34. **(a)** If $\mathbf{Y}(t) = (t, t^2/2)$, then $x(t) = t$ and $y(t) = t^2/2$. Then $dx/dt = 1$, and $dy/dt = t = x$. So $\mathbf{Y}(t)$ satisfies the differential equation.

(b) For $2\mathbf{Y}(t)$, we have $x(t) = 2t$, and $y(t) = t^2$. In this case, we need only consider $dx/dt = 2$ to see that the function is not a solution to the system.

35. **(a)** Using the Product Rule we compute

$$\frac{dW}{dt} = \frac{dx_1}{dt}y_2 + x_1\frac{dy_2}{dt} - \frac{dx_2}{dt}y_1 - x_2\frac{dy_1}{dt}.$$

(b) Since $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are solutions, we know that

$$\begin{aligned}\frac{dx_1}{dt} &= ax_1 + by_1\\ \frac{dy_1}{dt} &= cx_1 + dy_1\end{aligned}$$

and that

$$\begin{aligned}\frac{dx_2}{dt} &= ax_2 + by_2\\ \frac{dy_2}{dt} &= cx_2 + dy_2.\end{aligned}$$

Substituting these equations into the expression for dW/dt, we obtain

$$\frac{dW}{dt} = (ax_1 + by_1)y_2 + x_1(cx_2 + dy_2) - (ax_2 + by_2)y_1 - x_2(cx_1 + dy_1).$$

After we collect terms, we have

$$\frac{dW}{dt} = (a + d)W.$$

(c) This equation is a homogeneous, linear, first-order equation (as such it is also separable—see Sections 1.1, 1.2, and 1.8). Therefore, we know that the general solution is

$$W(t) = Ce^{(a+d)t}$$

where C is any constant (but note that $C = W(0)$).

(d) From Exercises 31 and 32, we know that $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are linearly independent if and only if $W(t) \neq 0$. But, $W(t) = Ce^{(a+d)t}$, so $W(t) = 0$ if and only if $C = W(0) = 0$. Hence, $W(t) = 0$ is zero for some t if and only if $C = W(0) = 0$.

EXERCISES FOR SECTION 3.2

1. **(a)** The characteristic polynomial is

$$(3 - \lambda)(-2 - \lambda) = 0,$$

and therefore the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$.

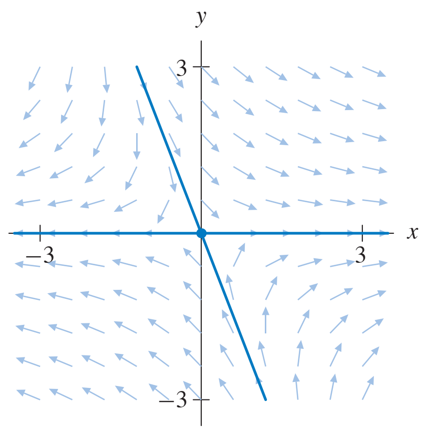
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -2$, we solve the system of equations

$$\begin{cases} 3x_1 + 2y_1 = -2x_1 \\ -2y_1 = -2y_1 \end{cases}$$

and obtain $5x_1 = -2y_1$.

Using the same procedure, we see that the eigenvectors (x_2, y_2) for $\lambda_2 = 3$ must satisfy the equation $y_2 = 0$.

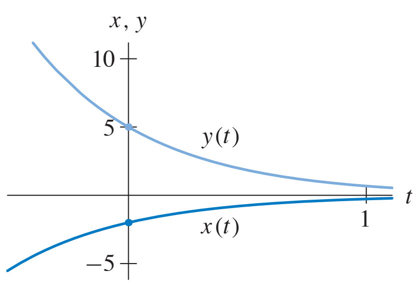
(c)



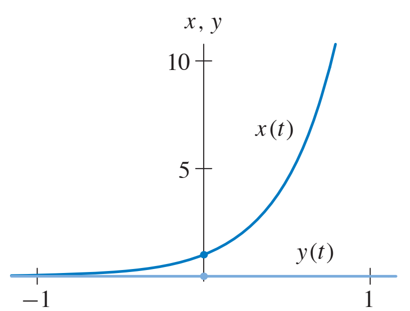
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (-2, 5)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (1, 0)$.

Given the eigenvalues and these eigenvectors, we have the two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-2t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-2t}\begin{pmatrix} -2 \\ 5 \end{pmatrix} + k_2 e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

2. **(a)** The characteristic polynomial is

$$(-4-\lambda)(-3-\lambda)-2=\lambda^2+7\lambda+10=0,$$

and therefore the eigenvalues are $\lambda_1=-2$ and $\lambda_2=-5$.

