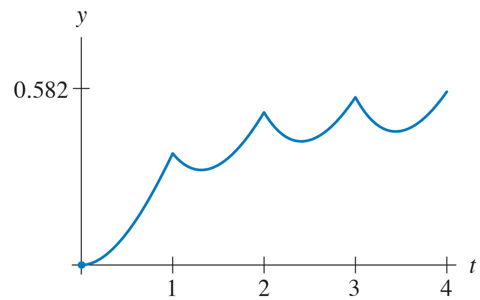

EXERCISES FOR SECTION 6.3

1. We use integration by parts twice to compute

$$\mathcal{L}[\sin \omega t] = \int_0^\infty \sin \omega t \, e^{-st} \, dt.$$

First, letting $u = \sin \omega t$ and $dv = e^{-st} \, dt$, we get

$$\begin{aligned}
\mathcal{L}[\sin \omega t] &= \sin \omega t \left. \frac{e^{-st}}{-s} \right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \omega \cos \omega t \, dt \\
&= \lim_{b \to \infty} \left[\left. \frac{e^{-st}}{-s} \sin \omega t \right|_0^b \right] + \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt \\
&= \frac{\omega}{s} \int_0^\infty e^{-st} \cos \omega t \, dt,
\end{aligned}$$

since the limit of $e^{-sb} \sin \omega b$ is 0 as $b \to \infty$ and $s > 0$.

Using integration by parts on

$$\int_0^\infty e^{-st} \cos \omega t \, dt,$$

with $u = \cos \omega t$ and $dv = e^{-st} \, dt$, we get

$$\begin{aligned}
\int_0^\infty e^{-st} \cos \omega t \, dt &= \frac{e^{-st}}{-s} \cos \omega t \Big|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} (-\omega \sin \omega t) \, dt \\
&= \lim_{b \to \infty} \left[\left. \frac{e^{-st}}{-s} \cos \omega t \right|_0^b \right] - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt \\
&= \lim_{b \to \infty} \left[\frac{e^{-sb}}{-s} \cos \omega b \right] + \frac{1}{s} - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt \\
&= \frac{1}{s} - \frac{\omega}{s} \int_0^\infty e^{-st} \sin \omega t \, dt,
\end{aligned}$$

since the limit of $e^{-sb}\cos\omega b$ is 0 as $b \to \infty$ and $s > 0$.

Thus,

$$\begin{aligned}\int_0^\infty \sin\omega t\, e^{-st}\, dt &= \frac{\omega}{s}\int_0^\infty e^{-st}\cos\omega t\, dt \\ &= \frac{\omega}{s}\left(\frac{1}{s} - \frac{\omega}{s}\int_0^\infty \sin\omega t\, e^{-st}\, dt\right) \\ &= \frac{\omega}{s^2} - \frac{\omega^2}{s^2}\int_0^\infty \sin\omega t\, e^{-st}\, dt.\end{aligned}$$

So

$$\frac{s^2+\omega^2}{s^2}\int_0^\infty \sin\omega t\, e^{-st}\, dt = \frac{\omega}{s^2},$$

and

$$\int_0^\infty \sin\omega t\, e^{-st}\, dt = \frac{\omega}{s^2+\omega^2}.$$

2. We use integration by parts twice to compute

$$\mathcal{L}[\cos\omega t] = \int_0^\infty \cos\omega t\, e^{-st}\, dt.$$

First, letting $u = \cos\omega t$ and $dv = e^{-st}\, dt$, we get

$$\begin{aligned}\mathcal{L}[\sin\omega t] &= \frac{e^{-st}}{-s}\cos\omega t\bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}(-\omega\sin\omega t)\, dt \\ &= \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\cos\omega t\bigg|_0^b\right] - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t\, dt \\ &= \frac{1}{s} - \int_0^\infty \frac{\omega}{s}e^{-st}\sin\omega t\, dt,\end{aligned}$$

since the limit of $e^{-sb}\cos\omega b$ is 0 as $b \to \infty$ and $s > 0$.

