p^2 - 4q < 0$.

(c) In order for $\mathbf{A}$ to have only one eigenvalue, p and q must satisfy $p^2 - 4q = 0$.

12. The characteristic polynomial of $\mathbf{A}$ is

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \lambda^2 - (a+d)\lambda + (ad - bc) = \lambda^2 - \operatorname{tr}(\mathbf{A})\lambda + \det(\mathbf{A})$$

(see Section 3.2). A quadratic polynomial has only one root if and only if its discriminant is 0. In this case, the discriminant of $\det(\mathbf{A} - \lambda\mathbf{I})$ is $\operatorname{tr}(\mathbf{A})^2 - 4\det(\mathbf{A})$.

13. Since every vector is an eigenvector with eigenvalue λ, we substitute $\mathbf{Y} = (1, 0)$ into the equation $\mathbf{AY} = \lambda\mathbf{Y}$ and get

$$\mathbf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Hence, $a = \lambda$ and $c = 0$. Similarly, letting $\mathbf{Y} = (0, 1)$, we have

$$\begin{pmatrix} b \\ d \end{pmatrix} = \lambda\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, $b = 0$ and $d = \lambda$.

14. First note that, because $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are independent, any vector $\mathbf{Y}_3$ can be written as a linear combination of $\mathbf{Y}_1$ and $\mathbf{Y}_2$. In other words, there exists k_1 and k_2 such that

$$\mathbf{Y}_3 = k_1\mathbf{Y}_1 + k_1\mathbf{Y}_2.$$

But then

$$\begin{aligned}
\mathbf{AY}_3 &= \mathbf{A}(k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2) \\
&= k_1\mathbf{AY}_1 + k_2\mathbf{AY}_2 \\
&= k_1\lambda\mathbf{Y}_1 + k_2\lambda\mathbf{Y}_2 \\
&= \lambda(k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2) \\
&= \lambda\mathbf{Y}_3.
\end{aligned}$$

That is, any $\mathbf{Y}_3$ is an eigenvector with eigenvalue λ.

Now use the result of Exercise 13 to conclude that $a = d = \lambda$ and $b = c = 0$.

15. Since $\mathbf{Y}_1(0) = \mathbf{V}_0$ and $\mathbf{Y}_2(0) = \mathbf{W}_0$, we see that $\mathbf{V}_0 = \mathbf{W}_0$.

Evaluating at $t = 1$ yields

$$\mathbf{Y}_1(1) = e^{\lambda}(\mathbf{V}_0 + \mathbf{V}_1) \quad \text{and} \quad \mathbf{Y}_2(1) = e^{\lambda}(\mathbf{W}_0 + \mathbf{W}_1).$$

Since $\mathbf{Y}_1(1) = \mathbf{Y}_2(1)$ and $\mathbf{V}_0 = \mathbf{W}_0$, we see that $\mathbf{V}_1 = \mathbf{W}_1$.

16. **(a)** Suppose that

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By assumption, we know that the characteristic polynomial of $\mathbf{A}$ has λ_0 as a root of multiplicity two. That is,

$$\begin{aligned}
\lambda^2 - (a+d)\lambda + (ad - bc) &= (\lambda - \lambda_0)^2 \\
&= \lambda^2 - (2\lambda_0)\lambda + \lambda_0^2.
\end{aligned}$$

Therefore, $a + d = 2\lambda_0$, and $ad - bc = \lambda_0^2$.

Now we compute $(\mathbf{A} - \lambda_0\mathbf{I})^2$ using the definition of matrix multiplication. We have

$$\begin{aligned}(\mathbf{A} - \lambda_0\mathbf{I})^2 &= \begin{pmatrix} a - \lambda_0 & b \\ c & d - \lambda_0 \end{pmatrix}\begin{pmatrix} a - \lambda_0 & b \\ c & d - \lambda_0 \end{pmatrix} \\ &= \begin{pmatrix} (a - \lambda_0)^2 + bc & b(a + d - 2\lambda_0) \\ c(a + d - 2\lambda_0) & bc + (d - \lambda_0)^2 \end{pmatrix}.\end{aligned}$$

Since $a + d = 2\lambda_0$, we see that the bottom-left and top-right entries are zero.

