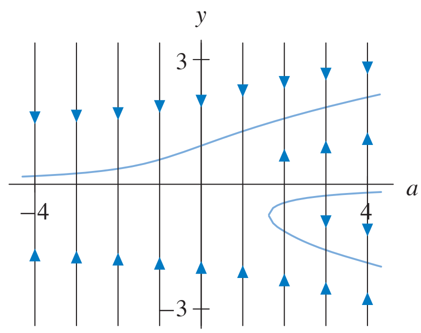
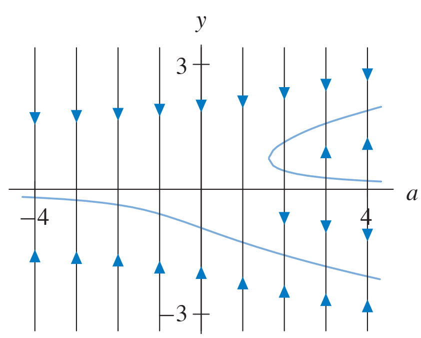


The bifurcation diagram for $r = 0.8$.

(d) If $r < 0$, there is a positive bifurcation value $a = a_0$. For $a < a_0$, the phase line has one equilibrium point, a negative sink. If $a > a_0$, there are two positive equilibria in addition to the negative sink. The larger of the two positive equilibria is a sink and the smaller is a source.



The bifurcation diagram for $r = -0.8$.

EXERCISES FOR SECTION 1.8

1. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-4t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{-t}$. Then

$$\begin{aligned}\frac{dy_p}{dt} + 4y_p &= -\alpha e^{-t} + 4\alpha e^{-t} \\ &= 3\alpha e^{-t}.\end{aligned}$$

Consequently, we must have $3\alpha = 9$ for $y_p(t)$ to be a solution. Hence, $\alpha = 3$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{-4t} + 3e^{-t}.$$

2. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-4t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{-t}$. Then

$$\begin{aligned}\frac{dy_p}{dt} + 4y_p &= -\alpha e^{-t} + 4\alpha e^{-t} \\ &= 3\alpha e^{-t}.\end{aligned}$$

Consequently, we must have $3\alpha = 3$ for $y_p(t)$ to be a solution. Hence, $\alpha = 1$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{-4t} + e^{-t}.$$

3. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-3t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} + 3y_p &= -2\alpha \sin 2t + 2\beta \cos 2t + (3\alpha \cos 2t + 3\beta \sin 2t) \\ &= (3\alpha + 2\beta) \cos 2t + (3\beta - 2\alpha) \sin 2t\end{aligned}$$

Consequently, we must have

$$(3\alpha + 2\beta) \cos 2t + (3\beta - 2\alpha) \sin 2t = 4 \cos 2t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} 3\alpha + 2\beta = 4 \\ 3\beta - 2\alpha = 0. \end{cases}$$

Hence, $\alpha = 12/13$ and $\beta = 8/13$. The general solution is

$$y(t) = ke^{-3t} + \tfrac{12}{13} \cos 2t + \tfrac{8}{13} \sin 2t.$$

4. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} - 2y_p &= -2\alpha \sin 2t + 2\beta \cos 2t - 2(\alpha \cos 2t + \beta \sin 2t) \\ &= (2\beta - 2\alpha) \cos 2t + (-2\alpha - 2\beta) \sin 2t.\end{aligned}$$

Consequently, we must have

$$(2\beta - 2\alpha) \cos 2t + (-2\alpha - 2\beta) \sin 2t = \sin 2t$$

for $y_p(t)$ to be a solution, that is, we must solve

$$\begin{cases} -2\alpha - 2\beta = 1 \\ -2\alpha + 2\beta = 0. \end{cases}$$

Hence, $\alpha = -1/4$ and $\beta = -1/4$. The general solution of the nonhomogeneous equation is

$$y(t) = ke^{2t} - \tfrac{1}{4}\cos 2t - \tfrac{1}{4}\sin 2t.$$

5. The general solution to the associated homogeneous equation is $y_h(t) = ke^{3t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha te^{3t}$ rather than αe^{3t} because αe^{3t} is a solution of the homogeneous equation. Then

$$\begin{aligned}\frac{dy_p}{dt} - 3y_p &= \alpha e^{3t} + 3\alpha te^{3t} - 3\alpha te^{3t}\\ &= \alpha e^{3t}.\end{aligned}$$

