42. **(a)** To find the general solution of the unforced equation, we note that the characteristic polynomial is $s^2 + 4$, which has roots $s = \pm 2i$. So the general solution of the unforced equation is

$$k_1 \cos 2t + k_2 \sin 2t.$$

To find a particular solution of the forced equation we guess

$$y_p(t) = a + bt + ct^2 + de^t.$$

Substituting this guess into the differential equation yields

$$(2c + de^t) + 4(a + bt + ct^2 + de^t) = 6 + t^2 + e^t,$$

which simplifies to

$$(2c + 4a) + (4b)t + (4c)t^2 + (5d)e^t = 6 + t^2 + e^t.$$

So $d = 1/5$, $c = 1/4$, $b = 0$, and $a = 1/8$ yield a solution, and the general solution of the forced equation is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t + \tfrac{11}{8} + \tfrac{1}{4}t^2 + \tfrac{1}{5}e^t.$$

(b) Note that

$$y'(t) = -2k_1 \sin 2t + 2k_2 \cos 2t + \tfrac{1}{2}t + \tfrac{1}{5}e^t.$$

To obtain the desired initial conditions we must solve

$$\begin{cases} k_1 + \frac{11}{8} + \frac{1}{5} = 0 \\ \quad 2k_2 + \frac{1}{5} = 0, \end{cases}$$

which yields $k_1 = -63/40$ and $k_2 = -1/10$. The solution of the initial-value problem is

$$y(t) = -\tfrac{63}{40} \cos 2t - \tfrac{1}{10} \sin 2t + \tfrac{11}{8} + \tfrac{1}{4}t^2 + \tfrac{1}{5}e^t.$$

(c) This solution tends to infinity at a rate that is determined by $e^t/5$ because this term dominates when t is large.

EXERCISES FOR SECTION 4.2

1. Recalling that the real part of e^{it} is $\cos t$, we see that the complex version of this equation is

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{it}.$$

To find a particular solution, we guess $y_c(t) = ae^{it}$. Then $dy_c/dt = iae^{it}$ and $d^2y_c/dt^2 = -ae^{it}$. Substituting these derivatives into the equation and collecting terms yields

$$(-a + 3ia + 2a)e^{-it} = e^{it},$$

which is satisfied if

$$(1+3i)a = 1.$$

Hence, we must have

$$a = \frac{1}{1+3i} = \frac{1}{10} - \frac{3}{10}i.$$

So

$$y_c(t) = \frac{1-3i}{10}e^{it} = \frac{1-3i}{10}(\cos t + i\sin t)$$

is a particular solution of the complex version of the equation. Taking the real part, we obtain the solution

$$y(t) = \tfrac{1}{10}\cos t + \tfrac{3}{10}\sin t.$$

To produce the general solution of the homogeneous equation, we note that the characteristic polynomial $s^2 + 3s + 2$ has roots $s = -2$ and $s = -1$. So the general solution is

$$y(t) = k_1 e^{-2t} + k_2 e^{-t} + \tfrac{1}{10}\cos t + \tfrac{3}{10}\sin t.$$

2. The only difference between this exercise and Exercise 1 is the coefficient of 5 on the right-hand side. Hence, the complex version is

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 5e^{it}.$$

The guess for the particular solution is the same $y_c(t) = ae^{it}$, and the same calculation yields that $y_c(t)$ is a solution if

$$a = \frac{5}{1+3i} = \frac{1}{2} - \frac{3}{2}i.$$

Hence,

$$y_c(t) = \frac{1-3i}{2}e^{it} = \frac{1-3i}{2}(\cos t + i\sin t).$$

Taking the real part and adding the general solution of the homogeneous equation (see Exercise 1), we obtain the general solution

$$y(t) = k_1 e^{-2t} + k_2 e^{-t} + \tfrac{1}{2}\cos t + \tfrac{3}{2}\sin t.$$

3. Recalling that the imaginary part of e^{it} is $\sin t$, the complex version of the equation is

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{it}.$$

This equation is precisely the same complex equation as in Exercise 1. Hence, we have already computed the solution

$$y_c(t) = \frac{1-3i}{10}(\cos t + i\sin t).$$

In this case we take the imaginary part

$$y(t) = -\tfrac{3}{10}\cos t + \tfrac{1}{10}\sin t$$

to obtain a solution of the original differential equation.