Now consider the top-left entry $(a - \lambda_0)^2 + bc$. We have

$$\begin{aligned}(a - \lambda_0)^2 + bc &= a^2 - 2a\lambda_0 + \lambda_0^2 + bc \\ &= a^2 - 2a\lambda_0 + ad - bc + bc,\end{aligned}$$

because $ad - bc = \lambda_0^2$. The right-hand side simplifies to

$$a^2 - 2a\lambda_0 + ad = a(a - 2\lambda_0 + d) = 0$$

because $a + d = 2\lambda_0$.

A similar argument is used to show that the bottom-right entry is zero.

(b) If $\mathbf{V}_0$ is an eigenvector, then $\mathbf{V}_1 = (\mathbf{A} - \lambda_0\mathbf{I})\mathbf{V}_0$ is the zero vector. If not, we use the result of part (a) to compute

$$(\mathbf{A} - \lambda_0\mathbf{I})\mathbf{V}_1 = (\mathbf{A} - \lambda_0\mathbf{I})^2\mathbf{V}_0 = 0 \text{ (the zero vector)}.$$

Consequently, $\mathbf{V}_1$ is an eigenvector.

17. **(a)** The characteristic polynomial is

$$(-\lambda)(-1 - \lambda) + 0 = \lambda^2 + \lambda,$$

so the eigenvalues are $\lambda = 0$ and $\lambda = -1$.

(b) To find the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue $\lambda = 0$, we must solve $\mathbf{A}\mathbf{V}_1 = 0\mathbf{V}_1 = 0$ where $\mathbf{A}$ is the matrix that defines this linear system. (Note that this is the same calculation we do if we want to locate the equilibrium points.) We get

$$\begin{cases} 2y_1 = 0 \\ -y_1 = 0, \end{cases}$$

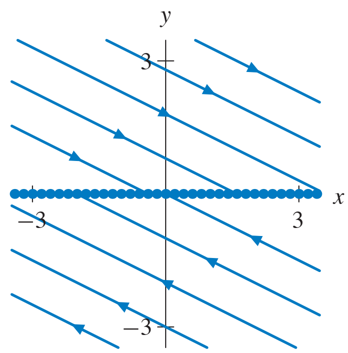
where $\mathbf{V}_1 = (x_1, y_1)$. Hence, the eigenvectors associated to $\lambda = 0$ (as well as the equilibrium points) must satisfy the equation $y_1 = 0$.

To find the eigenvectors $\mathbf{V}_2$ associated to the eigenvalue $\lambda = -1$, we must solve $\mathbf{A}\mathbf{V}_2 = -\mathbf{V}_2$. We get

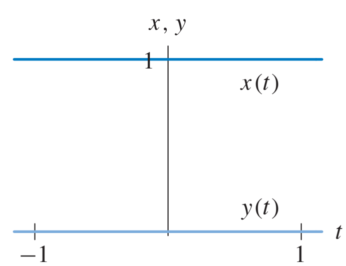
$$\begin{cases} 2y_2 = -x_2 \\ -y_2 = -y_2. \end{cases}$$

where $\mathbf{V}_2 = (x_2, y_2)$. Hence, the eigenvectors associated to $\lambda = -1$ must satisfy $2y_2 = -x_2$.

(c) The equation $y_1 = 0$ specifies a line of equilibrium points. Since the other eigenvalue is negative, solution curves not corresponding to equilibria move toward this line as t increases.



(d) Since $(1, 0)$ is an equilibrium point, it is easy to sketch the corresponding $x(t)$- and $y(t)$-graphs.



