EXERCISES FOR SECTION 1.1

1. Note that $dy/dt = 0$ if and only if $y = -3$. Therefore, the constant function $y(t) = -3$ for all t is the only equilibrium solution.

2. Note that $dy/dt = 0$ for all t only if $y^2 - 2 = 0$. Therefore, the only equilibrium solutions are $y(t) = -\sqrt{2}$ for all t and $y(t) = +\sqrt{2}$ for all t.

3. **(a)** The equilibrium solutions correspond to the values of P for which $dP/dt = 0$ for all t. For this equation, $dP/dt = 0$ for all t if $P = 0$ or $P = 230$.

(b) The population is increasing if $dP/dt > 0$. That is, $P(1 - P/230) > 0$. Hence, $0 < P < 230$.

(c) The population is decreasing if $dP/dt < 0$. That is, $P(1 - P/230) < 0$. Hence, $P > 230$ or $P < 0$. Since this is a population model, $P < 0$ might be considered "nonphysical."

4. **(a)** The equilibrium solutions correspond to the values of P for which $dP/dt = 0$ for all t. For this equation, $dP/dt = 0$ for all t if $P = 0$, $P = 50$, or $P = 200$.

(b) The population is increasing if $dP/dt > 0$. That is, $P < 0$ or $50 < P < 200$. Note, $P < 0$ might be considered "nonphysical" for a population model.

(c) The population is decreasing if $dP/dt < 0$. That is, $0 < P < 50$ or $P > 200$.

5. In order to answer the question, we first need to analyze the sign of the polynomial $y^3 - y^2 - 12y$. Factoring, we obtain

$$y^3 - y^2 - 12y = y(y^2 - y - 12) = y(y - 4)(y + 3).$$

(a) The equilibrium solutions correspond to the values of y for which $dy/dt = 0$ for all t. For this equation, $dy/dt = 0$ for all t if $y = -3$, $y = 0$, or $y = 4$.

(b) The solution $y(t)$ is increasing if $dy/dt > 0$. That is, $-3 < y < 0$ or $y > 4$.

(c) The solution $y(t)$ is decreasing if $dy/dt < 0$. That is, $y < -3$ or $0 < y < 4$.

6. **(a)** The rate of change of the amount of radioactive material is dr/dt. This rate is proportional to the amount r of material present at time t. With $-\lambda$ as the proportionality constant, we obtain the differential equation

$$\frac{dr}{dt} = -\lambda r.$$

Note that the minus sign (along with the assumption that λ is positive) means that the material decays.

(b) The only additional assumption is the initial condition $r(0) = r_0$. Consequently, the corresponding initial-value problem is

$$\frac{dr}{dt} = -\lambda r, \quad r(0) = r_0.$$

7. The general solution of the differential equation $dr/dt = -\lambda r$ is $r(t) = r_0 e^{-\lambda t}$ where $r(0) = r_0$ is the initial amount.

(a) We have $r(t) = r_0 e^{-\lambda t}$ and $r(5230) = r_0/2$. Thus

$$\frac{r_0}{2} = r_0 e^{-\lambda \cdot 5230}$$

$$\frac{1}{2} = e^{-\lambda \cdot 5230}$$

$$\ln \frac{1}{2} = -\lambda \cdot 5230$$

$$-\ln 2 = -\lambda \cdot 5230$$

because $\ln 1/2 = -\ln 2$. Thus,

$$\lambda = \frac{\ln 2}{5230} \approx 0.000132533.$$

(b) We have $r(t) = r_0 e^{-\lambda t}$ and $r(8) = r_0/2$. By a computation similar to the one in part (a), we have

$$\lambda = \frac{\ln 2}{8} \approx 0.0866434.$$

(c) If $r(t)$ is the number of atoms of C-14, then the units for dr/dt is number of atoms per year. Since $dr/dt = -\lambda r$, λ is "per year." Similarly, for I-131, λ is "per day." The unit of measurement of r does not matter.

(d) We get the same answer because the original quantity, r_0, cancels from each side of the equation. We are only concerned with the proportion remaining (one-half of the original amount).

8. We will solve for k percent. In other words, we want to find t such that $r(t) = (k/100)r_0$, and we know that $r(t) = r_0 e^{-\lambda t}$, where $\lambda = (\ln 2)/5230$ from Exercise 7. Thus we have

$$r_0 e^{-\lambda t} = \frac{k}{100} r_0$$

$$e^{-\lambda t} = \frac{k}{100}$$

$$-\lambda t = \ln\left(\frac{k}{100}\right)$$

$$t = \frac{-\ln\left(\frac{k}{100}\right)}{\lambda}$$

$$t = \frac{\ln 100 - \ln k}{\lambda}$$

$$t = \frac{5230(\ln 100 - \ln k)}{\ln 2}.$$

Thus, there is 88% left when $t \approx 964.54$ years; there is 12% left when $t \approx 15{,}998$ years; 2% left when $t \approx 29{,}517$ years; and 98% left when $t \approx 152.44$ years.