The general solution of the homogeneous equation is the same as in Exercise 1, so the general solution is

$$y(t) = k_1 e^{-2t} + k_2 e^{-t} - \tfrac{3}{10}\cos t + \tfrac{1}{10}\sin t.$$

4. This equation is the same as the equation in Exercise 3 except for the coefficient of 2 on the right-hand side. The complex version of the equation is

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2e^{it},$$

and the guess of the particular solution $y_c(t) = ae^{it}$ yields

$$a = \frac{1-3i}{5}$$

via the same steps as in Exercise 3. Taking the imaginary part of

$$y_c(t) = \frac{1-3i}{5}e^{it} = \frac{1-3i}{5}(\cos t + i\sin t)$$

and adding the general solution of the homogeneous equation yields

$$y(t) = k_1 e^{-2t} + k_2 e^{-t} + \tfrac{1}{5}\sin t - \tfrac{3}{5}\cos t.$$

5. The complex version of this equation is

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = e^{it}.$$

We guess a particular solution of the form $y_c(t) = ae^{it}$. Then $dy_c/dt = iae^{it}$ and $d^2y/dt^2 = -ae^{it}$. Substituting these derivatives into the complex differential equation yields

$$(-a + 6ia + 8a)e^{it} = e^{it},$$

which is satisfied if $(7 + 6i)a = 1$. Then $a = 1/(7 + 6i)$, and

$$y_c(t) = \frac{7-6i}{85}\,e^{it} = \frac{7-6i}{85}(\cos t + i\sin t).$$

The real part

$$y(t) = \tfrac{7}{85}\cos t + \tfrac{6}{85}\sin t$$

is a solution of the original differential equation.

To find the general solution of the homogeneous equation, we note that the characteristic polynomial $s^2 + 6s + 8$ has roots $s = -4$ and $s = -2$. Consequently, the general solution of the original nonhomogeneous equation is

$$y(t) = k_1 e^{-4t} + k_2 e^{-2t} + \tfrac{7}{85}\cos t + \tfrac{6}{85}\sin t.$$

6. The complex version of the equation is

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = -4e^{3it},$$

and to find a particular solution, we guess $y_c(t) = ae^{3it}$. Substituting this guess into the equation, we obtain

$$-9ae^{3it} + 18aie^{3it} + 8ae^{3it} = -4e^{3it},$$

which can be simplified to

$$(-1 + 18i)ae^{3it} = -4e^{3it}.$$

Thus, $y_c(t)$ is a solution if $a = -4/(-1 + 18i)$. We have

$$y_c(t) = \frac{-4}{-1 + 18i}\, e^{3it} = \frac{4 + 72i}{325}(\cos 3t + i \sin 3t),$$

and we take the real part to obtain a solution

$$y(t) = \tfrac{4}{325} \cos 3t - \tfrac{72}{325} \sin 3t$$

of the original equation.

To find the general solution of the unforced equation, we note that the characteristic polynomial is $s^2 + 6s + 8$, which has roots $s = -4$ and $s = -2$. Hence, the general solution of the original forced equation is

$$y(t) = k_1e^{-4t} + k_2e^{-2t} + \tfrac{4}{325} \cos 3t - \tfrac{72}{325} \sin 3t.$$