(e) To form the general solution, we must pick one eigenvector for each eigenvalue. Using part (b), we pick $\mathbf{V}_1 = (1, 0)$, and $\mathbf{V}_2 = (2, -1)$. We obtain the general solution

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

(f) To determine the solution whose initial condition is $(1, 0)$, we can substitute $t = 0$ in the general solution and solve for k_1 and k_2. However, since this initial condition is an equilibrium point, we need not make the effort. We simply observe that

$$\mathbf{Y}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

is the desired solution.

18. **(a)** The characteristic equation is

$$(2 - \lambda)(6 - \lambda) - 12 = \lambda^2 - 8\lambda = 0.$$

Therefore, the eigenvalues are $\lambda = 0$ and $\lambda = 8$.

(b) To find the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue $\lambda = 0$, we must solve $\mathbf{A}\mathbf{V}_1 = 0\mathbf{V}_1 = 0$ where $\mathbf{A}$ is the matrix that defines this linear system. (Note that this is the same calculation we do if we want to locate the equilibrium points.) We get

$$\begin{cases} 2x_1 + 4y_1 = 0 \\ 3x_1 + 6y_1 = 0, \end{cases}$$

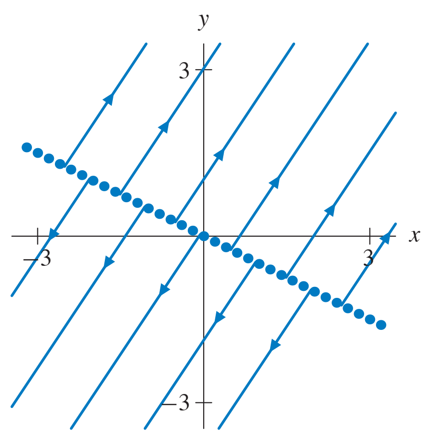
where $\mathbf{V}_1 = (x_1, y_1)$. Hence, the eigenvectors associated to $\lambda = 0$ (as well as the equilibrium points) must satisfy the equation $x_1 + 2y_1 = 0$.

To find the eigenvectors $\mathbf{V}_2$ associated to the eigenvalue $\lambda = 8$, we must solve $\mathbf{A}\mathbf{V}_2 = 8\mathbf{V}_2$. We get

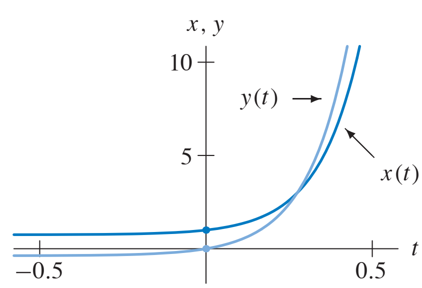
$$\begin{cases} 2x_2 + 4y_2 = 8x_2 \\ 3x_2 + 6y_2 = 8y_2, \end{cases}$$

where $\mathbf{V}_2 = (x_2, y_2)$. Hence, the eigenvectors associated to $\lambda = 8$ must satisfy $2y_2 = 3x_2$.

(c) The equation $x_1 + 2y_1 = 0$ specifies a line of equilibrium points. Since the other eigenvalue is positive, solution curves not corresponding to equilibria move away from this line as t increases.



(d) As t increases, both $x(t)$ and $y(t)$ increase exponentially. As t decreases, both x and y approach constants that are determined by the line of equilibrium points.



(e) To form the general solution, we must pick one eigenvector for each eigenvalue. Using part (b), we pick $\mathbf{V}_1 = (-2, 1)$, and $\mathbf{V}_2 = (2, 3)$. We obtain the general solution

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 e^{8t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

(f) To determine the solution whose initial condition is $(1, 0)$, we let $t = 0$ in the general solution and obtain the equations

$$k_1 \begin{pmatrix} -2 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Therefore, $k_1 = -3/8$ and $k_2 = 1/8$. The particular solution is

$$\mathbf{Y}(t) = \begin{pmatrix} \frac{3}{4} + \frac{1}{4}e^{8t} \\ -\frac{3}{8} + \frac{3}{8}e^{8t} \end{pmatrix}.$$

19. **(a)** The characteristic polynomial is

$$(4 - \lambda)(1 - \lambda) - 4 = \lambda^2 - 5\lambda,$$

so the eigenvalues are $\lambda = 0$ and $\lambda = 5$.

