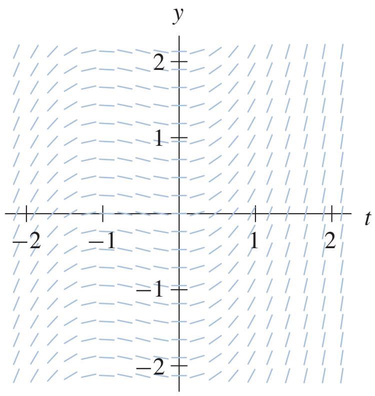
EXERCISES FOR SECTION 1.3

1.



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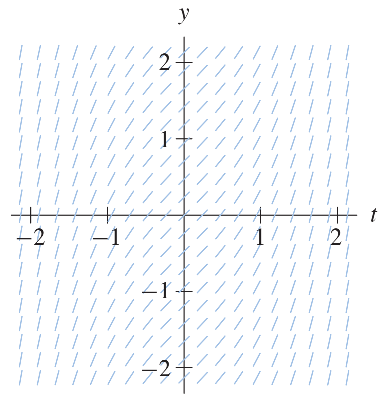

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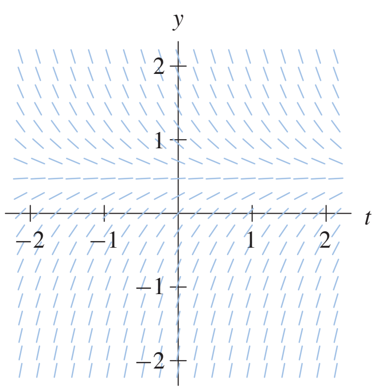

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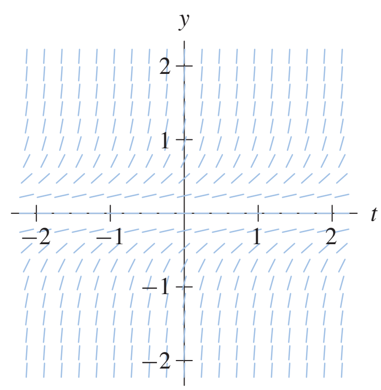

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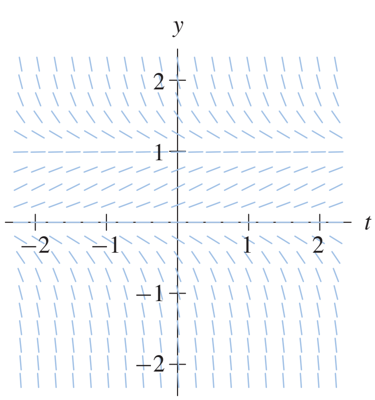

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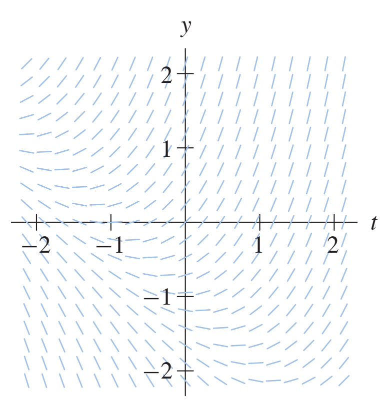
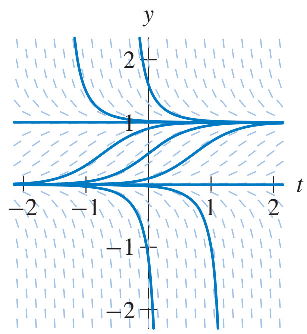
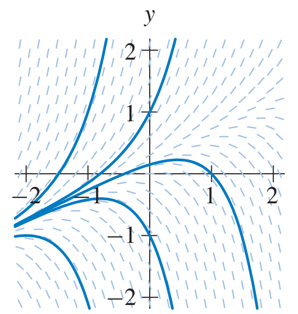


7. **(a)**



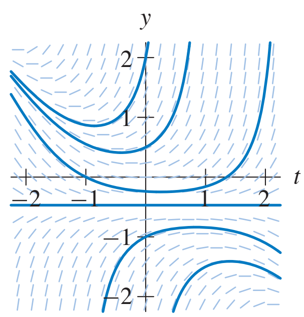
(b) The solution with $y(0) = 1/2$ approaches the equilibrium value $y = 1$ from below as t increases. It decreases toward $y = 0$ as t decreases.

