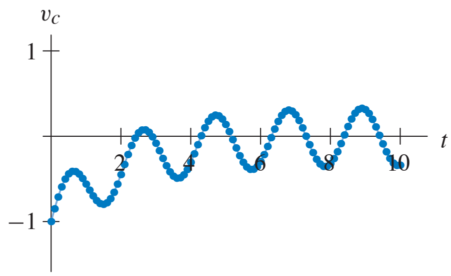
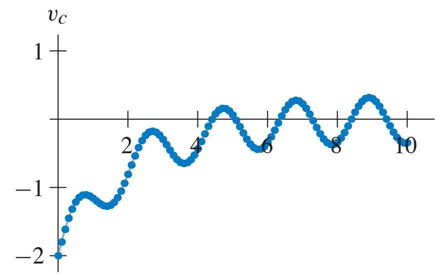
20.



Graph of approximate solution obtained using Euler's method with $\Delta t = 0.1$.

21.



Graph of approximate solution obtained using Euler's method with $\Delta t = 0.1$.

EXERCISES FOR SECTION 1.5

1. Since the constant function $y_1(t) = 3$ for all t is a solution, then the graph of any other solution $y(t)$ with $y(0) < 3$ cannot cross the line $y = 3$ by the Uniqueness Theorem. So $y(t) < 3$ for all t in the domain of $y(t)$.

2. Since $y(0) = 1$ is between the equilibrium solutions $y_2(t) = 0$ and $y_3(t) = 2$, we must have $0 < y(t) < 2$ for all t because the Uniqueness Theorem implies that graphs of solutions cannot cross (or even touch in this case).

3. Because $y_2(0) < y(0) < y_1(0)$, we know that

$$-t^2 = y_2(t) < y(t) < y_1(t) = t + 2$$

for all t. This restricts how large positive or negative $y(t)$ can be for a given value of t (that is, between $-t^2$ and $t + 2$). As $t \to -\infty$, $y(t) \to -\infty$ between $-t^2$ and $t + 2$ ($y(t) \to -\infty$ as $t \to -\infty$ at least linearly, but no faster than quadratically).

4. Because $y_1(0) < y(0) < y_2(0)$, the solution $y(t)$ must satisfy $y_1(t) < y(t) < y_2(t)$ for all t by the Uniqueness Theorem. Hence $-1 < y(t) < 1 + t^2$ for all t.

5. The Existence Theorem implies that a solution with this initial condition exists, at least for a small t-interval about $t = 0$. This differential equation has equilibrium solutions $y_1(t) = 0$, $y_2(t) = 1$, and $y_3(t) = 3$ for all t. Since $y(0) = 4$, the Uniqueness Theorem implies that $y(t) > 3$ for all t in the domain of $y(t)$. Also, $dy/dt > 0$ for all $y > 3$, so the solution $y(t)$ is increasing for all t in its domain. Finally, $y(t) \to 3$ as $t \to -\infty$.

6. Note that $dy/dt = 0$ if $y = 0$. Hence, $y_1(t) = 0$ for all t is an equilibrium solution. By the Uniqueness Theorem, this is the only solution that is 0 at $t = 0$. Therefore, $y(t) = 0$ for all t.

7. The Existence Theorem implies that a solution with this initial condition exists, at least for a small t-interval about $t = 0$. Because $1 < y(0) < 3$ and $y_1(t) = 1$ and $y_2(t) = 3$ are equilibrium solutions of the differential equation, we know that the solution exists for all t and that $1 < y(t) < 3$ for all t by the Uniqueness Theorem. Also, $dy/dt < 0$ for $1 < y < 3$, so dy/dt is always negative for this solution. Hence, $y(t) \to 1$ as $t \to \infty$, and $y(t) \to 3$ as $t \to -\infty$.

