22. **(a)** We consider dx/dt in each system. Setting $y = 0$ yields $dx/dt = 5x$ in system (i) and $dx/dt = x$ in system (ii). If the number x of prey is equal for both systems, dx/dt is larger in system (i). Therefore, the prey in system (i) reproduce faster if there are no predators.

(b) We must see what effect the predators (represented by the y-terms) have on dx/dt in each system. Since the magnitude of the coefficient of the xy-term is larger in system (ii) than in system (i), y has a greater effect on dx/dt in system (ii). Hence the predators have a greater effect on the rate of change of the prey in system (ii).

(c) We must see what effect the prey (represented by the x-terms) have on dy/dt in each system. Since x and y are both nonnegative, it follows that

$$-2y + \tfrac{1}{2}xy < -2y + 6xy,$$

and therefore, if the number of predators is equal for both systems, dy/dt is smaller in system (i). Hence more prey are required in system (i) than in system (ii) to achieve a certain growth rate.

23. **(a)** The independent variable is t, and x and y are dependent variables. Since each xy-term is positive, the presence of either species increases the rate of change of the other. Hence, these species cooperate. The parameter α is the growth-rate parameter for x, and γ is the growth-rate parameter for y. The parameter N represents the carrying capacity for x, but y has no carrying capacity. The parameter β measures the benefit to x of the interaction of the two species, and δ measures the benefit to y of the interaction.

(b) The independent variable is t, and x and y are the dependent variables. Since both xy-terms are negative, these species compete. The parameter γ is the growth-rate coefficient for x, and α is the growth-rate parameter for y. Neither population has a carrying capacity. The parameter δ measures the harm to x caused by the interaction of the two species, and β measures the harm to y caused by the interaction.

EXERCISES FOR SECTION 1.2

1. **(a)** Let's check Bob's solution first. Since $dy/dt = 1$ and

$$\frac{y(t)+1}{t+1} = \frac{t+1}{t+1} = 1,$$

Bob's answer is correct.

Now let's check Glen's solution. Since $dy/dt = 2$ and

$$\frac{y(t)+1}{t+1} = \frac{2t+2}{t+1} = 2,$$

Glen's solution is also correct.

Finally let's check Paul's solution. We have $dy/dt = 2t$ on one hand and

$$\frac{y(t)+1}{t+1} = \frac{t^2-1}{t+1} = t-1$$

on the other. Paul is wrong.

where c is an arbitrary constant. Exponentiating both sides gives

$$|P| = e^{kt+c} = e^c e^{kt}.$$

For population models we consider only $P \geq 0$, and the absolute value sign is unnecessary. Letting $P_0 = e^c$, we have

$$P(t) = P_0 e^{kt}.$$

In general, it is possible for $P(0)$ to be negative. In that case, $e^c = -P_0$, and $|P| = -P$. Once again we obtain

$$P(t) = P_0 e^{kt}.$$

5. **(a)** This equation is separable. (It is nonlinear and nonautonomous as well.)

(b) We separate variables and integrate to obtain

$$\int \frac{1}{y^2}\,dy = \int t^2\,dt$$

$$-\frac{1}{y} = \frac{t^3}{3} + c$$

$$y(t) = \frac{-1}{(t^3/3) + c},$$

where c is any real number. This function can also be written in the form

$$y(t) = \frac{-3}{t^3 + k}$$

where k is any constant. The constant function $y(t) = 0$ for all t is also a solution of this equation. It is the equilibrium solution at $y = 0$.

6. Separating variables and integrating, we obtain

$$\int \frac{1}{y}\,dy = \int t^4\,dt$$

$$\ln|y| = \frac{t^5}{5} + c$$

$$|y| = c_1 e^{t^5/5},$$

where $c_1 = e^c$. As in Exercise 22, we can eliminate the absolute values by replacing the positive constant c_1 with $k = \pm c_1$. Hence, the general solution is

$$y(t) = ke^{t^5/5},$$

where k is any real number. Note that $k = 0$ gives the equilibrium solution.