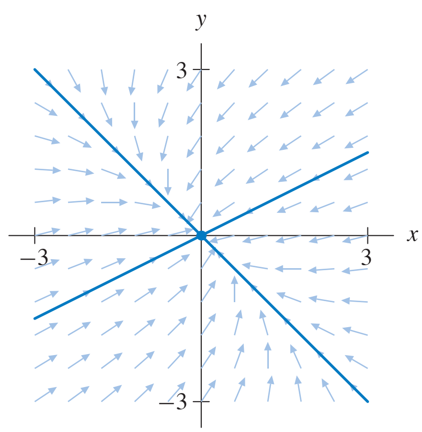
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1=-2$, we solve the system of equations

$$\begin{cases} -4x_1-2y_1=-2x_1 \\ -x_1-3y_1=-2y_1 \end{cases}$$

and obtain $y_1=-x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $x_2=2y_2$ for $\lambda_2=-5$.

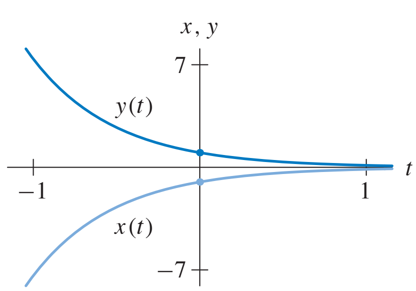
(c)



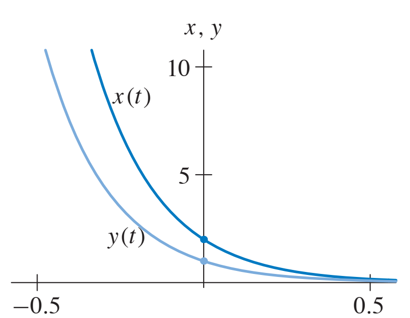
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1=(1,-1)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2=(2,1)$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t)=e^{-2t}\begin{pmatrix}1\\-1\end{pmatrix} \quad\text{and}\quad \mathbf{Y}_2(t)=e^{-5t}\begin{pmatrix}2\\1\end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

3. **(a)** The eigenvalues are the roots of the characteristic polynomial, so they are the solutions of

$$(-5 - \lambda)(-4 - \lambda) - 2 = \lambda^2 + 9\lambda + 18 = 0.$$

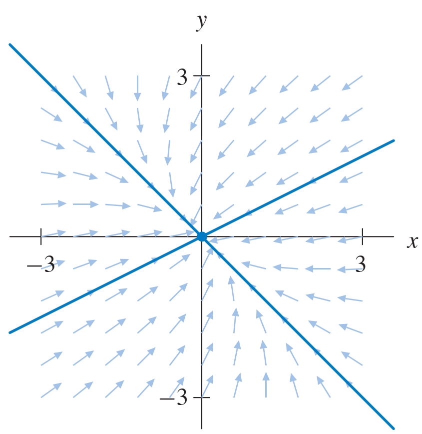
Therefore, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -6$.

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -3$, we solve the system of equations

$$\begin{cases} -5x_1 - 2y_1 = -3x_1 \\ -x_1 - 4y_1 = -3y_1 \end{cases}$$

and obtain $y_1 = -x_1$.
Using the same procedure, we obtain the eigenvalues (x_2, y_2) where $x_2 = 2y_2$ for $\lambda_2 = -6$.

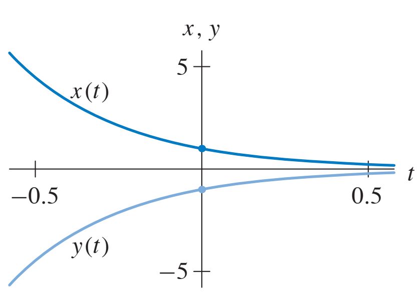
(c)



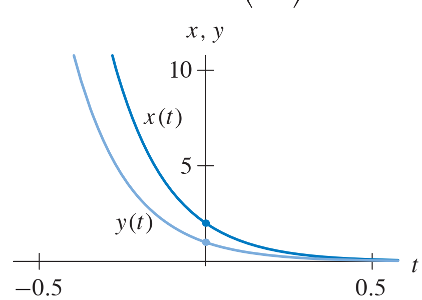
(d) One eigenvector $\mathbf{V}_1$ for $\lambda_1 = -3$ is $\mathbf{V}_1 = (1, -1)$, and one eigenvector $\mathbf{V}_2$ for $\lambda_2 = -6$ is $\mathbf{V}_2 = (2, 1)$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{-6t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



the $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-3t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{-6t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

4. **(a)** The characteristic polynomial is

$$(2 - \lambda)(4 - \lambda) + 1 = \lambda^2 - 6\lambda + 9 = 0,$$

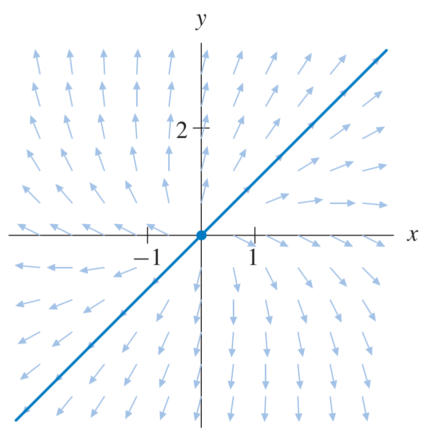
and therefore there is only one eigenvalue, $\lambda = 3$.