Using integration by parts on

$$\int_0^\infty e^{-st}\sin\omega t\, dt,$$

with $u = \sin\omega t$ and $dv = e^{-st}\, dt$, we get that

$$\begin{aligned}\int_0^\infty e^{-st}\sin\omega t\, dt &= \frac{e^{-st}}{-s}\sin\omega t\bigg|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\omega\cos\omega t\, dt \\ &= \lim_{b\to\infty}\left[\frac{e^{-st}}{-s}\sin\omega t\bigg|_0^b\right] + \frac{\omega}{s}\int_0^\infty e^{-st}\cos\omega t\, dt \\ &= \frac{\omega}{s}\int_0^\infty e^{-st}\cos\omega t\, dt,\end{aligned}$$

since the limit of $e^{-sb}\sin\omega b$ is 0 as $b \to \infty$ and $s > 0$.

Thus

$$\int_0^\infty \cos\omega t\, e^{-st}\, dt = \frac{1}{s} - \frac{\omega}{s}\int_0^\infty e^{-st}\sin\omega t\, dt$$

$$= \frac{1}{s} - \frac{\omega^2}{s^2}\int_0^\infty e^{-st}\cos\omega t\, dt.$$

So

$$\frac{s^2+\omega^2}{s^2}\int_0^\infty \cos\omega t\, e^{-st}\, dt = \frac{1}{s},$$

and

$$\int_0^\infty \cos\omega t\, e^{-st}\, dt = \frac{s}{s^2+\omega^2}.$$

3. We need to compute

$$\mathcal{L}[e^{at}\sin\omega t] = \int_0^\infty e^{at}\sin\omega t e^{-st}\, dt = \int_0^\infty \sin\omega t e^{-(s-a)t}\, dt.$$

We can do this using integration by parts twice and ending up with $\mathcal{L}[e^{at}\sin\omega t]$ on both sides of the equation. Alternately, if we let $r = s - a$, then

$$\int_0^\infty \sin\omega t\, e^{-(s-a)t}\, dt = \int_0^\infty \sin\omega t\, e^{-rt}\, dt$$

The integral on the right is the Laplace transform of $\sin\omega t$ with r as the new independent variable. From Exercise 1, we know

$$\int_0^\infty \sin\omega t\, e^{-rt}\, dt = \frac{\omega}{r^2+\omega^2}.$$

Substituting back we have

$$\mathcal{L}[e^{at}\sin\omega t] = \frac{\omega}{(s-a)^2+\omega^2}.$$

4. We need to compute

$$\mathcal{L}[e^{at}\cos\omega t] = \int_0^\infty e^{at}\cos\omega t\, e^{-st}\, dt = \int_0^\infty \cos\omega t\, e^{-(s-a)t}\, dt.$$

We can do this using integration by parts twice to end up with $\mathcal{L}[e^{at}\cos\omega t]$ on both sides of the equation. Alternately, if we let $r = s - a$, then

$$\int_0^\infty \cos\omega t\, e^{-(s-a)t}\, dt = \int_0^\infty \cos\omega t\, e^{-rt}\, dt.$$

The integral on the right is the Laplace transform of $\cos\omega t$ with r as the new independent variable. From the table, we know

$$\int_0^\infty \cos\omega t\, e^{-rt}\, dt = \frac{r}{r^2+\omega^2}.$$

Then substituting back we have

$$\mathcal{L}[e^{at}\cos\omega t] = \frac{s-a}{(s-a)^2+\omega^2}.$$

5. Using the formula

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] = s^2\mathcal{L}[y] - y'(0) - sy(0),$$

and the linearity of the Laplace transform, we get that

$$s^2\mathcal{L}[y] - y'(0) - sy(0) + \omega^2\mathcal{L}[y] = 0.$$

Substituting the initial conditions and solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{s}{s^2+\omega^2}.$$

6. Since

$$\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2},$$

we can compute that

$$\frac{d}{d\omega}\mathcal{L}[\cos\omega t] = \frac{-s(2\omega)}{(s^2+\omega^2)^2} = \frac{-2\omega s}{(s^2+\omega^2)^2},$$

but

$$\frac{d}{d\omega}\mathcal{L}[\cos\omega t] = \mathcal{L}\left[\frac{d}{d\omega}\cos\omega t\right] = \mathcal{L}[-t\sin\omega t].$$

We can bring the derivative with respect to ω inside the Laplace transform because the Laplace transform is an integral with respect to t, that is,

$$\frac{d}{d\omega}\mathcal{L}[\cos\omega t] = \frac{d}{d\omega}\int_0^\infty \cos\omega t\, e^{-st}\,dt = \int_0^\infty \frac{d}{d\omega}\left(\cos\omega t\, e^{-st}\right)\,dt.$$