Consequently, we must have $\alpha = -4$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{3t} - 4te^{3t}.$$

6. The general solution of the associated homogeneous equation is $y_h(t) = ke^{t/2}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha te^{t/2}$ rather than $\alpha e^{t/2}$ because $\alpha e^{t/2}$ is a solution of the homogeneous equation. Then

$$\begin{aligned}\frac{dy_p}{dt} - \frac{y_p}{2} &= \alpha e^{t/2} + \frac{\alpha}{2}te^{t/2} - \frac{\alpha te^{t/2}}{2}\\ &= \alpha e^{t/2}.\end{aligned}$$

Consequently, we must have $\alpha = 4$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{t/2} + 4te^{t/2}.$$

7. The general solution to the associated homogeneous equation is $y_h(t) = ke^{-2t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{t/3}$. Then

$$\begin{aligned}\frac{dy_p}{dt} + 2y_p &= \tfrac{1}{3}\alpha e^{t/3} + 2\alpha e^{t/3}\\ &= \tfrac{7}{3}\alpha e^{t/3}.\end{aligned}$$

Consequently, we must have $\frac{7}{3}\alpha = 1$ for $y_p(t)$ to be a solution. Hence, $\alpha = 3/7$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{-2t} + \tfrac{3}{7}e^{t/3}.$$

Since $y(0) = 1$, we have

$$1 = k + \tfrac{3}{7},$$

so $k = 4/7$. The function $y(t) = \frac{4}{7}e^{-2t} + \frac{3}{7}e^{t/3}$ is the solution of the initial-value problem.

8. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha e^{-2t}$. Then

$$\begin{aligned}\frac{dy_p}{dt} - 2y_p &= -2\alpha e^{-2t} - 2\alpha e^{-2t}\\ &= -4\alpha e^{-2t}.\end{aligned}$$

Consequently, we must have $-4\alpha = 3$ for $y_p(t)$ to be a solution. Hence, $\alpha = -3/4$, and the general solution to the nonhomogeneous equation is

$$y(t) = ke^{2t} - \tfrac{3}{4}e^{-2t}.$$

Since $y(0) = 10$, we have

$$10 = k - \tfrac{3}{4},$$

so $k = 43/4$. The function

$$y(t) = \tfrac{43}{4}e^{2t} - \tfrac{3}{4}e^{-2t}$$

is the solution of the initial-value problem.

9. The general solution of the associated homogeneous equation is $y_h(t) = ke^{-t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} + y_p &= -2\alpha \sin 2t + 2\beta \cos 2t + \alpha \cos 2t + \beta \sin 2t\\ &= (\alpha + 2\beta)\cos 2t + (-2\alpha + \beta)\sin 2t.\end{aligned}$$

Consequently, we must have

$$(\alpha + 2\beta)\cos 2t + (-2\alpha + \beta)\sin 2t = \cos 2t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} \alpha + 2\beta = 1 \\ -2\alpha + \beta = 0. \end{cases}$$

Hence, $\alpha = 1/5$ and $\beta = 2/5$. The general solution to the differential equation is

$$y(t) = ke^{-t} + \tfrac{1}{5}\cos 2t + \tfrac{2}{5}\sin 2t.$$

To find the solution of the given initial-value problem, we evaluate the general solution at $t = 0$ and obtain

$$y(0) = k + \tfrac{1}{5}.$$

Since the initial condition is $y(0) = 5$, we see that $k = 24/5$. The desired solution is

$$y(t) = \tfrac{24}{5}e^{-t} + \tfrac{1}{5}\cos 2t + \tfrac{2}{5}\sin 2t.$$

10. The general solution of the associated homogeneous equation is $y_h(t) = ke^{-3t}$. For a particular solution of the nonhomogeneous equation, we guess a solution of the form $y_p(t) = \alpha \cos 2t + \beta \sin 2t$. Then

$$\begin{aligned}\frac{dy_p}{dt} + 3y_p &= -2\alpha \sin 2t + 2\beta \cos 2t + 3\alpha \cos 2t + 3\beta \sin 2t \\ &= (3\alpha + 2\beta) \cos 2t + (-2\alpha + 3\beta) \sin 2t.\end{aligned}$$

