EXERCISES FOR SECTION 4.1

1. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 - s - 6,$$

so the eigenvalues are $s = -2$ and $s = 3$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-2t} + k_2 e^{3t}.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{4t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2 y_p}{dt^2} - \frac{dy_p}{dt} - 6y_p &= 16ke^{4t} - 4ke^{4t} - 6ke^{4t} \\ &= 6ke^{4t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = 1/6$. The general solution of the forced equation is

$$y(t) = k_1 e^{-2t} + k_2 e^{3t} + \tfrac{1}{6} e^{4t}.$$

2. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 6s + 8,$$

so the eigenvalues are $s = -2$ and $s = -4$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-2t} + k_2 e^{-4t}.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-3t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2 y_p}{dt^2} + 6\frac{dy_p}{dt} + 8y_p &= 9ke^{-3t} - 18ke^{-3t} + 8ke^{-3t} \\ &= -ke^{-3t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = -2$. The general solution of the forced equation is

$$y(t) = k_1 e^{-2t} + k_2 e^{-4t} - 2e^{-3t}.$$

3. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 - s - 2,$$

so the eigenvalues are $s = -1$ and $s = 2$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-t} + k_2 e^{2t}.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{3t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_p}{dt^2} - \frac{dy_p}{dt} - 2y_p &= 9ke^{3t} - 3ke^{3t} - 2ke^{3t} \\ &= 4ke^{3t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = 5/4$. The general solution of the forced equation is

$$y(t) = k_1e^{-t} + k_2e^{2t} + \tfrac{5}{4}e^{3t}.$$

4. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 4s + 13,$$

so the eigenvalues are $s = -2 \pm 3i$. Hence, the general solution of the homogeneous equation is

$$k_1e^{-2t}\cos 3t + k_2e^{-2t}\sin 3t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_p}{dt^2} + 4\frac{dy_p}{dt} + 13y_p &= ke^{-t} - 4ke^{-t} + 13ke^{-t} \\ &= 10ke^{-t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = 1/10$. The general solution of the forced equation is

$$y(t) = k_1e^{-2t}\cos 3t + k_2e^{-2t}\sin 3t + \tfrac{1}{10}e^{-t}.$$

5. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 4s + 13,$$

so the eigenvalues are $s = -2 \pm 3i$. Hence, the general solution of the homogeneous equation is

$$k_1e^{-2t}\cos 3t + k_2e^{-2t}\sin 3t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-2t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_p}{dt^2} + 4\frac{dy_p}{dt} + 13y_p &= 4ke^{-2t} - 8ke^{-2t} + 13ke^{-2t} \\ &= 9ke^{-2t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = -1/3$. The general solution of the forced equation is

$$y(t) = k_1e^{-2t}\cos 3t + k_2e^{-2t}\sin 3t - \tfrac{1}{3}e^{-2t}.$$

6. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + 7s + 10,$$

so the eigenvalues are $s = -2$ and $s = -5$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-2t} + k_2 e^{-5t}.$$

To find a particular solution of the forced equation, a reasonable looking guess is $y_p(t) = ke^{-2t}$. However, this guess is a solution of the homogeneous equation, so it is doomed to fail. We make the standard second guess of $y_p(t) = kte^{-2t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2 y_p}{dt^2} + 7\frac{dy_p}{dt} + 10y_p &= (-4ke^{-2t} + 4kte^{-2t}) + 7(ke^{-2t} - 2kte^{-2t}) + 10kte^{-2t} \\ &= 3ke^{-2t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = 1/3$. The general solution of the forced equation is

$$y(t) = k_1 e^{-2t} + k_2 e^{-5t} + \tfrac{1}{3} te^{-2t}.$$

7. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 - 5s + 4,$$

so the eigenvalues are $s = 1$ and $s = 4$. Hence, the general solution of the homogeneous equation is

$$k_1 e^t + k_2 e^{4t}.$$

To find a particular solution of the forced equation, a reasonable looking guess is $y_p(t) = ke^{4t}$. However, this guess is a solution of the homogeneous equation, so it is doomed to fail. We make the standard second guess of $y_p(t) = kte^{4t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2 y_p}{dt^2} - 5\frac{dy_p}{dt} + 4y_p &= (8ke^{4t} + 16kte^{4t}) - 5(ke^{4t} + 4kte^{4t}) + 4kte^{4t} \\ &= 3ke^{4t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = 1/3$. The general solution of the forced equation is

$$y(t) = k_1 e^t + k_2 e^{4t} + \tfrac{1}{3} te^{4t}.$$

8. To compute the general solution of the unforced equation, we use the method of Section 3.6. The characteristic polynomial is

$$s^2 + s - 6,$$

so the eigenvalues are $s = -3$ and $s = 2$. Hence, the general solution of the homogeneous equation is

$$k_1 e^{-3t} + k_2 e^{2t}.$$

To find a particular solution of the forced equation, a reasonable looking guess is $y_p(t) = ke^{-3t}$. However, this guess is a solution of the homogeneous equation, so it is doomed to fail. We make the standard second guess of $y_p(t) = kte^{-3t}$. Substituting into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_p}{dt^2} + \frac{dy_p}{dt} - 6y_p &= (-6ke^{-3t} + 9kte^{-3t}) + (ke^{-3t} - 3kte^{-3t}) - 6kte^{-3t}\\ &= -5ke^{-3t}.\end{aligned}$$

In order for $y_p(t)$ to be a solution of the forced equation, we must take $k = -4/5$. The general solution of the forced equation is

$$y(t) = k_1e^{-3t} + k_2e^{2t} - \tfrac{4}{5}te^{-3t}.$$

9. First we derive the general solution. The characteristic polynomial is

$$s^2 + 6s + 8,$$

so the eigenvalues are $s = -2$ and $s = -4$. To find the general solution of the forced equation, we also need a particular solution. We guess $y_p(t) = ke^{-t}$ and find that $y_p(t)$ is a solution only if $k = 1/3$. Therefore, the general solution is

$$y(t) = k_1e^{-2t} + k_2e^{-4t} + \tfrac{1}{3}e^{-t}.$$

To find the solution with the initial conditions $y(0) = y'(0) = 0$, we compute

$$y'(t) = -2k_1e^{-2t} - 4k_2e^{-4t} - \tfrac{1}{3}e^{-t}.$$

Then we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 + \frac{1}{3} = 0\\ -2k_1 - 4k_2 - \frac{1}{3} = 0.\end{cases}$$

Solving, we have $k_1 = -1/2$ and $k_2 = 1/6$, so the solution of the initial-value problem is

$$y(t) = -\tfrac{1}{2}e^{-2t} + \tfrac{1}{6}e^{-4t} + \tfrac{1}{3}e^{-t}.$$

10. First we derive the general solution. The characteristic polynomial is

$$s^2 + 7s + 12,$$

so the eigenvalues are $s = -3$ and $s = -4$. To find the general solution of the forced equation, we also need a particular solution. We guess $y_p(t) = ke^{-t}$ and find that $y_p(t)$ is a solution only if $k = 1/2$. Therefore, the general solution is

$$y(t) = k_1e^{-3t} + k_2e^{-4t} + \tfrac{1}{2}e^{-t}.$$

To find the solution with the initial conditions $y(0) = 2$ and $y'(0) = 1$, we compute

$$y'(t) = -3k_1 e^{-3t} - 4k_2 e^{-4t} - \tfrac{1}{2}e^{-t}.$$

Then we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 + \frac{1}{2} = 2 \\ -3k_1 - 4k_2 - \frac{1}{2} = 1. \end{cases}$$