7. We separate variables and integrate to obtain

$$\int \frac{dy}{2y+1} = \int dt.$$

We get

$$\frac{1}{2}\ln|2y+1| = t + c$$

$$|2y+1| = c_1 e^{2t},$$

where $c_1 = e^{2c}$. As in Exercise 22, we can drop the absolute value signs by replacing $\pm c_1$ with a new constant k_1. Hence, we have

$$2y + 1 = k_1 e^{2t}$$

$$\frac{1}{2}\left(k_1 e^{2t} - 1\right)$$

8. Separating variables and integrating, we obtain

$$\int \frac{1}{2-y}\,dy = \int dt$$

$$-\ln|2-y| = t + c$$

$$\ln|2-y| = -t + c_1,$$

where we have replaced $-c$ with c_1. Then

$$|2-y| = k_1 e^{-t},$$

where $k_1 = e_1^c$. We can drop the absolute value signs if we replace $\pm k_1$ with k_2, that is, if we allow k_2 to be either positive or negative. Then we have

$$\begin{aligned} 2 - y &= k_2 e^{-t} \\ y &= 2 - k_2 e^{-t}. \end{aligned}$$

This could also be written as $y(t) = ke^{-t} + 2$, where we replace $-k_2$ with k. Note that $k = 0$ gives the equilibrium solution.

9. We separate variables and integrate to obtain

$$\int e^y\,dy = \int dt$$

$$e^y = t + c,$$

where c is any constant. We obtain $y(t) = \ln(t + c)$.

10. We separate variables and obtain

$$\int \frac{dx}{1+x^2} = \int 1\,dt.$$

Integrating both sides, we get

$$\arctan x = t + c,$$

where c is a constant. Hence, the general solution is

$$x(t) = \tan(t + c).$$

11. **(a)** This equation is separable.

(b) We separate variables and integrate to obtain

$$\int \frac{1}{y^2}\,dy = \int (2t+3)\,dt$$

$$-\frac{1}{y} = t^2 + 3t + k$$

$$y(t) = \frac{-1}{t^2+3t+k},$$

where k is any constant. The constant function $y(t) = 0$ for all t is also a solution of this equation. It is the equilibrium solution at $y = 0$.

12. Separating variables and integrating, we obtain

$$\int y\,dy = \int t\,dt$$

$$\frac{y^2}{2} = \frac{t^2}{2} + k$$

$$y^2 = t^2 + c,$$

where $c = 2k$. Hence,

$$y(t) = \pm\sqrt{t^2 + c},$$

where the initial condition determines the choice of sign.

13. First note that the differential equation is not defined if $y = 0$.

In order to separate the variables, we write the equation as

$$\frac{dy}{dt} = \frac{t}{y(t^2+1)}$$

to obtain

$$\int y\,dy = \int \frac{t}{t^2+1}\,dt$$

$$\frac{y^2}{2} = \frac{1}{2}\ln(t^2+1) + c,$$

where c is any constant. So we get

$$y^2 = \ln\left(k(t^2+1)\right),$$

where $k = e^{2c}$ (hence any positive constant). We have

$$y(t) = \pm\sqrt{\ln\left(k(t^2+1)\right)},$$

where k is any positive constant and the sign is determined by the initial condition.

14. Separating variables and integrating, we obtain

$$\int y^{-1/3}\,dy = \int t\,dt$$

$$\frac{3}{2}y^{2/3} = \frac{t^2}{2} + k$$

$$y^{2/3} = \frac{t^2}{3} + c,$$

where $c = 2k/3$. Hence,

$$y(t) = \pm\left(\frac{t^2}{3} + c\right)^{3/2}.$$

Note that this form does not include the equilibrium solution $y = 0$.

15. First note that the differential equation is not defined for $y = -1/2$. We separate variables and integrate to obtain

$$\int (2y+1)\,dy = \int dt$$

$$y^2 + y = t + k,$$

where k is any constant. So

$$y(t) = \frac{-1 \pm \sqrt{4t+4k+1}}{2} = \frac{-1 \pm \sqrt{4t+c}}{2},$$

where c is any constant and the $\pm$ sign is determined by the initial condition.

We can rewrite the answer in the more simple form

$$y(t) = -\frac{1}{2} \pm \sqrt{t + c_1}$$

where $c_1 = k + 1/4$. If k can be any possible constant, then c_1 can be as well.

