Math 2410-10/15

Quiz #9

April 24^{th}**, 2017**

Note: Partial credit can not be awarded unless there is legible work to assess. Feel free to use the back of this page if you require additional space for your solutions.

1. In class we verified half of the extended linearity principle for linear second order differential equations. Verify the second half.
   That is, show that if $y_1(t)$ and $y_2(t)$ are solutions to

   $$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = g(t)$$

   then $y_1(t) - y_2(t)$ is a solution to the associated homogeneous equation.

   Solution: The associated homogeneous equation is

   $$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0.$$

   To show the function $y_1(t) - y_2(t)$ is a solution to this equation we simply need show that

   $$\frac{d^2(y_1 - y_2)}{dt^2} + p\frac{d(y_1 - y_2)}{dt} + q(y_1 - y_2) = 0$$

   .

   To this end, notice

   $$\begin{aligned}
   &\frac{d^2(y_1 - y_2)}{dt^2} + p\frac{d(y_1 - y_2)}{dt} + q(y_1 - y_2) \\
   &\qquad = \frac{d^2y_1}{dt^2} - \frac{d^2y_2}{dt^2} + p\left(\frac{dy_1}{dt} - \frac{d^2y_2}{dt^2}\right) + qy_1 - qy_2 && (1) \\
   &\qquad = \left(\frac{d^2y_1}{dt^2} + p\frac{dy_1}{dt} + qy_1\right) - \left(\frac{d^2y_2}{dt^2} + p\frac{dy_2}{dt} + qy_2\right) && (2) \\
   &\qquad = \qquad g(t) \qquad - \qquad g(t) && (3) \\
   &\qquad = 0. && (4)
   \end{aligned}$$

   Thus, we have shown what we wished to show and can conclude that $y_1(t) - y_2(t)$ is indeed a solution to the homogenoues equation. Note that line 3 follows from line 2 by the fact that both $y_1(t)$ and $y_2(t)$ are solutions to the original equation. That is

   $$\frac{d^2y_1}{dt^2} + p\frac{dy_1}{dt} + qy_1 = g(t) \text{ and } \frac{d^2y_2}{dt^2} + p\frac{dy_2}{dt} + qy_2 = g(t).$$

2. Find the general solution of the following nonhomogeneous linear second-order differential equation.

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 12y = 81e^{-3t}$$

Solution: To solve this differential equation we first find the general solution to the homogeneous equation

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 12y = 0,$$

then find one solution to the original equation via the method of undetermined coefficients and finish by combining these two solutions. First, to determine the general solution of the associated homogeneous equation we consider the polynomial

$$s^2 - 4s - 12 = (s+2)(s-6) = 0.$$

As this has two distinct roots $s = -2$ and $s = 6$. We can conclude that the general solution to the homogeneous equation is

$$y_h(t) = k_1e^{-2t} + k_2e^{6t}.$$

Now to find a particular solution we use undetermined coefficients. Analyzing the right-side of the equation motivates an attempt at a solution of the form ae^{-3t} for some constant a. We know ae^{-3t} will be a solution if

$$\begin{array}{rrcl} & \frac{d^2}{dt^2}(ae^{-3t}) - 4\frac{d}{dt}(ae^{-3t}) - 12(ae^{-3t}) & = & 81e^{-3t} \\ \Longrightarrow & 9ae^{-3t} + 12ae^{-3t} - 12ae^{-3t} & = & 81e^{-3t} \\ \Longrightarrow & 9a & = & 81 \\ \Longrightarrow & a & = & 9. \end{array}$$

Thus, we see $y_p(t) = 9e^{-3t}$ is a solution to the original equation. Hence, the general solution $y(t)$ is the sum of this solution with the general solution to the homogeneous equation, $y_h + y_p$. In conclusion,

$$y(t) = k_1e^{-2t} + k_2e^{6t} + 9e^{-3t}.$$