Solving, we have $k_1 = 15/2$ and $k_2 = -6$, so the solution of the initial-value problem is

$$y(t) = \tfrac{15}{2}e^{-3t} - 6e^{-4t} + \tfrac{1}{2}e^{-t}.$$

11. This is the same equation as Exercise 5. The general solution is

$$y(t) = k_1 e^{-2t}\cos 3t + k_2 e^{-2t}\sin 3t - \tfrac{1}{3}e^{-2t}.$$

To find the solution with the initial conditions $y(0) = y'(0) = 0$, we compute

$$y'(t) = -2k_1 e^{-2t}\cos 3t - 3k_1 e^{-2t}\sin 3t - 2k_2 e^{-2t}\sin 3t + 3k_2 e^{-2t}\cos 3t + \tfrac{2}{3}e^{-2t}.$$

Then we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 - \frac{1}{3} = 0 \\ -2k_1 + 3k_2 + \frac{2}{3} = 0. \end{cases}$$

Solving, we have $k_1 = 1/3$ and $k_2 = 0$, so the solution of the initial-value problem is

$$y(t) = \tfrac{1}{3}e^{-2t}\cos 3t - \tfrac{1}{3}e^{-2t}.$$

12. This is the same equation as Exercise 6. The general solution is

$$y(t) = k_1 e^{-2t} + k_2 e^{-5t} + \tfrac{1}{3}te^{-2t}.$$

To find the solution with the initial conditions $y(0) = y'(0) = 0$, we compute

$$y'(t) = -2k_1 e^{-2t} - 5k_2 e^{-5t} + \tfrac{1}{3}e^{-2t} - \tfrac{2}{3}te^{-2t}.$$

Then we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 = 0 \\ -2k_1 - 5k_2 + \frac{1}{3} = 0. \end{cases}$$

Solving, we have $k_1 = -1/9$ and $k_2 = 1/9$, so the solution of the initial-value problem is

$$y(t) = -\tfrac{1}{9}e^{-2t} + \tfrac{1}{9}e^{-5t} + \tfrac{1}{3}te^{-2t}.$$

13. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 4s + 3.$$

So the eigenvalues are $s = -1$ and $s = -3$, and the general solution of the unforced equation is

$$k_1 e^{-t} + k_2 e^{-3t}.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-t/2}$. Substituting $y_p(t)$ into the left-hand side of the differential equation gives

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 4\frac{dy_p}{dt} + 3y_p &= \tfrac{1}{4}ke^{-t/2} - 2ke^{-t/2} + 3ke^{-t/2} \\ &= \tfrac{5}{4}ke^{-t/2}. \end{aligned}$$

So $k = 4/5$ yields a solution of the forced equation.

The general solution of the forced equation is therefore

$$y(t) = k_1 e^{-t} + k_2 e^{-3t} + \tfrac{4}{5}e^{-t/2}.$$

(b) The derivative of the general solution is

$$y'(t) = -k_1 e^{-t} - 3k_2 e^{-3t} - \tfrac{2}{5}e^{-t/2}.$$

To find the solution with $y(0) = y'(0) = 0$, we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 + \frac{4}{5} = 0 \\ -k_1 - 3k_2 - \frac{2}{5} = 0. \end{cases}$$

Solving, we find that $k_1 = -1$ and $k_2 = 1/5$, so the solution of the initial-value problem is

$$y(t) = -e^{-t} + \tfrac{1}{5}e^{-3t} + \tfrac{4}{5}e^{-t/2}.$$

(c) Every solution tends to zero as t increases. Of the three terms that sum to the general solution, $\frac{4}{5}e^{-t/2}$ dominates when t is large, so all solutions are approximately $\frac{4}{5}e^{-t/2}$ for t large.

14. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 4s + 3.$$

So the eigenvalues are $s = -1$ and $s = -3$, and the general solution of the unforced equation is

$$k_1 e^{-t} + k_2 e^{-3t}.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-2t}$. Substituting $y_p(t)$ into the left-hand side of the differential equation gives

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 4\frac{dy_p}{dt} + 3y_p &= 4ke^{-2t} - 8ke^{-2t} + 3ke^{-2t} \\ &= -ke^{-2t}. \end{aligned}$$

So $k = -1$ yields a solution of the forced equation.

The general solution of the forced equation is therefore

$$y(t) = k_1 e^{-t} + k_2 e^{-3t} - e^{-2t}.$$

(b) The derivative of the general solution is

$$y'(t) = -k_1 e^{-t} - 3k_2 e^{-3t} + 2e^{-2t}.$$

To find the solution with $y(0) = y'(0) = 0$, we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 - 1 = 0 \\ -k_1 - 3k_2 + 2 = 0. \end{cases}$$

Solving, we find that $k_1 = 1/2$ and $k_2 = 1/2$, so the solution of the initial-value problem is

$$y(t) = \tfrac{1}{2}e^{-t} + \tfrac{1}{2}e^{-3t} - e^{-2t}.$$

(c) In the general solution, all three terms tend to zero, so the solution tends to zero. We can say a little more by noting that the term $k_1 e^{-t}$ is much larger (provided $k_1 \neq 0$). Hence, most solutions tend to zero at the rate of e^{-t}. If $k_1 = 0$, then solutions tend to zero at the rate of e^{-3t} provided $k_2 \neq 0$.

15. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 4s + 3.$$

So the eigenvalues are $s = -1$ and $s = -3$, and the general solution of the unforced equation is

$$k_1 e^{-t} + k_2 e^{-3t}.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-4t}$. Substituting $y_p(t)$ into the left-hand side of the differential equation gives

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 4\frac{dy_p}{dt} + 3y_p &= 16ke^{-4t} - 16ke^{-4t} + 3ke^{-4t} \\ &= 3ke^{-4t}. \end{aligned}$$

So $k = 1/3$ yields a solution of the forced equation.

The general solution of the forced equation is therefore

$$y(t) = k_1 e^{-t} + k_2 e^{-3t} + \tfrac{1}{3}e^{-4t}.$$

(b) The derivative of the general solution is

$$y'(t) = -k_1 e^{-t} - 3k_2 e^{-3t} - \tfrac{4}{3}e^{-4t}.$$

To find the solution with $y(0) = y'(0) = 0$, we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 + \frac{1}{3} = 0 \\ -k_1 - 3k_2 - \frac{4}{3} = 0. \end{cases}$$

Solving, we find that $k_1 = 1/6$ and $k_2 = -1/2$, so the solution of the initial-value problem is

$$y(t) = \tfrac{1}{6}e^{-t} - \tfrac{1}{2}e^{-3t} + \tfrac{1}{3}e^{-4t}.$$

(c) In the general solution, all three terms tend to zero, so the solution tends to zero. We can say a little more by noting that the term $k_1 e^{-t}$ is much larger (provided $k_1 \neq 0$). Hence, most solutions tend to zero at the rate of e^{-t}. If $k_1 = 0$, then solutions tend to zero at the rate of e^{-3t} provided $k_2 \neq 0$.

16. (a) The characteristic polynomial of the unforced equation is

$$s^2 + 4s + 20.$$

So the eigenvalues are $s = -2 \pm 4i$, and the general solution of the unforced equation is

$$k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-t/2}$. Substituting $y_p(t)$ into the left-hand side of the differential equation gives

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 4\frac{dy_p}{dt} + 20y_p &= \tfrac{1}{4}ke^{-t/2} - 2ke^{-t/2} + 20ke^{-t/2} \\ &= \tfrac{73}{4}ke^{-t/2}. \end{aligned}$$

So $k = 4/73$ yields a solution of the forced equation.