16. Note that there is an equilibrium solution of the form $y = -1/2$.

Separating variables and integrating, we have

$$\int \frac{1}{2y+1}\,dy = \int \frac{1}{t}\,dt$$

$$\frac{1}{2}\ln|2y+1| = \ln|t| + c$$

$$\ln|2y+1| = (\ln t^2) + c$$

$$|2y+1| = c_1 t^2,$$

where $c_1 = e^c$. We can eliminate the absolute value signs by allowing the constant c_1 to be either positive or negative. In other words, $2y + 1 = k_1 t^2$, where $k_1 = \pm c_1$. Hence,

$$y(t) = kt^2 - \tfrac{1}{2},$$

where $k = k_1/2$, or $y(t)$ is the equilibrium solution with $y = -1/2$.

17. First of all, the equilibrium solutions are $y = 0$ and $y = 1$. Now suppose $y \neq 0$ and $y \neq 1$. We separate variables to obtain

$$\int \frac{1}{y(1-y)}\,dy = \int dt = t + c,$$

where c is any constant. To integrate, we use partial fractions. Write

$$\frac{1}{y(1-y)} = \frac{A}{y} + \frac{B}{1-y}.$$

We must have $A = 1$ and $-A + B = 0$. Hence, $A = B = 1$ and

$$\frac{1}{y(1-y)} = \frac{1}{y} + \frac{1}{1-y}.$$

Consequently,

$$\int \frac{1}{y(1-y)}\,dy = \ln|y| - \ln|1-y| = \ln\left|\frac{y}{1-y}\right|.$$

After integration, we have

$$\ln\left|\frac{y}{1-y}\right| = t + c$$

$$\left|\frac{y}{1-y}\right| = c_1 e^t,$$

where $c_1 = e^c$ is any positive constant. To remove the absolute value signs, we replace the positive constant c_1 with a constant k that can be any real number and get

$$y(t) = \frac{ke^t}{1+ke^t},$$

where $k = \pm c_1$. If $k = 0$, we get the first equilibrium solution. The formula $y(t) = ke^t/(1+ke^t)$ yields all the solutions to the differential equation except for the equilibrium solution $y(t) = 1$.

18. Separating variables and integrating, we have

$$\int (1+3y^2)\,dy = \int 4t\,dt$$

$$y + y^3 = 2t^2 + c.$$

To express y as a function of t, we must solve a cubic. The equation for the roots of a cubic can be found in old algebra books or by asking a computer algebra program. But we do not learn a lot from the result.

19. The equation can be written in the form

$$\frac{dv}{dt} = (v+1)(t^2-2),$$

and we note that $v(t) = -1$ for all t is an equilibrium solution. Separating variables and integrating, we obtain

$$\int \frac{dv}{v+1} = \int t^2 - 2\,dt$$

$$\ln|v+1| = \frac{t^3}{3} - 2t + c,$$

where c is any constant. Thus,

$$|v+1| = c_1 e^{-2t+t^3/3},$$

where $c_1 = e^c$. We can dispose of the absolute value signs by allowing the constant c_1 to be any real number. In other words,

$$v(t) = -1 + ke^{-2t+t^3/3},$$

where $k = \pm c_1$. Note that, if $k = 0$, we get the equilibrium solution.

20. Rewriting the equation as

$$\frac{dy}{dt} = \frac{1}{(t+1)(y+1)}$$

we separate variables and obtain

$$\int (y+1)\,dy = \int \frac{1}{t+1}\,dt.$$

Hence,

$$\frac{y^2}{2} + y = \ln|t+1| + k.$$

We can solve using the quadratic formula. We have

$$\begin{aligned} y(t) &= -1 \pm \sqrt{1 + 2\ln|t+1| + 2k} \\ &= -1 \pm \sqrt{2\ln|t+1| + c}, \end{aligned}$$

where $c = 1 + 2k$ is any constant and the choice of sign is determined by the initial condition.

21. The function $y(t) = 0$ for all t is an equilibrium solution.

Suppose $y \neq 0$ and separate variables. We get

$$\int y + \frac{1}{y}\,dy = \int e^t\,dt$$

$$\frac{y^2}{2} + \ln|y| = e^t + c,$$

where c is any real constant. We cannot solve this equation for y, so we leave the expression for y in this implicit form. Note that the equilibrium solution $y = 0$ cannot be obtained from this implicit equation.

