If the time between trains is exactly the equilibrium value ($x = \alpha/\beta$), then theoretically $x(t)$ is constant. However, any disruption to x causes the solution to tend away from the source. Since it is very likely that some stops will have fewer than the expected number of passengers and some stops will have more, it is unlikely that the time between trains will remain constant for long.

48. If the trains are spaced too close together, then each train will catch up with the one in front of it. This phenomenon will continue until there is a very large time gap between two successive trains. When this happens, the time between these two trains will grow, and a second cluster of trains will form.

For the "B branch of the Green Line," the clusters seem to contain three or four trains during rush hour. For the "D branch of the Green Line," clusters seem to contain only two trains or three trains.

It is tempting to say that the trains should be spaced at time intervals of exactly α/β, and nothing else needs to be changed. In theory, this choice will result in equal spacing between trains, but we must remember that the equilibrium point, $x = \alpha/\beta$, is a source. Hence, anything that perturbs x will cause x to increase or decrease in an exponential fashion.

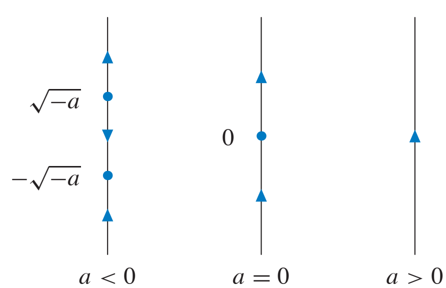
The only solution that is consistent with this model is to have the trains run to a schedule that allows for sufficient time for the loading of passengers. The trains will occasionally have to wait if they get ahead of schedule, but this plan avoids the phenomenon of one tremendously crowded train followed by two or three relatively empty ones.

EXERCISES FOR SECTION 1.7

1. The equilibrium points occur at solutions of $dy/dt = y^2 + a = 0$. For $a > 0$, there are no equilibrium points. For $a = 0$, there is one equilibrium point, $y = 0$. For $a < 0$, there are two equilibrium points, $y = \pm\sqrt{-a}$. Thus, $a = 0$ is a bifurcation value.

To draw the phase lines, note that:

- If $a > 0$, $dy/dt = y^2 + a > 0$, so the solutions are always increasing.
- If $a = 0$, $dy/dt > 0$ unless $y = 0$. Thus, $y = 0$ is a node.
- For $a < 0$, $dy/dt < 0$ for $-\sqrt{-a} < y < \sqrt{-a}$, and $dy/dt > 0$ for $y < -\sqrt{-a}$ and for $y > \sqrt{-a}$.

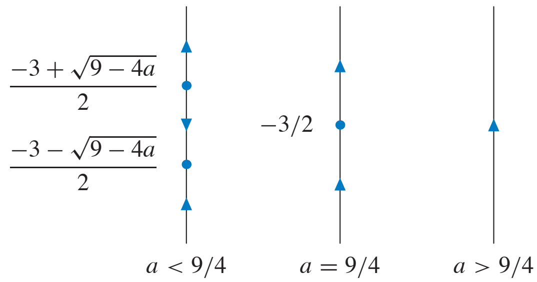


Phase lines for $a < 0$, $a = 0$, and $a > 0$.

2. The equilibrium points occur at solutions of $dy/dt = y^2 + 3y + a = 0$. From the quadratic formula, we have

$$y = \frac{-3 \pm \sqrt{9 - 4a}}{2}.$$

Hence, the bifurcation value of a is $9/4$. For $a < 9/4$, there are two equilibria, one source and one sink. For $a = 9/4$, there is one equilibrium which is a node, and for $a > 9/4$, there are no equilibria.



Phase lines for $a < 9/4$, $a = 9/4$, and $a > 9/4$.