8. The Existence Theorem implies that a solution with this initial condition exists, at least for a small t-interval about $t = 0$. Note that $y(0) < 0$. Since $y_1(t) = 0$ is an equilibrium solution, the Uniqueness Theorem implies that $y(t) < 0$ for all t. Also, $dy/dt < 0$ if $y < 0$, so $y(t)$ is decreasing for all t, and $y(t) \to -\infty$ as t increases. As $t \to -\infty$, $y(t) \to 0$.

9. **(a)** To check that $y_1(t) = t^2$ is a solution, we compute

$$\frac{dy_1}{dt} = 2t$$

and

$$\begin{aligned}-y_1^2 + y_1 + 2y_1t^2 + 2t - t^2 - t^4 &= -(t^2)^2 + (t^2) + 2(t^2)t^2 + 2t - t^2 - t^4 \\ &= 2t.\end{aligned}$$

To check that $y_2(t) = t^2 + 1$ is a solution, we compute

$$\frac{dy_2}{dt} = 2t$$

and

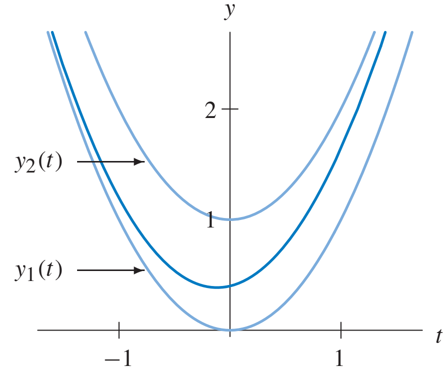
$$\begin{aligned}-y_2^2 + y_2 + 2y_2t^2 + 2t - t^2 - t^4 &= -(t^2+1)^2 + (t^2+1) + 2(t^2+1)t^2 \\ &\quad + 2t - t^2 - t^4 \\ &= 2t.\end{aligned}$$

(b) The initial values of the two solutions are $y_1(0) = 0$ and $y_2(0) = 1$. Thus if $y(t)$ is a solution and $y_1(0) = 0 < y(0) < 1 = y_2(0)$, then we can apply the Uniqueness Theorem to obtain

$$y_1(t) = t^2 < y(t) < t^2 + 1 = y_2(t)$$

for all t. Note that since the differential equation satisfies the hypothesis of the Existence and Uniqueness Theorem over the entire ty-plane, we can continue to extend the solution as long as it does not escape to $\pm\infty$ in finite time. Since it is bounded above and below by solutions that exist for all time, $y(t)$ is defined for all time also.

(c)



10. **(a)** If $y(t) = 0$ for all t, then $dy/dt = 0$ and $2\sqrt{|y(t)|} = 0$ for all t. Hence, the function that is constantly zero satisfies the differential equation.

(b) First, consider the case where $y > 0$. The differential equation reduces to $dy/dt = 2\sqrt{y}$. If we separate variables and integrate, we obtain

$$\sqrt{y} = t - c,$$

where c is any constant. The graph of this equation is the half of the parabola $y = (t - c)^2$ where $t \geq c$.

Next, consider the case where $y < 0$. The differential equation reduces to $dy/dt = 2\sqrt{-y}$. If we separate variables and integrate, we obtain

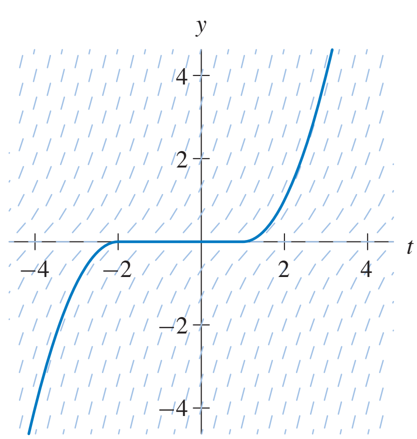
$$\sqrt{-y} = d - t,$$

where d is any constant. The graph of this equation is the half of the parabola $y = -(d - t)^2$ where $t \leq d$.