Canceling the minus signs on left and right gives

$$\mathcal{L}[t\sin\omega t] = \frac{2\omega s}{(s^2+\omega^2)^2}.$$

7. Since

$$\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2},$$

we can compute that

$$\frac{d}{d\omega}\mathcal{L}[\sin\omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2},$$

but

$$\frac{d}{d\omega}\mathcal{L}[\sin\omega t] = \mathcal{L}\left[\frac{d}{d\omega}\sin\omega t\right] = \mathcal{L}[t\cos\omega t].$$

So

$$\mathcal{L}[t\cos\omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}.$$

11. In this case, $b = 2$, and $(s + b/2)^2 = (s + 1)^2 = s^2 + 2s + 1$, so $s^2 + 2s + 10 = (s + 1)^2 + 3^2$.

12. In this case, $b = -4$, and $(s + b/2)^2 = (s - 2)^2 = s^2 - 4s + 4$, so $s^2 - 4s + 5 = (s - 2)^2 + 1^2$.

13. In this case, $b = 1$, and $(s + b/2)^2 = (s + 1/2)^2 = s^2 + s + 1/4$, so $s^2 + s + 1 = (s + 1/2)^2 + 3/4 = (s + 1/2)^2 + (\sqrt{3}/2)^2$.

14. In this case, $b = 6$, and $(s + b/2)^2 = (s + 3)^2 = s^2 + 6s + 9$, so $s^2 + 6s + 10 = (s + 3)^2 + 1^2$.

15. In Exercise 11, we completed the square and obtained $s^2 + 2s + 10 = (s + 1)^2 + 3^2$, so

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{1}{s^2 + 2s + 10}\right] &= \mathcal{L}^{-1}\left[\frac{1}{(s + 1)^2 + 3^2}\right] \\
&= \frac{1}{3}\mathcal{L}^{-1}\left[\frac{3}{(s + 1)^2 + 3^2}\right] \\
&= \frac{1}{3}e^{-t}\sin 3t.
\end{aligned}$$

16. In Exercise 12, we completed the square and obtained $s^2 - 4s + 5 = (s - 2)^2 + 1^2$, so

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{s}{s^2 - 4s + 5}\right] &= \mathcal{L}^{-1}\left[\frac{s}{(s - 2)^2 + 1^2}\right] \\
&= \mathcal{L}^{-1}\left[\frac{s - 2}{(s - 2)^2 + 1^2}\right] + \mathcal{L}^{-1}\left[\frac{2}{(s - 2)^2 + 1^2}\right] \\
&= e^{2t}\cos t + e^{2t}(2\sin t) = e^{2t}(\cos t + 2\sin t).
\end{aligned}$$

17. In Exercise 13, we completed the square and obtained $s^2 + s + 1 = (s + 1/2)^2 + (\sqrt{3}/2)^2$, so

$$\frac{2s + 3}{s^2 + s + 1} = \frac{2s + 3}{(s + 1/2)^2 + (\sqrt{3}/2)^2}.$$

We want to put this fraction in the right form so that we can use the formulas for $\mathcal{L}[e^{at}\cos\omega t]$ and $\mathcal{L}[e^{at}\sin\omega t]$. We see that

$$\begin{aligned}
\frac{2s + 3}{(s + 1/2)^2 + (\sqrt{3}/2)^2} &= \frac{2s + 1}{(s + 1/2)^2 + (\sqrt{3}/2)^2} + \frac{2}{(s + 1/2)^2 + (\sqrt{3}/2)^2} \\
&= \frac{2(s + 1/2)}{(s + 1/2)^2 + (\sqrt{3}/2)^2} + \frac{(4/\sqrt{3})(\sqrt{3}/2)}{(s + 1/2)^2 + (\sqrt{3}/2)^2}.
\end{aligned}$$

So

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{2s + 3}{s^2 + s + 1}\right] &= 2\mathcal{L}^{-1}\left[\frac{(s + 1/2)}{(s + 1/2)^2 + (\sqrt{3}/2)^2}\right] + \frac{4}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}/2}{(s + 1/2)^2 + (\sqrt{3}/2)^2}\right] \\
&= 2e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + \frac{4}{\sqrt{3}}e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right).
\end{aligned}$$

