Math 2410-10/15 # Quiz #7 **March 27^{th}, 2017**

Note: Partial credit can not be awarded unless there is legible work to assess. Feel free to use the back of this page if you require additional space for your solutions.

1. Consider the following first order system of differential equations

$$\frac{dx}{dt} = -2x - y$$
$$\frac{dy}{dt} = 2x - 5y$$

(i) Rewrite the system in matrix form.
(ii) Show that $\mathbf{Y}_1(t) = (e^{-3t}, e^{-3t})$ is a solution of the system.
(iii) The function $\mathbf{Y}_2(t) = (e^{-4t}, 2e^{-4t})$ is also a solution. (You **do not** need to verify this.) Show that $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are linearly independent solutions.
(iv) Give the general solution of the system.
Bonus: Find the particular solution to this system for which $\mathbf{Y}(0) = (2, 3)$.

To rewrite this system in matrix form we let $\mathbf{Y} = (x, y)$ and find the coefficient matrix $A = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix}$ so that

$$\frac{dx}{dt} = -2x - y$$
$$\frac{dy}{dt} = 2x - 5y$$

becomes

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Though we need only show $\mathbf{Y}_1$ is a solution, we verify both as they follow the exact same procedure. To show that $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are solutions we simply need show that $d\mathbf{Y}_1/dt = A\mathbf{Y}_1$ and $d\mathbf{Y}_2/dt = A\mathbf{Y}_2$. Notice

LHS: $\frac{d\mathbf{Y}}{dt}$		RHS: $A\mathbf{Y}$
$\frac{d\mathbf{Y}_1}{dt} = \frac{d}{dt}\begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix}$	$=$	$A\mathbf{Y}_1 = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} = \begin{pmatrix} -2e^{-3t} - e^{-3t} \\ 2e^{-3t} - 5e^{-3t} \end{pmatrix} = \begin{pmatrix} -3e^{-3t} \\ -3e^{-3t} \end{pmatrix}$
$\frac{d\mathbf{Y}_2}{dt} = \frac{d}{dt}\begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix} = \begin{pmatrix} -4e^{-4t} \\ -8e^{-4t} \end{pmatrix}$	$=$	$A\mathbf{Y}_2 = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix} = \begin{pmatrix} -2e^{-4t} - 2e^{-4t} \\ 2e^{-4t} - 10e^{-4t} \end{pmatrix} = \begin{pmatrix} -4e^{-4t} \\ -8e^{-4t} \end{pmatrix}$

Hence, we see both $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are solutions. We show next that these two solutions are linearly independent, that is, that their initial conditions are linearly independent. We have $\mathbf{Y}_1(0) = (e^{-3\cdot 0}, e^{-3\cdot 0}) = (1,1)$ and similarly, $\mathbf{Y}_2(0) = (1,2)$. To show these two vectors are linearly independent we consider the following determinant:

$$\det \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = 2 - 1 = 1 \neq 0.$$

As this determinant is nonzero we can conclude that the two vectors are linearly independent and as a consequence $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are linearly independent solutions. The general solution to a linear system is $\mathbf{Y}(t) = k_1\mathbf{Y}_1 + k_2\mathbf{Y}_2$ where k_1 and k_2 are arbitrary constants and $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are any two linearly independent solutions. We have that (e^{-3t}, e^{-3t}) and $(e^{-4t}, 2e^{-4t})$ are linearly independent solutions to the system. Thus, the general solution is

$$\mathbf{Y}(t) = k_1 \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} + k_2 \begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix}$$

where k_1 and k_2 are arbitrary constants. To answer the bonus question, that is, to solve the IVP

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \mathbf{Y}(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

we simply need determine k_1 and k_2 such that

$$\mathbf{Y}(0) = k_1 \begin{pmatrix} e^0 \\ e^0 \end{pmatrix} + k_2 \begin{pmatrix} e^0 \\ 2e^0 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

We can find the desired values of k_1 and k_2 by solving the system of equations

$$\begin{cases} k_1 + k_2 & = 2 \\ k_1 + 2k_2 & = 3 \end{cases}$$

It is easy to see that $k_1 = k_2 = 1$ solve this system. Therefore the function

$$\mathbf{Y}(t) = \begin{pmatrix} e^{-3t} \\ e^{-3t} \end{pmatrix} + \begin{pmatrix} e^{-4t} \\ 2e^{-4t} \end{pmatrix} = \begin{pmatrix} e^{-3t} + e^{-4t} \\ e^{-3t} + 2e^{-4t} \end{pmatrix}$$

is the desired solution to the given IVP.

2. Find two linearly independent straight-line solutions for the following system of differential equations.

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 4 & 6 \\ 0 & 7 \end{pmatrix} \mathbf{Y}$$

To find straight-line solutions we need determine the eigenvalues of the coefficient matrix and find an eigenvector for each of these. The eigenvalues are those λ such that

$$\det \begin{pmatrix} 4-\lambda & 6 \\ 0 & 7-\lambda \end{pmatrix} = (4-\lambda)(7-\lambda) - 6\cdot 0 = (4-\lambda)(7-\lambda) = 0.$$

Thus the eigenvalues are $\lambda = 4$ and $\lambda = 7$. To find an eigenvector with eigenvalue 4 we find one of the infinitely many solutions to the system

$$\begin{cases} (4-4)x + 6y &= 0 \\ 0x + (7-4)y &= 0 \end{cases}$$

Solutions to this system are (x, y) such that $y = 0$ and $x \neq 0$. Take $(1, 0)$ as our eigenvector with eigenvalue 4. We repeat this process to find an eigenvector with eigenvalue 7. Solutions to

$$\begin{cases} (4-7)x + 6y &= 0 \\ 0x + (7-7)y &= 0 \end{cases}$$

are (x, y) such that $x = 2y$ and $y \neq 0$. Take $(2, 1)$ as our eigenvector with eigenvalue 7. Thus two straight-line solutions to this system are

$$\mathbf{Y}_1(t) = e^{4t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{Y}_2(t) = e^{7t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

As these two solutions have initial conditions which are eigenvectors with differing eigenvalues, we conclude they are linearly independent.