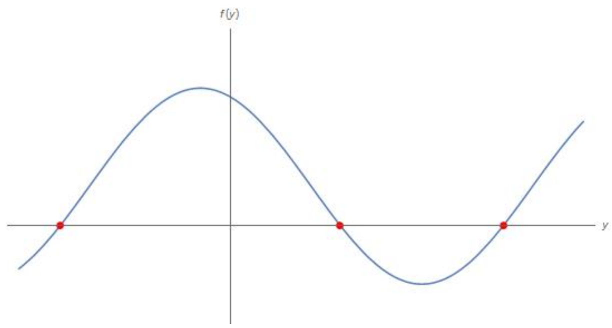
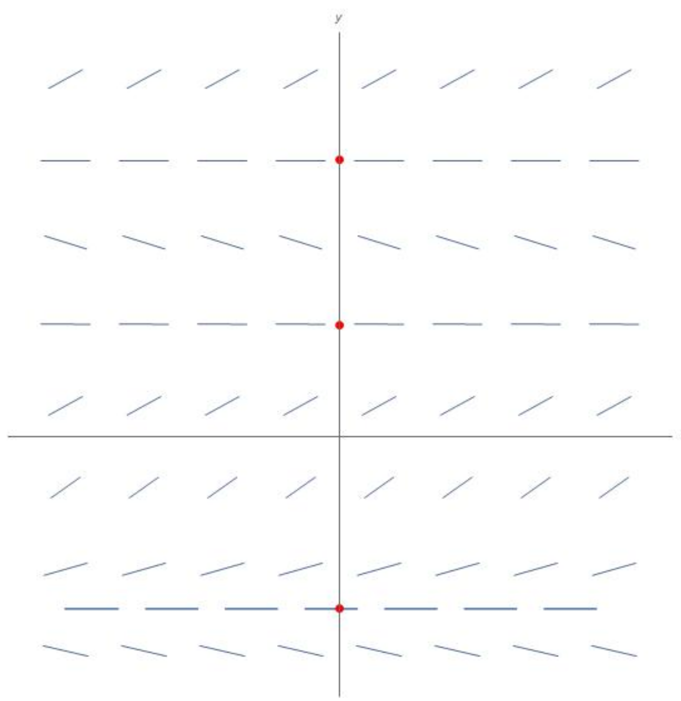
Math 2410-10/15 **Quiz #2** **January 30^{th}, 2017**

Note: Partial credit can not be awarded unless there is legible work to assess.

1. Suppose we know that the graph to the right is the graph of the right-hand side $f(y)$ of the differential equation
$$\frac{dy}{dt} = f(y).$$
Give a rough sketch of the slope field that corresponds to this differential equation.



Solution: To sketch the slope field, we note first that as this is an autonomous DE, the minitangents are all parallel along horizontal lines in the ty-plane. Next we identify locations in which $f(y)$ is zero (denoted by the red points) to find equilibrium solutions of the DE. At these points (note they are on the y-axis) minitangents of the slope field will all be horizontal. After this, we then plot minitangents in the intervals between these zeros where the direction and steepness is given by how large and positive or how small and negative the function is. To finish, we extrapolate what we've found across horizontal lines in the ty-plane.



2. A 5-gallon bucket is full of pure water. Suppose we begin dumping salt into the bucket continuously at a rate of 1/4 pounds per minute. Also, we open a spigot so that 1/2 gallons per minute leaves the bucket, and we add pure water to keep the bucket full. Finally, we ensure the salt water solution is always well mixed. Give a differential equation which models the change in the amount of salt in the bucket over time.
Note: You do not need to solve this differential equation.
Bonus: (1 point) How much salt do you expect to be in the solution after a very, very long time?

Solution: Let t be time in minutes, and $s(t)$ be the amount of salt in pounds contained in the bucket at time t. A reasonable model for the change in s over time, ds/dt, is:

$$\frac{ds}{dt} = (\text{rate at which salt enters the bucket}) - (\text{rate at which salt exits the bucket}).$$

Or more concisely:

$$\frac{ds}{dt} = (\text{rate in}) - (\text{rate out}).$$

Now, the only salt being added to the tank is the 1/4 pounds every minute. The fresh water will, of course, contribute no salt to the bucket. So

$$(\text{rate in}) = \frac{1}{4}\frac{\text{lbs}}{\text{min}}.$$

Notice that solution is leaving the bucket at a rate of 1/2 gallons per minute, so to find *rate out*, we simply need to find how much salt is in each gallon at a given time. As there is $s(t)$ pounds of salt at time t in a total of 5 gallons, we conclude there is $s(t)/5$ pounds per gallon in the tank at any given time. So,

$$(\text{rate out}) = \frac{s}{5}\frac{\text{lbs}}{\text{gal}} \cdot \frac{1}{2}\frac{\text{gal}}{\text{min}} = \frac{s}{10}\frac{\text{lbs}}{\text{min}}.$$

Thus, we obtain the following differential equation which models the rate of change in the amount of salt in the bucket over time:

$$\frac{ds}{dt} = \frac{1}{4} - \frac{s}{10}\frac{\text{lbs}}{\text{min}} = \frac{5-2s}{20}\frac{\text{lbs}}{\text{min}}.$$

Bonus: We see that $s_E(t) = 5/2$ is an equilibrium solution to the above DE. Furthermore, as our system has initial condition $s(0) = 0$, we see both $s(0) < s_E(t) = 5/2$ and that $ds/dt > 0$ for $s(0)$. Hence, as this DE satisfies the *existence and uniqueness theorem* we conclude that for all t in which the solution to our model $s(t)$ is defined, $s(t) < 5/2$ and $s(t)$ is increasing. This implies that $\lim_{t\to\infty} s(t) = 5/2$. In other words, after a very, very long time there should be about 5/2 pounds of salt in the system.