18. In Exercise 14, we completed the square and obtained $s^2 + 6s + 10 = (s + 3)^2 + 1^2$, so

$$\frac{s+1}{s^2+6s+10} = \frac{s+1}{(s+3)^2+1^2}.$$

We want to put this fraction in the right form so that we can use the formulas for $\mathcal{L}[e^{at}\cos\omega t]$ and $\mathcal{L}[e^{at}\sin\omega t]$. We see that

$$\frac{s+1}{(s+3)^2+1^2} = \frac{s+3}{(s+3)^2+1^2} - \frac{2}{(s+3)^2+1^2}.$$

So

$$\mathcal{L}^{-1}\left[\frac{s+1}{s^2+6s+10}\right] = e^{-3t}\cos t - 2e^{-3t}\sin t.$$

19. We compute

$$\begin{aligned}
\mathcal{L}\left[e^{(a+ib)t}\right] &= \int_0^\infty e^{(a+ib)t}e^{-st}\,dt \\
&= \int_0^\infty e^{-(s-(a+ib))t}\,dt \\
&= -\frac{1}{s-(a+ib)}\left(\lim_{u\to\infty}\left[e^{-(s-a)u}e^{-ibu}\right]-1\right).
\end{aligned}$$

The limit is zero as long as $s > a$. Hence,

$$\mathcal{L}\left[e^{(a+ib)t}\right] = \frac{1}{s-(a+ib)}$$

if $s > a$ and undefined otherwise. This is the same formula as for real exponentials. It can also be written

$$\mathcal{L}\left[e^{(a+ib)t}\right] = \frac{s-a+ib}{(s-a)^2+b^2}.$$

20. This follows from linearity:

$$\begin{aligned}
\mathcal{L}[y] &= \mathcal{L}[y_{\text{re}} + i\,y_{\text{im}}] \\
&= \int_0^\infty (y_{\text{re}} + i\,y_{\text{im}})\,e^{-st}\,dt \\
&= \int_0^\infty y_{\text{re}}(t)\,e^{-st}\,dt + i\int_0^\infty y_{\text{im}}(t)\,e^{-st}\,dt \\
&= \mathcal{L}[y_{\text{re}}] + i\,\mathcal{L}[y_{\text{im}}].
\end{aligned}$$

21. We recall that

$$e^{at} \cos \omega t = \mathrm{Re}(e^{(a+ib)t}).$$

So

$$\begin{aligned}
\mathcal{L}[e^{at} \cos \omega t] &= \mathrm{Re}(\mathcal{L}[e^{(a+ib)t}]) \\
&= \mathrm{Re}\left(\frac{s - a + i\omega}{(s-a)^2 + \omega^2}\right) \\
&= \frac{s-a}{(s-a)^2 + \omega^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{L}[e^{at} \sin \omega t] &= \mathrm{Im}(\mathcal{L}[e^{(a+ib)t}]) \\
&= \mathrm{Im}\left(\frac{s - a + i\omega}{(s-a)^2 + \omega^2}\right) \\
&= \frac{\omega}{(s-a)^2 + \omega^2}.
\end{aligned}$$

22. **(a)** The roots of $s^2 + 2s + 5$ are $-1 \pm 2i$, so the quadratic factors into

$$(s - (-1 + 2i))(s - (-1 - 2i)).$$

(b) We write

$$\frac{1}{s^2 + 2s + 5} = \frac{A}{s + 1 + 2i} + \frac{B}{s + 1 - 2i}.$$

So, finding common denominators (that is, usual partial fractions but with complex numbers) gives

$$\begin{cases} A + B = 0 \\ A + B + 2i(-A + B) = 1. \end{cases}$$

Solving, we get $A = i/4$ and $B = -i/4$, so

$$\frac{1}{s^2 + 2s + 5} = \frac{i/4}{s + 1 + 2i} + \frac{-i/4}{s + 1 - 2i}.$$

(c) We know that

$$\begin{aligned}
\mathcal{L}^{-1}\left[\frac{i/4}{s + 1 + 2i}\right] &= \frac{i}{4} e^{(-1-2i)t} \\
&= \frac{i}{4}\left(e^{-t} \cos(-2t) + i e^{-t} \sin(-2t)\right) \\
&= \frac{1}{4}\left(e^{-t} \sin 2t + i e^{-t} \cos 2t\right)
\end{aligned}$$

and

$$\mathcal{L}^{-1}\left[\frac{-i/4}{s+1-2i}\right] = -\frac{i}{4}e^{(-1+2i)t}$$

$$= -\frac{i}{4}\left(e^{-t}\cos 2t + ie^{-t}\sin 2t\right)$$

$$= \frac{1}{4}\left(e^{-t}\sin 2t - ie^{-t}\cos 2t\right).$$

(d) Adding, we get

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+5}\right] = \frac{1}{2}e^{-t}\sin 2t.$$

