Linear systems with complex eigenvalues
An example

Math 2410-010/015

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Our chief concern here is to find the general solution of the system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix} \mathbf{Y}. \tag{1}$$

After doing so, we will use this general solution to solve an initial value problem for this system.

1 Finding the general solution

Considering the section in which we are reading this handout (3.4), we have reason to suspect this system will have complex eigenvalues, we do not *need* this information to solve the system though. When presented with a linear system of any sort, we have methods for solving it regardless of the type of eigenvalues it has.[1] With this in mind, our first step in solving any linear system is to find the eigenvalues of the coefficient matrix. Here the coefficient matrix is

$$\begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix} \tag{2}$$

so to find its eigenvalues, we find numbers λ such that

$$\begin{aligned} \det \begin{pmatrix} 0-\lambda & 2 \\ -3 & 2-\lambda \end{pmatrix} &= (-\lambda)(2-\lambda) - (-6) \\ &= \lambda^2 - 2\lambda + 6 = 0. \end{aligned}$$

Applying the quadratic formula yields that

$$\lambda = \frac{2 \pm \sqrt{4-24}}{2} = \frac{2 \pm \sqrt{-20}}{2} = \frac{2 \pm \sqrt{-1}\sqrt{4}\sqrt{5}}{2} = \frac{2 \pm i2\sqrt{5}}{2} = 1 \pm i\sqrt{5}.$$

We see that 2 has complex eigenvalues. We know from section 3.4 that if we find one complex solution, we can decompose it into two linearly independent real solutions. To that end, let's

[1] 3.2 gives methods for distinct real eigenvalues, 3.4 gives methods for complex eigenvalues and 3.5 gives methods for repeated and zero eigenvalues.

find a complex solution by finding an eigenvector for one of $\lambda = 1 + i\sqrt{5}$ or $\lambda = 1 - i\sqrt{5}$. It doesn't matter which we choose, so I'll go with the one involving less negative signs: $\lambda = 1 + i\sqrt{5}$.

We seek a nontrivial $\mathbf{v} = (x_0, y_0)$ such that

$$\begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = (1 + i\sqrt{5}) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

To find such a $\mathbf{v}$ we solve the system

$$\begin{cases} 0x_0 + 2y_0 = (1 + i\sqrt{5})x_0 \\ -3x_0 + 2y_0 = (1 + i\sqrt{5})y_0 \end{cases}$$

Combining like terms yields

$$\begin{cases} 2y_0 &= (1 + i\sqrt{5})x_0 \\ (1 - i\sqrt{5})y_0 &= 3x_0 \end{cases}$$

Because $1 + i\sqrt{5}$ is an eigenvalue, this system should have infinitely many solutions. That is, the equations should be redundant. Let's check this fact. If they are redundant then there is some constant that we can multiply $(1 - i\sqrt{5})y = 3x_0$ by to obtain $2y = (1 + i\sqrt{5})x_0$. The only number we can multiply $3x_0$ by to get $(1 + i\sqrt{5})x_0$ is $(1 + i\sqrt{5})/3$. We see this constant verifies the two equations are redundant

$$\begin{aligned} \frac{1 + i\sqrt{5}}{3}(1 - i\sqrt{5})y_0 = \frac{1 + i\sqrt{5}}{3}3x_0 &\Longrightarrow \frac{1 - i\sqrt{5} + i\sqrt{5} - i^2(\sqrt{5})^2}{3}y_0 = (1 + i\sqrt{5})x_0 \\ &\Longrightarrow \frac{1 + 5}{3}y_0 = (1 + i\sqrt{5})x_0 \Longrightarrow 2y_0 = (1 + i\sqrt{5})x_0 \end{aligned}$$

Thus, we have that an eigenvector with eigenvalue $1 + i\sqrt{5}$ is any nonzero $\mathbf{v} = (x_0, y_0)$ such that

$$y_0 = \frac{(1 + i\sqrt{5})}{2}x_0.$$

Let's use $\mathbf{v} = (2, 1 + i\sqrt{5})$ as our eigenvector.