Consequently, we must have

$$(3\alpha + 2\beta) \cos 2t + (-2\alpha + 3\beta) \sin 2t = \cos 2t$$

for $y_p(t)$ to be a solution. We must solve

$$\begin{cases} 3\alpha + 2\beta = 1 \\ -2\alpha + 3\beta = 0. \end{cases}$$

Hence, $\alpha = 3/13$ and $\beta = 2/13$. The general solution to the differential equation is

$$y(t) = ke^{-3t} + \tfrac{3}{13} \cos 2t + \tfrac{2}{13} \sin 2t.$$

To find the solution of the given initial-value problem, we evaluate the general solution at $t = 0$ and obtain

$$y(0) = k + \tfrac{3}{13}.$$

Since the initial condition is $y(0) = -1$, we see that $k = -16/13$. The desired solution is

$$y(t) = -\tfrac{16}{13} e^{-3t} + \tfrac{3}{13} \cos 2t + \tfrac{2}{13} \sin 2t.$$

11. The general solution to the associated homogeneous equation is $y_h(t) = ke^{2t}$. For a particular solution of the nonhomogeneous equation, we guess $y_p(t) = \alpha te^{2t}$ rather than αe^{2t} because αe^{2t} is a solution of the homogeneous equation. Then

$$\begin{aligned}\frac{dy_p}{dt} - 2y_p &= \alpha e^{2t} + 2\alpha te^{2t} - 2\alpha te^{2t} \\ &= \alpha e^{2t}.\end{aligned}$$

Consequently, we must have $\alpha = 7$ for $y_p(t)$ to be a solution. Hence, the general solution to the nonhomogeneous equation is

$$y(t) = ke^{2t} + 7te^{2t}.$$

Note that $y(0) = k = 3$, so the solution to the initial-value problem is

$$y(t) = 3e^{2t} + 7te^{2t} = (3 + 7t)e^{2t}.$$

19. Let $y(t) = y_h(t) + y_1(t) + y_2(t)$. Then

$$\begin{aligned}\frac{dy}{dt} + a(t)y &= \frac{dy_h}{dt} + \frac{dy_1}{dt} + \frac{dy_2}{dt} + a(t)y_h + a(t)y_1 + a(t)y_2 \\ &= \frac{dy_h}{dt} + a(t)y_h + \frac{dy_1}{dt} + a(t)y_1 + \frac{dy_2}{dt} + a(t)y_2 \\ &= 0 + b_1(t) + b_2(t).\end{aligned}$$

This computation shows that $y_h(t) + y_1(t) + y_2(t)$ is a solution of the original differential equation.

20. If $y_p(t) = at^2 + bt + c$, then

$$\begin{aligned}\frac{dy_p}{dt} + 2y_p &= 2at + b + 2at^2 + 2bt + 2c \\ &= 2at^2 + (2a + 2b)t + (b + 2c).\end{aligned}$$

Then $y_p(t)$ is a solution if this quadratic is equal to $3t^2 + 2t - 1$. In other words, $y_p(t)$ is a solution if

$$\begin{cases} 2a = 3 \\ 2a + 2b = 2 \\ b + 2c = -1. \end{cases}$$

From the first equation, we have $a = 3/2$. Then from the second equation, we have $b = -1/2$. Finally, from the third equation, we have $c = -1/4$. The function

$$y_p(t) = \tfrac{3}{2}t^2 - \tfrac{1}{2}t - \tfrac{1}{4}$$

is a solution of the differential equation.

21. To find the general solution, we use the technique suggested in Exercise 19. We calculate two particular solutions—one for the right-hand side $t^2 + 2t + 1$ and one for the right-hand side e^{4t}.

With the right-hand side $t^2 + 2t + 1$, we guess a solution of the form

$$y_{p_1}(t) = at^2 + bt + c.$$

Then

$$\begin{aligned}\frac{dy_{p_1}}{dt} + 2y_{p_1} &= 2at + b + 2(at^2 + bt + c) \\ &= 2at^2 + (2a + 2b)t + (b + 2c).\end{aligned}$$

Then y_{p_1} is a solution if

$$\begin{cases} 2a = 1 \\ 2a + 2b = 2 \\ b + 2c = 1. \end{cases}$$

We get $a = 1/2, b = 1/2$, and $c = 1/4$.

With the right-hand side e^{4t}, we guess a solution of the form

$$y_{p_2}(t) = \alpha e^{4t}.$$

Then

$$\frac{dy_{p_2}}{dt} + 2y_{p_2} = 4\alpha e^{4t} + 2\alpha e^{4t} = 6\alpha e^{4t},$$

and y_{p_2} is a solution if $\alpha = 1/6$.