23. Using the quadratic formula, we see that the roots of $s^2 + 2s + 10 = 0$ are $s = -1 \pm 3i$. Thus $s^2 + 2s + 10 = (s + 1 + 3i)(s + 1 - 3i)$. So we want to find A and B so that

$$\frac{1}{s^2+2s+10} = \frac{A}{s+1+3i} + \frac{B}{s+1-3i}.$$

So, finding common denominators (that is, usual partial fractions only with complex numbers) gives

$$\begin{cases} A + B = 0 \\ A + B + 3i(-A + B) = 1. \end{cases}$$

Solving, we get $A = i/6$ and $B = -i/6$, so

$$\frac{1}{s^2+2s+10} = \frac{i/6}{s+1+3i} + \frac{-i/6}{s+1-3i}.$$

Thus

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2s+10}\right] = \mathcal{L}^{-1}\left[\frac{i/6}{s+1+3i} + \frac{-i/6}{s+1-3i}\right]$$

$$= \frac{i}{6}e^{(-1-3i)t} - \frac{i}{6}e^{(-1+3i)t}$$

$$= \frac{i}{6}\left(e^{-t}\cos(-3t) + ie^{-t}\sin(-3t)\right) - \frac{i}{6}\left(e^{-t}\cos 3t + ie^{-t}\sin 3t\right)$$

$$= -\frac{i}{6}\left(2ie^{-t}\sin 3t\right)$$

$$= \frac{1}{3}e^{-t}\sin 3t.$$

24. Using the quadratic formula, we find that the roots of the denominator are $2 \pm i$. Hence, we can factor the denominator into

$$s^2 - 4s + 5 = (s - (2 + i))(s - (2 - i)).$$

Using partial fraction decomposition, we get

$$\frac{s}{s - 4s + 5} = \frac{A}{s - (2 + i)} + \frac{B}{s - (2 - i)},$$

which gives the equations

$$\begin{cases} A + B = 1 \\ -(2 - i)A - (2 + i)B = 0. \end{cases}$$

Solving for A and B gives $A = 1/2 - i$ and $B = 1/2 + i$ so

$$\frac{s}{s - 4s + 5} = \frac{\frac{1}{2} - i}{s - (2 + i)} + \frac{\frac{1}{2} + i}{s - (2 - i)}.$$

Taking the inverse Laplace transform of the terms on the right gives

$$\left(\tfrac{1}{2} - i\right) e^{(2+i)t} + \left(\tfrac{1}{2} + i\right) e^{(2-i)t}.$$

Using Euler's formula to expand the exponentials and simplifying gives

$$e^{2t}(\cos t + 2\sin t).$$

25. Using the quadratic formula, the roots of the denominator are $(-1 \pm i\sqrt{3})/2$. Hence, we can factor the denominator into

$$\left(s - \left(\tfrac{-1+i\sqrt{3}}{2}\right)\right)\left(s - \left(\tfrac{-1-i\sqrt{3}}{2}\right)\right).$$

We then do the partial fractions decomposition

$$\frac{2s + 3}{s^2 + s + 1} = \frac{A}{s - \left(\frac{-1+i\sqrt{3}}{2}\right)} + \frac{B}{s - \left(\frac{-1-i\sqrt{3}}{2}\right)},$$

which gives rise to the equations

$$\begin{cases} A + B = 2 \\ \left(\frac{1+i\sqrt{3}}{2}\right) A + \left(\frac{1-i\sqrt{3}}{2}\right) B = 3. \end{cases}$$

Solving yields $A = 1 - \frac{2}{\sqrt{3}}i$ and $B = 1 + \frac{2}{\sqrt{3}}i$. So

$$\frac{2s + 3}{s^2 + s + 1} = \frac{1 - \frac{2}{\sqrt{3}}i}{s - \left(\frac{-1+i\sqrt{3}}{2}\right)} + \frac{1 + \frac{2}{\sqrt{3}}i}{s - \left(\frac{-1-i\sqrt{3}}{2}\right)}.$$