7. The complex version of this equation is

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 3e^{2it},$$

so we guess $y_c(t) = ae^{2it}$ to find a particular solution. Substituting $y_c(t)$ into the differential equation gives

$$(-4a + 8ai + 13a)e^{2it} = 3e^{2it},$$

which is satisfied if $(9 + 8i)a = 3$. Thus, $y_c(t)$ is a solution if

$$y_c(t) = \frac{3}{9 + 8i}\, e^{2it} = \frac{27 - 24i}{145}(\cos 2t + i \sin 2t).$$

A particular solution of the original equation is the real part

$$y(t) = \tfrac{27}{145} \cos 2t + \tfrac{24}{145} \sin 2t.$$

To find the general solution of the homogeneous equation, we note that the characteristic polynomial $s^2 + 4s + 13$ has roots $s = -2 \pm 3i$. Hence, the general solution of the original forced equation is

$$y(t) = k_1e^{-2t} \cos 3t + k_2e^{-2t} \sin 3t + \tfrac{27}{145} \cos 2t + \tfrac{24}{145} \sin 2t.$$

8. The complex version of the equation is

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = -e^{5it},$$

and we guess that there is a solution of the form $y_c(t) = ae^{5it}$. Substituting this guess into the differential equation yields

$$-25ae^{5it} + 20aie^{5it} + 20ae^{5it} = -e^{5it},$$

which can be simplified to

$$(-5 + 20i)ae^{5it} = -e^{5it}.$$

Thus, $y_c(t)$ is a solution if $a = -1/(-5 + 20i)$. We have

$$y_c(t) = \frac{-1}{-5 + 20i}\,e^{5it} = \frac{1 + 4i}{85}(\cos 5t + i \sin 5t).$$

We take the real part to obtain a solution

$$y(t) = \tfrac{1}{85}\cos 5t - \tfrac{4}{85}\sin 5t$$

of the original equation.

To find the general solution of the homogeneous equation, we note that the characteristic polynomial is $s^2 + 4s + 20$, which has roots $s = -2 \pm 4i$. Hence, the general solution of the original equation is

$$y(t) = k_1e^{-2t}\cos 4t + k_2e^{-2t}\sin 4t + \tfrac{1}{85}\cos 5t - \tfrac{4}{85}\sin 5t.$$

9. The complex version of this equation is

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = -3e^{2it},$$

and we guess that there is a solution of the form $y_c(t) = ae^{2it}$. Substituting this guess into the differential equation yields

$$(-4a + 8ia + 20a)e^{2it} = -3e^{2it},$$

which can be simplified to

$$(16 + 8i)ae^{2it} = -3e^{2it}.$$

Thus, $y_c(t)$ is a solution if $a = -3/(16 + 8i)$. We have

$$y_c(t) = \frac{-3}{16 + 8i}\,e^{2it} = \left(-\frac{3}{20} + \frac{3}{40}i\right)\left(\cos 2t + i \sin 2t\right).$$

We take the imaginary part to obtain a solution

$$y(t) = \tfrac{3}{40}\cos 2t - \tfrac{3}{20}\sin 2t$$

of the original equation.

To find the general solution of the homogeneous equation, we note that the characteristic polynomial is $s^2 + 4s + 20$, whose roots are $s = -2 \pm 4i$. Hence, the general solution of the original equation is

$$y(t) = k_1e^{-2t}\cos 4t + k_2e^{-2t}\sin 4t - \tfrac{3}{20}\sin 2t + \tfrac{3}{40}\cos 2t.$$

10. The complex version of the equation is

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = e^{3it},$$

and we guess there is a particular solution of the form $y_c(t) = ae^{3it}$. Substituting this guess into the differential equation yields

$$-9ae^{3it} + 6iae^{3it} + ae^{3it} = e^{3it},$$

which can be simplified to

$$(-8 + 6i)ae^{3it} = e^{3it}.$$

Thus, $y_c(t)$ is a solution if $a = 1/(-8 + 16i)$. We have

$$y_c(t) = \frac{1}{-8 + 16i}\, e^{3it} = -\left(\frac{2}{25} + \frac{3}{50}i\right)\left(\cos 3t + i \sin 3t\right).$$

We take the real part to obtain a solution

$$y(t) = -\tfrac{2}{25}\cos 3t + \tfrac{3}{50}\sin 3t$$

of the original equation.