22. Since $y^2 - 4 = (y+2)(y-2)$, there are two equilibrium solutions, $y_1(t) = -2$ for all t and $y_2(t) = 2$ for all t. If $y \neq \pm 2$, we separate variables and obtain

$$\int \frac{dy}{y^2 - 4} = \int dt.$$

To integrate the left-hand side, we use partial fractions. If

$$\frac{1}{y^2 - 4} = \frac{A}{y+2} + \frac{B}{y-2},$$

then $A + B = 0$ and $2(B - A) = 1$. Hence, $A = -1/4$ and $B = 1/4$, and

$$\frac{1}{(y+2)(y-2)} = \frac{-1/4}{y+2} + \frac{1/4}{y-2}.$$

Consequently,

$$\int \frac{dy}{y^2 - 4} = -\frac{1}{4}\ln|y+2| + \frac{1}{4}\ln|y-2|.$$

Using this integral on the separated equation above, we get

$$\frac{1}{4}\ln\left|\frac{y-2}{y+2}\right| = t + c,$$

which yields

$$\left|\frac{y-2}{y+2}\right| = c_1 e^{4t},$$

where $c_1 = e^{4c}$. As in Exercise 22, we can drop the absolute value signs by replacing $\pm c_1$ with a new constant k. Hence, we have

$$\frac{y-2}{y+2} = ke^{4t}.$$

Solving for y, we obtain

$$y(t) = \frac{2(1+ke^{4t})}{1-ke^{4t}}.$$

Note that, if $k = 0$, we get the equilibrium solution $y_2(t)$. The formula $y(t) = 2(1+ke^{4t})/(1-ke^{4t})$ provides all of the solutions to the differential equation except the equilibrium solution $y_1(t)$.

23. The constant function $w(t) = 0$ is an equilibrium solution. Suppose $w \neq 0$ and separate variables. We get

$$\begin{aligned}\int \frac{dw}{w} &= \int \frac{dt}{t} \\ \ln|w| &= \ln|t| + c \\ &= \ln c_1|t|,\end{aligned}$$

where c is any constant and $c_1 = e^c$. Therefore,

$$|w| = c_1|t|.$$

We can eliminate the absolute value signs by allowing the constant to assume positive or negative values. We have

$$w = kt,$$

where $k = \pm c_1$. Moreover, if $k = 0$ we get the equilibrium solution.

24. Separating variables and integrating, we have

$$\begin{aligned}\int \cos y\, dy &= \int dx \\ \sin y &= x + c \\ y(x) &= \arcsin(x + c),\end{aligned}$$

where c is any real number. The branch of the inverse sine function that we use depends on the initial condition.

25. Separating variables and integrating, we have

$$\int \frac{1}{x}\,dx = \int -t\,dt$$

$$\ln|x| = -\frac{t^2}{2} + c$$

$$|x| = k_1 e^{-t^2/2},$$

where $k_1 = e^c$. We can eliminate the absolute value signs by allowing the constant k_1 to be either positive or negative. Thus, the general solution is

$$x(t) = ke^{-t^2/2}$$

where $k = \pm k_1$. Using the initial condition to solve for k, we have

$$\frac{1}{\sqrt{\pi}} = x(0) = ke^0 = k.$$

Therefore,

$$x(t) = \frac{e^{-t^2/2}}{\sqrt{\pi}}.$$

26. Separating variables and integrating, we have

$$\int \frac{1}{y}\,dy = \int t\,dt$$

$$\ln|y| = \frac{t^2}{2} + c$$

$$|y| = k_1 e^{t^2/2},$$

where $k_1 = e^c$. We can eliminate the absolute value signs by allowing the constant k_1 to be either positive or negative. Thus, the general solution can be written as

$$y(t) = ke^{t^2/2}.$$

Using the initial condition to solve for k, we have

$$3 = y(0) = ke^0 = k.$$

Therefore, $y(t) = 3e^{t^2/2}$.