The general solution of the forced equation is therefore

$$y(t) = k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t + \tfrac{4}{73} e^{-t/2}.$$

(b) The derivative of the general solution is

$$\begin{aligned} y'(t) = &-k_1 e^{-2t} \cos 4t - 4k_1 e^{-2t} \sin 4t \\ &-2k_2 e^{-2t} \sin 4t + 4k_2 e^{-2t} \cos 4t - \tfrac{2}{73} e^{-t/2}. \end{aligned}$$

To find the solution with $y(0) = y'(0) = 0$, we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + \frac{4}{73} = 0 \\ -2k + 4k_2 - \frac{2}{73} = 0. \end{cases}$$

Solving, we find that $k_1 = -4/73$ and $k_2 = -3/146$, so the solution of the initial-value problem is

$$y(t) = -\tfrac{4}{73} e^{-2t} \cos 4t - \tfrac{3}{146} e^{-2t} \sin 4t + \tfrac{4}{73} e^{-t/2}.$$

(c) Every solution tends to zero at the rate $e^{-t/2}$. The terms involving sine and cosine have e^{-4t} as a coefficient, so they tend to zero much more quickly than the exponential $\frac{4}{73} e^{-t/2}$.

17. (a) The characteristic polynomial of the unforced equation is

$$s^2 + 4s + 20.$$

So the eigenvalues are $s = -2 \pm 4i$, and the general solution of the unforced equation is

$$k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-2t}$. Substituting $y_p(t)$ into the left-hand side of the differential equation gives

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 4\frac{dy_p}{dt} + 20y_p &= 4ke^{-2t} - 8ke^{-2t} + 20ke^{-2t} \\ &= 16ke^{-2t}. \end{aligned}$$

So $k = 1/16$ yields a solution of the forced equation.

The general solution of the forced equation is therefore

$$y(t) = k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t + \tfrac{1}{16} e^{-2t}.$$

(b) The derivative of the general solution is

$$y'(t) = -2k_1 e^{-2t} \cos 4t - 4k_1 e^{-2t} \sin 4t - 2k_2 e^{-2t} \sin 4t + 4k_2 e^{-2t} \cos 4t - \tfrac{1}{8} e^{-2t}.$$

To find the solution with $y(0) = y'(0) = 0$, we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + \frac{1}{16} = 0 \\ -2k_1 + 4k_2 - \frac{1}{8} = 0. \end{cases}$$

Solving, we find that $k_1 = -1/16$ and $k_2 = 0$, so the solution of the initial-value problem is

$$y(t) = -\tfrac{1}{16} e^{-2t} \cos 4t + \tfrac{1}{16} e^{-2t}.$$

(c) Every solution tends to zero like e^{-2t} and all but one exponential solution oscillates with frequency $2/\pi$.

18. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 4s + 20.$$

So the eigenvalues are $s = -2 \pm 4i$, and the general solution of the unforced equation is

$$k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^{-4t}$. Substituting $y_p(t)$ into the left-hand side of the differential equation gives

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 4\frac{dy_p}{dt} + 20y_p &= 16ke^{-4t} - 16ke^{-4t} + 20ke^{-4t} \\ &= 20ke^{-4t}. \end{aligned}$$

So $k = 1/20$ yields a solution of the forced equation.

The general solution of the forced equation is therefore

$$y(t) = k_1 e^{-2t} \cos 4t + k_2 e^{-2t} \sin 4t + \tfrac{1}{20} e^{-4t}.$$

(b) The derivative of the general solution is

$$\begin{aligned} y'(t) = -k_1 e^{-2t} &\cos 4t - 4k_1 e^{-2t} \sin 4t \\ &-2k_2 e^{-2t} \sin 4t + 4k_2 e^{-2t} \cos 4t - \tfrac{1}{5} e^{-4t}. \end{aligned}$$

To find the solution with $y(0) = y'(0) = 0$, we evaluate at $t = 0$ and obtain the simultaneous equations

$$\begin{cases} k_1 + \frac{1}{20} = 0 \\ -2k_1 + 4k_2 - \frac{1}{5} = 0. \end{cases}$$

Solving, we find that $k_1 = -1/20$ and $k_2 = 1/40$, so the solution of the initial-value problem is

$$y(t) = -\tfrac{1}{20} e^{-2t} \cos 4t + \tfrac{1}{40} e^{-2t} \sin 4t + \tfrac{1}{20} e^{-4t}.$$

(c) From the formula for the general solution, we see that every solution tends to zero. The e^{-4t} term in the general solution tends to zero quickest, so for large t, the solution is very close to the unforced solution. All solutions tend to zero and all but the purely exponential one oscillates with frequency $2/\pi$ and an amplitude that decreases at the rate of e^{-2t}.

19. The natural guesses of $y_p(t) = ke^{-t}$ and $y_p(t) = kte^{-t}$ fail to be solutions of the forced equation because they are both solutions of the unforced equation. (The characteristic polynomial of the unforced equation is

$$s^2 + 2s + 1,$$

which has -1 as a double root.)

So we guess $y_p(t) = kt^2e^{-t}$. Substituting this guess into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y_p}{dt^2} + 2\frac{dy_p}{dt} + y_p &= (2ke^{-t} - 4kte^{-t} + kt^2e^{-t}) + 2(2kte^{-t} - kt^2e^{-t}) + kt^2e^{-t} \\ &= 2ke^{-t}.\end{aligned}$$

So $k = 1/2$ yields the solution

$$y_p(t) = \tfrac{1}{2}t^2e^{-t}.$$

From the characteristic polynomial, we know that the general solution of the unforced equation is

$$k_1e^{-t} + k_2te^{-t}.$$

Consequently, the general solution of the forced equation is

$$y(t) = k_1e^{-t} + k_2te^{-t} + \tfrac{1}{2}t^2e^{-t}.$$

20. If we guess a constant function of the form $y_p(t) = k$, then substituting $y_p(t)$ into the left-hand side of the differential equation yields

$$\begin{aligned}\frac{d^2(k)}{dt^2} + p\frac{d(k)}{dt} + qk &= 0 + 0 + qk \\ &= qk.\end{aligned}$$

Since the right-hand side of the differential equation is simply the constant c, $k = c/q$ yields a constant solution.

21. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 - 5s + 4.$$

So the eigenvalues are $s = 1$ and $s = 4$, and the general solution of the unforced equation is

$$k_1e^t + k_2e^{4t}.$$

To find one solution of the forced equation, we guess the constant function $y_p(t) = k$. Substituting $y_p(t)$ into the left-hand side of the differential equation, we obtain

$$\frac{d^2y_p}{dt^2} - 5\frac{dy_p}{dt} + 4y_p = 0 - 5 \cdot 0 + 4k = 4k.$$

Hence, $k = 5/4$ yields a solution of the forced equation. The general solution of the forced equation is

$$y(t) = k_1 e^t + k_2 e^{4t} + \tfrac{5}{4}.$$

(b) To find the solution satisfying the initial conditions $y(0) = y'(0) = 0$, we compute the derivative of the general solution

$$y'(t) = k_1 e^t + 4k_2 e^{4t}.$$

Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 + \frac{5}{4} = 0 \\ k_1 + 4k_2 = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = -5/3$ and $k_2 = 5/12$. The solution of the initial-value problem is

$$y(t) = \tfrac{5}{4} - \tfrac{5}{3} e^t + \tfrac{5}{12} e^{4t}.$$

22. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 5s + 6.$$

So the eigenvalues are $s = -2$ and $s = -3$, and the general solution of the unforced equation is

$$k_1 e^{-2t} + k_2 e^{-3t}.$$

To find one solution of the forced equation, we guess the constant function $y_p(t) = k$. Substituting $y_p(t)$ into the left-hand side of the differential equation, we obtain

$$\frac{d^2 y_p}{dt^2} + 5\frac{dy_p}{dt} + 6y_p = 0 + 5 \cdot 0 + 6k = 6k.$$

Hence, $k = 1/3$ yields a solution of the forced equation. The general solution of the forced equation is

$$y(t) = k_1 e^{-2t} + k_2 e^{-3t} + \tfrac{1}{3}.$$

(b) To find the solution satisfying the initial conditions $y(0) = y'(0) = 0$, we compute the derivative of the general solution

$$y'(t) = -2k_1 e^{-2t} - 3k_2 e^{-3t}.$$

Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 + \frac{1}{3} = 0 \\ -2k_1 - 3k_2 = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = -1$ and $k_2 = 2/3$. The solution of the initial-value problem is

$$y(t) = -e^{-2t} + \tfrac{2}{3} e^{-3t} + \tfrac{1}{3}.$$

23. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 2s + 10.$$

So the eigenvalues are $s = -1 \pm 3i$, and the general solution of the unforced equation is

$$k_1 e^{-t} \cos 3t + k_2 e^{-t} \sin 3t.$$

To find one solution of the forced equation, we guess the constant function $y_p(t) = k$. Substituting $y_p(t)$ into the left-hand side of the differential equation, we obtain

$$\frac{d^2 y_p}{dt^2} + 2\frac{dy_p}{dt} + 10y_p = 0 + 2 \cdot 0 + 10k = 10k.$$

Hence, $k = 1$ yields a solution of the forced equation. The general solution of the forced equation is

$$y(t) = k_1 e^{-t} \cos 3t + k_2 e^{-t} \sin 3t + 1.$$

(b) To find the solution satisfying the initial conditions $y(0) = y'(0) = 0$, we compute the derivative of the general solution

$$y'(t) = -k_1 e^{-t} \cos 3t - 3k_1 e^{-t} \sin 3t - k_2 e^{-t} \sin 3t + 3k_2 e^{-t} \cos 3t.$$

Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 + 1 = 0 \\ -k_1 + 3k_2 = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = -1$ and $k_2 = -1/3$. The solution of the initial-value problem is

$$y(t) = -e^{-t} \cos 3t - \tfrac{1}{3} e^{-t} \sin 3t + 1.$$

24. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 4s + 6.$$

So the eigenvalues are $s = -2 \pm i\sqrt{2}$, and the general solution of the unforced equation is

$$k_1 e^{-2t} \cos \sqrt{2}\, t + k_2 e^{-2t} \sin \sqrt{2}\, t.$$

To find one solution of the forced equation, we guess the constant function $y_p(t) = k$. Substituting $y_p(t)$ into the left-hand side of the differential equation, we obtain

$$\frac{d^2 y_p}{dt^2} + 4\frac{dy_p}{dt} + 6y_p = 0 + 4 \cdot 0 + 6k = 6k.$$

Hence, $k = -4/3$ yields a solution of the forced equation. The general solution of the forced equation is

$$y(t) = k_1 e^{-2t} \cos \sqrt{2}\, t + k_2 e^{-2t} \sin \sqrt{2}\, t - \tfrac{4}{3}.$$

(b) To find the solution satisfying the initial conditions $y(0) = y'(0) = 0$, we compute the derivative of the general solution

$$\begin{aligned} y'(t) = -2k_1 e^{-2t} \cos \sqrt{2}\, t - \sqrt{2}\, k_1 e^{-2t} \sin \sqrt{2}\, t \\ -2k_2 e^{-2t} \sin \sqrt{2}\, t + \sqrt{2}\, k_2 e^{-2t} \cos \sqrt{2}\, t. \end{aligned}$$

Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 - \frac{4}{3} = 0 \\ -2k_1 + \sqrt{2}\, k_2 = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = 4/3$ and $k_2 = 4\sqrt{2}/3$. The solution of the initial-value problem is

$$y(t) = \tfrac{4}{3} e^{-2t} \cos \sqrt{2}\, t - \tfrac{4\sqrt{2}}{3} e^{-2t} \sin \sqrt{2}\, t - \tfrac{4}{3}.$$

25. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 9.$$

So the eigenvalues are $s = \pm 3i$, and the general solution of the unforced equation is

$$k_1 \cos 3t + k_2 \sin 3t.$$

To find one solution of the forced equation, we guess $y_p(t) = ke^{-t}$. Substituting $y_p(t)$ into the left-hand side of the differential equation, we obtain

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 9y_p &= ke^{-t} + 9ke^{-t} \\ &= 10ke^{-t}. \end{aligned}$$

Hence, $k = 1/10$ yields a solution of the forced equation. The general solution of the forced equation is

$$y(t) = k_1 \cos 3t + k_2 \sin 3t + \tfrac{1}{10} e^{-t}.$$

(b) To find the solution satisfying the initial conditions $y(0) = y'(0) = 0$, we compute the derivative of the general solution

$$y'(t) = -3k_1 \sin 3t + 3k_2 \cos 3t - \tfrac{1}{10} e^{-t}.$$

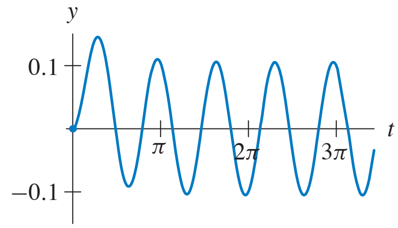
Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 + \frac{1}{10} = 0 \\ 3k_2 - \frac{1}{10} = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = -1/10$ and $k_2 = 1/30$. The solution of the initial-value problem is

$$y(t) = -\tfrac{1}{10} \cos 3t + \tfrac{1}{30} \sin 3t + \tfrac{1}{10} e^{-t}.$$

(c) Since the function $e^{-t}/10 \to 0$ quickly, the solution quickly approaches a solution of the unforced oscillator.



26. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 4.$$

So the eigenvalues are $s = \pm 2i$, and the general solution of the unforced equation is

$$k_1 \cos 2t + k_2 \sin 2t.$$

To find one solution of the forced equation, we guess $y_p(t) = ke^{-2t}$. Substituting into the left-hand side of the differential equation, we obtain

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 4y_p &= 4ke^{-2t} + 4ke^{-2t} \\ &= 8ke^{-2t}. \end{aligned}$$

Hence, $k = 1/4$ yields a solution of the forced equation. The general solution of the forced equation is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t + \tfrac{1}{4} e^{-2t}.$$

(b) To find the solution satisfying the initial conditions $y(0) = y'(0) = 0$, we compute the derivative of the general solution

$$y'(t) = -2k_1 \sin 2t + 2k_2 \cos 2t - \tfrac{1}{2} e^{-2t}.$$

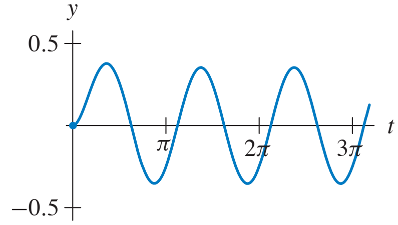
Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 + \frac{1}{4} = 0 \\ 2k_2 - \frac{1}{2} = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = -1/4$ and $k_2 = 1/4$. The solution of the initial-value problem is

$$y(t) = -\tfrac{1}{4} \cos 2t + \tfrac{1}{4} \sin 2t + \tfrac{1}{4} e^{-2t}.$$

(c) Since $e^{-2t}/4 \to 0$ quickly, the solution quickly approaches a solution of the unforced oscillator.



27. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 2.$$

So the eigenvalues are $s = \pm i\sqrt{2}$, and the general solution of the unforced equation is

$$k_1 \cos \sqrt{2}\, t + k_2 \sin \sqrt{2}\, t.$$

To find one solution of the forced equation, we guess $y_p(t) = k$. Substituting into the left-hand side of the differential equation, we obtain

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 2y_p &= 0 + 2k \\ &= 2k. \end{aligned}$$

Hence, $k = -3/2$ yields a solution of the forced equation. The general solution of the forced equation is

$$y(t) = k_1 \cos \sqrt{2}\, t + k_2 \sin \sqrt{2}\, t - \tfrac{3}{2}.$$

(b) To find the solution satisfying the initial conditions $y(0) = y'(0) = 0$, we compute the derivative of the general solution

$$y'(t) = -\sqrt{2}\, k_1 \sin \sqrt{2}\, t + \sqrt{2}\, k_2 \cos \sqrt{2}\, t.$$

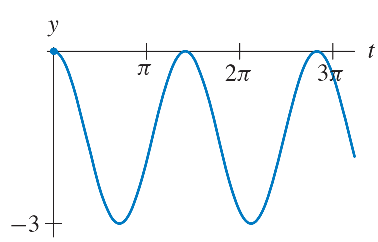
Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 - \frac{3}{2} = 0 \\ \sqrt{2}\, k_2 = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = 3/2$ and $k_2 = 0$. The solution of the initial-value problem is

$$y(t) = \tfrac{3}{2} \cos \sqrt{2}\, t - \tfrac{3}{2}.$$

(c) The solution oscillates about the constant $y = -3/2$ with oscillations of amplitude $3/2$.



28. **(a)** The characteristic polynomial of the unforced equation is

$$\lambda^2 + 4 = 0.$$

So the eigenvalues are $\lambda = \pm 2i$, and the general solution of the unforced equation is

$$k_1 \cos 2t + k_1 \sin 2t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^t$. Substituting into the differential equation, we obtain

$$ke^t + 4ke^t = e^t,$$

which is satisfied if $5k = 1$. Hence, $k = 1/5$ yields a solution of the forced equation.

The general solution of the forced equation is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t + \tfrac{1}{5}e^t.$$

(b) To find the solution with $y(0) = y'(0) = 0$, we note that

$$y'(t) = -2k_1 \sin 2t + 2k_2 \cos 2t + \tfrac{1}{5}e^t.$$

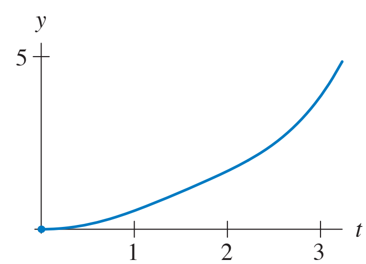
Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 + \frac{1}{5} = 0 \\ 2k_2 + \frac{1}{5} = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = -1/5$ and $k_2 = -1/10$. The solution of the initial-value problem is

$$y(t) = -\tfrac{1}{5}\cos 2t - \tfrac{1}{10}\sin 2t + \tfrac{1}{5}e^t.$$

(c) Since $e^t \to \infty$, the solution tends to infinity, but it oscillates about the values of $\frac{1}{5}e^t$ as it does so.



29. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 9.$$

So the eigenvalues are $s = \pm 3i$, and the general solution of the unforced equation is

$$k_1 \cos 3t + k_2 \sin 3t.$$

To find one solution of the forced equation, we guess $y_p(t) = k$, where k is a constant. Substituting this guess into the left-hand side of the differential equation, we obtain

$$\frac{d^2y_p}{dt^2} + 9y_p = 9k.$$

Hence, $k = 2/3$ yields a solution of the forced equation. The general solution of the forced equation is

$$y(t) = k_1 \cos 3t + k_2 \sin 3t + \tfrac{2}{3}.$$

(b) To find the solution satisfying the initial conditions $y(0) = y'(0) = 0$, we compute the derivative of the general solution

$$y'(t) = -3k_1 \sin 3t + 3k_2 \cos 3t.$$

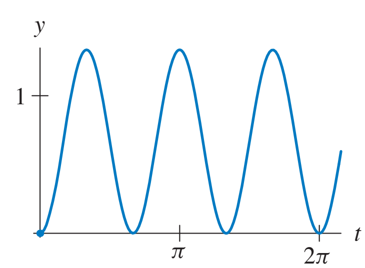
Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 + \frac{2}{3} = 0 \\ 3k_2 = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = -2/3$ and $k_2 = 0$. The solution of the initial-value problem is

$$y(t) = -\tfrac{2}{3} \cos 3t + \tfrac{2}{3}.$$

(c) The solution oscillates about the constant function $y = 2/3$ with amplitude $2/3$.



30. **(a)** The characteristic polynomial of the unforced equation is

$$s^2 + 2 = 0.$$

So $s = \pm i\sqrt{2}$ are the eigenvalues, and the general solution of the unforced equation is

$$k_1 \cos \sqrt{2}\, t + k_2 \sin \sqrt{2}\, t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ke^t$. Substituting this guess into the differential equation yields

$$ke^t + 2ke^t = -e^t,$$

which is satisfied if $3k = -1$. Hence, $k = -1/3$ yields a solution of the forced equation. The general solution of the forced equation is

$$y(t) = k_1 \cos \sqrt{2}\, t + k_2 \sin \sqrt{2}\, t - \tfrac{1}{3}e^t.$$

(b) To satisfy the initial conditions $y(0) = y'(0) = 0$, we note that

$$y'(t) = -\sqrt{2}\, k_1 \sin \sqrt{2}\, t + \sqrt{2}\, k_2 \cos \sqrt{2}\, t - \tfrac{1}{3}e^t.$$

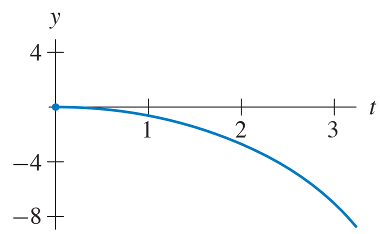
Using the initial conditions and evaluating $y(t)$ and $y'(t)$ at $t = 0$, we obtain the simultaneous equations

$$\begin{cases} k_1 - \frac{1}{3} = 0 \\ \sqrt{2}\,k_2 - \frac{1}{3} = 0. \end{cases}$$

Solving for k_1 and k_2 gives $k_1 = 1/3$ and $k_2 = \sqrt{2}/6$. The solution of the initial-value problem is

$$y(t) = \tfrac{1}{3}\cos\sqrt{2}\,t + \tfrac{\sqrt{2}}{6}\sin\sqrt{2}\,t - \tfrac{1}{3}e^t.$$

(c) Since $e^t \to \infty$ quickly, the solution tends to $-\infty$ at an exponential rate.



