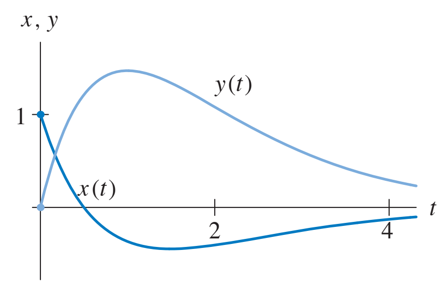
The solution corresponding to the initial condition $D = (1, 0)$ moves to the left and up through the first quadrant in the phase plane before it enters the second quadrant and heads toward the origin tangent to the line $y = -2x$. Thus the $y(t)$-graph is always positive for $t > 0$, and it attains a unique maximum value before it tends to 0. Initially the $x(t)$-graph decreases. It crosses the $y(t)$-graph, becomes negative, and attains a minimum value before it tends to 0 as $t \to \infty$.



EXERCISES FOR SECTION 3.6

1. The characteristic polynomial is

$$s^2 - 6s - 7,$$

so the eigenvalues are $s = -1$ and $s = 7$. Hence, the general solution is

$$y(t) = k_1 e^{-t} + k_2 e^{7t}.$$

2. The characteristic polynomial is

$$s^2 - s - 12,$$

so the eigenvalues are $s = -3$ and $s = 4$. Hence, the general solution is

$$y(t) = k_1 e^{-3t} + k_2 e^{4t}.$$

3. The characteristic polynomial is

$$s^2 + 6s + 9,$$

so $s = -3$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{-3t} + k_2 t e^{-3t}.$$

4. The characteristic polynomial is

$$s^2 - 4s + 4,$$

so $s = 2$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{2t} + k_2 t e^{2t}.$$

5. The characteristic polynomial is

$$s^2 + 8s + 25,$$

so the complex eigenvalues are $s = -4 \pm 3i$. Hence, the general solution is

$$y(t) = k_1 e^{-4t} \cos 3t + k_2 e^{-4t} \sin 3t.$$

6. The characteristic polynomial is

$$s^2 - 4s + 29,$$

so the complex eigenvalues are $s = 2 \pm 5i$. Hence, the general solution is

$$y(t) = k_1 e^{2t} \cos 5t + k_2 e^{2t} \sin 5t.$$

7. The characteristic polynomial is

$$s^2 + 2s - 3,$$

so the eigenvalues are $s = 1$ and $s = -3$. Hence, the general solution is

$$y(t) = k_1 e^t + k_2 e^{-3t},$$

and we have

$$y'(t) = k_1 e^t - 3k_2 e^{-3t}.$$

From the initial conditions, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 = 6 \\ k_1 - 3k_2 = -2. \end{cases}$$

Solving for k_1 and k_2 yields $k_1 = 4$ and $k_2 = 2$. Hence, the solution to our initial-value problem is $y(t) = 4e^t + 2e^{-3t}$.

8. The characteristic polynomial is

$$s^2 + 4s - 5,$$

so the eigenvalues are $s = 1$ and $s = -5$. Hence, the general solution is

$$y(t) = k_1 e^t + k_2 e^{-5t},$$

and we have

$$y'(t) = k_1 e^t - 5k_2 e^{-5t}.$$

From the initial conditions, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 = 11 \\ k_1 - 5k_2 = -7. \end{cases}$$

Solving for k_1 and k_2 yields $k_1 = 8$ and $k_2 = 3$. Hence, the solution to our initial-value problem is $y(t) = 8e^t + 3e^{-5t}$.

9. The characteristic polynomial is

$$s^2 - 4s + 13,$$

so the eigenvalues are $s = 2 \pm 3i$. Hence, the general solution is

$$y(t) = k_1 e^{2t} \cos 3t + k_2 e^{2t} \sin 3t.$$

From the initial condition $y(0) = 1$, we see that $k_1 = 1$. Differentiating

$$y(t) = e^{2t} \cos 3t + k_2 e^{2t} \sin 3t$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = 2 + 3k_2$. Since $y'(0) = -4$, we have $k_2 = -2$. Hence, the solution to our initial-value problem is

$$y(t) = e^{2t} \cos 3t - 2e^{2t} \sin 3t.$$

10. The characteristic polynomial is

$$s^2 + 4s + 20,$$

so the eigenvalues are $s = -2 \pm 4i$. Hence, the general solution is

$$y(t) = k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t.$$

From the initial condition $y(0) = 2$, we see that $k_1 = 2$. Differentiating

$$y(t) = 2e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = -4 + 4k_2$. Since $y'(0) = -8$, we have $k_2 = -1$. Hence, the solution to our initial-value problem is

$$y(t) = 2e^{-2t} \cos 4t - e^{-2t} \sin 4t.$$

11. The characteristic polynomial is

$$s^2 - 8s + 16,$$

so $s = 4$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{4t} + k_2 t e^{4t}.$$