Taking inverse Laplace transforms of the right-hand side gives

$$\left(1 - \tfrac{2}{\sqrt{3}} i\right) e^{(-1+i\sqrt{3})t/2} + \left(1 + \tfrac{2}{\sqrt{3}} i\right) e^{(-1-i\sqrt{3})t/2}.$$

Using Euler's formula to replace the complex exponentials and simplifying yields

$$2e^{-t/2} \cos\left(\tfrac{\sqrt{3}}{2} t\right) + \tfrac{4}{\sqrt{3}} e^{-t/2} \sin\left(\tfrac{\sqrt{3}}{2} t\right).$$

26. Using the quadratic formula, we find the roots of the denominator are $-3 \pm i$ so the denominator can be factored

$$s^2 + 6s + 10 = (s - (-3 + i))(s - (-3 - i)).$$

The partial fractions decomposition is

$$\frac{s+1}{s^2 + 6s + 10} = \frac{A}{s - (-3 - i)} + \frac{B}{s - (-3 + i)},$$

which leads to the equations

$$\begin{cases} A + B = 1 \\ (3 - i)A + (3 + i)B = 1. \end{cases}$$

Solving, we find $A = \frac{1}{2} - i$ and $B = \frac{1}{2} + i$, so

$$\frac{s+1}{s^2 + 6s + 10} = \frac{\frac{1}{2} - i}{s - (-3 - i)} + \frac{\frac{1}{2} + i}{s - (-3 + i)}.$$

Taking inverse Laplace transform of the right-hand side gives

$$\left(\tfrac{1}{2} - i\right) e^{(-3-i)t} + \left(\tfrac{1}{2} + i\right) e^{(-3+i)t}$$

and using Euler's formula and simplifying gives

$$e^{-3t} \cos t - 2e^{-3t} \sin t.$$

27. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2 y}{dt^2}\right] + 4\mathcal{L}[y] = \frac{8}{s},$$

and using the fact that $\mathcal{L}[d^2y/dt^2] = s^2\mathcal{L}[y] - sy(0) - y'(0)$, we have

$$(s^2 + 4)\mathcal{L}[y] - sy(0) - y'(0) = \frac{8}{s}.$$

(b) Substituting the initial conditions yields

$$(s^2+4)\mathcal{L}[y] - 11s - 5 = \frac{8}{s},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{11s+5}{s^2+4} + \frac{8}{s(s^2+4)}.$$

The partial fractions decomposition of $8/(s(s^2+4))$ is

$$\frac{8}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4}.$$

Putting the right-hand side over a common denominator gives us

$$(A+B)s^2 + Cs + 4A = 8,$$

and consequently, $A = 2$, $B = -2$, and $C = 0$. In other words,

$$\frac{8}{s(s^2+4)} = \frac{2}{s} + \frac{-2s}{s^2+4}.$$

We obtain

$$\mathcal{L}[y] = \frac{2}{s} + \frac{9s+5}{s^2+4}.$$

(c) To take the inverse Laplace transform, we rewrite $\mathcal{L}[y]$ in the form

$$\mathcal{L}[y] = \frac{2}{s} + 9\left(\frac{s}{s^2+4}\right) + \frac{5}{2}\left(\frac{2}{s^2+4}\right).$$

Therefore, $y(t) = 2 + 9\cos 2t + \frac{5}{2}\sin 2t$.

28. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] - \mathcal{L}[y] = \frac{1}{s-2},$$

and using the fact that $\mathcal{L}[d^2y/dt^2] = s^2\mathcal{L}[y] - sy(0) - y'(0)$, we have

$$(s^2-1)\mathcal{L}[y] - sy(0) - y'(0) = \frac{1}{s-2}.$$

(b) Substituting the initial conditions yields

$$(s^2-1)\mathcal{L}[y] - s + 1 = \frac{1}{s-2},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{1}{s+1} + \frac{1}{(s-2)(s^2-1)}.$$

Using the partial fractions decomposition

$$\frac{1}{(s-2)(s^2-1)} = \frac{\frac{1}{3}}{s-2} + \frac{-\frac{1}{2}}{s-1} + \frac{\frac{1}{6}}{s+1},$$

we obtain

$$\mathcal{L}[y] = \frac{\frac{1}{3}}{s-2} + \frac{-\frac{1}{2}}{s-1} + \frac{\frac{7}{6}}{s+1}.$$

