EXERCISES FOR SECTION 2.1

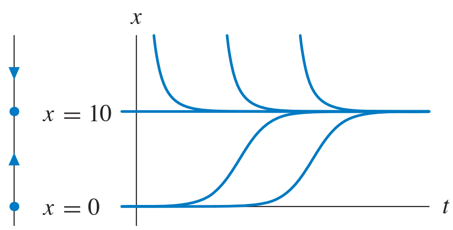
1. In the case where it takes many predators to eat one prey, the constant in the negative effect term of predators on the prey is small. Therefore, (ii) corresponds the system of large prey and small predators. On the other hand, one predator eats many prey for the system of large predators and small prey, and, therefore, the coefficient of negative effect term on predator-prey interaction on the prey is large. Hence, (i) corresponds to the system of small prey and large predators.

2. For (i), the equilibrium points are $x = y = 0$ and $x = 10$, $y = 0$. For the latter equilibrium point prey alone exist; there are no predators. For (ii), the equilibrium points are $(0, 0)$, $(0, 15)$, and $(3/5, 30)$. For the latter equilibrium point, both species coexist. For $(0, 15)$, the prey are extinct but the predators survive.

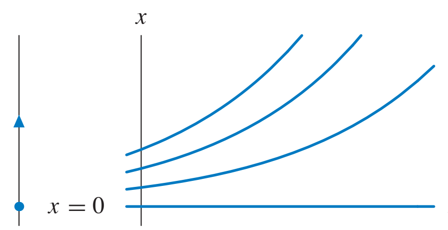
3. Substitution of $y = 0$ into the equation for dy/dt yields $dy/dt = 0$ for all t. Therefore, $y(t)$ is constant, and since $y(0) = 0$, $y(t) = 0$ for all t.

Note that to verify this assertion rigorously, we need a uniqueness theorem (see Section 2.5).

4. For (i), the prey obey a logistic model. The population tends to the equilibrium point at $x = 10$. For (ii), the prey obey an exponential growth model, so the population grows unchecked.



Phase line and graph for (i).

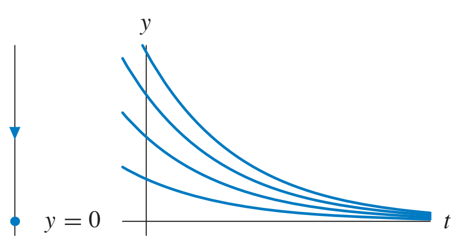


Phase line and graph for (ii).

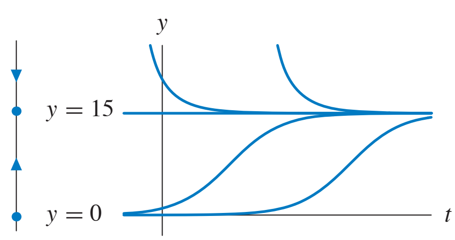
5. Substitution of $x = 0$ into the equation for dx/dt yields $dx/dt = 0$ for all t. Therefore, $x(t)$ is constant, and since $x(0) = 0$, $x(t) = 0$ for all t.

Note that to verify this assertion rigorously, we need a uniqueness theorem (see Section 2.5).

6. For (i), the predators obey an exponential decay model, so the population tends to 0. For (ii), the predators obey a logistic model. The population tends to the equilibrium point at $y = 15$.



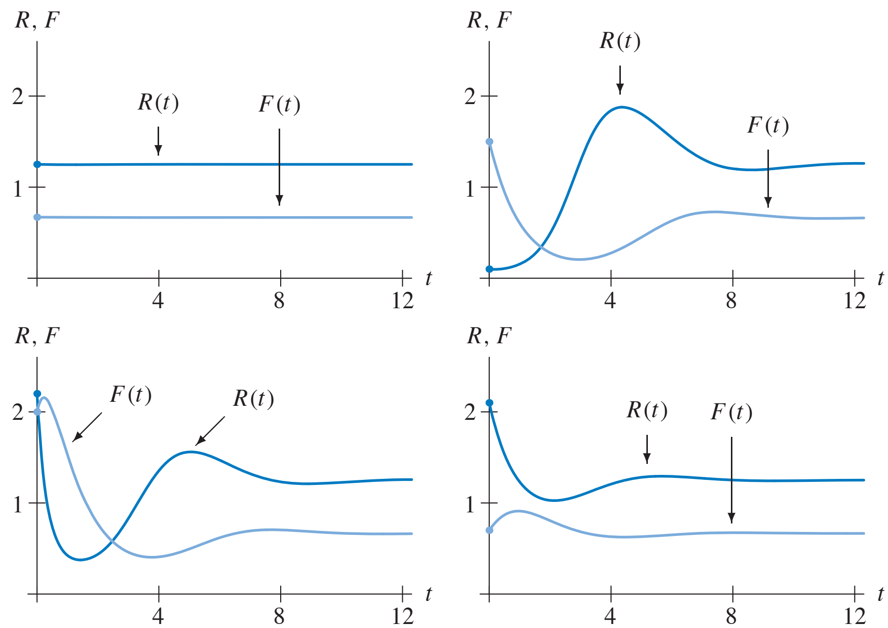
Phase line and graph for (i).