Now, whenever we have an eigenvalue λ and eigenvector $\mathbf{v}$, we know $e^{\lambda t}\mathbf{v}$ is a solution to the system. So,

$$\mathbf{Y}_1(t) = e^{(1+i\sqrt{5})t} \begin{pmatrix} 2 \\ 1 + i\sqrt{5} \end{pmatrix}$$

is a solution to the system. We seek the general solution to the system and we wish for it to not involve complex numbers, so we must use $\mathbf{Y}_1$ to obtain two linearly independent real valued solutions. We do this by using Euler's formula

$$e^{a+bi} = e^a(\cos(b) + i\sin(b))$$

to decompose $\mathbf{Y}_1$ into a real and imaginary part.

$$
\begin{aligned}
\mathbf{Y}_1(t) &= e^{(1+i\sqrt{5})t}\begin{pmatrix} 2 \\ 1+i\sqrt{5} \end{pmatrix} && (3)\\
&= e^{t+i\sqrt{5}t}\begin{pmatrix} 2 \\ 1+i\sqrt{5} \end{pmatrix} && (4)\\
&= e^{t}e^{i\sqrt{5}t}\begin{pmatrix} 2 \\ 1+i\sqrt{5} \end{pmatrix} && (5)\\
&= e^{t}(\cos(\sqrt{5}t)+i\sin(\sqrt{5}t))\begin{pmatrix} 2 \\ 1+i\sqrt{5} \end{pmatrix} && (6)\\
&= e^{t}\begin{pmatrix} 2(\cos(\sqrt{5}t)+i\sin(\sqrt{5}t)) \\ (1+i\sqrt{5})(\cos(\sqrt{5}t)+i\sin(\sqrt{5}t)) \end{pmatrix} && (7)\\
&= e^{t}\begin{pmatrix} 2\cos(\sqrt{5}t)+i2\sin(\sqrt{5}t) \\ (\cos(\sqrt{5}t)+i\sin(\sqrt{5}t))+i\sqrt{5}(\cos(\sqrt{5}t)+i\sin(\sqrt{5}t)) \end{pmatrix} && (8)\\
&= e^{t}\begin{pmatrix} 2\cos(\sqrt{5}t)+i2\sin(\sqrt{5}t) \\ (\cos(\sqrt{5}t)+i\sin(\sqrt{5}t))+i\sqrt{5}\cos(\sqrt{5}t)+i^2\sqrt{5}\sin(\sqrt{5}t)) \end{pmatrix} && (9)\\
&= e^{t}\begin{pmatrix} 2\cos(\sqrt{5}t)+i2\sin(\sqrt{5}t) \\ \cos(\sqrt{5}t)+i\sin(\sqrt{5}t)+i\sqrt{5}\cos(\sqrt{5}t)-\sqrt{5}\sin(\sqrt{5}t)) \end{pmatrix} && (10)\\
&= e^{t}\left[\begin{pmatrix} 2\cos(\sqrt{5}t) \\ \cos(\sqrt{5}t)-\sqrt{5}\sin(\sqrt{5}t) \end{pmatrix}+i\begin{pmatrix} 2\sin(\sqrt{5}t) \\ \sin(\sqrt{5}t)+\sqrt{5}\cos(\sqrt{5}t) \end{pmatrix}\right] && (11)\\
&= \underbrace{e^{t}\begin{pmatrix} 2\cos(\sqrt{5}t) \\ \cos(\sqrt{5}t)-\sqrt{5}\sin(\sqrt{5}t) \end{pmatrix}}_{\mathbf{Y}_{re}(t)}+i\underbrace{\left[e^{t}\begin{pmatrix} 2\sin(\sqrt{5}t) \\ \sin(\sqrt{5}t)+\sqrt{5}\cos(\sqrt{5}t) \end{pmatrix}\right]}_{\mathbf{Y}_{im}(t)} && (12)
\end{aligned}
$$

Note that step 6 is where we applied Euler's formula and step 11 is where we gathered our real and imaginary terms, split them into two vectors and factored out i.