To obtain all solutions, we observe that any choice of constants c and d where $c \geq d$ leads to a solution of the form

$$y(t) = \begin{cases} -(d-t)^2, & \text{if } t \leq d; \\ 0, & \text{if } d \leq t \leq c; \\ (t-c)^2, & \text{if } t \geq c. \end{cases}$$

(See the following figure for the case where $d = -2$ and $c = 1$.)



(c) The partial derivative $\partial f/\partial y$ of $f(t, y) = \sqrt{|y|}$ does not exist along the t-axis.

(d) If $y_0 = 0$, `HPGSolver` plots the equilibrium solution that is constantly zero. If $y_0 \neq 0$, it plots a solution whose graph crosses the t-axis. This is a solution where $c = d$ in the formula given above.

11. The key observation is that the differential equation is not defined when $t = 0$.

(a) Note that $dy_1/dt = 0$ and $y_1/t^2 = 0$, so $y_1(t)$ is a solution.

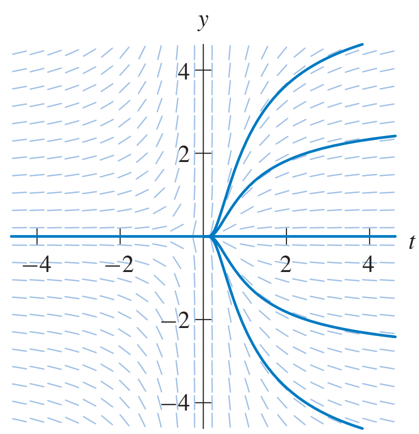
(b) Separating variables, we have

$$\int \frac{dy}{y} = \int \frac{dt}{t^2}.$$

Solving for y we obtain $y(t) = ce^{-1/t}$, where c is any constant. Thus, for any real number c, define the function $y_c(t)$ by

$$y_c(t) = \begin{cases} 0 & \text{for } t \le 0; \\ ce^{-1/t} & \text{for } t > 0. \end{cases}$$

For each c, $y_c(t)$ satisfies the differential equation for all $t \neq 0$.



There are infinitely many solutions of the form $y_c(t)$ that agree with $y_1(t)$ for $t < 0$.

(c) Note that $f(t, y) = y/t^2$ is not defined at $t = 0$. Therefore, we *cannot* apply the Uniqueness Theorem for the initial condition $y(0) = 0$. The "solution" $y_c(t)$ given in part (b) actually represents two solutions, one for $t < 0$ and one for $t > 0$.

12. **(a)** Note that

$$\frac{dy_1}{dt} = \frac{d}{dt}\left(\frac{1}{t-1}\right) = -\frac{1}{(t-1)^2} = -(y_1(t))^2$$

and

$$\frac{dy_2}{dt} = \frac{d}{dt}\left(\frac{1}{t-2}\right) = -\frac{1}{(t-2)^2} = -(y_2(t))^2,$$

so both $y_1(t)$ and $y_2(t)$ are solutions.

(b) Note that $y_1(0) = -1$ and $y_2(0) = -1/2$. If $y(t)$ is another solution whose initial condition satisfies $-1 < y(0) < -1/2$, then $y_1(t) < y(t) < y_2(t)$ for all t by the Uniqueness Theorem. Also, since $dy/dt < 0$, $y(t)$ is decreasing for all t in its domain. Therefore, $y(t) \to 0$ as $t \to -\infty$, and the graph of $y(t)$ has a vertical asymptote between $t = 1$ and $t = 2$.

15. **(a)** The equation is separable. We separate, integrate

$$\int (y+2)^2\,dy = \int dt,$$

and solve for y to obtain the general solution

$$y(t) = (3t + c)^{1/3} - 2,$$

where c is any constant. To obtain the desired solution, we use the initial condition $y(0) = 1$ and solve

$$1 = (3 \cdot 0 + c)^{1/3} - 2$$

for c to obtain $c = 27$. So the solution to the given initial-value problem is

$$y(t) = (3t + 27)^{1/3} - 2.$$

(b) This function is defined for all t. However, $y(-9) = -2$, and the differential equation is not defined at $y = -2$. Strictly speaking, the solution exists only for $t > -9$.