From the initial condition $y(0) = 3$, we see that $k_1 = 3$. Differentiating

$$y(t) = 3e^{4t} + k_2 t e^{4t}$$

and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = 12 + k_2$. Since $y'(0) = 11$, we have $k_2 = -1$. Hence, the solution to our initial-value problem is

$$y(t) = 3e^{4t} - te^{4t}.$$

12. The characteristic polynomial is

$$s^2 - 4s + 4,$$

so $s = 2$ is a repeated eigenvalue. Hence, the general solution is

$$y(t) = k_1 e^{2t} + k_2 t e^{2t}.$$

From the initial condition $y(0) = 1$, we see that $k_1 = 1$. Differentiating $y(t) = e^{2t} + k_2 t e^{2t}$ and evaluating $y'(t)$ at $t = 0$ yields $y'(0) = 2 + k_2$. Since $y'(0) = 1$, we have $k_2 = -1$. Hence, the solution to our initial-value problem is

$$y(t) = e^{2t} - te^{2t}.$$

13. (a) The resulting second-order equation is

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 7y = 0,$$

and the corresponding system is

$$\begin{aligned}\frac{dy}{dt} &= v\\ \frac{dv}{dt} &= -7y - 8v.\end{aligned}$$

(b) Recall that we can read off the characteristic equation of the second-order equation straight from the equation without having to revert to the corresponding system. We obtain

$$\lambda^2 + 8\lambda + 7 = 0.$$

Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -7$.

To find the eigenvectors associated to the eigenvalue λ_1, we solve the simultaneous system of equations

$$\begin{cases} v = -y \\ -7y - 8v = -v. \end{cases}$$

From the first equation, we immediately see that the eigenvectors associated to this eigenvalue must satisfy $v = -y$. Similarly, the eigenvectors associated to the eigenvalue $\lambda_2 = -7$ must satisfy the equation $v = -7y$.

(c) Since the eigenvalues are real and negative, the equilibrium point at the origin is a sink, and the system is overdamped.

29. Note: We assume that m, k and b are nonnegative—the physically relevant case. All references to graphs and phase portraits are from Sections 3.5 and 3.6.

Table 3.1
Possible harmonic oscillators.

name	eigenvalues	parameters	decay rate	phase portrait and graphs
undamped	pure imaginary	$b = 0$	no decay	Figure 3.41
underdamped	complex with negative real part	$b^2 - 4mk < 0$	$e^{-bt/(2m)}$	Figure 3.42
critically damped	only one eigenvalue	$b^2 - 4mk = 0$	$e^{-bt/(2m)}$	Figure 3.34
overdamped	two negative real	$b^2 - 4mk > 0$	$e^{\lambda t}$ where $\lambda = \dfrac{-b + \sqrt{b^2 - 4mk}}{2m}$	Figures 3.43–3.45 and Exercise 13

30. Note that

$$\frac{dy}{dt} = \frac{d}{dt}(k_1 y_1 + k_2 y_2) = k_1 \frac{dy_1}{dt} + k_2 \frac{dy_2}{dt}$$

and

$$\frac{d^2y}{dt^2} = \frac{d^2}{dt^2}(k_1 y_1 + k_2 y_2) = k_1 \frac{d^2y_1}{dt^2} + k_2 \frac{d^2y_2}{dt^2}.$$

Therefore,

$$\begin{aligned}
\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy &= k_1 \frac{d^2y_1}{dt^2} + k_2 \frac{d^2y_2}{dt^2} + p\left(k_1 \frac{dy_1}{dt} + k_2 \frac{dy_2}{dt}\right) + q\,(k_1 y_1 + k_2 y_2) \\
&= k_1 \left(\frac{d^2y_1}{dt^2} + p\frac{dy_1}{dt} + qy_1\right) + k_2 \left(\frac{d^2y_2}{dt^2} + p\frac{dy_2}{dt} + qy_2\right) \\
&= 0.
\end{aligned}$$

31. Note that

$$\frac{dy}{dt} = \frac{d}{dt}(y_{\mathrm{re}} + i y_{\mathrm{im}}) = \frac{dy_{\mathrm{re}}}{dt} + i\frac{dy_{\mathrm{im}}}{dt}$$

and

$$\frac{d^2y}{dt^2} = \frac{d^2}{dt^2}(y_{\mathrm{re}} + i y_{\mathrm{im}}) = \frac{d^2y_{\mathrm{re}}}{dt^2} + i\frac{d^2y_{\mathrm{im}}}{dt^2}.$$

Then note that

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = \left(\frac{d^2y_{\mathrm{re}}}{dt^2} + p\frac{dy_{\mathrm{re}}}{dt} + qy_{\mathrm{re}}\right) + i\left(\frac{d^2y_{\mathrm{im}}}{dt^2} + p\frac{dy_{\mathrm{im}}}{dt} + qy_{\mathrm{im}}\right).$$

Both

$$\frac{d^2y_{\mathrm{re}}}{dt^2} + p\frac{dy_{\mathrm{re}}}{dt} + qy_{\mathrm{re}} = 0 \quad \text{and} \quad \frac{d^2y_{\mathrm{im}}}{dt^2} + p\frac{dy_{\mathrm{im}}}{dt} + qy_{\mathrm{im}} = 0$$

because a complex number is zero only if both its real and imaginary parts vanish. In other words, $y_{\mathrm{re}}(t)$ and $y_{\mathrm{im}}(t)$ are solutions of the original equation.