We're almost ready to present the general solution! We know from class (or pg. 300-301 of our textbook) that

$$
\mathbf{Y}_{re}(t) = e^{t}\begin{pmatrix} 2\cos(\sqrt{5}t) \\ \cos(\sqrt{5}t)-\sqrt{5}\sin(\sqrt{5}t) \end{pmatrix} \text{ and } \mathbf{Y}_{im}(t) = e^{t}\begin{pmatrix} 2\sin(\sqrt{5}t) \\ \sin(\sqrt{5}t)+\sqrt{5}\cos(\sqrt{5}t) \end{pmatrix}
$$

are linearly independent solutions to the system devoid of any complex numbers.[2] Since we have two linearly independent solutions to the system we know

$$
\begin{aligned}
\mathbf{Y}(t) &= k_1\mathbf{Y}_{re}(t)+k_2\mathbf{Y}_{re}(t) && (13)\\
&= k_1e^{t}\begin{pmatrix} 2\cos(\sqrt{5}t) \\ \cos(\sqrt{5}t)-\sqrt{5}\sin(\sqrt{5}t) \end{pmatrix}+k_2e^{t}\begin{pmatrix} 2\sin(\sqrt{5}t) \\ \sin(\sqrt{5}t)+\sqrt{5}\cos(\sqrt{5}t) \end{pmatrix} && (14)
\end{aligned}
$$

is the general solution to the system where k_1 and k_2 are arbitrary constants.

[2] Though we should be able to verify these statements.

2 Solving an initial value problem

To finish up, let's solve the following initial value problem

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 2 \\ -3 & 2 \end{pmatrix} \mathbf{Y} \qquad \mathbf{Y}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{15}$$

In order to do this, we require the general solution. Luckily, we found it in the previous section. From 14, we know the solution to 15 must be of the form

$$\mathbf{Y}(t) = k_1 e^t \begin{pmatrix} 2\cos(\sqrt{5}t) \\ \cos(\sqrt{5}t) - \sqrt{5}\sin(\sqrt{5}t) \end{pmatrix} + k_2 e^t \begin{pmatrix} 2\sin(\sqrt{5}t) \\ \sin(\sqrt{5}t) + \sqrt{5}\cos(\sqrt{5}t) \end{pmatrix}$$

for some constants k_1 and k_2 such that

$$\begin{aligned} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \mathbf{Y}(0) &= k_1 e^0 \begin{pmatrix} 2\cos(\sqrt{5}(0)) \\ \cos(\sqrt{5}(0)) - \sqrt{5}\sin(\sqrt{5}(0)) \end{pmatrix} + k_2 e^0 \begin{pmatrix} 2\sin(\sqrt{5}(0)) \\ \sin(\sqrt{5}(0)) + \sqrt{5}\cos(\sqrt{5}(0)) \end{pmatrix} \\ &= k_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ \sqrt{5} \end{pmatrix} \\ &= \begin{pmatrix} 2k_1 \\ k_1 + \sqrt{5}k_2 \end{pmatrix} \end{aligned}$$

which implies

$$\begin{cases} 2k_1 & = 1 \\ \;k_1 + \sqrt{5}k_2 & = 1 \end{cases}$$

So, we see the solution to the IVP must be such that $k_1 = 1/2$ and $k_2 = 1/(2\sqrt{5})$. That is

$$\mathbf{Y}(t) = \frac{1}{2} e^t \begin{pmatrix} 2\cos(\sqrt{5}t) \\ \cos(\sqrt{5}t) - \sqrt{5}\sin(\sqrt{5}t) \end{pmatrix} + \frac{1}{2\sqrt{5}} e^t \begin{pmatrix} 2\sin(\sqrt{5}t) \\ \sin(\sqrt{5}t) + \sqrt{5}\cos(\sqrt{5}t) \end{pmatrix}$$

is the desired solution to the initial value problem 15.