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda = 3$, we solve the system of equations

$$\begin{cases} 2x_1 + y_1 = 3x_1 \\ -x_1 + 4y_1 = 3y_1 \end{cases}$$

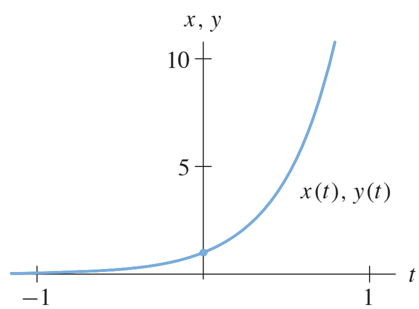
and obtain $y_1 = x_1$.

(c)



(d) One eigenvector $\mathbf{V}$ for λ is $\mathbf{V} = (1, 1)$. Given this eigenvector, we have the solution

$$\mathbf{Y}(t) = e^{3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs (which are identical) for $\mathbf{Y}(t)$

(e) Since the method of eigenvalues and eigenvectors does not give us a second solution that is linearly independent from $\mathbf{Y}(t)$, we cannot form the general solution.

5. **(a)** The characteristic polynomial is

$$\left(-\tfrac{1}{2}-\lambda\right)^2 = 0,$$

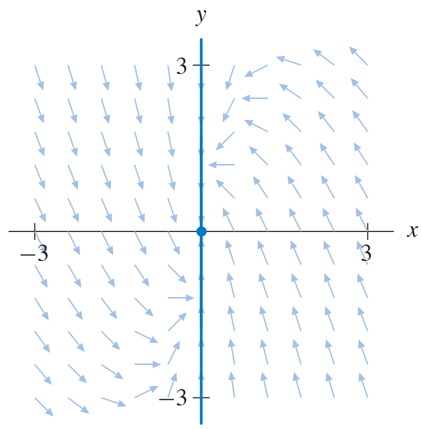
and therefore there is only one eigenvalue, $\lambda = -1/2$.

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda = -1/2$, we solve the system of equations

$$\begin{cases} -\tfrac{1}{2}x_1 = -\tfrac{1}{2}x_1 \\ x_1 - \tfrac{1}{2}y_1 = -\tfrac{1}{2}y_1 \end{cases}$$

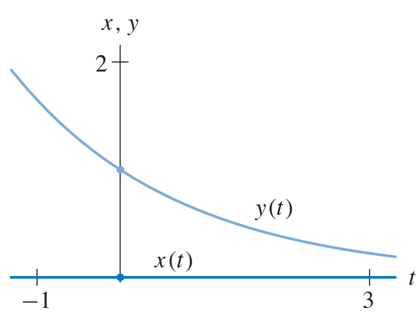
and obtain $x_1 = 0$.

(c)



(d) Given the eigenvalue $\lambda = -1/2$ and the eigenvector $\mathbf{V} = (0, 1)$, we have the solution

$$\mathbf{Y}(t) = e^{-t/2}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}(t)$.

(e) Since the method of eigenvalues and eigenvectors does not give us a second solution that is linearly independent from $\mathbf{Y}(t)$, we cannot form the general solution.

6. **(a)** The characteristic polynomial is

$$(5-\lambda)(-\lambda)-36=0,$$

and therefore the eigenvalues are $\lambda_1=-4$ and $\lambda_2=9$.

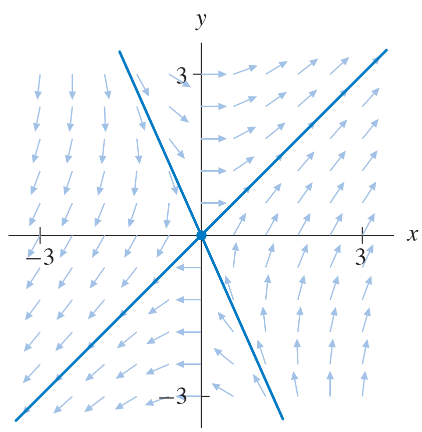
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1=-4$, we solve the system of equations

$$\begin{cases} 5x_1+4y_1=-4x_1 \\ 9x_1=-4y_1 \end{cases}$$

and obtain $9x_1=-4y_1$.

Using the same procedure, we see that the eigenvectors (x_2, y_2) for $\lambda_2=9$ must satisfy the equation $y_2=x_2$.