8. **(a)**



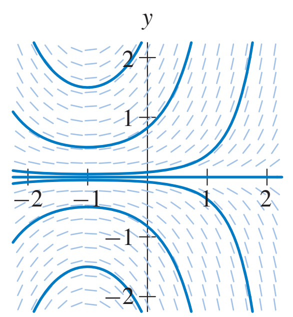
(b) The solution $y(t)$ with $y(0) = 1/2$ increases with $y(t) \to \infty$ as t increases. As t decreases, $y(t) \to -\infty$.

9. **(a)**



(b) The solution $y(t)$ with $y(0) = 1/2$ has $y(t) \to \infty$ both as t increases and as t decreases.

10. **(a)**



(b) The solution $y(t)$ with $y(0) = 1/2$ has $y(t) \to \infty$ both as t increases and as t decreases.

11. **(a)** On the line $y = 3$ in the ty-plane, all of the slope marks have slope -1.

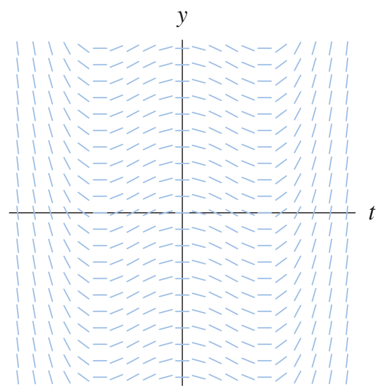
(b) Because f is continuous, if y is close to 3, then $f(t, y) < 0$. So any solution close to $y = 3$ must be decreasing. Therefore, solutions $y(t)$ that satisfy $y(0) < 3$ can never be larger than 3 for $t > 0$, and consequently $y(t) < 3$ for all t.

12. **(a)** Since $y(t) = 2$ for all t is a solution and $dy/dt = 0$ for all t, $f(t, y(t)) = f(t, 2) = 0$ for all t.

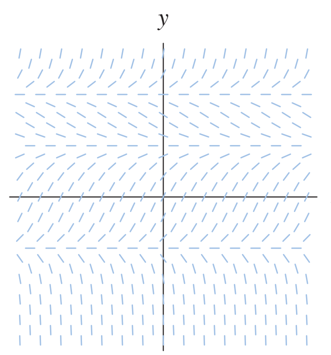
(b) Therefore, the slope marks all have zero slope along the horizontal line $y = 2$.

(c) If the graphs of solutions cannot cross in the ty-plane, then the graph of a solution must stay on the same side of the line $y = 2$ as it is at time $t = 0$. In Section 1.5, we discuss conditions that guarantee that graphs of solutions do not cross.

13. The slope field in the ty-plane is constant along vertical lines.



14. Because f depends only on y (the equation is autonomous), the slope field is constant along horizontal lines in the ty-plane. The roots of f correspond to equilibrium solutions. If $f(y) > 0$, the corresponding lines in the slope field have positive slope. If $f(y) < 0$, the corresponding lines in the slope field have negative slope.



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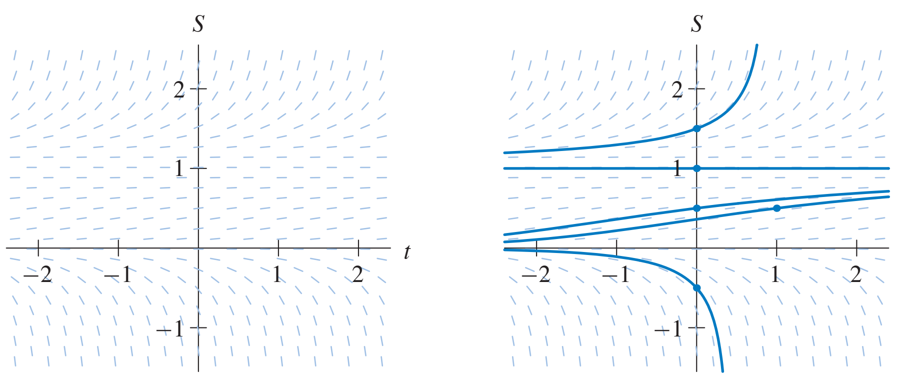