To find the general solution of the unforced equation we note that the characteristic polynomial is $s^2 + 2s + 1$, which has $s = -1$ as a double root. Hence, the general solution of the original equation is

$$y(t) = k_1e^{-t} + k_2te^{-t} - \tfrac{2}{25}\cos 3t + \tfrac{3}{50}\sin 3t.$$

11. From Exercise 5, we know that the general solution of this equation is

$$y(t) = k_1e^{-4t} + k_2e^{-2t} + \tfrac{7}{85}\cos t + \tfrac{6}{85}\sin t.$$

To find the desired solution, we must solve for k_1 and k_2 using the initial conditions. We have

$$\begin{cases} k_1 + k_2 + \frac{7}{85} = 0 \\ -4k_1 - 2k_2 + \frac{6}{85} = 0. \end{cases}$$

We obtain $k_1 = 2/17$ and $k_2 = -1/5$. The desired solution is

$$y(t) = \tfrac{2}{17}e^{-4t} - \tfrac{1}{5}e^{-2t} + \tfrac{7}{85}\cos t + \tfrac{6}{85}\sin t.$$

12. We can solve this initial-value problem by finding the general solution in many ways. One method involves producing a particular solution to the differential equation using the guess-and-test technique described in the text. Another way to find a particular solution is to note that the left-hand side of this equation is the same as the left-hand side of the equation in Exercise 6 and the right-hand side of this equation differs from the right-hand side of that equation by a factor of $-1/2$. Since these equations are linear, we can use the Linearity Principle to derive a particular solution to this equation by multiplying the particular solution we found in Exercise 6 by $-1/2$. Hence the general solution of the differential equation in this exercise is

$$y(t) = k_1 e^{-4t} + k_2 e^{-2t} - \tfrac{2}{325}\cos 3t + \tfrac{36}{325}\sin 3t.$$

To obtain the desired initial conditions, we must solve for k_1 and k_2. We have

$$\begin{cases} k_1 + k_2 - \frac{2}{325} = 0 \\ -4k_1 - 2k_2 + \frac{108}{325} = 0. \end{cases}$$

We obtain $k_1 = 4/25$ and $k_2 = -2/13$. The desired solution is

$$y(t) = \tfrac{4}{25}e^{-4t} - \tfrac{2}{13}e^{-2t} - \tfrac{2}{325}\cos 3t + \tfrac{36}{325}\sin 3t.$$

13. From Exercise 9, we know that the general solution of this equation is

$$y(t) = k_1 e^{-2t}\cos 4t + k_2 e^{-2t}\sin 4t - \tfrac{3}{20}\sin 2t + \tfrac{3}{40}\cos 2t.$$

To find the desired solution, we must solve for k_1 and k_2 using the initial conditions. We have

$$\begin{cases} k_1 + \frac{3}{40} = 0 \\ -2k_1 + 4k_2 - \frac{6}{20} = 0. \end{cases}$$

We obtain $k_1 = -3/40$ and $k_2 = 3/80$. The desired solution is

$$y(t) = -\tfrac{3}{40}e^{-2t}\cos 4t + \tfrac{3}{80}e^{-2t}\sin 4t - \tfrac{3}{20}\sin 2t + \tfrac{3}{40}\cos 2t.$$

14. First we find the general solution of the differential equation using the Extended Linearity Principle and the standard guess-and-test technique. The complex version of the equation is

