Math 2410-10/15 **Quiz #8 Key** **April 16^{th}, 2017**

Note: Partial credit can not be awarded unless there is legible work to assess. Feel free to use the back of this page if you require additional space for your solutions.

1. Consider the following linear system of differential equations.

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \mathbf{Y}$$

(i) Find the general solution to this system;
(ii) and sketch its phase portrait.

Solution:

(i) The first step in solving any linear system is to find the eigenvalues of the coefficient matrix $\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}$. To that end, note

$$\det \begin{pmatrix} -2-\lambda & -1 \\ 1 & -4-\lambda \end{pmatrix} = (-2-\lambda)(-4-\lambda)-(-1)(1) = \lambda^2+6\lambda+9 = (\lambda+3)^2 = 0$$

if and only if $\lambda = -3$. So this system has a repeated eigenvalue $\lambda = -3$. Thus, the general solution of the system is

$$\mathbf{Y}(t) = e^{-3t}\mathbf{V}_0 + te^{-3t}\mathbf{V}_1$$

where $\mathbf{V}_0 = (x_0, y_0)$ is an arbitrary initial condition and $\mathbf{V}_1$ is given by

$$\mathbf{V}_1 = \begin{pmatrix} -2-(-3) & -1 \\ 1 & -4-(-3) \end{pmatrix} \mathbf{V}_0 = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}.$$

We conclude that the general solution is

$$\mathbf{Y}(t) = e^{-3t}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-3t}\begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}.$$

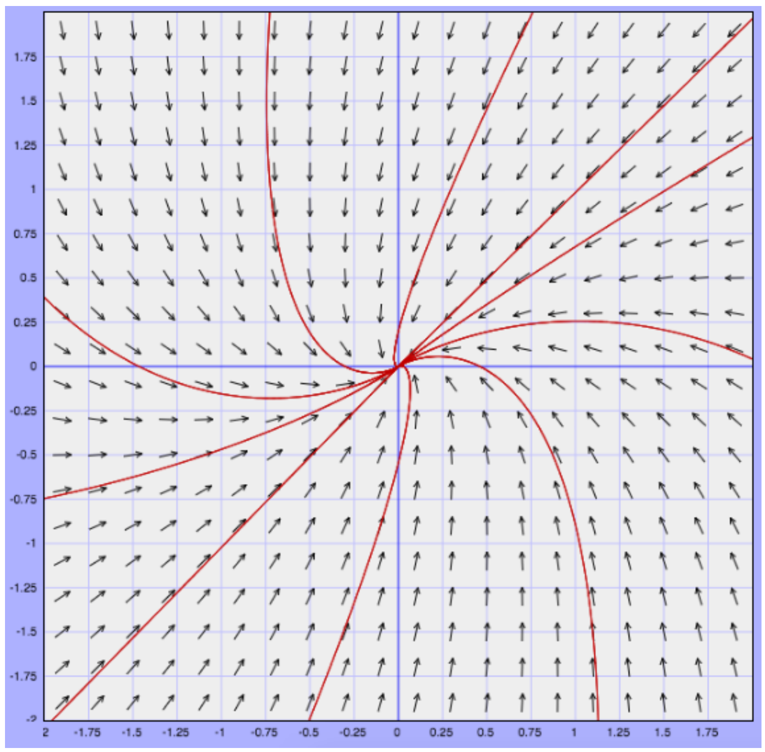
(ii) Note: This was not a part of the in-class quiz due to time constraints, I am including it here for sake of reference. The phase portrait is given below. To sketch this for the system we first plot the straight-line solutions and then determine the direction of the "pseudo-spirals" using the vectorfield to complete the phase portrait. To plot straight-line solutions, we require an eigenvector. Recall that $\mathbf{V}_1$ is either $(0, 0)$ or an eigenvector. So, for any choice of x_0 and y_0, we have that $(x_0 - y_0, x_0 - y_0)$ is an eigenvector. Let's use $x_0 = 1$ and $y_0 = 0$ to have $(1, 1)$ as our eigenvector. Thus straight-line solutions to this system are of the form

$$ke^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for any constant k. Hence there are straight-line solutions approaching the origin along the line that passes through $(1,1)$ and $(0,0)$. We know every other curve is a straight-line solution approaching the origin. It is only left to determine if they approach in a clockwise or counterclockwise manner. To do this, let us consider the vector field at $(1,0)$ and $(0,1)$. We have

$$\begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}.$$

So solutions are traveling to the left and up at $(1,0)$ and to the left and down at $(0,1)$. Thus, the solution curves are approaching the origin in a counterclockwise fashion. With this, we are able to arrive at a sketch resembling the following.



2. Find the general solution of the following linear second-order differential equation.

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 29y = 0$$

Solution: First, we find solutions of the form e^{st}. We know such a function is a solution if and only if $s^2 - 4s + 29 = 0$. Thus if

$$s = \frac{4 \pm \sqrt{16 - 4(29)}}{2} = \frac{4 \pm \sqrt{16 - 116}}{2} = \frac{4 \pm 10i}{2} = 2 \pm 5i.$$

Hence $e^{(2+5i)t}$ is a complex valued solution which we can rewrite using Euler's formula to obtain two real valued solutions y_{re} and y_{im}:

$$e^{(2+5i)t} = e^{2t}e^{i(5t)} = e^{2t}\cos(5t) + i(e^{2t}\sin(5t))$$

where $y_{re}(t) = e^{2t}\cos(5t)$ is the real part of $e^{(2+5i)t}$ and $y_{im}(t) = e^{2t}\sin(5t)$ is the imaginary part. By our knowledge of linear systems with complex eigenvalues, we can conclude that the general solution to the system is

$$y(t) = k_1 e^{2t}\cos(5t) + k_2 e^{2t}\sin(5t)$$

for constants k_1 and k_2.