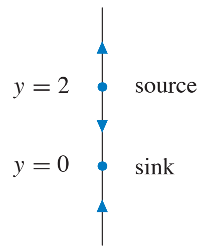
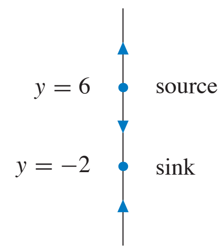
EXERCISES FOR SECTION 1.6

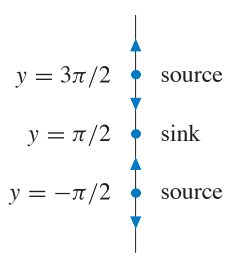
1. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = 3y(y-2)$, the equilibrium points are $y = 0$ and $y = 2$. Since $f(y)$ is positive for $y < 0$, negative for $0 < y < 2$, and positive for $y > 2$, the equilibrium point $y = 0$ is a sink and the equilibrium point $y = 2$ is a source.



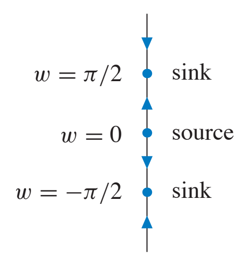
2. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = y^2 - 4y - 12 = (y-6)(y+2)$, the equilibrium points are $y = -2$ and $y = 6$. Since $f(y)$ is positive for $y < -2$, negative for $-2 < y < 6$, and positive for $y > 6$, the equilibrium point $y = -2$ is a sink and the equilibrium point $y = 6$ is a source.



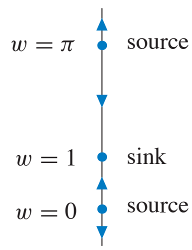
3. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = \cos y$, the equilibrium points are $y = \pi/2 + n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. Since $\cos y > 0$ for $-\pi/2 < y < \pi/2$ and $\cos y < 0$ for $\pi/2 < y < 3\pi/2$, we see that the equilibrium point at $y = \pi/2$ is a sink. Since the sign of $\cos y$ alternates between positive and negative in a period fashion, we see that the equilibrium points at $y = \pi/2 + 2n\pi$ are sinks and the equilibrium points at $y = 3\pi/2 + 2n\pi$ are sources.



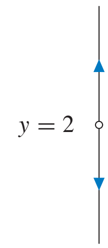
4. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = w \cos w$, the equilibrium points are $w = 0$ and $w = \pi/2 + n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. The sign of $w \cos w$ alternates positive and negative at successive zeros. It is negative for $-\pi/2 < w < 0$ and positive for $0 < w < \pi/2$. Therefore, $w = 0$ is a source, and the equilibrium points alternate back and forth between sources and sinks.



5. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = (1 - w)\sin w$, the equilibrium points are $w = 1$ and $w = n\pi$, where $n = 0, \pm 1, \pm 2, \ldots$. The sign of $(1 - w)\sin w$ alternates between positive and negative at successive zeros. It is negative for $-\pi < w < 0$ and positive for $0 < w < 1$. Therefore, $w = 0$ is a source, and the equilibrium points alternate between sinks and sources.

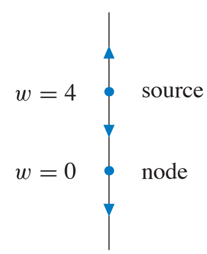


6. This equation has no equilibrium points, but the equation is not defined at $y = 2$. For $y > 2$, $dy/dt > 0$, so solutions increase. If $y < 2$, $dy/dt < 0$, so solutions decrease. The solutions approach the point $y = 2$ as time decreases and actually arrive there in finite time.

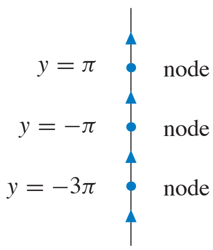


7. The derivative dv/dt is always negative, so there are no equilibrium points, and all solutions are decreasing.

8. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = 3w^3 - 12w^2$, the equilibrium points are $w = 0$ and $w = 4$. Since $f(w) < 0$ for $w < 0$ and $0 < w < 4$, and $f(w) > 0$ for $w > 4$, the equilibrium point at $w = 0$ is a node and the equilibrium point at $w = 4$ is a source.