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 2e^{2it},$$

and we guess $y_c(t) = ae^{2it}$ as a particular solution. Substituting this guess into the equation yields

$$a = \frac{2}{-3+4i} = \frac{-6-8i}{25}.$$

Hence, a particular solution is the real part of

$$y_c(t) = \frac{-6-8i}{25}\,(\cos 2t + i\sin 2t).$$

We have

$$y(t) = -\tfrac{6}{25}\cos 2t + \tfrac{8}{25}\sin 2t.$$

To find the general solution of the homogeneous equation, we note that the characteristic polynomial is $s^2 + 2s + 1$, which has $s = -1$ as a double root. Hence, the general solution of the original equation is

$$y(t) = k_1 e^{-t} + k_2 t e^{-t} - \tfrac{6}{25}\cos 2t + \tfrac{8}{25}\sin 2t.$$

To obtain the desired initial conditions, we solve for k_1 and k_2 using

$$\begin{cases} k_1 - \frac{6}{25} = 0 \\ -k_1 + k_2 + \frac{16}{25} = 0. \end{cases}$$

We see that $k_1 = 6/25$ and $k_2 = -2/5$, so the desired solution is

$$y(t) = \tfrac{6}{25}e^{-t} - \tfrac{2}{5}te^{-t} - \tfrac{6}{25}\cos 2t + \tfrac{8}{25}\sin 2t.$$

15. **(a)** If we guess

$$y_p(t) = a\cos 3t + b\sin 3t,$$

then

$$y_p'(t) = -3a\sin 3t + 3b\cos 3t$$

and

$$y_p''(t) = -9a\cos 3t - 9b\sin 3t.$$

Substituting this guess and its derivatives into the differential equation gives

$$(-8a + 9b)\cos 3t + (-9a - 8b)\sin 3t = \cos 3t.$$

Thus $y_p(t)$ is a solution if a and b satisfy the simultaneous equations

$$\begin{cases} -8a + 9b = 1 \\ -9a - 8b = 0. \end{cases}$$

Solving these equations for a and b, we obtain $a = -8/145$ and $b = 9/145$, so

$$y_p(t) = -\tfrac{8}{145}\cos 3t + \tfrac{9}{145}\sin 3t$$

is a solution.

(b) If we guess

$$y_p(t) = A\cos(3t + \phi),$$

then

$$y_p'(t) = -3A\sin(3t + \phi)$$

and

$$y_p''(t) = -9A\cos(3t + \phi).$$

Substituting this guess and its derivatives into the differential equation yields

$$-8A\cos(3t + \phi) - 9A\sin(3t + \phi) = \cos 3t.$$

Using the trigonometric identities for the sine and cosine of the sum of two angles, we have

$$-8A\,(\cos 3t \cos\phi - \sin 3t \sin\phi) - 9A\,(\sin 3t \cos\phi + \cos 3t \sin\phi) = \cos 3t.$$

This equation can be rewritten as

$$(-8A\cos\phi - 9A\sin\phi)\cos 3t + (8A\sin\phi - 9A\cos\phi)\sin 3t = \cos 3t.$$

It holds if

$$\begin{cases} -8A\cos\phi - 9A\sin\phi = 1 \\ \;\;9A\cos\phi - 8A\sin\phi = 0. \end{cases}$$

Multiplying the first equation by 9 and the second by 8 and adding yields

$$145A\sin\phi = -9.$$

Similarly, multiplying the first equation by -8 and the second by 9 and adding yields

$$145A\cos\phi = -8.$$

Taking the ratio gives

$$\frac{\sin\phi}{\cos\phi} = \tan\phi = \frac{9}{8}.$$

Also, squaring both $145A\sin\phi = -9$ and $145A\cos\phi = -8$ yields

$$145^2A^2\cos^2\phi + 145^2A^2\sin^2\phi = 145,$$

so $A^2 = 1/145$.