(b) To find the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue $\lambda = 0$, we must solve $\mathbf{A}\mathbf{V}_1 = 0\mathbf{V}_1 = 0$ where $\mathbf{A}$ is the matrix that defines this linear system. (Note that this is the same calculation we do if we want to locate the equilibrium points.) We get

$$\begin{cases} 4x_1 + 2y_1 = 0 \\ 2x_1 + y_1 = 0, \end{cases}$$

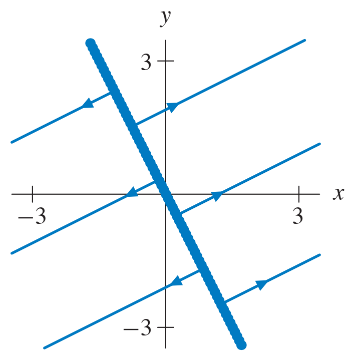
where $\mathbf{V}_1 = (x_1, y_1)$. Hence, the eigenvectors associated to $\lambda = 0$ (as well as the equilibrium points) must satisfy the equation $y_1 = -2x_1$.

To find the eigenvectors $\mathbf{V}_2$ associated to the eigenvalue $\lambda = 5$, we must solve $\mathbf{A}\mathbf{V}_2 = 5\mathbf{V}_2$. We get

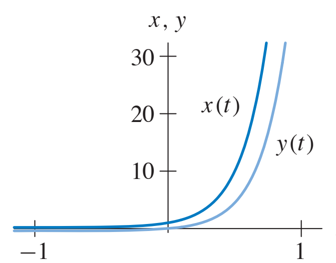
$$\begin{cases} 4x_2 + 2y_2 = 5x_2 \\ 2x_2 + y_2 = 5y_2. \end{cases}$$

where $\mathbf{V}_2 = (x_2, y_2)$. Hence, the eigenvectors associated to $\lambda = 5$ must satisfy $x_2 = 2y_2$.

(c) The equation $y_1 = -2x_1$ specifies a line of equilibrium points. Since the other eigenvalue is positive, solution curves not corresponding to equilibria move away from this line as t increases.



(d) As t increases, both $x(t)$ and $y(t)$ increase exponentially. As t decreases, both x and y approach constants that are determined by the line of equilibrium points.



(e) To form the general solution, we must pick one eigenvector for each eigenvalue. Using part (b), we pick $\mathbf{V}_1 = (1, -2)$, and $\mathbf{V}_2 = (2, 1)$. We obtain the general solution

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(f) To determine the solution whose initial condition is $(1, 0)$, we let $t = 0$ in the general solution and obtain the equations

$$k_1 \begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

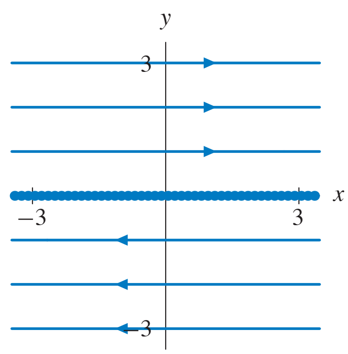
Therefore, $k_1 = 1/5$ and $k_2 = 2/5$, and the particular solution is

$$\mathbf{Y}(t) = \begin{pmatrix} \frac{1}{5} + \frac{4}{5}e^{5t} \\ -\frac{2}{5} + \frac{2}{5}e^{5t} \end{pmatrix}.$$

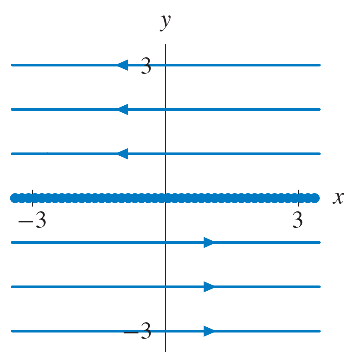
20. **(a)** The characteristic equation is $\lambda^2 - (a + d)\lambda + ad - bc = 0$. If 0 is an eigenvalue of $\mathbf{A}$, then 0 is a root of the characteristic polynomial. Thus, the constant term in the above equation must be 0—that is, $ad - bc = \det \mathbf{A} = 0$.