27. Separating variables and integrating, we obtain

$$\int \frac{dy}{y^2} = -\int dt$$

$$-\frac{1}{y} = -t + c.$$

So we get

$$y = \frac{1}{t - c}.$$

Now we need to find the constant c so that $y(0) = 1/2$. To do this we solve

$$\frac{1}{2} = \frac{1}{0 - c}$$

and get $c = -2$. The solution of the initial-value problem is

$$y(t) = \frac{1}{t + 2}.$$

28. First we separate variables and integrate to obtain

$$\int y^{-3}\, dy = \int t^2\, dt,$$

which yields

$$-\frac{y^{-2}}{2} = \frac{t^3}{3} + c.$$

Solving for y gives

$$y^2 = \frac{1}{c_1 - 2t^3/3},$$

where $c_1 = -2c$. So

$$y(t) = \pm\frac{1}{\sqrt{c_1 - 2t^3/3}}.$$

The initial value $y(0)$ is negative, so we choose the negative square root and obtain

$$y(t) = -\frac{1}{\sqrt{c_1 - 2t^3/3}}.$$

Using $-1 = y(0) = -1/\sqrt{c_1}$, we see that $c_1 = 1$ and the solution of the initial-value problem is

$$y(t) = -\frac{1}{\sqrt{1 - 2t^3/3}}.$$

29. We do not need to do any computations to solve this initial-value problem. We know that the constant function $y(t) = 0$ for all t is an equilibrium solution, and it satisfies the initial condition.

30. Rewriting the equation as

$$\frac{dy}{dt} = \frac{t}{(1 - t^2)y},$$

we separate variables and integrate obtaining

$$\begin{aligned}\int y\,dy &= \int \frac{t}{1-t^2}\,dt \\ \frac{y^2}{2} &= -\frac{1}{2}\ln|1-t^2| + c \\ y &= \pm\sqrt{-\ln|1-t^2| + k}.\end{aligned}$$

Since $y(0) = 4$ is positive, we use the positive square root and solve

$$4 = y(0) = \sqrt{-\ln|1| + k} = \sqrt{k}$$

for k. We obtain $k = 16$. Hence,

$$y(t) = \sqrt{16 - \ln(1-t^2)}.$$

We may replace $|1-t^2|$ with $(1-t^2)$ because the solution is only defined for $-1 < t < 1$.

31. From Exercise 7, we already know that the general solution is

$$y(t) = ke^{2t} - \tfrac{1}{2},$$

so we need only find the constant k for which $y(0) = 3$. We solve

$$3 = ke^0 - \tfrac{1}{2}$$

for k and obtain $k = 7/2$. The solution of the initial-value problem is

$$y(t) = \tfrac{7}{2}e^{2t} - \tfrac{1}{2}.$$

32. First we find the general solution by writing the differential equation as

$$\frac{dy}{dt} = (t+2)y^2,$$

separating variables, and integrating. We have

$$\begin{aligned}\int \frac{1}{y^2}\,dy &= \int (t+2)\,dt \\ -\frac{1}{y} &= \frac{t^2}{2} + 2t + c \\ &= \frac{t^2 + 4t + c_1}{2},\end{aligned}$$

where $c_1 = 2c$. Inverting and multiplying by -1 produces

$$y(t) = \frac{-2}{t^2 + 4t + c_1}.$$

Setting

$$1 = y(0) = \frac{-2}{c_1}$$

and solving for c_1, we obtain $c_1 = -2$. So

$$y(t) = \frac{-2}{t^2 + 4t - 2}.$$

33. We write the equation in the form

$$\frac{dx}{dt} = \frac{t^2}{x(t^3 + 1)}$$

and separate variables to obtain

$$\int x\,dx = \int \frac{t^2}{t^3 + 1}\,dt$$

$$\frac{x^2}{2} = \frac{1}{3}\ln|t^3 + 1| + c,$$

where c is a constant. Hence,

$$x^2 = \frac{2}{3}\ln|t^3 + 1| + 2c.$$

The initial condition $x(0) = -2$ implies

$$4 = (-2)^2 = \frac{2}{3}\ln|1| + 2c.$$

Thus, $c = 2$. Solving for $x(t)$, we choose the negative square root because $x(0)$ is negative, and we drop the absolute value sign because $t^3 + 1 > 0$ for t near 0. The result is

$$x(t) = -\sqrt{\tfrac{2}{3}\ln(t^3 + 1) + 4}.$$

34. Separating variables, we have

$$\int \frac{y\,dy}{1 - y^2} = \int dt$$

$$= t + c,$$

where c is any constant. To integrate the left-hand side, we substitute $u = 1 - y^2$. Then $du = -2y\,dy$. We get

$$\int \frac{y\,dy}{1 - y^2} = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| = -\frac{1}{2}\ln|1 - y^2|.$$