We can use either $A = 1/\sqrt{145}$ or $A = -1/\sqrt{145}$, but this choice of sign for A effects the value of ϕ. If we pick $A = -1/\sqrt{145}$, then $\sqrt{145}\,\sin\phi = 9$, $\sqrt{145}\,\cos\phi = 8$, and $\tan\phi = 9/8$. In this case, $\phi = \arctan(9/8)$. Hence, a particular solution of the original equation is

$$y_p(t) = \frac{1}{\sqrt{145}}\,\cos\left(3t + \arctan\frac{9}{8}\right).$$

16. **(a)** Substituting $ky_p(t)$ into the left-hand side of the differential equation and simplifying yields

$$\frac{d^2(ky_p)}{dt^2} + p\frac{d(ky_p)}{dt} + q(ky_p) = k\frac{d^2y_p}{dt^2} + pk\frac{dy_p}{dt} + qky_p$$

since the derivative of ky is $k(dy/dt)$ if k is a constant. Consequently,

$$\begin{aligned}\frac{d^2(ky_p)}{dt^2} + p\frac{d(ky_p)}{dt} + q(ky_p) &= k\left(\frac{d^2y_p}{dt^2} + p\frac{dy_p}{dt} + qy_p\right) \\ &= k\,g(t)\end{aligned}$$

because $y_p(t)$ is a solution of

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t).$$

(b) By Exercise 5 we know that one solution of

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = \cos t$$

is

$$y_1(t) = \tfrac{7}{85}\cos t + \tfrac{6}{85}\sin t.$$

Using the result of part (a), a particular solution of the given equation is $y_2(t) = 5y_1(t)$. In other words,

$$y_2(t) = \tfrac{7}{17}\cos t + \tfrac{6}{17}\sin t$$

is a particular solution to the equation in this exercise.

The general solution of the homogeneous equation is the same as in Exercise 5, so the general solution for this exercise is

$$y(t) = k_1e^{-4t} + k_2e^{-2t} + \tfrac{7}{17}\cos t + \tfrac{6}{17}\sin t.$$

17. Since p and q are both positive, the solution of the homogeneous equation (the unforced response) tends to zero. Hence, we can match solutions to equations by considering the period (or frequency) and the amplitude of the steady-state solution (forced response). We also need to consider the rate at which solutions tend to the steady-state solution.

(a) The steady-state solution has period $2\pi/3$, and since the period of the steady-state solution is the same as the period of the forcing function, these solutions correspond to equations (v) or (vi). Moreover, this observation applies to the solutions in part (d) as well. Therefore, we need to match equations (v) and (vi) with the solutions in parts (a) and (d).

Solutions approach the steady-state faster in (d) than in (a). To distinguish (v) from (vi), we consider their characteristic polynomials. The characteristic polynomial for (v) is

$$s^2 + 5s + 1,$$

which has eigenvalues $(-5 \pm \sqrt{21})/2$. The characteristic polynomial for (vi) is

$$s^2 + s + 1,$$

which has eigenvalues $(-1 \pm i\sqrt{3})/2$. The rate of approach to the steady-state for (v) is determined by the slow eigenvalue $(-5+\sqrt{21})/2 \approx -0.21$. The rate of approach to the steady-state for (vi) is determined by the real part of the eigenvalue, -0.5. Therefore, the graphs in part (a) come from equation (v), and the graphs in part (d) come from equation (vi).

(b) The steady-state solution has period 2π, and since the period of the steady-state solution is the same as the period of the forcing function, these solutions correspond to equations (i) or (ii). Moreover, this observation applies to the solutions in part (c) as well. Therefore, we need to match equations (i) and (ii) with the solutions in parts (b) and (c).

The amplitude of the steady-state solution is larger in (b) than in (c). To distinguish (i) from (ii), we calculate the amplitudes of the steady-state solutions for (i) and (ii). If we complexify these equations, we get

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = e^{it}.$$

Guessing a solution of the form $y_c(t) = ae^{it}$, we see that

$$a = \frac{1}{(q-1) + pi}.$$

The amplitude of the steady-state solution is $|a|$. For equation (i), $|a| = 1/\sqrt{29} \approx 0.19$, and for equation (ii), it is $1/\sqrt{5} \approx 0.44$. Therefore, the graphs in part (b) correspond to equation (ii), and the graphs in part (c) correspond to equation (i).