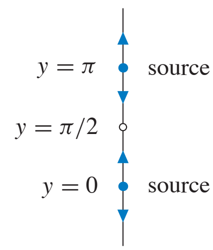


9. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = 1 + \cos y$, the equilibrium points are $y = n\pi$, where $n = \pm 1, \pm 3, \ldots$. Since $f(y)$ is non-negative for all values of y, all of the equilibrium points are nodes.

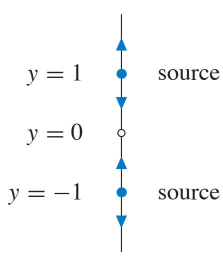


10. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = \tan y$, the equilibrium points are $y = n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$. Since $\tan y$ changes from negative to positive at each of its zeros, all of these equilibria are sources.

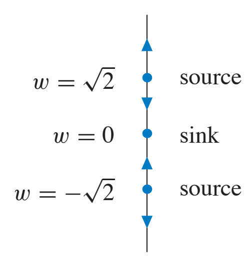
The differential equation is not defined at $y = \pi/2 + n\pi$ for $n = 0, \pm 1, \pm 2, \ldots$. Solutions increase or decrease toward one of these points as t increases and reach it in finite time.



11. The equilibrium points of $dy/dt = f(y)$ are the numbers y where $f(y) = 0$. For $f(y) = y \ln |y|$, there are equilibrium points at $y = \pm 1$. In addition, although the function $f(y)$ is technically undefined at $y = 0$, the limit of $f(y)$ as $y \to 0$ is 0. Thus we can treat $y = 0$ as another equilibrium point. Since $f(y) < 0$ for $y < -1$ and $0 < y < 1$, and $f(y) > 0$ for $y > 1$ and $-1 < y < 0$, $y = -1$ is a source, $y = 0$ is a sink, and $y = 1$ is a source.



12. The equilibrium points of $dw/dt = f(w)$ are the numbers w where $f(w) = 0$. For $f(w) = (w^2 - 2) \arctan w$, there are equilibrium points at $w = \pm\sqrt{2}$ and $w = 0$. Since $f(w) > 0$ for $w > \sqrt{2}$ and $-\sqrt{2} < w < 0$, and $f(w) < 0$ for $w < -\sqrt{2}$ and $0 < w < \sqrt{2}$, the equilibrium points at $w = \pm\sqrt{2}$ are sources, and the equilibrium point at $w = 0$ is a sink.



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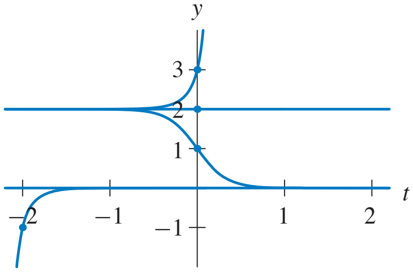

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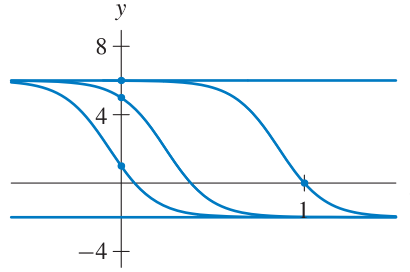

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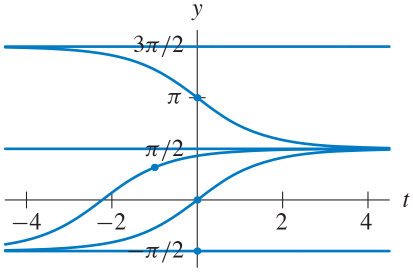

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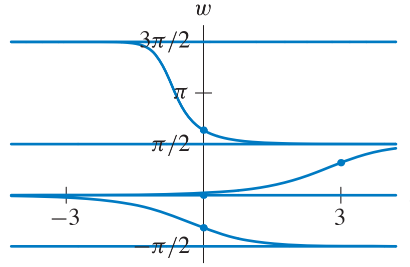