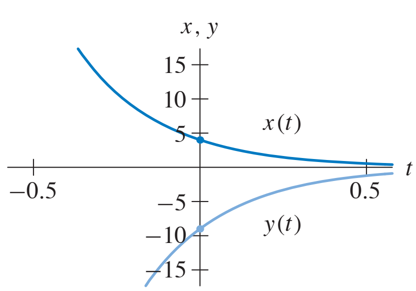
(c)



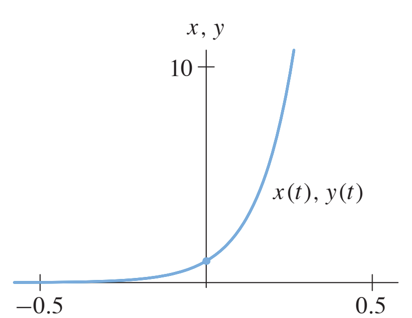
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1=(4,-9)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2=(1,1)$.

Given the eigenvalues and these eigenvectors, we have the two linearly independent solutions

$$\mathbf{Y}_1(t)=e^{-4t}\begin{pmatrix}4\\-9\end{pmatrix} \quad\text{and}\quad \mathbf{Y}_2(t)=e^{9t}\begin{pmatrix}1\\1\end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The (identical) $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-4t} \begin{pmatrix} 4 \\ -9 \end{pmatrix} + k_2 e^{9t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

7. **(a)** The characteristic polynomial is

$$(3-\lambda)(-\lambda) - 4 = \lambda^2 - 3\lambda - 4 = (\lambda - 4)(\lambda + 1) = 0,$$

and therefore the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 4$.

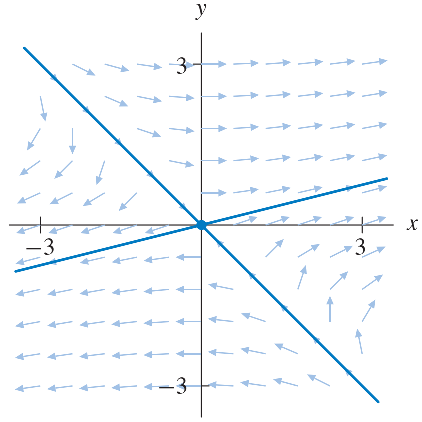
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -1$, we solve the system of equations

$$\begin{cases} 3x_1 + 4y_1 = -x_1 \\ x_1 = -y_1 \end{cases}$$

and obtain $y_1 = -x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $x_2 = 4y_2$ for $\lambda_2 = 4$.

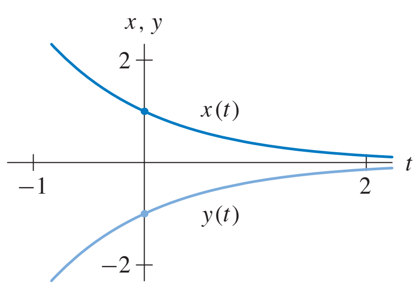
(c)



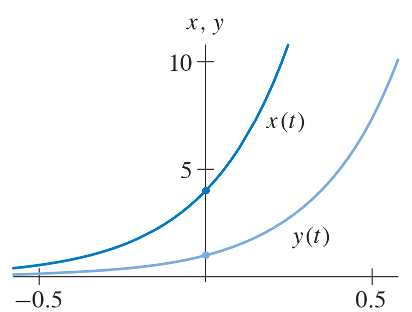
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (1, -1)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (4, 1)$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{4t} \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 e^{4t}\begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

8. **(a)** The characteristic polynomial is

$$(2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0,$$

and therefore the eigenvalues are

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\sqrt{5}}{2}.$$

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = (3+\sqrt{5})/2$, we solve the system of equations

$$\begin{cases} 2x_1 - y_1 = \dfrac{3+\sqrt{5}}{2}x_1 \\ -x_1 + y_1 = \dfrac{3+\sqrt{5}}{2}y_1 \end{cases}$$

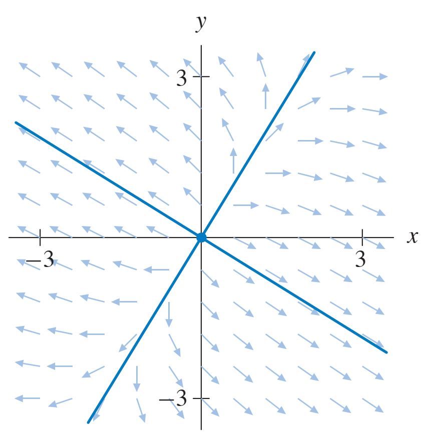
and obtain

$$y_1 = \frac{1-\sqrt{5}}{2}x_1,$$

which is equivalent to the equation $2y_1 = (1-\sqrt{5})x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $2y_2 = (1+\sqrt{5})x_2$ for $\lambda_2 = (3-\sqrt{5})/2$.