31. **(a)** The general solution for the homogeneous equation is

$$k_1 \cos 2t + k_2 \sin 2t.$$

Suppose $y_p(t) = at^2 + bt + c$. Substituting $y_p(t)$ into the differential equation, we get

$$\frac{d^2y_p}{dt^2} + 4y_p = -3t^2 + 2t + 3$$

$$2a + 4(at^2 + bt + c) = -3t^2 + 2t + 3$$

$$4at^2 + 4bt + (2a + 4c) = -3t^2 + 2t + 3.$$

Therefore, $y_p(t)$ is a solution if and only if

$$\begin{cases} 4a = -3 \\ 4b = 2 \\ 2a + 4c = 3. \end{cases}$$

Therefore, $a = -3/4$, $b = 1/2$, and $c = 9/8$. The general solution is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t - \tfrac{3}{4}t^2 + \tfrac{1}{2}t + \tfrac{9}{8}.$$

(b) To solve the initial-value problem, we use the initial conditions $y(0) = 2$ and $y'(0) = 0$ along with the general solution to form the simultaneous equations

$$\begin{cases} k_1 + \frac{9}{8} = 2 \\ 2k_2 + \frac{1}{2} = 0. \end{cases}$$

Therefore, $k_1 = 7/8$ and $k_2 = -1/4$. The solution is

$$y(t) = \tfrac{7}{8}\cos 2t - \tfrac{1}{4}\sin 2t - \tfrac{3}{4}t^2 + \tfrac{1}{2}t + \tfrac{9}{8}.$$

32. **(a)** The general solution for the homogeneous equation is

$$k_1 + k_2 e^{-2t}.$$

Suppose $y_p(t) = at^2 + bt$. Substituting $y_p(t)$ into the differential equation, we get

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 2\frac{dy_p}{dt} &= 3t + 2 \\ 2a + 2(2at + b) &= 3t + 2 \\ 4at + (2a + 2b) &= 3t + 2. \end{aligned}$$

Therefore, $y_p(t)$ is a solution only if $4a = 3$ and $2a + 2b = 2$. These two equations imply that $a = 3/4$ and $b = 1/4$. The general solution is

$$y(t) = k_1 + k_2 e^{-2t} + \tfrac{3}{4}t^2 + \tfrac{1}{4}t.$$

(b) To solve the initial-value problem, we compute

$$y'(t) = -2k_2 e^{-2t} + \tfrac{3}{2}t + \tfrac{1}{4}.$$

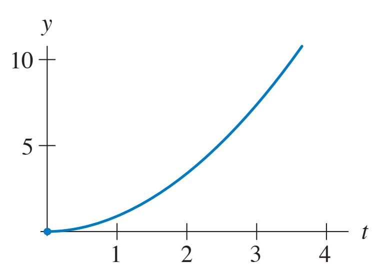
Evaluating $y(t)$ and $y'(t)$ at $t = 0$ and using the initial conditions, we obtain the simultaneous equations

$$\begin{cases} k_1 + k_2 = 0 \\ -2k_2 + \frac{1}{4} = 0. \end{cases}$$

Hence, $k_1 = -1/8$ and $k_2 = 1/8$ provide the desired solution

$$y(t) = -\tfrac{1}{8} + \tfrac{1}{8}e^{-2t} + \tfrac{3}{4}t^2 + \tfrac{1}{4}t.$$

(c) Since $e^{-2t} \to 0$ quickly, the solution tends to infinity at a rate that is determined by $\frac{3}{4}t^2$.



33. **(a)** For the unforced equation, the general solution is

$$k_1 \cos 2t + k_2 \sin 2t.$$

To find a particular solution of the forced equation, we guess $y_p(t) = at + b$. Substituting this guess into the differential equation, we get

$$\begin{aligned} \frac{d^2 y_p}{dt} + 4y_p &= 3t + 2 \\ 0 + 4(at + b) &= 3t + 2 \\ 4at + 4b &= 3t + 2. \end{aligned}$$

Therefore, $a = 3/4$ and $b = 1/2$ yield a solution. The general solution for the forced equation is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t + \tfrac{3}{4}t + \tfrac{1}{2}.$$

(b) To solve the initial-value problem, we compute

$$y'(t) = -2k_1 \sin 2t + 2k_2 \cos 2t + \tfrac{3}{4}.$$

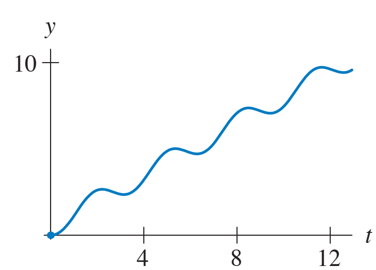
Evaluating $y(t)$ and $y'(t)$ at $t = 0$ and using the initial conditions, we obtain the simultaneous equations

$$\begin{cases} k_1 + \frac{1}{2} = 0 \\ 2k_2 + \frac{3}{4} = 0. \end{cases}$$

Hence, $k_1 = -1/2$ and $k_2 = -3/8$ provide the desired solution

$$y(t) = -\tfrac{1}{2} \cos 2t - \tfrac{3}{8} \cos 2t + \tfrac{3}{4}t + \tfrac{1}{2}.$$

(c) The solution tends to ∞ as it oscillates about the line $y = \frac{3}{4}t + \frac{1}{2}$.



34. **(a)** To find a particular solution of the forced equation, we guess

$$y_p(t) = at^2 + bt + c.$$

Substituting this guess into the equation yields

$$(2a) + 3(2at + b) + 2(at^2 + bt + c) = t^2,$$

which can be rewritten as

$$(2a)t^2 + (6a + 2b)t + (2a + 3b + 2c) = t^2.$$

Equating coefficients, we have

$$\begin{cases} 2a = 1 \\ 6a + 2b = 0 \\ 2a + 3b + 2c = 0, \end{cases}$$

which gives $a = 1/2$, $b = -3/2$ and $c = 7/4$. So

$$y_p(t) = \tfrac{1}{2}t^2 - \tfrac{3}{2}t + \tfrac{7}{4}.$$

To find the general solution of the unforced equation, we note that the characteristic polynomial

$$s^2 + 3s + 2$$

has roots $s = -2$ and $s = -1$, so the general solution for the forced equation is

$$y(t) = k_1 e^{-2t} + k_2 e^{-t} + \tfrac{1}{2}t^2 - \tfrac{3}{2}t + \tfrac{7}{4}.$$

(b) Note that

$$y'(t) = -2k_1 e^{-2t} - k_2 e^{-t} + t - \tfrac{3}{2}.$$

To satisfy the desired initial conditions, we compute

$$y(0) = k_1 + k_2 + \tfrac{7}{4}$$

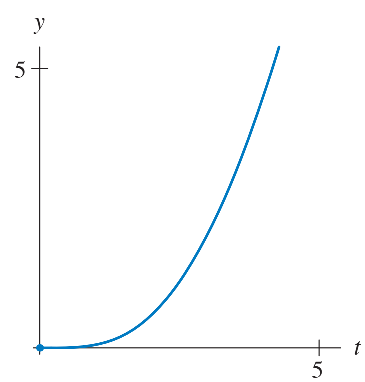
and

$$y'(0) = -2k_1 - k_2 - \tfrac{3}{2}.$$

Using the initial conditions $y(0) = 0$ and $y'(0) = 0$, we have $k_1 = 1/4$ and $k_2 = -2$. So the desired solution is

$$y(t) = \tfrac{1}{4}e^{-2t} - 2e^{-t} + \tfrac{1}{2}t^2 - \tfrac{3}{2}t + \tfrac{7}{4}.$$

(c) The solution $\frac{1}{4}e^{-2t} - 2e^{-t}$ of the unforced equation tends to zero quickly, so the solution of the original equation tends to infinity at a rate that is determined by the quadratic $t^2/2 - 3t/2 + 7/4$. This rate is essentially the same as that of t^2.