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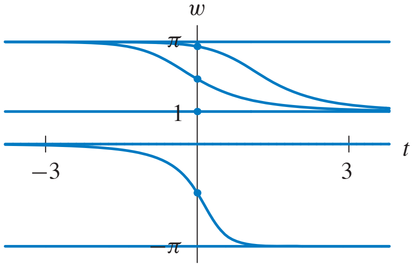

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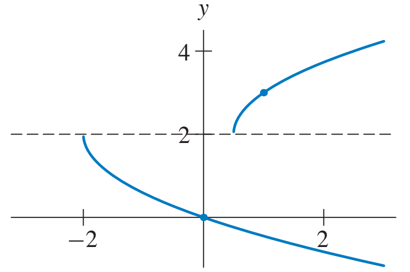
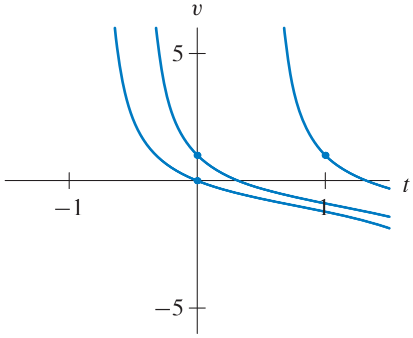


The equation is undefined at $y = 2$.

19.



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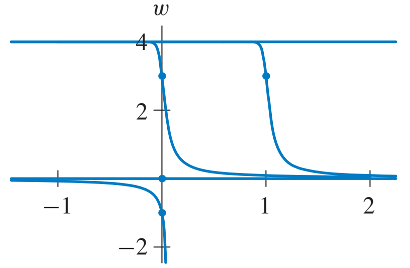

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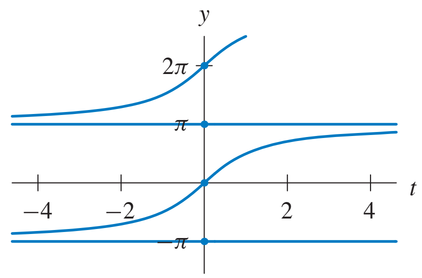


22. Because $y(0) = -1 < 2 - \sqrt{2}$, this solution increases toward $2 - \sqrt{2}$ as t increases and decreases as t decreases.

23. The initial value $y(0) = 2$ is between the equilibrium points $y = 2 - \sqrt{2}$ and $y = 2 + \sqrt{2}$. Also, $dy/dt < 0$ for $2 - \sqrt{2} < y < 2 + \sqrt{2}$. Hence the solution is decreasing and tends toward $y = 2 - \sqrt{2}$ as $t \to \infty$. It tends toward $y = 2 + \sqrt{2}$ as $t \to -\infty$.

24. The initial value $y(0) = -2$ is below both equilibrium points. Since $dy/dt > 0$ for $y < 2 - \sqrt{2}$, the solution is increasing for all t and tends to the equilibrium point $y = 2 - \sqrt{2}$ as $t \to \infty$. As t decreases, $y(t) \to -\infty$ in finite time. In fact, because $y(0) = -2 < -1$, this solution is always below the solution in Exercise 22.

25. The initial value $y(0) = -4$ is below both equilibrium points. Since $dy/dt > 0$ for $y < 2 - \sqrt{2}$, the solution is increasing for all t and tends to the equilibrium point $y = 2 - \sqrt{2}$ as $t \to \infty$. As t decreases, $y(t) \to -\infty$ in finite time.

26. The initial value $y(0) = 4$ is greater than the largest equilibrium point $2 + \sqrt{2}$, and $dy/dt > 0$ if $y > 2 + \sqrt{2}$. Hence, this solution increases without bound as t increases. (In fact, it blows up in finite time). As $t \to -\infty$, $y(t) \to 2 + \sqrt{2}$.

27. The initial value $y(3) = 1$ is between the equilibrium points $y = 2 - \sqrt{2}$ and $y = 2 + \sqrt{2}$. Also, $dy/dt < 0$ for $2 - \sqrt{2} < y < 2 + \sqrt{2}$. Hence the solution is decreasing and tends toward the smaller equilibrium point $y = 2 - \sqrt{2}$ as $t \to \infty$. It tends toward the larger equilibrium point $y = 2 + \sqrt{2}$ as $t \to -\infty$.