The general solution of the associated homogeneous equation is $y_h(t) = ke^{-2t}$, so the general solution of the original equation is

$$ke^{-2t} + \tfrac{1}{2}t^2 + \tfrac{1}{2}t + \tfrac{1}{4} + \tfrac{1}{6}e^{4t}.$$

To find the solution that satisfies the initial condition $y(0) = 0$, we evaluate the general solution at $t = 0$ and obtain

$$k + \tfrac{1}{4} + \tfrac{1}{6} = 0.$$

Hence, $k = -5/12$.

22. To find the general solution, we use the technique suggested in Exercise 19. We calculate two particular solutions—one for the right-hand side t^3 and one for the right-hand side $\sin 3t$.

With the right-hand side t^3, we are tempted to guess that there is a solution of the form at^3, but there isn't. Instead we guess a solution of the form

$$y_{p_1}(t) = at^3 + bt^2 + ct + d.$$

Then

$$\begin{aligned}\frac{dy_{p_1}}{dt} + y_{p_1} &= 3at^2 + 2bt + c + at^3 + bt^2 + ct + d \\ &= at^3 + (3a + b)t^2 + (2b + c)t + (c + d)\end{aligned}$$

Then y_{p_1} is a solution if

$$\begin{cases} a = 1 \\ 3a + b = 0 \\ 2b + c = 0 \\ c + d = 0. \end{cases}$$

We get $a = 1, b = -3, c = 6$, and $d = -6$.

With the right-hand side $\sin 3t$, we guess a solution of the form

$$y_{p_2}(t) = \alpha \cos 3t + \beta \sin 3t.$$

Then

$$\begin{aligned}\frac{dy_{p_2}}{dt} + y_{p_1} &= -3\alpha \sin 3t + 3\beta \cos 3t + \alpha \cos 3t + \beta \sin 3t \\ &= (\alpha + 3\beta) \cos 3t + (-3\alpha + \beta) \sin 3t.\end{aligned}$$

Then y_{p_2} is a solution if

$$\begin{cases} \alpha + 3\beta = 0 \\ -3\alpha + \beta = 1. \end{cases}$$

We get $\alpha = -3/10$ and $\beta = 1/10$.

The general solution of the associated homogeneous equation is $y_h(t) = ke^{-t}$, so the general solution of the original equation is

$$ke^{-t} + t^3 - 3t^2 + 6t - 6 - \tfrac{3}{10}\cos 3t + \tfrac{1}{10}\sin 3t.$$

To find the solution that satisfies the initial condition $y(0) = 0$, we evaluate the general solution at $t = 0$ and obtain

$$k - 6 - \tfrac{3}{10} = 0.$$

Hence, $k = 63/10$.

23. To find the general solution, we use the technique suggested in Exercise 19. We calculate two particular solutions—one for the right-hand side $2t$ and one for the right-hand side $-e^{4t}$.

With the right-hand side $2t$, we guess a solution of the form

$$y_{p_1}(t) = at + b.$$

Then

$$\begin{aligned} \frac{dy_{p_1}}{dt} - 3y_{p_1} &= a - 3(at + b) \\ &= -3at + (a - 3b). \end{aligned}$$

Then y_{p_1} is a solution if

$$\begin{cases} -3a = 2 \\ a - 3b = 0. \end{cases}$$

We get $a = -2/3$, and $b = -2/9$.

With the right-hand side $-e^{4t}$, we guess a solution of the form

$$y_{p_2}(t) = \alpha e^{4t}.$$

Then

$$\frac{dy_{p_2}}{dt} - 3y_{p_2} = 4\alpha e^{4t} - 3\alpha e^{4t} = \alpha e^{4t},$$

and y_{p_2} is a solution if $\alpha = -1$.

The general solution of the associated homogeneous equation is $y_h(t) = ke^{3t}$, so the general solution of the original equation is

$$y(t) = ke^{3t} - \tfrac{2}{3}t - \tfrac{2}{9} - e^{4t}.$$

To find the solution that satisfies the initial condition $y(0) = 0$, we evaluate the general solution at $t = 0$ and obtain

$$y(0) = k - \tfrac{2}{9} - 1.$$

Hence, $k = 11/9$ if $y(0) = 0$.

29. **(a)** The differential equation modeling the problem is

$$\frac{dP}{dt} = .011P + 1{,}040,$$

where \$1,040 is the amount of money added to the account per year (assuming a "continuous deposit").