(b) If $\det \mathbf{A} = 0$, then the characteristic equation becomes $\lambda^2 - (a + d)\lambda = 0$, and this equation has 0 as a root. Therefore 0 is an eigenvalue of $\mathbf{A}$.

21. **(a)** The characteristic polynomial is $\lambda^2 = 0$, so $\lambda = 0$ is the sole eigenvalue. To sketch the phase portrait we note that $dy/dt = 0$, so $y(t)$ is always a constant function. Moreover, $dx/dt = 2y$, so $x(t)$ is increasing if $y > 0$, and it is decreasing if $y < 0$.



(b) This system is exactly the same as the one in part (a) except that the sign of dx/dt has changed. Hence, the phase portrait is the identical except for the fact that the arrows point the other way.



22. **(a)** This system has only one eigenvalue, $\lambda = 0$, and the eigenvectors lie along the x-axis (the line $y = 0$).

To find the general solution, we start with an arbitrary initial condition $\mathbf{V}_0 = (x_0, y_0)$. Then

$$\mathbf{V}_1 = \left[\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} - 0\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right]\mathbf{V}_0$$
$$= \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 2y_0 \\ 0 \end{pmatrix}.$$

We obtain the general solution

$$\mathbf{Y}(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t\begin{pmatrix} 2y_0 \\ 0 \end{pmatrix}.$$

(b) Following the procedure in part (a) we obtain

$$\mathbf{V}_1 = \begin{pmatrix} -2y_0 \\ 0 \end{pmatrix},$$

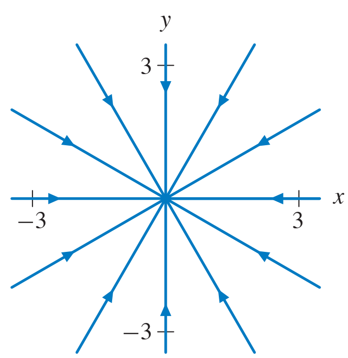
and consequently, the general solution is

$$\mathbf{Y}(t) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + t \begin{pmatrix} -2y_0 \\ 0 \end{pmatrix}.$$

23. **(a)** The characteristic polynomial is $(a - \lambda)(d - \lambda)$, so the eigenvalues are a and d.

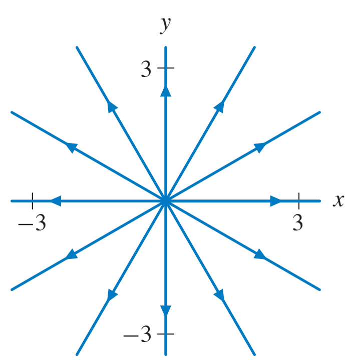
(b) If $a \neq d$, the lines of eigenvectors for a and d are the x- and y-axes respectively.

(c) If $a = d < 0$, every nonzero vector is an eigenvector (see Exercise 14), and all the vectors point toward the origin. Hence, every solution curve is asymptotic to the origin along a straight line.



The general solution is $\mathbf{Y}(t) = e^{at}\mathbf{Y}_0$, where $\mathbf{Y}_0$ is the initial condition.

(d) The only difference between this case and part (c) is that the arrows in the vector field are reversed. Every solution tends away from the origin along a straight line.



Again the general solution is $\mathbf{Y}(t) = e^{at}\mathbf{Y}_0$, where $\mathbf{Y}_0$ is the initial condition.