holds even as y tends to zero.

If we define $a(t) = f(t, 1)$, we have

$$f(t, y) = a(t)y.$$

The differential equation is linear and homogeneous.

EXERCISES FOR SECTION 1.9

1. We rewrite the equation in the form

$$\frac{dy}{dt} + \frac{y}{t} = 2$$

and note that the integrating factor is

$$\mu(t) = e^{\int (1/t)\,dt} = e^{\ln t} = t.$$

Multiplying both sides by $\mu(t)$, we obtain

$$t\frac{dy}{dt} + y = 2t.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(ty)}{dt} = 2t,$$

and integrating both sides with respect to t, we obtain

$$ty = t^2 + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = \frac{1}{t}(t^2 + c) = t + \frac{c}{t}.$$

2. We rewrite the equation in the form

$$\frac{dy}{dt} - \frac{3}{t}y = t^5$$

and note that the integrating factor is

$$\mu(t) = e^{\int (-3/t)\,dt} = e^{-3\ln t} = e^{\ln(t^{-3})} = t^{-3}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$t^{-3}\frac{dy}{dt} - 3t^{-4}y = t^2.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(t^{-3}y)}{dt} = t^2$$

and integrating both sides with respect to t, we obtain

$$t^{-3}y = \frac{t^3}{3} + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = \frac{t^6}{3} + ct^3.$$

3. We rewrite the equation in the form

$$\frac{dy}{dt} + \frac{y}{1+t} = t^2$$

and note that the integrating factor is

$$\mu(t) = e^{\int (1/(1+t))\,dt} = e^{\ln(1+t)} = 1 + t.$$

Multiplying both sides by $\mu(t)$, we obtain

$$(1+t)\frac{dy}{dt} + y = (1+t)t^2.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d((1+t)y)}{dt} = t^3 + t^2,$$

and integrating both sides with respect to t, we obtain

$$(1+t)y = \frac{t^4}{4} + \frac{t^3}{3} + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = \frac{3t^4 + 4t^3 + 12c}{12(t+1)}.$$

4. We rewrite the equation in the form

$$\frac{dy}{dt} + 2ty = 4e^{-t^2}$$

and note that the integrating factor is

$$\mu(t) = e^{\int 2t\,dt} = e^{t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{t^2}\frac{dy}{dt} + 2te^{t^2}y = 4.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(e^{t^2}y)}{dt} = 4,$$

and integrating both sides with respect to t, we obtain

$$e^{t^2}y = 4t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = 4te^{-t^2} + ce^{-t^2}.$$

5. Note that the integrating factor is

$$\mu(t) = e^{\int(-2t/(1+t^2))\,dt} = e^{-\ln(1+t^2)} = \left(e^{\ln(1+t^2)}\right)^{-1} = \frac{1}{1+t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{1+t^2}\frac{dy}{dt} - \frac{2t}{(1+t^2)^2}\,y = \frac{3}{1+t^2}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{1+t^2}\right) = \frac{3}{1+t^2}.$$

Integrating both sides with respect to t, we obtain

$$\frac{y}{1+t^2} = 3\arctan(t) + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = (1+t^2)(3\arctan(t) + c).$$

6. Note that the integrating factor is

$$\mu(t) = e^{\int(-2/t)\,dt} = e^{-2\ln t} = e^{\ln(t^{-2})} = t^{-2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$t^{-2}\frac{dy}{dt} - 2t^{-3}y = te^t.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(t^{-2}y)}{dt} = te^t,$$

and integrating both sides with respect to t, we obtain

$$t^{-2}y = (t-1)e^t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = t^2(t-1)e^t + ct^2.$$

7. We rewrite the equation in the form

$$\frac{dy}{dt} + \frac{y}{1+t} = 2$$

and note that the integrating factor is

$$\mu(t) = e^{\int (1/(1+t))\,dt} = e^{\ln(1+t)} = 1+t.$$

Multiplying both sides by $\mu(t)$, we obtain

$$(1+t)\frac{dy}{dt} + y = 2(1+t).$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d((1+t)y)}{dt} = 2(1+t),$$

and integrating both sides with respect to t, we obtain

$$(1+t)y = 2t + t^2 + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = \frac{t^2+2t+c}{1+t}.$$