(b) To find the general solution, we first compute the general solution of the associated homogeneous equation. It is $P_h(t) = ke^{0.011t}$.

To find a particular solution of the nonhomogeneous equation, we observe that the equation is autonomous, and we calculate its equilibrium solution. It is $P(t) = -1{,}040/.011 \approx -94{,}545.46$ for all t. (This equilibrium solution is what we would have calculated if we had guessed a constant.)

Hence, the general solution is

$$P(t) = -94{,}545.46 + ke^{0.011t}.$$

Since the account initially has \$1,000 in it, the initial condition is $P(0) = 1{,}000$. Solving

$$1000 = -94{,}545.46 + ke^{0.011(0)}$$

yields $k = 95{,}545.46$. Therefore, our model is

$$P(t) = -94{,}545.46 + 95{,}545.46e^{0.011t}.$$

To find the amount on deposit after 5 years, we evaluate $P(5)$ and obtain

$$-94{,}545.46 + 95{,}545.46e^{0.011(5)} \approx 6{,}402.20.$$

30. Let $M(t)$ be the amount of money left at time t. Then, we have the initial condition $M(0) = \$70{,}000$. Money is being added to the account at a rate of 1.5% and removed from the account at a rate of \$30,000 per year, so

$$\frac{dM}{dt} = 0.015M - 30{,}000.$$

To find the general solution, we first compute the general solution of the associated homogeneous equation. It is $M_h(t) = ke^{0.015t}$.

To find a particular solution of the nonhomogeneous equation, we observe that the equation is autonomous, and we calculate its equilibrium solution. It is $M(t) = 30{,}000/.015 = \$2{,}000{,}000$ for all t. (This equilibrium solution is what we would have calculated if we had guessed a constant.)

Therefore we have

$$M(t) = 2{,}000{,}000 + ke^{0.015t}.$$

Using the initial condition $M(0) = 70{,}000$, we have

$$2{,}000{,}000 + k = 70{,}000,$$

so $k = -1{,}930{,}000$ and

$$M(t) = 2{,}000{,}000 - 1{,}930{,}000e^{0.015t}.$$

Solving for the value of t when $M(t) = 0$, we have

$$2{,}000{,}000 - 1{,}930{,}000e^{0.015t} = 0,$$

32. Note that $dy/dt = 1/5$ for this function. Substituting $y(t) = t/5$ in the right-hand side of the differential equation yields

$$(\cos t)\left(\frac{t}{5}\right) + \frac{1}{5}(1 - t\cos t),$$

which also equals $1/5$. Hence, $y(t) = t/5$ is a solution.

33. **(a)** We know that

$$\frac{dy_h}{dt} = a(t)y_h \quad \text{and} \quad \frac{dy_p}{dt} = a(t)y_p + b(t).$$

Then

$$\begin{aligned}\frac{d(y_h + y_p)}{dt} &= a(t)y_h + a(t)y_p + b(t)\\ &= a(t)(y_h + y_p) + b(t).\end{aligned}$$

(b) We know that

$$\frac{dy_p}{dt} = a(t)y_p + b(t) \quad \text{and} \quad \frac{dy_q}{dt} = a(t)y_q + b(t).$$

Then

$$\begin{aligned}\frac{d(y_p - y_q)}{dt} &= (a(t)y_p + b(t)) - (a(t)y_q + b(t))\\ &= a(t)(y_p - y_q).\end{aligned}$$

34. Suppose k is a constant and $y_1(t)$ is a solution. Then we know that $ky_1(t)$ is also a solution. Hence,

$$\frac{d(ky_1)}{dt} = f(t, ky_1)$$

for all t. Also,

$$\frac{d(ky_1)}{dt} = k\frac{dy_1}{dt} = kf(t, y_1)$$

because $y_1(t)$ is a solution. Therefore, we have

$$f(t, ky_1) = kf(t, y_1)$$

for all t. In particular, if $y_1(t) \neq 0$, we can pick $k = 1/y_1(t)$, and we get

$$f(t, 1) = \frac{1}{y_1(t)} f(t, y_1(t)).$$

In other words,

$$y_1(t) f(t, 1) = f(t, y_1(t))$$

for all t for which $y_1(t) \neq 0$. If we ignore the dependence on t, we have

$$yf(t, 1) = f(t, y)$$

for all $y \neq 0$ because we know that there is a solution $y_1(t)$ that solves the initial-value problem $y_1(t) = y$. By continuity, we know that the equality

$$yf(t, 1) = f(t, y)$$