28. **(a)** Any solution that has an initial value between the equilibrium points at $y = -1$ and $y = 2$ must remain between these values for all t, so $-1 < y(t) < 2$ for all t.

(b) The extra assumption implies that the solution is increasing for all t such that $-1 < y(t) < 2$. Again assuming that the Uniqueness Theorem applies, we conclude that $y(t) \to 2$ as $t \to \infty$ and $y(t) \to -1$ as $t \to -\infty$.

29. The function $f(y)$ has two zeros $\pm y_0$, where y_0 is some positive number. So the differential equation $dy/dt = f(y)$ has two equilibrium solutions, one for each zero. Also, $f(y) < 0$ if $-y_0 < y < y_0$ and $f(y) > 0$ if $y < -y_0$ or if $y > y_0$. Hence y_0 is a source and $-y_0$ is a sink.

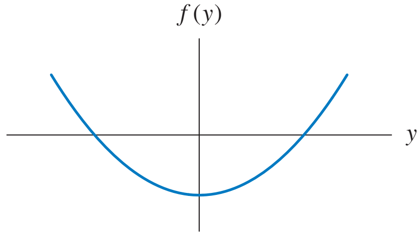
30. The function $f(y)$ has two zeros, one positive and one negative. We denote them as y_1 and y_2, where $y_1 < y_2$. So the differential equation $dy/dt = f(y)$ has two equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y_1 < y < y_2$ and $f(y) < 0$ if $y < y_1$ or if $y > y_2$. Hence y_1 is a source and y_2 is a sink.

31. The function $f(y)$ has three zeros. We denote them as y_1, y_2, and y_3, where $y_1 < 0 < y_2 < y_3$. So the differential equation $dy/dt = f(y)$ has three equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y < y_1$, $f(y) < 0$ if $y_1 < y < y_2$, and $f(y) > 0$ if $y_2 < y < y_3$ or if $y > y_3$. Hence y_1 is a sink, y_2 is a source, and y_3 is a node.

32. The function $f(y)$ has four zeros, which we denote $y_1, \ldots, y_4$ where $y_1 < 0 < y_2 < y_3 < y_4$. So the differential equation $dy/dt = f(y)$ has four equilibrium solutions, one for each zero. Also, $f(y) > 0$ if $y < y_1$, if $y_2 < y < y_3$, or if $y_3 < y < y_4$; and $f(y) < 0$ if $y_1 < y < y_2$ or if $y > y_4$. Hence y_1 is a sink, y_2 is a source, y_3 is a node, and y_4 is a sink.

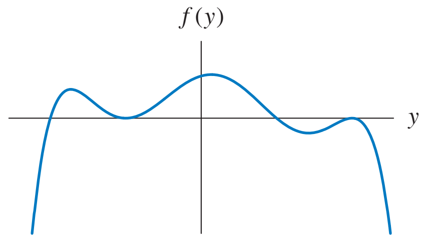
33. Since there are two equilibrium points, the graph of $f(y)$ must touch the y-axis at two distinct numbers y_1 and y_2. Assume that $y_1 < y_2$. Since the arrows point up if $y < y_1$ and if $y > y_2$, we must have $f(y) > 0$ for $y < y_1$ and for $y > y_2$. Similarly, $f(y) < 0$ for $y_1 < y < y_2$.

The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of $f(y)$. So the following graph is one of many possible answers.



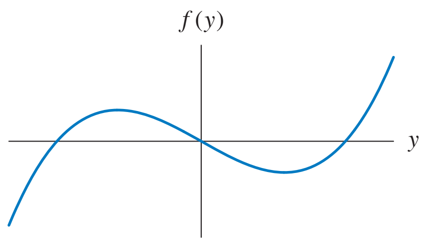
34. Since there are four equilibrium points, the graph of $f(y)$ must touch the y-axis at four distinct numbers y_1, y_2, y_3, and y_4. We assume that $y_1 < y_2 < y_3 < y_4$. Since the arrows point up only if $y_1 < y < y_2$ or if $y_2 < y < y_3$, we must have $f(y) > 0$ for $y_1 < y < y_2$ and for $y_2 < y < y_3$. Moreover, $f(y) < 0$ if $y < y_1$, if $y_3 < y < y_4$, or if $y > y_4$. Therefore, the graph of f crosses the y-axis at y_1 and y_3, but it is tangent to the y-axis at y_2 and y_4.