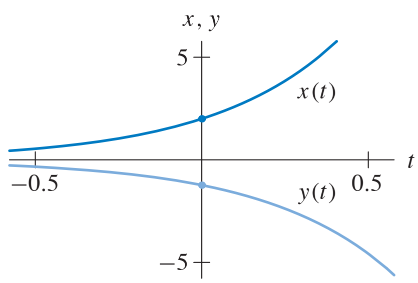
(c)



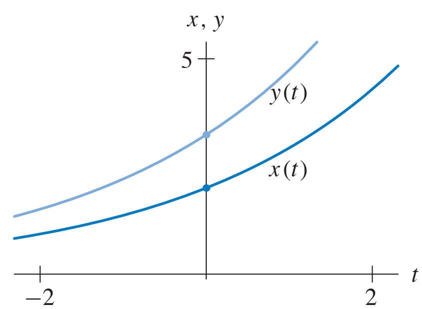
(d) One eigenvector $\mathbf{V}_1$ for the eigenvalue λ_1 is $\mathbf{V}_1 = (2, 1-\sqrt{5})$, and one eigenvector $\mathbf{V}_2$ for the eigenvalue λ_2 is $\mathbf{V}_2 = (2, 1+\sqrt{5})$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{(3+\sqrt{5})t/2}\begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{(3-\sqrt{5})t/2}\begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{(3+\sqrt{5})t/2} \begin{pmatrix} 2 \\ 1-\sqrt{5} \end{pmatrix} + k_2 e^{(3-\sqrt{5})t/2} \begin{pmatrix} 2 \\ 1+\sqrt{5} \end{pmatrix}.$$

9. **(a)** The characteristic polynomial is

$$(2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0,$$

and therefore the eigenvalues are

$$\lambda_1 = \frac{3+\sqrt{5}}{2} \quad \text{and} \quad \lambda_2 = \frac{3-\sqrt{5}}{2}.$$

(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = (3+\sqrt{5})/2$, we solve the system of equations

$$\begin{cases} 2x_1 + y_1 = \dfrac{3+\sqrt{5}}{2} x_1 \\[2ex] x_1 + y_1 = \dfrac{3+\sqrt{5}}{2} y_1 \end{cases}$$

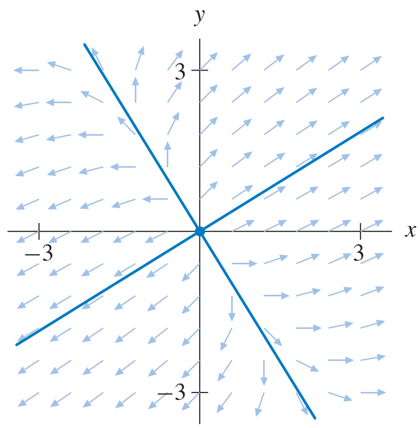
and obtain

$$y_1 = \frac{-1+\sqrt{5}}{2} x_1,$$

which is equivalent to the equation $2y_1 = (-1+\sqrt{5})x_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $2y_2 = (-1-\sqrt{5})x_2$ for $\lambda_2 = (3-\sqrt{5})/2$.

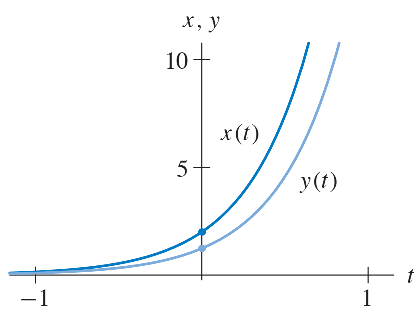
(c)



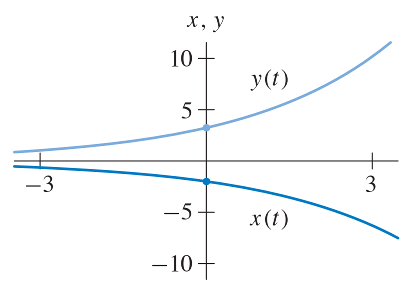
(d) One eigenvector $\mathbf{V}_1$ for the eigenvalue λ_1 is $\mathbf{V}_1 = (2, -1 + \sqrt{5})$, and one eigenvector $\mathbf{V}_2$ for the eigenvalue λ_2 is $\mathbf{V}_2 = (-2, 1 + \sqrt{5})$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{(3+\sqrt{5})t/2} \begin{pmatrix} 2 \\ -1 + \sqrt{5} \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{(3-\sqrt{5})t/2} \begin{pmatrix} -2 \\ 1 + \sqrt{5} \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{(3+\sqrt{5})t/2} \begin{pmatrix} 2 \\ -1 + \sqrt{5} \end{pmatrix} + k_2 e^{(3-\sqrt{5})t/2} \begin{pmatrix} -2 \\ 1 + \sqrt{5} \end{pmatrix}.$$

10. **(a)** The characteristic polynomial is

$$(-1 - \lambda)(-4 - \lambda) + 2 = \lambda^2 + 5\lambda + 6 = (\lambda + 3)(\lambda + 2) = 0,$$

and therefore the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -3$.