35. **(a)** The general solution of the homogeneous equation is

$$k_1 \cos 2t + k_2 \sin 2t.$$

To find a particular solution to the nonhomogeneous equation, we guess

$$y_p(t) = at^2 + bt + c.$$

Substituting $y_p(t)$ into the differential equation yields

$$\begin{aligned} \frac{d^2 y_p}{dt^2} + 4y_p &= t - \frac{t}{20} \\ 2a + 4(at^2 + bt + c) &= t - \frac{t}{20} \\ (4a)t^2 + (4b)t + (2a + 4c) &= t - \frac{t}{20}. \end{aligned}$$

Equating coefficients, we obtain the simultaneous equations

$$\begin{cases} 4a = -\frac{1}{20} \\ 4b = 1 \\ 2a + 4c = 0. \end{cases}$$

Therefore, $a = -1/80$, $b = 1/4$, and $c = 1/160$ yield a solution to the nonhomogeneous equation, and the general solution of the nonhomogeneous equation is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t - \tfrac{1}{80}t^2 + \tfrac{1}{4}t + \tfrac{1}{160}.$$

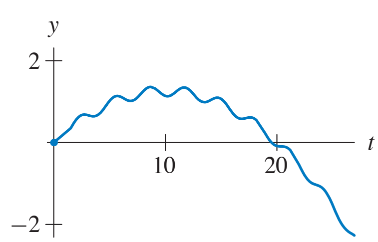
(b) To solve the initial-value problem with $y(0) = 0$ and $y'(0) = 0$, we have

$$\begin{cases} k_1 + \frac{1}{160} = 0 \\ 2k_2 + \frac{1}{4} = 0. \end{cases}$$

Therefore, $k_1 = -1/160$ and $k_2 = -1/8$, and the solution is

$$y(t) = -\tfrac{1}{160} \cos 2t - \tfrac{1}{8} \sin 2t - \tfrac{1}{80}t^2 + \tfrac{1}{4}t + \tfrac{1}{160}.$$

(c) Since the solution to the homogeneous equation is periodic with a small amplitude and since the solution to the nonhomogeneous equation goes to $-\infty$ at a rate determined by $-t^2/80$, the solution tends to $-\infty$.



36. Substituting $y_1 + y_2$ into the differential equation, we obtain

$$\begin{aligned}\frac{d^2(y_1+y_2)}{dt^2} &+ p\frac{d(y_1+y_2)}{dt} + q(y_1+y_2) \\ &= \left(\frac{d^2y_1}{dt^2} + p\frac{dy_1}{dt} + qy_1\right) + \left(\frac{d^2y_2}{dt^2} + p\frac{dy_2}{dt} + qy_2\right) \\ &= g(t) + h(t)\end{aligned}$$

since y_1 and y_2 are solutions of $y'' + py' + qy = g(t)$ and $y'' + py' + qy = h(t)$ respectively. Therefore, $y_1 + y_2$ is a solution of

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t) + h(t).$$

37. **(a)** We must find a particular solution. Using the result of Exercise 36, we guess $y_p(t) = ae^{-t} + b$, where a and b are constants to be determined. (We could solve two separate problems and add the answers, but this approach is more efficient.) Hence we have $dy_p/dt = -ae^{-t}$ and $d^2y_p/dt^2 = ae^{-t}$. Substituting these derivatives into the differential equation, we obtain

$$(a - 5a + 6a)e^{-t} + 6b = e^{-t} + 4,$$

which is satisfied if $2a = 1$ and $6b = 4$. Hence, $a = 1/2$ and $b = 2/3$ yield the particular solution $y_p(t) = e^{-t}/2 + 2/3$.

The general solution of the homogeneous equation is obtained from the characteristic polynomial

$$s^2 + 5s + 6,$$

whose roots are $s = -2$ and $s = -3$.

Hence the general solution is

$$y(t) = k_1e^{-2t} + k_2e^{-3t} + \tfrac{1}{2}e^{-t} + \tfrac{2}{3}.$$

(b) To obtain the solution to the initial-value problem specified, we note that

$$y(0) = k_1 + k_2 + 1/2 + 2/3 \quad \text{and} \quad y'(0) = -2k_1 - 3k_2 - 1/2.$$

Using the initial conditions $y(0) = 0$ and $y'(0) = 0$, we have $k_1 = -3$ and $k_2 = 11/6$. The solution is

$$y(t) = -3e^{-2t} + \tfrac{11}{6}e^{-3t} + \tfrac{1}{2}e^{-t} + \tfrac{2}{3}.$$

(c) All of the exponential terms in the solution to the initial-value problem tend to 0. Hence, the solution tends to the constant $y = 2/3$. The rate that this solution tends to the constant is determined by $e^{-t}/2$, which is the largest of the terms that tend to zero when t is large.

38. **(a)** By Exercise 34, the general solution of the unforced equation is

$$k_1e^{-2t} + k_2e^{-t}.$$

To find a particular solution of the forced equation, we guess $y_p(t) = ae^{-t} + b$. Substituting this guess into the equation yields

$$ae^{-t} + 3(-ae^{-t}) + 2(ae^{-t} + b) = e^{-t} - 4,$$

which unfortunately reduces to

$$0 \cdot e^{-t} + 2b = e^{-t} - 4.$$

This guess does not produce a solution to the forced equation. (The difficulty is caused by the fact that ae^{-t} is a solution of the unforced equation.)

We must make a second guess of $y_p(t) = ate^{-t} + b$. Substituting this second guess into the forced equation yields

$$(-2ae^{-t} + ate^{-t}) + 3(ae^{-t} - ate^{-t}) + 2(ate^{-t} + b) = e^{-t} - 4,$$

which can be simplified to

$$ae^{-t} + 2b = e^{-t} - 4.$$

Hence, $a = 1$ and $b = -2$ yield the solution

$$y_p(t) = te^{-t} - 2,$$

and

$$y(t) = k_1 e^{-2t} + k_2 e^{-t} + te^{-t} - 2$$

is the general solution of the forced equation.

(b) Note that

$$y'(t) = -2k_1 e^{-2t} - k_2 e^{-t} + e^{-t} - te^{-t}.$$

To satisfy the initial conditions $y(0) = 0$ and $y'(0) = 0$, we must have

$$\begin{cases} k_1 + k_2 - 2 = 0 \\ -2k_1 - k_2 + 1 = 0, \end{cases}$$

which hold if $k_1 = -1$ and $k_2 = 3$. Hence, the solution of the initial-value problem is

$$y(t) = -e^{-2t} + 3e^{-t} + te^{-t} - 2.$$

(c) Since all three terms that include an exponential tend to 0 relatively quickly, the solution tends to $y = -2$.