(c) Taking the inverse Laplace transform, we have

$$y(t) = \tfrac{1}{3}e^{2t} - \tfrac{1}{2}e^{t} + \tfrac{7}{6}e^{-t}.$$

29. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] - 4\mathcal{L}\left[\frac{dy}{dt}\right] + 5\mathcal{L}[y] = \frac{2}{s-1},$$

and using the formulas for $\mathcal{L}[dy/dt]$ and $\mathcal{L}[d^2y/dt^2]$ in terms of $\mathcal{L}[y]$, we have

$$(s^2 - 4s + 5)\mathcal{L}[y] - sy(0) - y'(0) + 4y(0) = \frac{2}{s-1}.$$

(b) Substituting the initial conditions yields

$$(s^2 - 4s + 5)\mathcal{L}[y] - 3s + 11 = \frac{2}{s-2},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{3s-11}{s^2-4s+5} + \frac{2}{(s-1)(s^2-4s+5)}.$$

Using the partial fractions decomposition

$$\frac{2}{(s-1)(s^2-4s+5)} = \frac{1}{s-1} + \frac{-s+3}{s^2-4s+5},$$

we obtain

$$\mathcal{L}[y] = \frac{1}{s-1} + \frac{2s-8}{s^2-4s+5}.$$

(c) In order to compute the inverse Laplace transform, we first write

$$s^2 - 4s + 5 = (s-2)^2 + 1$$

by completing the square, and then we write

$$\frac{2s-8}{s^2-4s+5} = \frac{2(s-2)}{(s-2)^2+1} - \frac{4}{(s-2)^2+1}.$$

Taking the inverse Laplace transform, we have

$$y(t) = e^t + 2e^{2t}\cos t - 4e^{2t}\sin t.$$

30. **(a)** Taking the Laplace transform of both sides of the equation, we obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 6\mathcal{L}\left[\frac{dy}{dt}\right] + 13\mathcal{L}[y] = 13\frac{e^{-4s}}{s},$$

and using the formulas for $\mathcal{L}[dy/dt]$ and $\mathcal{L}[d^2y/dt^2]$ in terms of $\mathcal{L}[y]$, we have

$$(s^2 + 6s + 13)\mathcal{L}[y] - sy(0) - y'(0) - 6y(0) = 13\frac{e^{-4s}}{s}.$$

(b) Substituting the initial conditions yields

$$(s^2 + 6s + 13)\mathcal{L}[y] - 3s - 19 = 13\frac{e^{-4s}}{s},$$

and solving for $\mathcal{L}[y]$ we get

$$\mathcal{L}[y] = \frac{3s + 19}{s^2 + 6s + 13} + \left(\frac{13}{s(s^2 + 6s + 13)}\right) e^{-4s}.$$

Using the partial fractions decomposition

$$\frac{13}{s(s^2 + 6s + 13)} = \frac{1}{s} - \frac{s + 6}{s^2 + 6s + 13},$$

we obtain

$$\mathcal{L}[y] = \frac{3s + 19}{s^2 + 6s + 13} + \left(\frac{1}{s} - \frac{s + 6}{s^2 + 6s + 13}\right) e^{-4s}.$$

(c) In order to compute the inverse Laplace transform, we first write

$$s^2 + 6s + 13 = (s + 3)^2 + 4$$

by completing the square, and then we write

$$\frac{3s + 19}{s^2 + 6s + 13} = 3\left(\frac{s + 3}{(s + 3)^2 + 4}\right) + 5\left(\frac{2}{(s + 3)^2 + 4}\right)$$

and

$$\frac{s + 6}{s^2 + 6s + 13} = \left(\frac{s + 3}{(s + 3)^2 + 4}\right) + \frac{3}{2}\left(\frac{2}{(s + 3)^2 + 4}\right).$$

Taking the inverse Laplace transform, we have

$$y(t) = 3e^{-3t}\cos 2t + 5e^{-3t}\sin 2t + u_4(t)\left(1 - e^{-3(t-4)}\cos 2(t - 4) - \tfrac{3}{2}e^{-3(t-4)}\sin 2(t - 4)\right).$$

31. **(a)** Note that this is resonant forcing of an undamped oscillator. We take the Laplace transform of both sides

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 4\mathcal{L}[y] = \mathcal{L}[\cos 2t]$$

and obtain

$$s^2\mathcal{L}[y] + 2s + 4\mathcal{L}[y] = \frac{s}{s^2 + 4}.$$

(b) Solving for $\mathcal{L}[y]$, we get

$$\mathcal{L}[y] = -\frac{2s}{s^2+4} + \frac{s}{(s^2+4)^2}.$$

(c) To take the inverse Laplace transform, we note that

$$\mathcal{L}^{-1}\left[-\frac{2s}{s^2+4}\right] = -2\cos 2t$$

and

$$\mathcal{L}^{-1}\left[\frac{s}{(s^2+4)^2}\right] = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{4s}{(s^2+4)^2}\right] = \frac{t}{4}\sin 2t.$$

So

$$y(t) = -2\cos 2t + \frac{t}{4}\sin 2t,$$

which is of the form we would expect for a resonant response.