16. **(a)** This slope field is constant along horizontal lines, so it corresponds to an autonomous equation. The autonomous equations are (i), (ii), and (iii). This field does not correspond to equation (ii) because it has the equilibrium solution $y = -1$. The slopes are negative for $y < -1$. Consequently, this field corresponds to equation (iii).

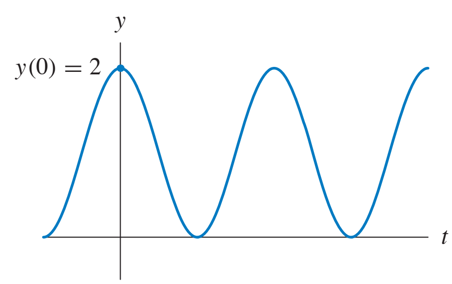
(b) Note that the slopes are constant along vertical lines—lines along which t is constant, so the right-hand side of the corresponding equation depends only on t. The only choices are equations (iv) and (viii). Since the slopes are negative for $-\sqrt{2} < t < \sqrt{2}$, this slope field corresponds to equation (viii).

(c) This slope field depends both on y and on t, so it can only correspond to equations (v), (vi), or (vii). Since this field has the equilibrium solution $y = 0$, this slope field corresponds to equation (v).

(d) This slope field also depends on both y and on t, so it can only correspond to equations (v), (vi), or (vii). This field does not correspond to equation (v) because $y = 0$ is not an equilibrium solution. Since the slopes are nonnegative for $y > -1$, this slope field corresponds to equation (vi).

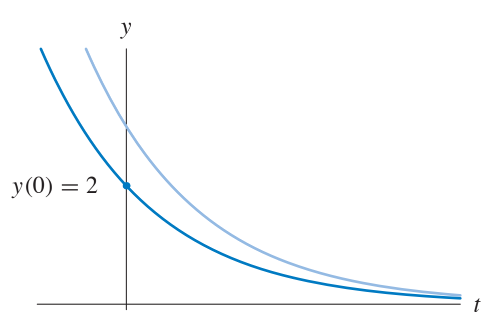
17. **(a)** Because the slope field is constant on vertical lines, the given information is enough to draw the entire slope field.

(b) The solution with initial condition $y(0) = 2$ is a vertical translation of the given solution. We only need change the "constant of integration" so that $y(0) = 2$.



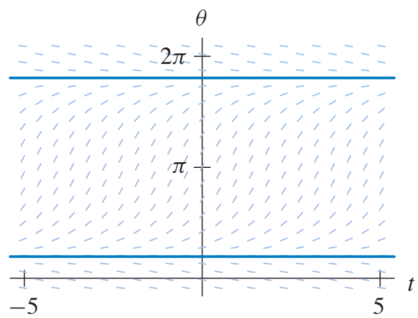
18. **(a)** Because the equation is autonomous, the slope field is constant on horizontal lines, so this solution provides enough information to sketch the slope field on the entire upper half plane. Also, if we assume that f is continuous, then the slope field on the line $y = 0$ must be horizontal.

(b) The solution with initial condition $y(0) = 2$ is a translate to the left of the given solution.

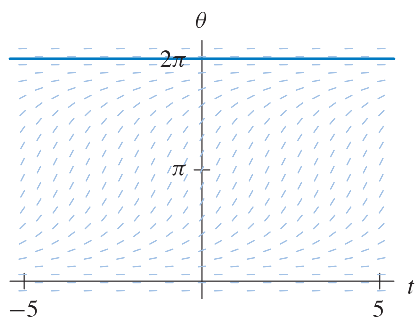


19. **(a)** Even though the question only asks for slope fields in this part, we superimpose the graphs of the equilibrium solutions on the fields to illustrate the equilibrium solutions (see part (b)).