39. **(a)** First, to find a particular solution of the forced equation, we guess

$$y_p(t) = at + b + ce^{-t}.$$

For y_p, $dy_p/dt = a - ce^{-t}$ and $d^2y_p/dt^2 = ce^{-t}$. Substituting these derivatives into the differential equation and collecting terms gives

$$(c - 6c + 8c)e^{-t} + (8a)t + (6a + 8b) = 2t + e^{-t},$$

which holds if $3c = 1$, $8a = 2$, and $6a + 8b = 0$. Hence, $c = 1/3$, $a = 1/4$, and $b = -3/16$ yield the solution

$$y_p(t) = -\tfrac{3}{16} + \tfrac{1}{4}t + \tfrac{1}{3}e^{-t}.$$

The characteristic polynomial of the homogeneous equation is

$$s^2 + 6s + 8,$$

which has roots $s = -4$ and $s = -2$, so the general solution of the forced equation is

$$y(t) = k_1 e^{-4t} + k_2 e^{-2t} - \tfrac{3}{16} + \tfrac{1}{4}t + \tfrac{1}{3}e^{-t}.$$

(b) To find the solution for the initial conditions $y(0) = 0$ and $y'(0) = 0$, we solve

$$\begin{cases} k_1 + k_2 - \frac{3}{16} + \frac{1}{3} = 0 \\ -4k_1 - 2k_2 + \frac{1}{4} - \frac{1}{3} = 0. \end{cases}$$

Thus, $k_1 = 5/48$ and $k_2 = -1/4$ yield the solution

$$y(t) = \tfrac{5}{48}e^{-4t} - \tfrac{1}{4}e^{-2t} - \tfrac{3}{16} + \tfrac{1}{4}t + \tfrac{1}{3}e^{-t}$$

of the initial-value problem.

(c) All exponential terms in the solution tend to zero. Hence, the solution tends to infinity linearly in t and is close to $t/4$ for large t.

40. **(a)** From Exercise 39 we know that the general solution of the unforced equation is

$$y(t) = k_1 e^{-4t} + k_2 e^{-2t}.$$

To find a particular solution of the forced equation, we guess

$$y_p(t) = at + b + ce^t.$$

Substituting this guess into the differential equation, we obtain

$$ce^t + 6(a + ce^t) + 8(at + b + ce^t) = 2t + e^t,$$

which simplifies to

$$(15c)e^t + (8a)t + (6a + 8b) = 2t + e^t.$$

Hence, $c = 1/15$, $a = 1/4$, and $b = -3/16$ yield the solution

$$y_p(t) = \tfrac{1}{4}t - \tfrac{3}{16} + \tfrac{1}{15}e^t.$$

The general solution of the forced equation is

$$y(t) = k_1 e^{-4t} + k_2 e^{-2t} + \tfrac{1}{4}t - \tfrac{3}{16} + \tfrac{1}{15}e^t.$$

(b) Note that

$$y'(t) = -4k_1 e^{-4t} - 2k_2 e^{-2t} + \tfrac{1}{4} + \tfrac{1}{15}e^t.$$

Hence, to obtain the desired initial conditions we must solve

$$\begin{cases} k_1 + k_2 - \frac{3}{16} + \frac{1}{15} = 0 \\ -4k_1 - 2k_2 + \frac{1}{4} + \frac{1}{15} = 0. \end{cases}$$

We obtain $k_1 = 3/80$ and $k_2 = 1/12$. Hence, the desired solution is

$$y(t) = \tfrac{3}{80}e^{-4t} + \tfrac{1}{12}e^{-2t} + \tfrac{1}{4}t - \tfrac{3}{16} + \tfrac{1}{15}e^{t}.$$

(c) For large t, the term $e^t/15$ dominates, so the solution tends to infinity at a rate determined by $e^t/15$.

41. **(a)** To find the general solution, we first guess

$$y_p(t) = ae^{-t} + bt + c,$$

where a, b and c are constants to be determined. For y_p,

$$\frac{dy_p}{dt} = -ae^{-t} + b \quad \text{and} \quad \frac{d^2y_p}{dt^2} = ae^{-t}.$$

Substituting these derivatives into the differential equation and collecting terms gives

$$(a + 4a)e^{-t} + (4b)t + (4c) = t + e^{-t},$$

which is satisfied if $5a = 1$, $4b = 1$, and $4c = 0$. Hence, a solution is

$$y_p(t) = \tfrac{1}{5}e^{-t} + \tfrac{1}{4}t.$$

To find the general solution of the homogeneous equation, we note that the characteristic polynomial $s^2 + 4$ has roots $s = \pm 2i$. Hence, the general solution of the forced equation is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t + \tfrac{1}{5}e^{-t} + \tfrac{1}{4}t.$$

(b) To find the solution with the desired initial conditions, we note that $y(0) = k_1 + 0 + 1/5$ and $y'(0) = 2k_2 - 1/5 + 1/4$. We must solve the simultaneous equations

$$\begin{cases} k_1 + \frac{1}{5} = 0 \\ 2k_2 + \frac{1}{20} = 0. \end{cases}$$

Thus, $k_1 = -1/5$ and $k_2 = -1/40$ yield the solution

$$y(t) = -\tfrac{1}{5}\cos 2t - \tfrac{1}{40}\sin 2t + \tfrac{1}{5}e^{-t} + \tfrac{1}{4}t.$$

(c) Since all of the terms in the solution except $t/4$ are bounded for $t > 0$, the solution tends to infinity at a rate that is determined by $t/4$.

42. **(a)** To find the general solution of the unforced equation, we note that the characteristic polynomial is $s^2 + 4$, which has roots $s = \pm 2i$. So the general solution of the unforced equation is

$$k_1 \cos 2t + k_2 \sin 2t.$$

To find a particular solution of the forced equation we guess

$$y_p(t) = a + bt + ct^2 + de^t.$$

Substituting this guess into the differential equation yields

$$(2c + de^t) + 4(a + bt + ct^2 + de^t) = 6 + t^2 + e^t,$$

which simplifies to

$$(2c + 4a) + (4b)t + (4c)t^2 + (5d)e^t = 6 + t^2 + e^t.$$

So $d = 1/5$, $c = 1/4$, $b = 0$, and $a = 1/8$ yield a solution, and the general solution of the forced equation is

$$y(t) = k_1 \cos 2t + k_2 \sin 2t + \tfrac{11}{8} + \tfrac{1}{4}t^2 + \tfrac{1}{5}e^t.$$

(b) Note that

$$y'(t) = -2k_1 \sin 2t + 2k_2 \cos 2t + \tfrac{1}{2}t + \tfrac{1}{5}e^t.$$

To obtain the desired initial conditions we must solve

$$\begin{cases} k_1 + \frac{11}{8} + \frac{1}{5} = 0 \\ \quad 2k_2 + \frac{1}{5} = 0, \end{cases}$$

which yields $k_1 = -63/40$ and $k_2 = -1/10$. The solution of the initial-value problem is

$$y(t) = -\tfrac{63}{40} \cos 2t - \tfrac{1}{10} \sin 2t + \tfrac{11}{8} + \tfrac{1}{4}t^2 + \tfrac{1}{5}e^t.$$

(c) This solution tends to infinity at a rate that is determined by $e^t/5$ because this term dominates when t is large.

EXERCISES FOR SECTION 4.2

1. Recalling that the real part of e^{it} is $\cos t$, we see that the complex version of this equation is

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{it}.$$

To find a particular solution, we guess $y_c(t) = ae^{it}$. Then $dy_c/dt = iae^{it}$ and $d^2y_c/dt^2 = -ae^{it}$. Substituting these derivatives into the equation and collecting terms yields

$$(-a + 3ia + 2a)e^{-it} = e^{it},$$