9. (a) The general solution of the exponential decay model $dr/dt = -\lambda r$ is $r(t) = r_0 e^{-\lambda t}$, where $r(0) = r_0$ is the initial amount. Since $r(\tau) = r_0/e$, we have

$$\frac{r_0}{e} = r_0 e^{-\lambda \tau}$$

11. The solution of $dR/dt = kR$ with $R(0) = 4{,}000$ is

$$R(t) = 4{,}000\, e^{kt}.$$

Setting $t = 6$, we have $R(6) = 4{,}000\, e^{(k)(6)} = 130{,}000$. Solving for k, we obtain

$$k = \tfrac{1}{6} \ln\left(\tfrac{130{,}000}{4{,}000}\right) \approx 0.58.$$

Therefore, the rabbit population in the year 2010 would be $R(10) = 4{,}000\, e^{(0.58 \cdot 10)} \approx 1{,}321{,}198$ rabbits.

12. **(a)** In this analysis, we consider only the case where v is positive. The right-hand side of the differential equation is a quadratic in v, and it is zero if $v = \sqrt{mg/k}$. Consequently, the solution $v(t) = \sqrt{mg/k}$ for all t is an equilibrium solution. If $0 \le v < \sqrt{mg/k}$, then $dv/dt > 0$, and consequently, $v(t)$ is an increasing function. If $v > \sqrt{mg/k}$, then $dv/dt < 0$, and $v(t)$ is a decreasing function. In either case, $v(t) \to \sqrt{mg/k}$ as $t \to \infty$.

(b) See part (a).

13. The rate of learning is dL/dt. Thus, we want to know the values of L between 0 and 1 for which dL/dt is a maximum. As $k > 0$ and $dL/dt = k(1 - L)$, dL/dt attains it maximum value at $L = 0$.

14. **(a)** Let $L_1(t)$ be the solution of the model with $L_1(0) = 1/2$ (the student who starts out knowing one-half of the list) and $L_2(t)$ be the solution of the model with $L_2(0) = 0$ (the student who starts out knowing none of the list). At time $t = 0$,

$$\frac{dL_1}{dt} = 2\,(1 - L_1(0)) = 2\left(1 - \tfrac{1}{2}\right) = 1,$$

and

$$\frac{dL_2}{dt} = 2\,(1 - L_2(0)) = 2.$$

Hence, the student who starts out knowing none of the list learns faster at time $t = 0$.

(b) The solution $L_2(t)$ with $L_2(0) = 0$ will learn one-half the list in some amount of time $t_* > 0$. For $t > t_*$, $L_2(t)$ will increase at exactly the same rate that $L_1(t)$ increases for $t > 0$. In other words, $L_2(t)$ increases at the same rate as $L_1(t)$ at t_* time units later. Hence, $L_2(t)$ will never catch up to $L_1(t)$ (although they both approach 1 as t increases). In other words, after a very long time $L_2(t) \approx L_1(t)$, but $L_2(t) < L_1(t)$.

15. **(a)** We have $L_B(0) = L_A(0) = 0$. So Aly's rate of learning at $t = 0$ is dL_A/dt evaluated at $t = 0$. At $t = 0$, we have

$$\frac{dL_A}{dt} = 2(1 - L_A) = 2.$$

Beth's rate of learning at $t = 0$ is

$$\frac{dL_B}{dt} = 3(1 - L_B)^2 = 3.$$

Hence Beth's rate is larger.

17. Let $P(t)$ be the population at time t, k be the growth-rate parameter, and N be the carrying capacity. The modified models are

(a) $dP/dt = k(1 - P/N)P - 100$

(b) $dP/dt = k(1 - P/N)P - P/3$

(c) $dP/dt = k(1 - P/N)P - a\sqrt{P}$, where a is a positive parameter.

18. **(a)** The differential equation is $dP/dt = 0.3P(1 - P/2500) - 100$. The equilibrium solutions of this equation correspond to the values of P for which $dP/dt = 0$ for all t. Using the quadratic formula, we obtain two such values, $P_1 \approx 396$ and $P_2 \approx 2104$. If $P > P_2$, $dP/dt < 0$, so $P(t)$ is decreasing. If $P_1 < P < P_2$, $dP/dt > 0$, so $P(t)$ is increasing. Hence the solution that satisfies the initial condition $P(0) = 2500$ decreases toward the equilibrium $P_2 \approx 2104$.