Using this integral, we have

$$-\frac{1}{2}\ln|1 - y^2| = t + c$$

$$|1 - y^2| = c_1 e^{-2t},$$

where $c_1 = e^{-2c}$. As in Exercise 22, we can drop the absolute value signs by replacing $\pm c_1$ with a new constant k. Hence, we have

$$y(t) = \pm\sqrt{1 - ke^{-2t}}$$

Because $y(0)$ is negative, we use the negative square root and solve

$$-2 = y(0) = -\sqrt{1 - ke^0} = -\sqrt{1 - k}$$

for k. We obtain $k = -3$. Hence, $y(t) = -\sqrt{1 + 3e^{-2t}}$.

35. We separate variables to obtain

$$\int \frac{dy}{1 + y^2} = \int t\,dt$$

$$\arctan y = \frac{t^2}{2} + c,$$

where c is a constant. Hence the general solution is

$$y(t) = \tan\left(\frac{t^2}{2} + c\right).$$

Next we find c so that $y(0) = 1$. Solving

$$1 = \tan\left(\frac{0^2}{2} + c\right)$$

yields $c = \pi/4$, and the solution to the initial-value problem is

$$y(t) = \tan\left(\frac{t^2}{2} + \frac{\pi}{4}\right).$$

36. Separating variables and integrating, we obtain

$$\int (2y + 3)\,dy = \int dt$$

$$y^2 + 3y = t + c$$

$$y^2 + 3y - (t + c) = 0.$$

We can use the quadratic formula to obtain

$$y = -\tfrac{3}{2} \pm \sqrt{t + c_1},$$

where $c_1 = c + 9/4$. Since $y(0) = 1 > -3/2$ we take the positive square root and solve

$$1 = y(0) = -\tfrac{3}{2} + \sqrt{c_1},$$

so $c_1 = 25/4$. The solution to the initial-value problem is

$$y(t) = -\tfrac{3}{2} + \sqrt{t + \tfrac{25}{4}}.$$

37. Separating variables and integrating, we have

$$\int \frac{1}{y^2}\,dy = \int 2t + 3t^2\,dt$$

$$-\frac{1}{y} = t^2 + t^3 + c$$

$$y = \frac{-1}{t^2 + t^3 + c}.$$

Using $y(1) = -1$ we have

$$-1 = y(1) = \frac{-1}{1+1+c} = \frac{-1}{2+c},$$

so $c = -1$. The solution to the initial-value problem is

$$y(t) = \frac{-1}{t^2 + t^3 - 1}.$$

38. Separating variables and integrating, we have

$$\int \frac{y}{y^2+5}\,dy = \int dt$$

$$= t + c,$$

where c is any constant. To integrate the left-hand side, we substitute $u = y^2 + 5$. Then $du = 2y\,dy$. We have

$$\int \frac{y}{y^2+5}\,dy = \tfrac{1}{2}\int \frac{du}{u} = \tfrac{1}{2}\ln|u| = \tfrac{1}{2}\ln|y^2+5|.$$

Using this integral, we have

$$\tfrac{1}{2}\ln|y^2+5| = t + c$$

$$|y^2+5| = c_1 e^{2t},$$

where $c_1 = e^{2c}$. As in Exercise 26, we can drop the absolute value signs by replacing $\pm c_1$ with a new constant k. Hence, we have

$$y(t) = \pm\sqrt{ke^{2t} - 5}$$

Because $y(0)$ is negative, we use the negative square root and solve

$$-2 = y(0) = -\sqrt{ke^0 - 5} = -\sqrt{k-5}$$

for k. We obtain $k = 9$. Hence, $y(t) = -\sqrt{9e^{2t} - 5}$.