(c) As $t \to \infty$, $y(t) \to \infty$. As $t \to -9^+$, $y(t) \to -2$.

16. **(a)** The equation is separable. Separating variables we obtain

$$\int (y-2)\,dy = \int t\,dt.$$

Solving for y with help from the quadratic formula yields the general solution

$$y(t) = 2 \pm \sqrt{t^2 + c}\,.$$

To find c, we let $t = -1$ and $y = 0$, and we obtain $c = 3$. The desired solution is therefore $y(t) = 2 - \sqrt{t^2 + 3}$

(b) Since $t^2 + 2$ is always positive and $y(t) < 2$ for all t, the solution $y(t)$ is defined for all real numbers.

(c) As $t \to \pm\infty$, $t^2 + 3 \to \infty$. Therefore,

$$\lim_{t \to \pm\infty} y(t) = -\infty.$$

17. This exercise shows that solutions of autonomous equations cannot have local maximums or minimums. Hence they must be either constant or monotonically increasing or monotonically decreasing. A useful corollary is that a function $y(t)$ that oscillates cannot be the solution of an autonomous differential equation.

(a) Note $dy_1/dt = 0$ at $t = t_0$ because $y_1(t)$ has a local maximum. Because $y_1(t)$ is a solution, we know that $dy_1/dt = f(y_1(t))$ for all t in the domain of $y_1(t)$. In particular,

$$0 = \left.\frac{dy_1}{dt}\right|_{t=t_0} = f(y_1(t_0)) = f(y_0),$$

so $f(y_0) = 0$.

(b) This differential equation is autonomous, so the slope marks along any given horizontal line are parallel. Hence, the slope marks along the line $y = y_0$ must all have zero slope.

(c) For all t,

$$\frac{dy_2}{dt} = \frac{d(y_0)}{dt} = 0$$

because the derivative of a constant function is zero, and for all t

$$f(y_2(t)) = f(y_0) = 0.$$

So $y_2(t)$ is a solution.

(d) By the Uniqueness Theorem, we know that two solutions that are in the same place at the same time are the same solution. We have $y_1(t_0) = y_0 = y_2(t_0)$. Moreover, $y_1(t)$ is assumed to be a solution, and we showed that $y_2(t)$ is a solution in parts (a) and (b) of this exercise. So $y_1(t) = y_2(t)$ for all t. In other words, $y_1(t) = y_0$ for all t.

(e) Follow the same four steps as before. We still have $dy_1/dt = 0$ at $t = t_0$ because y_1 has a local minimum at $t = t_0$.

18. **(a)** Solving for r, we get

$$r = \left(\frac{3v}{4\pi}\right)^{1/3}.$$

Consequently,

$$\begin{aligned} s(t) &= 4\pi\left(\frac{3v}{4\pi}\right)^{2/3} \\ &= cv(t)^{2/3}, \end{aligned}$$

where c is a constant. Since we are assuming that the rate of growth of $v(t)$ is proportional to its surface area $s(t)$, we have

$$\frac{dv}{dt} = kv^{2/3},$$

where k is a constant.

(b) The partial derivative with respect to v of dv/dt does not exist at $v = 0$. Hence the Uniqueness Theorem tells us nothing about the uniqueness of solutions that involve $v = 0$. In fact, if we use the techniques described in the section related to the uniqueness of solutions for $dy/dt = 3y^{2/3}$, we can find infinitely many solutions with this initial condition.

(c) Since it does not make sense to talk about rain drops with negative volume, we always have $v \geq 0$. Once $v > 0$, the evolution of the drop is completely determined by the differential equation.

What is the physical significance of a drop with $v = 0$? It is tempting to interpret the fact that solutions can have $v = 0$ for an arbitrary amount of time before beginning to grow as a statement that the rain drops can spontaneously begin to grow at any time. Since the model gives no information about when a solution with $v = 0$ starts to grow, it is not very useful for the understanding the initial formation of rain drops. The safest assertion is to say is the model breaks down if $v = 0$.