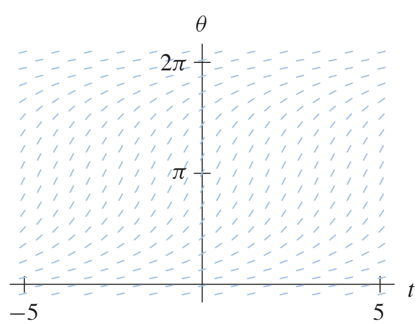
$I_1 = -0.1$



$I_2 = 0.0$



$I_3 = 0.1$



(b) For $I_1 = -0.1$, the equilibrium values satisfy the equation

$$1 - \cos\theta + (1 + \cos\theta)(-0.1) = 0.$$

We have

$$\begin{aligned} 0.9 - 1.1\cos\theta &= 0 \\ \cos\theta &= \tfrac{0.9}{1.1} \\ \theta &\approx \pm 0.613. \end{aligned}$$

Therefore, the equilibrium values are $\theta \approx 2\pi n \pm 0.613$ radians, where n is any integer. There are two equilibrium solutions with values $\theta \approx 0.613$ and $\theta \approx 5.670$ between 0 and 2π.

For $I_2 = 0.0$, similar calculations yield equilibrium values at even multiples of 2π, and for $I_3 = 0.1$, there are no equilibrium values.

(c) For $I_1 = -0.1$, the graphs of the equilibrium solutions divide the $t\theta$-plane into horizontal strips in which the signs of the slopes do not change. For example, if $0.613 < \theta < 5.670$ (approximately), then the slopes are positive. If $5.670 < \theta < 6.896$ (approximately), then the slopes are negative. Therefore, any solution $\theta(t)$ with an initial condition θ_0 that is between 0.613 and 6.896 (approximately) satisfies the limit $\theta(t) \to 5.670$ (approximately) as $t \to \infty$. Moreover, any solution $\theta(t)$ with an initial condition θ_0 that is between -0.613 and 5.670 (approximately) satisfies the limit $\theta(t) \to 0.613$ (approximately) as $t \to -\infty$.

For $I_2 = 0.0$, the graphs of the equilibrium solutions also divide the $t\theta$-plane into horizontal strips in which the signs of the slopes do not change. However, in this case, the slopes are always positive (or zero in the case of the equilibrium solutions). Therefore, for example, any solution $\theta(t)$ with an initial condition θ_0 that is between 0 and 2π satisfies the limits $\theta(t) \to 2\pi$ as $t \to \infty$ and $\theta(t) \to 0$ as $t \to -\infty$.

Lastly, if $I_3 = 0.1$, all of the slopes are positive, so all solutions are increasing for all t. The fact that $\theta(t) \to \infty$ as $t \to \infty$ requires an analytic estimate in addition to a qualitative analysis.

20. Separating variables, we have

$$\int \frac{dv_c}{v_c} = \int -\frac{1}{RC}\,dt$$

$$\ln|v_c| = -\frac{t}{RC} + c_1$$

$$|v_c| = c_2 e^{-t/RC}$$

where $c_2 = e^{c_1}$. We can eliminate the absolute value signs by allowing c_2 to be positive or negative. If we let $v_c(0) = c_2 e^0 = v_0$, then we obtain $c_2 = v_0$. Therefore $v_c(t) = v_0 e^{-t/RC}$ where $v_0 = v_c(0)$.

To check that this function is a solution, we calculate the left-hand side of the equation

$$\frac{dv_c}{dt} = \frac{d}{dt} v_0 e^{-t/RC} = -\frac{v_0}{RC} e^{-t/RC}.$$

The result agrees with the right-hand side because

$$-\frac{v_c}{RC} = -\frac{v_0 e^{-t/RC}}{RC} = -\frac{v_0}{RC} e^{-t/RC}.$$

21. Separating variables, we obtain

$$\int \frac{dv_c}{K - v_c} = \int \frac{dt}{RC}.$$

Integrating both sides, we have

$$-\ln|K - v_c| = \frac{t}{RC} + c_1,$$