3. The equilibrium points occur at solutions of $dy/dt = y^2 - ay + 1 = 0$. From the quadratic formula, we have

$$y = \frac{a \pm \sqrt{a^2 - 4}}{2}.$$

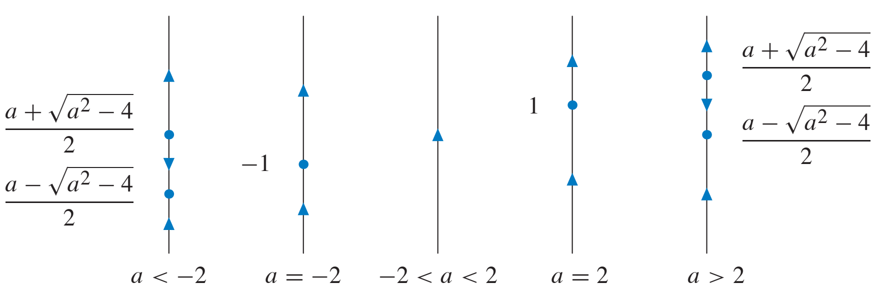
If $-2 < a < 2$, then $a^2 - 4 < 0$, and there are no equilibrium points. If $a > 2$ or $a < -2$, there are two equilibrium points. For $a = \pm 2$, there is one equilibrium point at $y = a/2$. The bifurcations occur at $a = \pm 2$.

To draw the phase lines, note that:

- For $-2 < a < 2$, $dy/dt = y^2 - ay + 1 > 0$, so the solutions are always increasing.
- For $a = 2$, $dy/dt = (y-1)^2 \geq 0$, and $y = 1$ is a node.
- For $a = -2$, $dy/dt = (y+1)^2 \geq 0$, and $y = -1$ is a node.
- For $a < -2$ or $a > 2$, let

$$y_1 = \frac{a - \sqrt{a^2 - 4}}{2} \quad \text{and} \quad y_2 = \frac{a + \sqrt{a^2 - 4}}{2}.$$

Then $dy/dt < 0$ if $y_1 < y < y_2$, and $dy/dt > 0$ if $y < y_1$ or $y > y_2$.



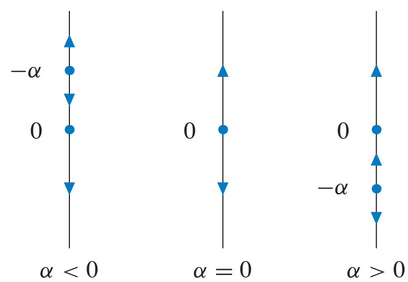
The five possible phase lines.

4. The equilibrium points occur at solutions of $dy/dt = y^3 + \alpha y^2 = 0$. For $\alpha = 0$, there is one equilibrium point, $y = 0$. For $\alpha \neq 0$, there are two equilibrium points, $y = 0$ and $y = -\alpha$. Thus, $\alpha = 0$ is a bifurcation value.

To draw the phase lines, note that:

- If $\alpha < 0$, $dy/dt > 0$ only if $y > -\alpha$.
- If $\alpha = 0$, $dy/dt > 0$ if $y > 0$, and $dy/dt < 0$ if $y < 0$.
- If $\alpha > 0$, $dy/dt < 0$ only if $y < -\alpha$.

Hence, as α increases from negative to positive, the source at $y = -\alpha$ moves from positive to negative as it "passes through" the node at $y = 0$.



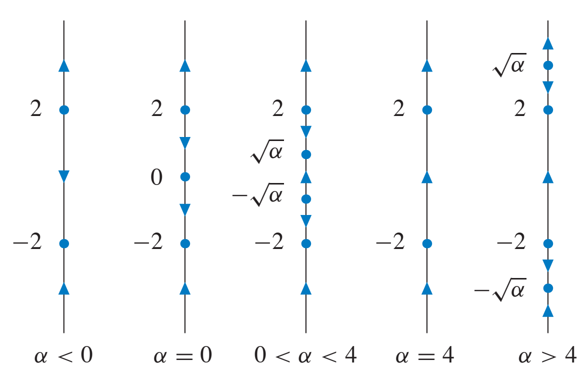
5. To find the equilibria we solve

$$(y^2 - \alpha)(y^2 - 4) = 0,$$

obtaining $y = \pm 2$ and $y = \pm\sqrt{\alpha}$ if $\alpha \geq 0$. Hence, there are two bifurcation values of α, $\alpha = 0$ and $\alpha = 4$.

For $\alpha < 0$, there are only two equilibria. The point $y = -2$ is a sink and $y = 2$ is a source. At $\alpha = 0$, there are three equilibria. There is a sink at $y = -2$, a source at $y = 2$, and a node at $y = 0$. For $0 < \alpha < 4$, there are four equilibria. The point $y = -2$ is still a sink, $y = -\sqrt{\alpha}$ is a source, $y = \sqrt{\alpha}$ is a sink, and $y = 2$ is still a source.