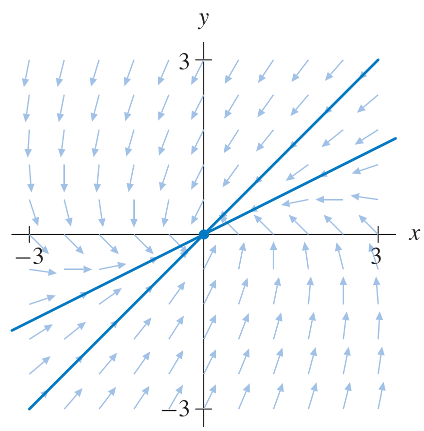
(b) To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -2$, we solve the system of equations

$$\begin{cases} -x_1 - 2y_1 = -2x_1 \\ x_1 - 4y_1 = -2y_1 \end{cases}$$

and obtain $x_1 = 2y_1$.

Using the same procedure, we obtain the eigenvectors (x_2, y_2) where $x_2 = y_2$ for $\lambda_2 = -3$.

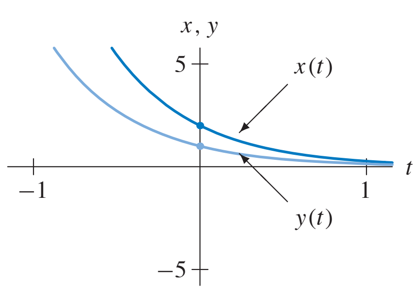
(c)



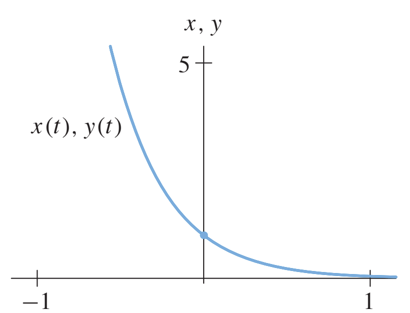
(d) One eigenvector $\mathbf{V}_1$ for λ_1 is $\mathbf{V}_1 = (2, 1)$, and one eigenvector $\mathbf{V}_2$ for λ_2 is $\mathbf{V}_2 = (1, 1)$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$



The $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_1(t)$.



The identical) $x(t)$- and $y(t)$-graphs for $\mathbf{Y}_2(t)$.

(e) The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{-2t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

11. The eigenvalues are the roots of the characteristic polynomial, so they are solutions of

$$(-2-\lambda)(1-\lambda)-4=\lambda^2+\lambda-6=0.$$

Hence, $\lambda_1 = 2$ and $\lambda_2 = -3$ are the eigenvalues.

To find the eigenvectors for the eigenvalue $\lambda_1 = 2$, we solve

$$\left\{\begin{array}{l} -2x_1 - 2y_1 = 2x_1 \\ -2x_1 + y_1 = 2y_1, \end{array}\right.$$

so $y_1 = -2x_1$ is the line of eigenvectors. In particular, $(1, -2)$ is an eigenvector for $\lambda_1 = 2$.

Similarly, the line of eigenvectors for $\lambda_2 = -3$ is given by $x_1 = 2y_1$. In particular, $(2, 1)$ is an eigenvector for $\lambda_2 = -3$.

Given the eigenvalues and these eigenvectors, we have the two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{-3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

The general solution is

$$\mathbf{Y}(t) = k_1 e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2 e^{-3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(a) Given the initial condition $\mathbf{Y}(0) = (1, 0)$, we must solve

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

for k_1 and k_2. This vector equation is equivalent to the two scalar equations

$$\left\{\begin{array}{r} k_1 + 2k_2 = 1 \\ -2k_1 + k_2 = 0. \end{array}\right.$$

Solving these equations, we obtain $k_1 = 1/5$ and $k_2 = 2/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = \tfrac{1}{5}e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \tfrac{2}{5}e^{-3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(b) Given the initial condition $\mathbf{Y}(0) = (0, 1)$ we must solve

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 1 \\ -2 \end{pmatrix} + k_2\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

for k_1 and k_2. This vector equation is equivalent to the two scalar equations

$$\left\{\begin{array}{r} k_1 + 2k_2 = 0 \\ -2k_1 + k_2 = 1. \end{array}\right.$$

Solving these equations, we obtain $k_1 = -2/5$ and $k_2 = 1/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = -\tfrac{2}{5}e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix} + \tfrac{1}{5}e^{-3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(c) The initial condition $\mathbf{Y}(0) = (1, -2)$ is an eigenvector for the eigenvalue $\lambda_1 = 2$. Hence, the solution with this initial condition is

$$\mathbf{Y}(t) = e^{2t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

12. The characteristic polynomial is

$$(3 - \lambda)(-2 - \lambda) = 0,$$

and therefore the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$.

To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = 3$, we solve the system of equations

$$\begin{cases} 3x_1 = 3x_1 \\ x_1 - 2y_1 = 3y_1 \end{cases}$$

and obtain

$$5y_1 = x_1.$$

Therefore, an eigenvector for the eigenvalue $\lambda_1 = 3$ is $\mathbf{V}_1 = (5, 1)$.