32. **(a)** We take Laplace transform of both sides to obtain

$$\mathcal{L}\left[\frac{d^2y}{dt^2}\right] + 3\mathcal{L}[y] = \mathcal{L}[u_4(t)\cos(5(t-4))],$$

which is equivalent to

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 3\mathcal{L}[y] = \frac{e^{-4s}s}{s^2+25}.$$

Using the given initial conditions, we have

$$(s^2+3)\mathcal{L}[y] + 2 = \frac{e^{-4s}s}{s^2+25}.$$

(b) Solving for $\mathcal{L}[y]$ gives

$$\mathcal{L}[y] = \frac{-2}{s^2+3} + \frac{e^{-4s}s}{(s^2+3)(s^2+25)}.$$

(c) Now to find the inverse Laplace transform, we first note that

$$\mathcal{L}^{-1}\left[\frac{-2}{s^2+3}\right] = \frac{-2}{\sqrt{3}}\mathcal{L}^{-1}\left[\frac{\sqrt{3}}{s^2+3}\right] = \frac{-2}{\sqrt{3}}\sin\sqrt{3}\,t.$$

For the second term, we first use partial fractions to write

$$\frac{s}{(s^2+3)(s^2+25)} = \frac{1}{22}\left(\frac{s}{s^2+3} - \frac{s}{s^2+25}\right).$$

Hence

$$\mathcal{L}^{-1}\left[\frac{e^{-4s}s}{(s^2+3)(s^2+25)}\right] = \frac{1}{22}u_4(t)\left(\cos(\sqrt{3}\,(t-4)) - \cos(5(t-4))\right).$$

Combining the two results, we obtain the solution of the initial-value problem

$$y(t) = -\frac{2}{\sqrt{3}}\sin\sqrt{3}\,t + \frac{1}{22}u_4(t)\left(\cos(\sqrt{3}\,(t-4)) - \cos(5(t-4))\right).$$

35. **(a)** Consider

$$\mathcal{L}[f] = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt.$$

We can calculate dF/ds by differentiating under the integral sign. That is,

$$\begin{aligned} \frac{dF}{ds} &= \int_0^\infty \frac{\partial}{\partial s}\left(f(t)\, e^{-st}\right) dt \\ &= \int_0^\infty f(t)(-t)e^{-st}\, dt \\ &= -\mathcal{L}[tf(t)]. \end{aligned}$$

(b) If we apply this result to

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2} = \omega(s^2+\omega^2)^{-1},$$

we obtain

$$\begin{aligned} \mathcal{L}[t \sin \omega t] &= -\omega(-1)(s^2+\omega^2)^{-2}(2s) \\ &= \frac{2\omega s}{(s^2+\omega^2)^2}. \end{aligned}$$

Compare this result with the result of Exercise 6.

EXERCISES FOR SECTION 6.4

1. This is the $\frac{0}{0}$ case of L'Hôpital's Rule. Differentiating numerator and denominator with respect to Δt, we obtain

$$\frac{se^{s\Delta t} - (-s)e^{-s\Delta t}}{2},$$

which simplifies to

$$\frac{s(e^{s\Delta t} + e^{-s\Delta t})}{2}.$$

Since both $e^{s\Delta t}$ and $e^{-s\Delta t}$ tend to 1 as $\Delta t \to 0$, the desired limit is s.

2. Taking Laplace transforms of both sides and applying the rules yields

$$s^2\mathcal{L}[y] - sy(0) - y'(0) + 3\mathcal{L}[y] = 5\mathcal{L}[\delta_2].$$

Simplifying, using the initial conditions, and the fact that $\mathcal{L}[\delta_2] = e^{-2s}$, we get

$$(s^2+3)\mathcal{L}[y] = 5e^{-2s}.$$