The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of $f(y)$. So the following graph is one of many possible answers.



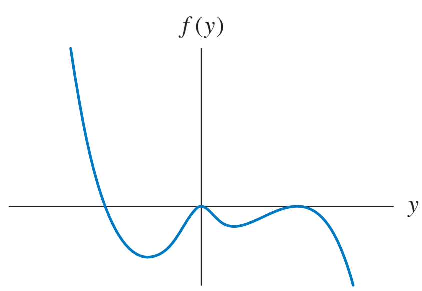
35. Since there are three equilibrium points (one appearing to be at $y = 0$), the graph of $f(y)$ must touch the y-axis at three numbers y_1, y_2, and y_3. We assume that $y_1 < y_2 = 0 < y_3$. Since the arrows point down for $y < y_1$ and $y_2 < y < y_3$, $f(y) < 0$ for $y < y_1$ and for $y_2 < y < y_3$. Similarly, $f(y) > 0$ if $y_1 < y < y_2$ and if $y > y_3$.

The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of $f(y)$. So the following graph is one of many possible answers.



36. Since there are three equilibrium points (one appearing to be at $y = 0$), the graph of $f(y)$ must touch the y-axis at three numbers y_1, y_2, and y_3. We assume that $y_1 < y_2 = 0 < y_3$. Since the arrows point up only for $y < y_1$, $f(y) > 0$ only if $y < y_1$. Otherwise, $f(y) \le 0$.

The precise location of the equilibrium points is not given, and the direction of the arrows on the phase line is determined only by the sign (and not the magnitude) of $f(y)$. So the following graph is one of many possible answers.



37. **(a)** This phase line has two equilibrium points, $y = 0$ and $y = 1$. Equations (ii), (iv), (vi), and (viii) have exactly these equilibria. There exists a node at $y = 0$. Only equations (iv) and (viii) have a node at $y = 0$. Moreover, for this phase line, $dy/dt < 0$ for $y > 1$. Only equation (viii) satisfies this property. Consequently, the phase line corresponds to equation (viii).

(b) This phase line has two equilibrium points, $y = 0$ and $y = 1$. Equations (ii), (iv), (vi) and (viii) have exactly these equilibria. Moreover, for this phase line, $dy/dt > 0$ for $y > 1$. Only equations (iv) and (vi) satisfy this property. Lastly, $dy/dt > 0$ for $y < 0$. Only equation (vi) satisfies this property. Consequently, the phase line corresponds to equation (vi).

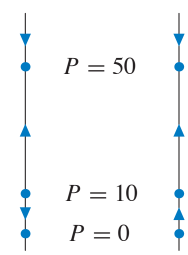
(c) This phase line has an equilibrium point at $y = 3$. Only equations (i) and (v) have this equilibrium point. Moreover, this phase line has another equilibrium point at $y = 0$. Only equation (i) satisfies this property. Consequently, the phase line corresponds to equation (i).

(d) This phase line has an equilibrium point at $y = 2$. Only equations (iii) and (vii) have this equilibrium point. Moreover, there exists a node at $y = 0$. Only equation (vii) satisfies this property. Consequently, the phase line corresponds to equation (vii).

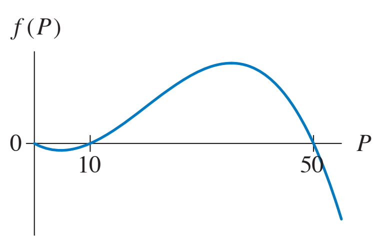
38. **(a)** Because $f(y)$ is continuous we can use the Intermediate Value Theorem to say that there must be a zero of $f(y)$ between -10 and 10. This value of y is an equilibrium point of the differential equation. In fact, $f(y)$ must cross from positive to negative, so if there is a single equilibrium point, it must be a sink (see part (b)).