39. Let $S(t)$ denote the amount of salt (in pounds) in the bucket at time t (in minutes). We derive a differential equation for S by considering the difference between the rate that salt is entering the bucket and the rate that salt is leaving the bucket. Salt is entering the bucket at the rate of 1/4 pounds per minute. The rate that salt is leaving the bucket is the product of the concentration of salt in the mixture and the rate that the mixture is leaving the bucket. The concentration is $S/5$, and the mixture is leaving the bucket at the rate of $1/2$ gallons per minute. We obtain the differential equation

$$\frac{dS}{dt} = \frac{1}{4} - \frac{S}{5} \cdot \frac{1}{2},$$

which can be rewritten as

$$\frac{dS}{dt} = \frac{5 - 2S}{20}.$$

This differential equation is separable, and we can find the general solution by integrating

$$\int \frac{1}{5 - 2S}\, dS = \int \frac{1}{20}\, dt.$$

We have

$$\begin{aligned} -\frac{\ln|5 - 2S|}{2} &= \frac{t}{20} + c \\ \ln|5 - 2S| &= -\frac{t}{10} + c_1 \\ |5 - 2S| &= c_2 e^{-t/10}. \end{aligned}$$

We can eliminate the absolute value signs and determine c_2 using the initial condition $S(0) = 0$ (the water is initially free of salt). We have $c_2 = 5$, and the solution is

$$S(t) = 2.5 - 2.5e^{-t/10} = 2.5(1 - e^{-t/10}).$$

(a) When $t = 1$, we have $S(1) = 2.5(1 - e^{-0.1}) \approx 0.238$ lbs.

(b) When $t = 10$, we have $S(10) = 2.5(1 - e^{-1}) \approx 1.58$ lbs.

(c) When $t = 60$, we have $S(60) = 2.5(1 - e^{-6}) \approx 2.49$ lbs.

(d) When $t = 1000$, we have $S(1000) = 2.5(1 - e^{-100}) \approx 2.50$ lbs.

(e) When t is very large, the $e^{-t/10}$ term is close to zero, so $S(t)$ is very close to 2.5 lbs. In this case, we can also reach the same conclusion by doing a qualitative analysis of the solutions of the equation. The constant solution $S(t) = 2.5$ is the only equilibrium solution for this equation, and by examining the sign of dS/dt, we see that all solutions approach $S = 2.5$ as t increases.

40. Rewrite the equation as

$$\frac{dC}{dt} = -k_1 C + (k_1 N + k_2 E),$$

separate variables, and integrate to obtain

$$\begin{aligned} \int \frac{1}{-k_1 C + (k_1 N + k_2 E)}\, dC &= \int dt \\ -\frac{1}{k_1} \ln|-k_1 C + k_1 N + k_2 E| &= t + c \\ -k_1 C + k_1 N + k_2 E &= c_1 e^{-k_1 t}, \end{aligned}$$

where c_1 is a constant determined by the initial condition. Hence,

$$C(t) = N + \frac{k_2}{k_1}E - c_2 e^{-k_1 t},$$

where c_2 is a constant.

(a) Substituting the given values for the parameters, we obtain

$$C(t) = 600 - c_2 e^{-0.1t},$$

and the initial condition $C(0) = 150$ gives $c_2 = 450$, which implies that

$$C(t) = 600 - 450e^{-0.1t}.$$

Hence, $C(2) \approx 232$.

(b) Using part (a), $C(5) \approx 328$.

(c) When t is very large, $e^{-0.1t}$ is very close to zero, so $C(t) \approx 600$. (We could also obtain this conclusion by doing a qualitative analysis of the solutions.)

(d) Using the new parameter values and $C(0) = 600$ yields

$$C(t) = 300 + 300e^{-0.1t},$$

so $C(1) \approx 571$, $C(5) \approx 482$, and $C(t) \to 300$ as $t \to \infty$.

(e) Again changing the parameter values and using $C(0) = 600$, we have

$$C(t) = 500 + 100e^{-0.1t},$$

so $C(1) \approx 590$, $C(5) \approx 560$, and $C(t) \to 500$ as $t \to \infty$.

41. **(a)** If we let k denote the proportionality constant in Newton's law of cooling, the differential equation satisfied by the temperature T of the chocolate is

$$\frac{dT}{dt} = k(T - 70).$$

We also know that $T(0) = 170$ and that $dT/dt = -20$ at $t = 0$. Therefore, we obtain k by evaluating the differential equation at $t = 0$. We have

$$-20 = k(170 - 70),$$

so $k = -0.2$. The initial-value problem is

$$\frac{dT}{dt} = -0.2(T - 70), \quad T(0) = 170.$$

(b) We can solve the initial-value problem in part (a) by separating variables. We have