32. If we let $v = dy/dt$, then the corresponding first-order system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv,$$

and the corresponding matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

If λ is an eigenvalue, then it is a root of the characteristic polynomial. In other words,

$$\lambda^2 + p\lambda + q = 0.$$

Now consider

$$\mathbf{A}\begin{pmatrix} 1 \\ \lambda \end{pmatrix} = \begin{pmatrix} \lambda \\ -q - p\lambda \end{pmatrix}$$
$$= \begin{pmatrix} \lambda \\ \lambda^2 \end{pmatrix}$$
$$= \lambda \begin{pmatrix} 1 \\ \lambda \end{pmatrix}.$$

33. **(a)** If we let $v = dy/dt$, then the corresponding first-order system is

$$\frac{dy}{dt} = v$$
$$\frac{dv}{dt} = -qy - pv,$$

and the corresponding matrix is

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$

If λ_0 is a repeated eigenvalue, then the characteristic polynomial is

$$\lambda^2 + p\lambda + q = (\lambda - \lambda_0^2) = \lambda^2 - 2\lambda_0\lambda + \lambda_0^2.$$

Consequently, $p = -2\lambda_0$, $q = \lambda_0^2$, and

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -\lambda_0^2 & 2\lambda_0 \end{pmatrix}.$$

(b) To compute the general solution of the corresponding first-order system, we consider an arbitrary initial condition $\mathbf{V}_0 = (y_0, v_0)$ and calculate

$$(\mathbf{A} - \lambda_0 \mathbf{I})\mathbf{V}_0 = \begin{pmatrix} -\lambda_0 & 1 \\ -\lambda_0^2 & \lambda_0 \end{pmatrix} \begin{pmatrix} y_0 \\ v_0 \end{pmatrix}$$

$$= \begin{pmatrix} -\lambda_0 y_0 + v_0 \\ -\lambda_0^2 y_0 + \lambda_0 v_0 \end{pmatrix}.$$

The general solution of the first-order system is

$$\mathbf{Y}(t) = e^{\lambda_0 t} \begin{pmatrix} y_0 \\ v_0 \end{pmatrix} + t e^{\lambda_0 t} \begin{pmatrix} -\lambda_0 y_0 + v_0 \\ -\lambda_0^2 y_0 + \lambda_0 v_0 \end{pmatrix}.$$

(c) From the first component of the result in part (b), we obtain the general solution of the original second-order equation in the form

$$y(t) = y_0 e^{\lambda_0 t} + (-\lambda_0 y_0 + v_0) t e^{\lambda_0 t}.$$

(d) Let $k_1 = y_0$ and $k_2 = -\lambda_0 y_0 + v_0$. Clearly, all k_1 are possible. Moreover, once the value of k_1 is determined, k_2 can be determined from v_0 using $k_2 = -\lambda_0 k_1 + v_0$, and v_0 can be determined by k_2 using $v_0 = k_2 + \lambda_0 k_1$. Hence, k_1 and k_2 are arbitrary constants because y_0 and v_0 are arbitrary.

34. We must first find out how fast the "typical" solution of this equation approaches the origin.

The characteristic equation for this harmonic oscillator is

$$s^2 + bs + 3 = 0,$$

and the roots are

$$\frac{-b \pm \sqrt{b^2 - 12}}{2}.$$

These roots are complex if $b^2 < 12$, and all solutions tend to the equilibrium at the rate of $e^{(-b/2)t}$.

If $b^2 > 12$, the roots are real, and the general solution is

$$y(t) = k_1 e^{((-b+\sqrt{b^2-12})/2)t} + k_2 e^{((-b-\sqrt{b^2-12})/2)t}.$$

For the typical solution, both k_1 and k_2 are nonzero, so the typical solution tends to the origin at a rate determined by the slower of these two exponentials. The second of these exponential terms tends to 0 most quickly since $-b - \sqrt{b^2 - 12} < -b + \sqrt{b^2 - 12} < 0$. So the typical solution tends to 0 at the rate determined by the exponential of the form $e^{((-b+\sqrt{b^2-12})/2)t}$.

We must determine which of the two exponentials

$$e^{(-b/2)t}$$

(for $b < 2\sqrt{3}$) and

$$e^{((-b+\sqrt{b^2-12})/2)t}$$