Phase line and graph for (ii).

7. The population starts with a relatively large rabbit (R) and a relatively small fox (F) population. The rabbit population grows, then the fox population grows while the rabbit population decreases. Next the fox population decreases until both populations are close to zero. Then the rabbit population grows again and the cycle starts over. Each repeat of the cycle is less dramatic (smaller total oscillation) and both populations oscillate toward an equilibrium which is approximately $(R, F) = (1/2, 3/2)$.

8. **(a)**



(b) Each of the solutions tends to the equilibrium point at $(R, F) = (5/4, 2/3)$. The populations of both species tend to a limit and the species coexist. For curve B, note that the F-population initially decreases while R increases. Eventually F bottoms out and begins to rise. Then R peaks and begins to fall. Then both populations tend to the limit.

9. By hunting, the number of prey decreases α units per unit of time. Therefore, the rate of change dR/dt of the number of prey has the term $-\alpha$. Only the equation for dR/dt needs modification.

(i) $dR/dt = 2R - 1.2RF - \alpha$

(ii) $dR/dt = R(2 - R) - 1.2RF - \alpha$

10. Hunting decreases the number of predators by an amount proportional to the number of predators alive (that is, by a term of the form $-kF$), so we have $dF/dt = -F + 0.9RF - kF$ in each case.

11. Since the second food source is unlimited, if $R = 0$ and k is the growth parameter for the predator population, F obeys an exponential growth model, $dF/dt = kF$. The only change we have to make is in the rate of F, dF/dt. For both (i) and (ii), $dF/dt = kF + 0.9RF$.

12. In the absence of prey, the predators would obey a logistic growth law. So we could modify both systems by adding a term of the form $-kF/N$, where k is the growth-rate parameter and N is the carrying capacity of predators. That is, we have $dF/dt = kF(1 - F/N) + 0.9RF$.

13. If $R - 5F > 0$, the number of predators increases and, if $R - 5F < 0$, the number of predators decreases. Since the condition on prey is same, we modify only the predator part of the system. the modified rate of change of the predator population is

$$\frac{dF}{dt} = -F + 0.9RF + k(R - 5F)$$

where $k > 0$ is the immigration parameter for the predator population.

14. In both cases the rate of change of population of prey decreases by a factor of kF. Hence we have

(i) $dR/dt = 2R - 1.2RF - kF$

(ii) $dR/dt = 2R - R^2 - 1.2RF - kF$

15. Suppose $y = 1$. If we can find a value of x such that $dy/dt = 0$, then for this x and $y = 1$ the predator population is constant. (This point may not be an equilibrium point because we do not know if $dx/dt = 0$.) The required value of x is $x = 0.05$ in system (i) and $x = 20$ in system (ii). Survival for one unit of predators requires 0.05 units of prey in (i) and 20 units of prey in (ii). Therefore, (i) is a system of inefficient predators and (ii) is a system of efficient predators.

16. At first, the number of rabbits decreases while the number of foxes increases. Then the foxes have too little food, so their numbers begin to decrease. Eventually there are so few foxes that the rabbits begin to multiply. Finally, the foxes become extinct and the rabbit population tends to the constant population $R = 3$.

17. **(a)** For the initial condition close to zero, the pest population increases much more rapidly than the predator. After a sufficient increase in the predator population, the pest population starts to decrease while the predator population keeps increasing. After a sufficient decrease in the pest population, the predator population starts to decrease. Then, the population comes back to the initial point.

(b) After applying the pest control, you may see the increase of the pest population due to the absence of the predator. So in the short run, this sort of pesticide can cause an explosion in the pest population.

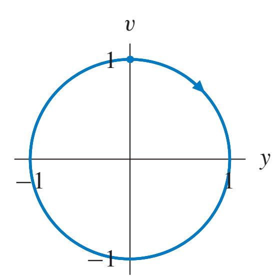
18. One way to consider this type of predator-prey interaction is to raise the growth rate of the prey population. If only weak or sick prey are removed, the remaining population may be assumed to be able to reproduce at a higher rate.

19. **(a)** Substituting $y(t) = \sin t$ into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y}{dt^2} + y &= \frac{d^2(\sin t)}{dt^2} + \sin t \\ &= -\sin t + \sin t \\ &= 0,\end{aligned}$$

so the left-hand side equals the right-hand side for all t.

(b)



(c) These two solutions trace the same curve in the yv-plane—the unit circle.