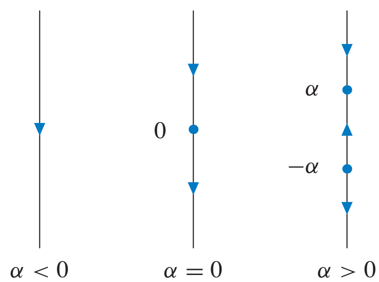
For $\alpha = 4$, there are only two equilibria, $y = \pm 2$. Both are nodes. For $\alpha > 4$, there are four equilibria again. The point $y = -\sqrt{\alpha}$ is a sink, $y = -2$ is now a source, $y = 2$ is now a sink, and $y = \sqrt{\alpha}$ is a source.



6. The equilibrium points occur at solutions of $dy/dt = \alpha - |y| = 0$. For $\alpha < 0$, there are no equilibrium points. For $\alpha = 0$, there is one equilibrium point, $y = 0$. For $\alpha > 0$, there are two equilibrium points, $y = \pm\alpha$. Therefore, $\alpha = 0$ is a bifurcation value.

To draw the phase lines, note that:

- If $\alpha < 0$, $dy/dt = \alpha - |y| < 0$, so the solutions are always decreasing.
- If $\alpha = 0$, $dy/dt < 0$ unless $y = 0$. Thus, $y = 0$ is a node.
- For $\alpha > 0$, $dy/dt > 0$ for $-\alpha < y < \alpha$, and $dy/dt < 0$ for $y < -\alpha$ and for $y > \alpha$.



7. We have

$$\frac{dy}{dt} = y^4 + \alpha y^2 = y^2(y^2 + \alpha).$$

If $\alpha > 0$, there is one equilibrium point at $y = 0$, and $dy/dt > 0$ otherwise. Hence, $y = 0$ is a node.

If $\alpha < 0$, there are equilibria at $y = 0$ and $y = \pm\sqrt{-\alpha}$. From the sign of $y^4 + \alpha y^2$, we know that $y = 0$ is a node, $y = -\sqrt{-\alpha}$ is a sink, and $y = \sqrt{-\alpha}$ is a source.

The bifurcation value of α is $\alpha = 0$. As α increases through 0, a sink and a source come together with the node at $y = 0$, leaving only the node. For $\alpha < 0$, there are three equilibria, and for $\alpha \geq 0$, there is only one equilibrium.

8. The equilibrium points occur at solutions of

$$\frac{dy}{dt} = y^6 - 2y^3 + \alpha = (y^3)^2 - 2(y^3) + \alpha = 0.$$

Using the quadratic formula to solve for y^3, we obtain

$$y^3 = \frac{2 \pm \sqrt{4 - 4\alpha}}{2}.$$

Thus the equilibrium points are at

$$y = \left(1 \pm \sqrt{1 - \alpha}\right)^{1/3}.$$

If $\alpha > 1$, there are no equilibrium points because this equation has no real solutions. If $\alpha < 1$, the differential equation has two equilibrium points. A bifurcation occurs at $\alpha = 1$ where the differential equation has one equilibrium point at $y = 1$.

9. The bifurcations occur at values of α for which the graph of $\sin y + \alpha$ is tangent to the y-axis. That is, $\alpha = -1$ and $\alpha = 1$.

For $\alpha < -1$, there are no equilibria, and all solutions become unbounded in the negative direction as t increases.

If $\alpha = -1$, there are equilibrium points at $y = \pi/2 \pm 2n\pi$ for every integer n. All equilibria are nodes, and as $t \to \infty$, all other solutions decrease toward the nearest equilibrium solution below the given initial condition.

For $-1 < \alpha < 1$, there are infinitely many sinks and infinitely many sources, and they alternate along the phase line. Successive sinks differ by 2π. Similarly, successive sources are separated by 2π.

As α increases from -1 to $+1$, nearby sink and source pairs move apart. This separation continues until α is close to 1 where each source is close to the next sink with larger value of y.