(c) See the answer to part (b).

(d) See the answer to part (a).

18. **(a)** Due to the result in Exercise 36 in Section 4.1, we consider each forcing term separately. That is, we consider the two differential equations

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 3 \quad \text{and} \quad \frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 2\cos 2t.$$

To find a particular solution of the first equation, we guess a constant function $y_1(t) = a$. Substituting this guess into the equation yields $20a = 3$, so $y_1(t) = 3/20$ is a solution.

To find a particular solution of the second equation, we consider the complex version

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = 2e^{2it}$$

and guess a solution of the form $y_c(t) = ae^{2it}$. Substituting $y_c(t)$ into the equation yields

$$(16 + 8i)ae^{2it} = 2e^{2it},$$

which is satisfied if $a = 2/(16 + 8i)$. A solution $y_2(t)$ of the second equation is obtained by taking the real part of $y_c(t)$. Since

$$y_c(t) = \left(\tfrac{1}{10} - \tfrac{1}{20}i\right)\left(\cos 2t + i \sin 2t\right),$$

we have

$$y_2(t) = \tfrac{1}{10}\cos 2t + \tfrac{1}{20}\sin 2t.$$

To find the solution of the unforced equation, we note that the characteristic polynomial is $s^2 + 4s + 20$, which has roots $s = -2 \pm 4i$.

Hence, the general solution is

$$y(t) = k_1 e^{-2t}\cos 4t + k_2 e^{-2t}\sin 4t + \tfrac{3}{20} + \tfrac{1}{10}\cos 2t + \tfrac{1}{20}\sin 2t.$$

(b) The first two terms of the general solution tend quickly to zero, so all solutions eventually oscillate about $y = 3/20$. The period and amplitude of the oscillations is determined by the period and amplitude of the oscillations of $y_2(t)$. The period of $y_2(t)$ is π.

19. **(a)** Using the fact that the real part of $e^{(-1+i)t}$ is $e^{-t}\cos t$, the complex version of this equation is

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{(-1+i)t}.$$

Guessing $y_c(t) = ae^{(-1+i)t}$ yields

$$a(-1+i)^2 e^{(-1+i)t} + 4a(-1+i)e^{(-1+i)t} + 20ae^{(-1+i)t} = e^{(-1+i)t}.$$

Simplifying we have

$$a(16+2i)e^{(-1+i)t} = e^{(-1+i)t}.$$

Thus, $y_c(t)$ is a solution of the complex differential equation if $a = 1/(16+2i)$, and we have

$$y_c(t) = \left(\tfrac{4}{65} - \tfrac{1}{130}i\right) e^{-t} (\cos t + i \sin t).$$

So one solution of the original equation is

$$y_p(t) = \tfrac{4}{65}e^{-t}\cos t + \tfrac{1}{130}e^{-t}\sin t.$$

To find the general solution of the homogeneous equation, we note that the characteristic polynomial $s^2 + 4s + 20$ has roots $s = -2 \pm 4i$.

Hence, the general solution of the original equation is

$$y(t) = k_1 e^{-2t}\cos 4t + k_2 e^{-2t}\sin 4t + \tfrac{4}{65}e^{-t}\cos t + \tfrac{1}{130}e^{-t}\sin t.$$

(b) All four terms in the general solution tend to zero as $t \to \infty$. Hence, all solutions tend to zero as $t \to \infty$. The terms with factors of e^{-2t} tend to zero very quickly, which leaves the terms of the particular solution $y_p(t)$ as the largest terms, so all solutions are asymptotic to $y_p(t)$. Since the solution $y_p(t)$ oscillates with period 2π and the amplitude of its oscillations decreases at the rate of e^{-t}, all solutions oscillate with this period and decaying amplitude.

20. (a) To find the general solution of the unforced equation, we note that the characteristic polynomial is $s^2 + 4s + 20$, which has roots $s = -2 \pm 4i$.