Using the same procedure, we obtain the eigenvector $\mathbf{V}_2 = (0, 1)$ for $\lambda_2 = -2$.

The general solution to this linear system is therefore

$$\mathbf{Y}(t) = k_1 e^{3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2 e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(a) We have $\mathbf{Y}(0) = (1, 0)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 5k_1 = 1 \\ k_1 + k_2 = 0. \end{cases}$$

Solving these equations, we obtain $k_1 = 1/5$ and $k_2 = -1/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = \tfrac{1}{5}e^{3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} - \tfrac{1}{5}e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) We have $\mathbf{Y}(0) = (0, 1)$. Since this initial condition is an eigenvector associated to the $\lambda = -2$ eigenvalue, we do not need to do any additional calculation. The desired solution to the initial-value problem is

$$\mathbf{Y}(t) = e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(c) We have $\mathbf{Y}(0) = (2, 2)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 5 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 5k_1 = 2 \\ k_1 + k_2 = 2. \end{cases}$$

Solving these equations, we obtain $k_1 = 2/5$ and $k_2 = 8/5$. Thus, the particular solution is

$$\mathbf{Y}(t) = \tfrac{2}{5}e^{3t}\begin{pmatrix} 5 \\ 1 \end{pmatrix} + \tfrac{8}{5}e^{-2t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

13. The characteristic polynomial is

$$(-4 - \lambda)(-3 - \lambda) - 2 = \lambda^2 + 7\lambda + 10 = 0,$$

and therefore the eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = -2$.

To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = -5$, we solve the system of equations

$$\begin{cases} -4x_1 + y_1 = -5x_1 \\ 2x_1 - 3y_1 = -5y_1 \end{cases}$$

and obtain

$$y_1 = -x_1.$$

Therefore, an eigenvector for the eigenvalue $\lambda_1 = -5$ is $\mathbf{V}_1 = (1, -1)$.

Using the same procedure, we obtain the eigenvector $\mathbf{V}_2 = (1, 2)$ for $\lambda_2 = -2$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

The general solution to this linear system is

$$\mathbf{Y}(t) = k_1e^{-5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(a) We have $\mathbf{Y}(0) = (1, 0)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{Y}(0) = k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} k_1 + k_2 = 1 \\ -k_1 + 2k_2 = 0. \end{cases}$$

Solving these equations, we obtain $k_1 = 2/3$ and $k_2 = 1/3$. Thus, the particular solution is

$$\mathbf{Y}(t) = \tfrac{2}{3} e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \tfrac{1}{3} e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(b) We have $\mathbf{Y}(0) = (2, 1)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \mathbf{Y}(0) = k_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} k_1 + k_2 = 2 \\ -k_1 + 2k_2 = 1. \end{cases}$$

Solving these equations, we obtain $k_1 = 1$ and $k_2 = 1$. Thus, the particular solution is

$$\mathbf{Y}(t) = e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(c) We have $\mathbf{Y}(0) = (-1, -2)$. Since this initial condition is an eigenvector associated to the $\lambda = -2$ eigenvalue, we do not need to do any additional calculation. The desired solution to the initial-value problem is

$$\mathbf{Y}(t) = -e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

14. The characteristic polynomial is

$$(4 - \lambda)(1 - \lambda) + 2 = \lambda^2 - 5\lambda + 6 = 0,$$

and therefore the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 2$.

To obtain the eigenvectors (x_1, y_1) for the eigenvalue $\lambda_1 = 3$, we solve the system of equations

$$\begin{cases} 4x_1 - 2y_1 = 3x_1 \\ x_1 + y_1 = 3y_1 \end{cases}$$

and obtain

$$x_1 = 2y_1.$$

Therefore, an eigenvector for the eigenvalue $\lambda_1 = 3$ is $\mathbf{V}_1 = (2, 1)$.

Using the same procedure, we obtain the eigenvector $\mathbf{V}_2 = (1, 1)$ for $\lambda_2 = 2$.

Given the eigenvalues and these eigenvectors, we have two linearly independent solutions

$$\mathbf{Y}_1(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2(t) = e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The general solution to this linear system is

$$\mathbf{Y}(t) = k_1 e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2 e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(a) We have $\mathbf{Y}(0) = (1, 0)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 2k_1 + k_2 = 1 \\ k_1 + k_2 = 0. \end{cases}$$

Solving these equations, we obtain $k_1 = 1$ and $k_2 = -1$. Thus, the particular solution is

$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b) We have $\mathbf{Y}(0) = (2, 1)$. Since this initial condition is an eigenvector associated to the $\lambda = 3$ eigenvalue, we do not need to do any additional calculation. The desired solution to the initial-value problem is

$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(c) We have $\mathbf{Y}(0) = (-1, -2)$, so we must find k_1 and k_2 so that

$$\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \mathbf{Y}(0) = k_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} + k_2\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

This vector equation is equivalent to the simultaneous system of linear equations

$$\begin{cases} 2k_1 + k_2 = -1 \\ k_1 + k_2 = -2. \end{cases}$$

Solving these equations, we obtain $k_1 = 1$ and $k_2 = -3$. Thus, the particular solution is

$$\mathbf{Y}(t) = e^{3t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} - 3e^{2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

15. Given any vector $\mathbf{Y}_0 = (x_0, y_0)$, we have

$$\mathbf{A}\mathbf{Y}_0 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} ax_0 \\ ay_0 \end{pmatrix} = a \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = a\mathbf{Y}_0.$$

Therefore, every nonzero vector is an eigenvector associated to the eigenvalue a.