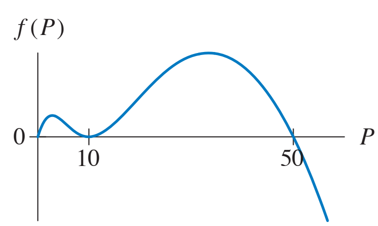
(b) We know that $f(y)$ must cross the y-axis between -10 and 10. Moreover, it must cross from positive to negative because $f(-10)$ is positive and $f(10)$ is negative. Where $f(y)$ crosses the y-axis from positive to negative, we have a sink. If $y = 1$ is a source, then crosses the y-axis from negative to positive at $y = 1$. Hence, $f(y)$ must cross the y-axis from positive to negative at least once between $y = -10$ and $y = 1$ and at least once between $y = 1$ and $y = 10$. There must be at least one sink in each of these intervals. (We need the assumption that the number of equilibrium points is finite to prevent cases where $f(y) = 0$ along an entire interval.)

39. **(a)** In terms of the phase line with $P \geq 0$, there are three equilibrium points. If we assume that $f(P)$ is differentiable, then a decreasing population at $P = 100$ implies that $f(P) < 0$ for $P > 50$. An increasing population at $P = 25$ implies that $f(P) > 0$ for $10 < P < 50$. These assumptions leave two possible phase lines since the arrow between $P = 0$ and $P = 10$ is undetermined.



(b) Given the observations in part (a), we see that there are two basic types of graphs that go with the assumptions. However, there are many graphs that correspond to each possibility. The following two graphs are representative.





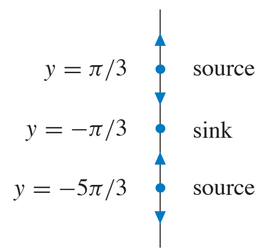
(c) The functions $f(P) = P(P-10)(50-P)$ and $f(P) = P(P-10)^2(50-P)$ respectively are two examples but there are many others.

40. **(a)** The equilibrium points of $d\theta/dt = f(\theta)$ are the numbers θ where $f(\theta) = 0$. For

$$f(\theta) = 1 - \cos\theta + (1+\cos\theta)\left(-\tfrac{1}{3}\right) = \tfrac{2}{3}(1 - 2\cos\theta),$$

the equilibrium points are $\theta = 2\pi n \pm \pi/3$, where $n = 0, \pm 1, \pm 2, \ldots$.

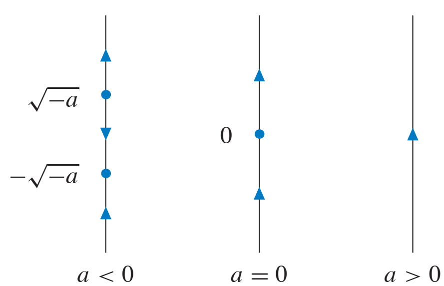
(b) The sign of $d\theta/dt$ alternates between positive and negative at successive equilibrium points. It is negative for $-\pi/3 < \theta < \pi/3$ and positive for $\pi/3 < \theta < 5\pi/3$. Therefore, $\pi/3 = 0$ is a source, and the equilibrium points alternate back and forth between sources and sinks.



41. The equilibrium points occur at solutions of $dy/dt = y^2 + a = 0$. For $a > 0$, there are no equilibrium points. For $a = 0$, there is one equilibrium point, $y = 0$. For $a < 0$, there are two equilibrium points, $y = \pm\sqrt{-a}$.

To draw the phase lines, note that:

- If $a > 0$, $dy/dt = y^2 + a > 0$, so the solutions are always increasing.
- If $a = 0$, $dy/dt > 0$ unless $y = 0$. Thus, $y = 0$ is a node.
- For $a < 0$, $dy/dt < 0$ for $-\sqrt{-a} < y < \sqrt{-a}$, and $dy/dt > 0$ for $y < -\sqrt{-a}$ and for $y > \sqrt{-a}$.



(a) The phase lines for $a < 0$ are qualitatively the same, and the phase lines for $a > 0$ are qualitatively the same.

(b) The phase line undergoes a qualitative change at $a = 0$.