At $\alpha = 1$, there are infinitely many nodes, and they are located at $y = 3\pi/2 \pm 2n\pi$ for every integer n. For $\alpha > 1$, there are no equilibria, and all solutions become unbounded in the positive direction as t increases.

10. Note that $0 < e^{-y^2} \leq 1$ for all y, and its maximum value occurs at $y = 0$. Therefore, for $\alpha < -1$, dy/dt is always negative, and the solutions are always decreasing.

If $\alpha = -1$, $dy/dt = 0$ if and only if $y = 0$. For $y \neq 0$, $dy/dt < 0$, and the equilibrium point at $y = 0$ is a node.

If $-1 < \alpha < 0$, then there are two equilibrium points which we compute by solving

$$e^{-y^2} + \alpha = 0.$$

We get $-y^2 = \ln(-\alpha)$. Consequently, $y = \pm\sqrt{\ln(-1/\alpha)}$. As $\alpha \to 0$ from below, $\ln(-1/\alpha) \to \infty$, and the two equilibria tend to $\pm\infty$.

If $\alpha \geq 0$, dy/dt is always positive, and the solutions are always increasing.

11. For $\alpha = 0$, there are three equilibria. There is a sink to the left of $y = 0$, a source at $y = 0$, and a sink to the right of $y = 0$.

As α decreases, the source and sink on the right move together. A bifurcation occurs at $\alpha \approx -2$. At this bifurcation value, there is a sink to the left of $y = 0$ and a node to the right of $y = 0$. For α below this bifurcation value, there is only the sink to the left of $y = 0$.

As α increases from zero, the sink to the left of $y = 0$ and the source move together. There is a bifurcation at $\alpha \approx 2$ with a node to the left of $y = 0$ and a sink to the right of $y = 0$. For α above this bifurcation value, there is only the sink to the right of $y = 0$.

12. Note that if α is very negative, then the equation $g(y) = -\alpha y$ has only one solution. It is $y = 0$. Furthermore, $dy/dt > 0$ for $y < 0$, and $dy/dt < 0$ for $y > 0$. Consequently, the equilibrium point at $y = 0$ is a sink.

In the figure, it appears that the tangent line to the graph of g at the origin has slope 1 and does not intersect the graph of g other than at the origin. If so, $\alpha = -1$ is a bifurcation value. For $\alpha \leq -1$, the differential equation has one equilibrium, which is a sink. For $\alpha > -1$, the equation has three equilibria, $y = 0$ and two others, one on each side of $y = 0$. The equilibrium point at the origin is a source, and the other two equilibria are sinks.

13. **(a)** Each phase line has an equilibrium point at $y = 0$. This corresponds to equations (i), (iii), and (vi). Since $y = 0$ is the only equilibrium point for $A < 0$, this only corresponds to equation (iii).

(b) The phase line corresponding to $A = 0$ is the only phase line with $y = 0$ as an equilibrium point, which corresponds to equations (ii), (iv), and (v). For the phase lines corresponding to

$A < 0$, there are no equilibrium points. Only equations (iv) and (v) satisfy this property. For the phase lines corresponding to $A > 0$, note that $dy/dt < 0$ for $-\sqrt{A} < y < \sqrt{A}$. Consequently, the bifurcation diagram corresponds to equation (v).

(c) The phase line corresponding to $A = 0$ is the only phase line with $y = 0$ as an equilibrium point, which corresponds to equations (ii), (iv), and (v). For the phase lines corresponding to $A < 0$, there are no equilibrium points. Only equations (iv) and (v) satisfy this property. For the phase lines corresponding to $A > 0$, note that $dy/dt > 0$ for $-\sqrt{A} < y < \sqrt{A}$. Consequently, the bifurcation diagram corresponds to equation (iv).

(d) Each phase line has an equilibrium point at $y = 0$. This corresponds to equations (i), (iii), and (vi). The phase lines corresponding to $A > 0$ only have two nonnegative equilibrium points. Consequently, the bifurcation diagram corresponds to equation (i).