(b) The differential equation is $dP/dt = 0.3P(1 - P/2500) - P/3$. The equilibrium solutions of this equation are $P_1 \approx -277$ and $P_2 = 0$. If $P > 0$, $dP/dt < 0$, so $P(t)$ is decreasing. Hence, for $P(0) = 2500$, the population decreases toward $P = 0$ (extinction).

19. Several different models are possible. Let $R(t)$ denote the rhinoceros population at time t. The basic assumption is that there is a minimum threshold that the population must exceed if it is to survive. In terms of the differential equation, this assumption means that dR/dt must be negative if R is close to zero. Three models that satisfy this assumption are:

- If k is a growth-rate parameter and M is a parameter measuring when the population is "too small", then
$$\frac{dR}{dt} = kR\left(\frac{R}{M} - 1\right).$$
- If k is a growth-rate parameter and b is a parameter that determines the level the population will start to decrease ($R < b/k$), then
$$\frac{dR}{dt} = kR - b.$$
- If k is a growth-rate parameter and b is a parameter that determines the extinction threshold, then
$$\frac{dR}{dt} = kR - \frac{b}{R}.$$

In each case, if R is below a certain threshold, dR/dt is negative. Thus, the rhinos will eventually die out. The choice of which model to use depends on other assumptions. There are other equations that are also consistent with the basic assumption.

20. **(a)** The relative growth rate for the year 1990 is

$$\frac{1}{s(t)}\frac{ds}{dt} = \frac{1}{5.3}\left(\frac{7.6 - 3.5}{1991 - 1989}\right) \approx 0.387.$$

Hence, the relative growth rate for the year 1990 is 38.7%.

(b) If the quantity $s(t)$ grows exponentially, then we can model it as $s(t) = s_0 e^{kt}$, where s_0 and k are constants. Calculating the relative growth rate, we have

$$\frac{1}{s(t)}\frac{ds}{dt} = \frac{1}{s_0 e^{kt}}\left(ks_0 e^{kt}\right) = k.$$

Therefore, if a quantity grows exponentially, its relative growth rate is constant for all t.

22. **(a)** We consider dx/dt in each system. Setting $y = 0$ yields $dx/dt = 5x$ in system (i) and $dx/dt = x$ in system (ii). If the number x of prey is equal for both systems, dx/dt is larger in system (i). Therefore, the prey in system (i) reproduce faster if there are no predators.

(b) We must see what effect the predators (represented by the y-terms) have on dx/dt in each system. Since the magnitude of the coefficient of the xy-term is larger in system (ii) than in system (i), y has a greater effect on dx/dt in system (ii). Hence the predators have a greater effect on the rate of change of the prey in system (ii).

(c) We must see what effect the prey (represented by the x-terms) have on dy/dt in each system. Since x and y are both nonnegative, it follows that

$$-2y + \tfrac{1}{2}xy < -2y + 6xy,$$

and therefore, if the number of predators is equal for both systems, dy/dt is smaller in system (i). Hence more prey are required in system (i) than in system (ii) to achieve a certain growth rate.

23. **(a)** The independent variable is t, and x and y are dependent variables. Since each xy-term is positive, the presence of either species increases the rate of change of the other. Hence, these species cooperate. The parameter α is the growth-rate parameter for x, and γ is the growth-rate parameter for y. The parameter N represents the carrying capacity for x, but y has no carrying capacity. The parameter β measures the benefit to x of the interaction of the two species, and δ measures the benefit to y of the interaction.

(b) The independent variable is t, and x and y are the dependent variables. Since both xy-terms are negative, these species compete. The parameter γ is the growth-rate coefficient for x, and α is the growth-rate parameter for y. Neither population has a carrying capacity. The parameter δ measures the harm to x caused by the interaction of the two species, and β measures the harm to y caused by the interaction.

EXERCISES FOR SECTION 1.2

1. **(a)** Let's check Bob's solution first. Since $dy/dt = 1$ and

$$\frac{y(t)+1}{t+1} = \frac{t+1}{t+1} = 1,$$

Bob's answer is correct.

Now let's check Glen's solution. Since $dy/dt = 2$ and

$$\frac{y(t)+1}{t+1} = \frac{2t+2}{t+1} = 2,$$

Glen's solution is also correct.

Finally let's check Paul's solution. We have $dy/dt = 2t$ on one hand and

$$\frac{y(t)+1}{t+1} = \frac{t^2-1}{t+1} = t-1$$

on the other. Paul is wrong.