16. The characteristic polynomial of $\mathbf{A}$ is

$$(a - \lambda)(d - \lambda) = 0,$$

and thus the eigenvalues of $\mathbf{A}$ are $\lambda_1 = a$ and $\lambda_2 = d$.

To find the eigenvectors $\mathbf{V}_1 = (x_1, y_1)$ associated to $\lambda_1 = a$, we need to solve the equation

$$\mathbf{A}\mathbf{V}_1 = a\mathbf{V}_1$$

for all possible vectors $\mathbf{V}_1$. Rewritten in terms of components, this equation is equivalent to

$$\begin{cases} ax_1 + by_1 = ax_1 \\ \qquad\;\; dy_1 = ay_1. \end{cases}$$

Since $a \neq d$, the second equation implies that $y_1 = 0$. If so, then the first equation is satisfied for all x_1. In other words, the eigenvectors $\mathbf{V}_1$ associated to the eigenvalue a are the vectors of the form $(x_1, 0)$.

To find the eigenvectors $\mathbf{V}_2 = (x_2, y_2)$ associated to $\lambda_2 = d$, we need to solve the equation

$$\mathbf{A}\mathbf{V}_2 = d\mathbf{V}_2$$

for all possible vectors $\mathbf{V}_2$. Rewritten in terms of components, this equation is equivalent to

$$\begin{cases} ax_2 + by_2 = dx_2 \\ \qquad\;\; dy_2 = dy_2. \end{cases}$$

The second equation always holds, so the eigenvectors $\mathbf{V}_2$ are those vectors that satisfy the equation $ax_2 + by_2 = dx_2$, which can be rewritten as

$$by_2 = (d - a)x_2.$$

These vectors form a line through the origin of slope $(d - a)/b$.

17. The characteristic polynomial of $\mathbf{B}$ is

$$\lambda^2 - (a + d)\lambda + ad - b^2.$$

The roots of this polynomial are

$$\begin{aligned} \lambda &= \frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - b^2)}}{2} \\ &= \frac{a + d \pm \sqrt{a^2 + 2ad + d^2 - 4ad + 4b^2}}{2} \\ &= \frac{a + d \pm \sqrt{(a - d)^2 + 4b^2}}{2}. \end{aligned}$$

Since the discriminant $D = (a - d)^2 + 4b^2$ is always nonnegative, the roots λ are real. Therefore, the matrix $\mathbf{B}$ has real eigenvalues. If $b \neq 0$, then D is positive and hence $\mathbf{B}$ has two distinct eigenvalues. (The only way to have only one eigenvalue is for $D = 0$).

18. The characteristic equation is

$$(a - \lambda)(-\lambda) - bc = \lambda^2 - a\lambda - bc = 0.$$

Finding the roots via the quadratic formula, we obtain the eigenvalues

$$\frac{a \pm \sqrt{a^2 + 4bc}}{2}.$$

Note that these eigenvalues are very different from the case where the matrix is upper triangular (see Exercise 16). For example, they are not necessarily real numbers because $a^2 + 4bc$ can be negative.

19. **(a)** To form the system, we introduce the new dependent variable $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -p\frac{dy}{dt} - qy = -pv - qy.$$

Written in matrix form this system where $\mathbf{Y} = (y, v)$, we have

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix} \mathbf{Y}.$$

(b) The characteristic polynomial is

$$(0 - \lambda)(-p - \lambda) + q = \lambda^2 + p\lambda + q.$$

(c) The roots of this polynomial (the eigenvalues) are

$$\frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

(d) The roots are distinct real numbers if the discriminant $D = p^2 - 4q$ is positive. In other words, the roots are distinct real numbers if $p^2 > 4q$.

(e) Since q is positive, $p^2 - 4q < p^2$, so we know that $\sqrt{p^2 - 4q} < \sqrt{p^2} = p$. Since the numerator in the expression for the eigenvalues is $-p \pm \sqrt{p^2 - 4q}$, we see that it must be negative. Since the denominator is positive, the eigenvalues must be negative.

20. **(a)** The parameters $m = 1$, $k = 4$, and $b = 5$ yield the second-order equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = 0.$$

Given $v = dy/dt$, the corresponding system is

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -4y - 5v. \end{aligned}$$