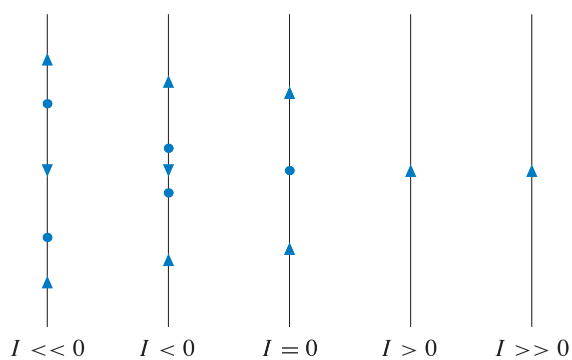
14. To find the equilibria we solve

$$1 - \cos\theta + (1 + \cos\theta)(I) = 0$$

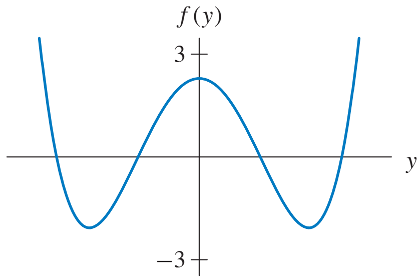
$$1 + I - (1 - I)\cos\theta = 0$$

$$\cos\theta = \frac{1 + I}{1 - I}.$$

For $I > 0$, the fraction on the right-hand side is greater than 1. Therefore, there are no equilibria. For $I = 0$, the equilbria correspond to the solutions of $\cos\theta = 1$, that is, $\theta = 2\pi n$ for integer values of n. For $I < 0$, the fraction on the right-hand side is between -1 and 1. As $I \to -\infty$, the fraction on the right-hand side approaches -1. Therefore the equilibria approach $\pm\pi$.



15. The graph of f needs to cross the y-axis exactly four times so that there are exactly four equilibria if $\alpha = 0$. The function must be greater than -3 everywhere so that there are no equilibria if $\alpha \geq 3$. Finally, the graph of f must cross horizontal lines three or more units above the y-axis exactly twice so that there are exactly two equilibria for $\alpha \leq -3$. The following graph is an example of the graph of such a function.

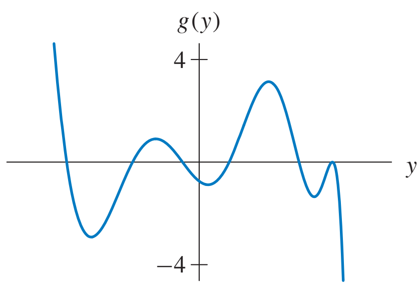


16. The graph of g can only intersect horizontal lines above 4 once, and it must go from above to below as y increases. Then there is exactly one sink for $\alpha \leq -4$.

Similarly, the graph of g can only intersect horizontal lines below -4 once, and it must go from above to below as y increases. Then there is exactly one sink for $\alpha \geq 4$.

Finally, the graph of g must touch the y-axis at exactly six points so that there are exactly six equilibria for $\alpha = 0$.

The following graph is the graph of one such function.



17. No such $f(y)$ exists. To see why, suppose that there is exactly one sink y_0 for $\alpha = 0$. Then, $f(y) > 0$ for $y < y_0$, and $f(y) < 0$ for $y > y_0$. Now consider the system $dy/dt = f(y) + 1$. Then $dy/dt \geq 1$ for $y < y_0$. If this system has an equilibrium point y_1 that is a source, then $y_1 > y_0$ and $dy/dt < 0$ for y slightly less than y_1. Since $f(y)$ is continuous and $dy/dt \geq 1$ for $y \leq y_0$, then dy/dt must have another zero between y_0 and y_1.

18. **(a)** For all $C \geq 0$, the equation has a source at $P = C/k$, and this is the only equilibrium point. Hence all of the phase lines are qualitatively the same, and there are no bifurcation values for C.

(b) If $P(0) > C/k$, the corresponding solution $P(t) \to \infty$ at an exponential rate as $t \to \infty$, and if $P(0) < C/k$, $P(t) \to -\infty$, passing through "extinction" ($P = 0$) after a finite time.

19. **(a)** A model of the fish population that includes fishing is

$$\frac{dP}{dt} = 2P - \frac{P^2}{50} - 3L,$$

where L is the number of licenses issued. The coefficient of 3 represents the average catch of 3 fish per year. As L is increased, the two equilibrium points for $L = 0$ (at $P = 0$ and $P = 100$) will move together. If L is sufficiently large, there are no equilibrium points. Hence we wish to