To find a particular solution of the forced equation, we note that the complex version of the equation is

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 20y = e^{(-2+4i)t}.$$

We could guess $y_c(t) = ae^{(-2+4i)t}$ as a particular solution, but with perfect hindsight, we recall that the roots of the characteristic polynomial of the unforced equation are $-2 \pm 4i$. Hence, $e^{(-2+4i)t}$ is already a solution of the homogeneous equation. In other words, no value of a will make this $y_c(t)$ a solution. (Why?)

So we second guess $y_c(t) = ate^{(-2+4i)t}$ and substitute this guess into the equation to obtain

$$a\left([-4+8i+(-12-16i)t] + 4[1+(-2+4i)t] + 20t\right)e^{(-2+4i)t} = e^{(-2+4i)t},$$

which simplifies to

$$a(8i)e^{(-2+4i)t} = e^{(-2+4i)t}.$$

Hence, $y_c(t) = ate^{(-2+4i)t}$ is a solution if $a = 1/(8i) = -i/8$. To find a particular solution of the original equation, we compute the imaginary part of

$$y_c(t) = -\tfrac{1}{8}ite^{-2t}(\cos 4t + i\sin 4t).$$

We have

$$y(t) = -\tfrac{1}{8}te^{-2t}\cos 4t.$$

The general solution of the original equation is

$$y(t) = k_1 e^{-2t}\cos 4t + k_2 e^{-2t}\sin 4t - \tfrac{1}{8}te^{-2t}\cos 4t.$$

(b) All terms of the general solution tend to zero. The term that tends to zero most slowly is $-(te^{-2t}\cos 4t)/8$, so for large t, all solutions are approximately equal to this term.

21. Note that the real part of

$$(a - bi)(\cos\omega t + i\sin\omega t)$$

is $g(t)$. Hence, we must find k and ϕ such that

$$ke^{i\phi} = a - bi.$$

Using the polar form of the complex number $z = a - bi$, we see that

$$ke^{i\phi} = a - bi = z = |z|e^{i\theta},$$

where θ is the polar angle for z (see Appendix C). Therefore, we can choose

$$k = |z| = \sqrt{a^2 + b^2} \quad \text{and} \quad \phi = \theta.$$

22. Note that $g_1(t)$ is the real part of $k_1 e^{i\phi_1}e^{i\omega t}$ and $g_2(t)$ is the real part of $k_2 e^{i\phi_2}e^{i\omega t}$, so $g_1(t) + g_2(t)$ is the real part of

$$k_1 e^{i\phi_1}e^{i\omega t} + k_2 e^{i\phi_2}e^{i\omega t},$$

which can be rewritten as

$$\left(k_1 e^{i\phi_1} + k_2 e^{i\phi_2}\right)e^{i\omega t}.$$

Thus, the phasor for $g_1(t) + g_2(t)$ is $k_1 e^{i\phi_1} + k_2 e^{i\phi_2}$.

23. Note that the real part of

$$(k_1 - ik_2)e^{i\beta t} = (k_1 - ik_2)(\cos\beta t + i\sin\beta t)$$

is

$$y(t) = k_1\cos\beta t + k_2\sin\beta t.$$

Let $Ke^{i\phi}$ be the polar form of the complex number $k_1 + ik_2$. Then the polar form of $k_1 - ik_2$ is $Ke^{-i\phi}$. Using the Laws of Exponents and Euler's formula, we have

$$\begin{aligned}(k_1 - ik_2)e^{i\beta t} &= Ke^{-i\phi}e^{i\beta t}\\ &= Ke^{i(\beta t - \phi)}\\ &= K(\cos(\beta t - \phi) + i\sin(\beta t - \phi)),\end{aligned}$$

and the real part is $K\cos(\beta t - \phi)$. Hence, we see that

$$y(t) = k_1\cos\beta t + k_2\sin\beta t$$

can be rewritten as

$$y(t) = K\cos(\beta t - \phi).$$