To find the solution that satisfies the initial condition $y(0) = 3$, we evaluate the general solution at $t = 0$ and obtain

$$c = 3.$$

The desired solution is

$$y(t) = \frac{t^2+2t+3}{1+t}.$$

8. We rewrite the equation in the form

$$\frac{dy}{dt} - \frac{1}{t+1}\, y = 4t^2 + 4t$$

and note that the integrating factor is

$$\mu(t) = e^{\int (-1/(t+1))\, dt} = e^{-\ln(t+1)} = \left(e^{\ln((t+1)^{-1})}\right) = \frac{1}{t+1}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{t+1}\frac{dy}{dt} - \frac{1}{(t+1)^2}\, y = \frac{4t^2+4t}{t+1}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{t+1}\right) = 4t.$$

Integrating both sides with respect to t, we obtain

$$\frac{y}{t+1} = 2t^2 + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = (2t^2 + c)(t+1) = 2t^3 + 2t^2 + ct + c.$$

To find the solution that satisfies the initial condition $y(1) = 10$, we evaluate the general solution at $t = 1$ and obtain $c = 3$. The desired solution is

$$y(t) = 2t^3 + 2t^2 + 3t + 3.$$

9. In Exercise 1, we derived the general solution

$$y(t) = t + \frac{c}{t}.$$

To find the solution that satisfies the initial condition $y(1) = 3$, we evaluate the general solution at $t = 1$ and obtain $c = 2$. The desired solution is

$$y(t) = t + \frac{2}{t}.$$

10. In Exercise 4, we derived the general solution

$$y(t) = 4te^{-t^2} + ce^{-t^2}.$$

To find the solution that satisfies the initial condition $y(0) = 3$, we evaluate the general solution at $t = 0$ and obtain $c = 3$. The desired solution is

$$y(t) = 4te^{-t^2} + 3e^{-t^2}.$$

11. Note that the integrating factor is

$$\mu(t) = e^{\int -(2/t)\,dt} = e^{-2\int (1/t)\,dt} = e^{-2\ln t} = e^{\ln(t^{-2})} = \frac{1}{t^2}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$\frac{1}{t^2}\frac{dy}{dt} - \frac{2y}{t^3} = 2.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d}{dt}\left(\frac{y}{t^2}\right) = 2,$$

and integrating both sides with respect to t, we obtain

$$\frac{y}{t^2} = 2t + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = 2t^3 + ct^2.$$

To find the solution that satisfies the initial condition $y(-2) = 4$, we evaluate the general solution at $t = -2$ and obtain

$$-16 + 4c = 4.$$

Hence, $c = 5$, and the desired solution is

$$y(t) = 2t^3 + 5t^2.$$

12. Note that the integrating factor is

$$\mu(t) = e^{\int (-3/t)\,dt} = e^{-3\ln t} = e^{\ln(t^{-3})} = t^{-3}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$t^{-3}\frac{dy}{dt} - 3t^{-4}y = 2e^{2t}.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(t^{-3}y)}{dt} = 2e^{2t},$$

and integrating both sides with respect to t, we obtain

$$t^{-3}y = e^{2t} + c,$$

where c is an arbitrary constant. The general solution is

$$y(t) = t^3(e^{2t} + c).$$

To find the solution that satisfies the initial condition $y(1) = 0$, we evaluate the general solution at $t = 1$ and obtain $c = -e^2$. The desired solution is

$$y(t) = t^3(e^{2t} - e^2).$$

15. We rewrite the equation in the form

$$\frac{dy}{dt} - \frac{y}{t^2} = 4\cos t$$

and note that the integrating factor is

$$\mu(t) = e^{\int (-1/t^2)\,dt} = e^{1/t}.$$

Multiplying both sides by $\mu(t)$, we obtain

$$e^{1/t}\frac{dy}{dt} - \frac{e^{1/t}}{t^2}\,y = 4e^{1/t}\cos t.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(e^{1/t}y)}{dt} = 4e^{1/t}\cos t,$$

and integrating both sides with respect to t, we obtain

$$e^{1/t}y = \int 4e^{1/t}\cos t\,dt.$$

Since the integral on the right-hand side is impossible to express using elementary functions, we write the general solution as

$$y(t) = 4e^{-1/t}\int e^{1/t}\cos t\,dt.$$

16. We rewrite the equation in the form

$$\frac{dy}{dt} - y = 4\cos t^2$$

and note that the integrating factor is

$$\mu(t) = e^{\int -1\,dt} = e^{-t}.$$

Multiplying both sides of the equation by $\mu(t)$, we obtain

$$e^{-t}\frac{dy}{dt} - e^{-t}y = 4e^{-t}\cos t^2.$$

Applying the Product Rule to the left-hand side, we see that this equation is the same as