$$\int \frac{dT}{T - 70} = \int -0.2\,dt$$

$$\ln|T - 70| = -0.2t + k$$

$$|T - 70| = ce^{-0.2t}.$$

Since the temperature of the chocolate cannot become lower than the temperature of the room, we can ignore the absolute value and conclude

$$T(t) = 70 + ce^{-0.2t}.$$

Now we use the initial condition $T(0) = 170$ to find the constant c because

$$170 = T(0) = 70 + ce^{-0.2(0)},$$

which implies that $c = 100$. The solution is

$$T = 70 + 100e^{-0.2t}.$$

In order to find t so that the temperature is 110° F, we solve

$$110 = 70 + 100e^{-0.2t}$$

for t obtaining

$$\tfrac{2}{5} = e^{-0.2t}$$

$$\ln \tfrac{2}{5} = -0.2t$$

so that

$$t = \frac{\ln(2/5)}{-0.2} \approx 4.6.$$

42. Let t be time measured in minutes and let $H(t)$ represent the hot sauce in the chili measured in teaspoons at time t. Then $H(0) = 12$.

The pot contains 32 cups of chili, and chili is removed from the pot at the rate of 1 cup per minute. Since each cup of chili contains $H/32$ teaspoons of hot sauce, the differential equation is

$$\frac{dH}{dt} = -\frac{H}{32}.$$

The general solution of this equation is

$$H(t) = ke^{-t/32}.$$

(We could solve this differential equation by separation of variables, but this is also the equation for which we guessed solutions in Section 1.1.) Since $H(0) = 12$, we get the solution

$$H(t) = 12e^{-t/32}.$$

We wish to find t such that $H(t) = 4$ (two teaspoons per gallon in two gallons). We have

$$12e^{-t/32} = 4$$

$$-\frac{t}{32} = \ln\frac{1}{3}$$

$$t = 32\ln 3.$$

So, $t \approx 35.16$ minutes. A reasonable approximation is 35 minutes and in that time 35 cups will have been eaten.

43. **(a)** We rewrite the differential equation as

$$\frac{dv}{dt} = g\left(1 - \frac{k}{mg}v^2\right).$$

Letting $\alpha = \sqrt{k/(mg)}$ and separating variables, we have

$$\int \frac{dv}{1 - \alpha^2 v^2} = \int g\,dt.$$

Now we use the partial fractions decomposition

$$\frac{1}{1 - \alpha^2 v^2} = \frac{1/2}{1 + \alpha v} + \frac{1/2}{1 - \alpha v}$$

to obtain

$$\int \frac{dv}{1 + \alpha v} + \int \frac{dv}{1 - \alpha v} = 2gt + c,$$

where c is an arbitrary constant. Integrating the left-hand side, we get

$$\frac{1}{\alpha}\left(\ln|1 + \alpha v| - \ln|1 - \alpha v|\right) = 2gt + c.$$

Multiplying through by α and using the properties of logarithms, we have

$$\ln\left|\frac{1 + \alpha v}{1 - \alpha v}\right| = 2\alpha g t + c.$$

Exponentiating and eliminating the absolute value signs yields

$$\frac{1 + \alpha v}{1 - \alpha v} = Ce^{2\alpha g t}.$$

Solving for v, we have

$$v = \frac{1}{\alpha}\frac{Ce^{2\alpha g t} - 1}{Ce^{2\alpha g t} + 1}.$$

Recalling that $\alpha = \sqrt{k/(mg)}$, we see that $\alpha g = \sqrt{kg/m}$, and we get

$$v(t) = \sqrt{\frac{mg}{k}}\left(\frac{Ce^{2\sqrt{(kg/m)}\,t} - 1}{Ce^{2\sqrt{(kg/m)}\,t} + 1}\right).$$

Note: If we assume that $v(0) = 0$, then $C = 1$. The solution to this initial-value problem is often expressed in terms of the hyperbolic tangent function as

$$v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}\,t\right).$$

(b) The fraction in the parentheses of the general solution

$$v(t) = \sqrt{\frac{mg}{k}}\left(\frac{Ce^{2\sqrt{(kg/m)}\,t} - 1}{Ce^{2\sqrt{(kg/m)}\,t} + 1}\right),$$

tends to 1 as $t \to \infty$, so the limit of $v(t)$ as $t \to \infty$ is $\sqrt{mg/k}$.