$$\frac{d(e^{-t}y)}{dt} = 4e^{-t}\cos t^2,$$

and integrating both sides with respect to t, we obtain

$$e^{-t}y = \int 4e^{-t}\cos t^2\,dt.$$

Since the integral on the right-hand side is impossible to express using elementary functions, we write the general solution as

$$y(t) = 4e^{t}\int e^{-t}\cos t^2\,dt.$$

This equation is linear, and we can rewrite it as

$$\frac{dS}{dt} + \frac{S}{15+t} = 2.$$

The integrating factor is

$$\mu(t) = e^{\int 1/(15+t)\,dt} = e^{\ln(15+t)} = 15+t.$$

Multiplying both sides of the equation by $\mu(t)$, we obtain

$$(15+t)\frac{dS}{dt} + S = 2(15+t),$$

which via the Product Rule is equivalent to

$$\frac{d((15+t)S)}{dt} = 30 + 2t.$$

Integration and simplification yields

$$S(t) = \frac{t^2 + 30t + c}{15+t}.$$

Using the initial condition $S(0) = 6$, we have $c/15 = 6$, which implies that $c = 90$ and

$$S(t) = \frac{t^2 + 30t + 90}{15+t}.$$

The tank is full when $t = 15$, and the amount of salt at that time is $S(15) = 51/2$ pounds.

25. We will use the term "parts" as shorthand for the product of parts per billion of dioxin and the volume of water in the tank. Basically this product represents the total amount of dioxin in the tank. The tank initially contains 200 gallons at a concentration of 2 parts per billion, which results in 400 parts of dioxin.

Let $y(t)$ be the amount of dioxin in the tank at time t. Since water with 4 parts per billion of dioxin flows in at the rate of 5 gallons per minute, 20 parts of dioxin enter the tank each minute. Also, the volume of water in the tank at time t is $200 + 2t$, so the concentration of dioxin in the tank is $y/(200 + 2t)$. Since well-mixed water leaves the tank at the rate of 2 gallons per minute, the differential equation that represents the change in the amount of dioxin in the tank is

$$\frac{dy}{dt} = 20 - 2\left(\frac{y}{200+2t}\right),$$

which can be simplified and rewritten as

$$\frac{dy}{dt} + \left(\frac{1}{100+t}\right) y = 20.$$

The integrating factor is

$$\mu(t) = e^{\int (1/(100+t))\,dt} = e^{\ln(100+t)} = 100+t.$$

Multiplying both sides by $\mu(t)$, we obtain

$$(100+t)\frac{dy}{dt} + y = 20(100+t),$$

which is equivalent to

$$\frac{d((100+t)y)}{dt} = 20(100+t)$$

by the Product Rule. Integrating both sides with respect to t, we obtain

$$(100+t)y = 2000t + 10t^2 + c.$$

Since $y(0) = 400$, we see that $c = 40,000$. Therefore,

$$y(t) = \frac{10t^2 + 2000t + 40,000}{t+100}.$$

The tank fills up at $t = 100$, and $y(100) = 1,700$. To express our answer in terms of concentration, we calculate $y(100)/400 = 4.25$ parts per billion.

26. Let $S(t)$ denote the amount of sugar in the tank at time t. Sugar is added to the tank at the rate of p pounds per minute. The amount of sugar that leaves the tank is the product of the concentration of the sugar in the water and the rate that the water leaves the tank. At time t, there are $100 - t$ gallons of sugar water in the tank, so the concentration of sugar is $S(t)/(100-t)$. Since sugar water leaves the tank at the rate of 1 gallon per minute, the differential equation for S is

$$\frac{dS}{dt} = p - \frac{S}{100-t}.$$

Since this equation is linear, we rewrite it as

$$\frac{dS}{dt} + \frac{S}{100-t} = p,$$

and the integrating factor is

$$\mu(t) = e^{\int (1/(100-t))\,dt} = e^{-\ln(100-t)} = \frac{1}{100-t}.$$

Multiplying both sides of the differential equation by $\mu(t)$ yields

$$\left(\frac{1}{100-t}\right)\frac{dS}{dt} + \frac{S}{(100-t)^2} = \frac{p}{100-t},$$

which is equivalent to

$$\frac{d}{dt}\left(\frac{S}{100-t}\right) = \frac{p}{100-t}$$

by the Product Rule. We integrate both sides and obtain

$$\frac{S}{100-t} = -p\ln(100-t) + c,$$