Math 2410-010/015 | # Exam II | **April 7$^{\text{th}}$, 2017**

Name: ____________________________________

Instructions:

- Answer each question to the best of your ability.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- You may use a calculator, but you must show all your work in order to receive credit.

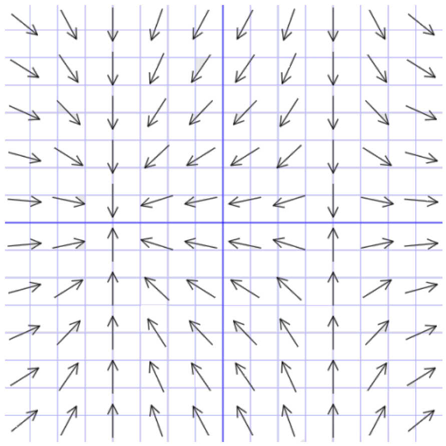
Academic Integrity Agreement

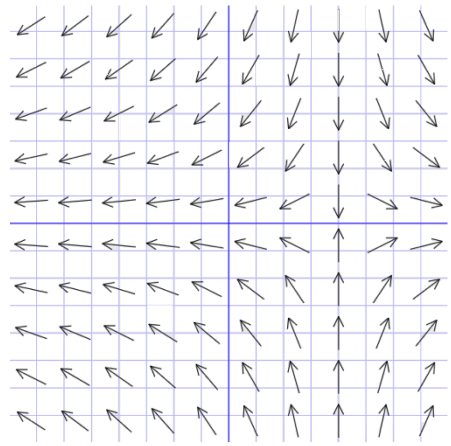
I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

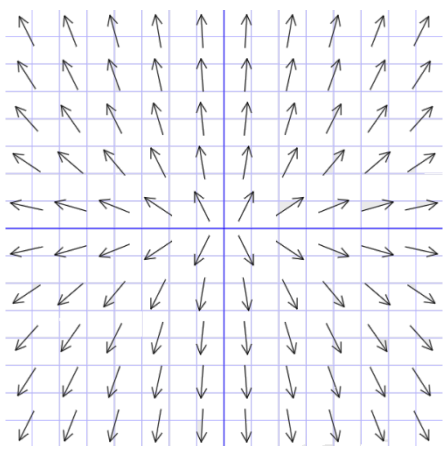
Signed:____
(full name)

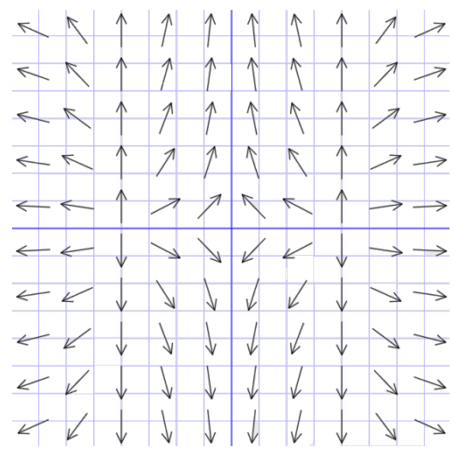
Questions:	1	2	3	4	5	6	7	**Total**
Points:	20	10	10	15	25	10	10	100
Score:								

1. (20 points) The direction fields of four systems of differential equations are given below. They are plotted for $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$.

I 

II 

III 

IV 

Each is generated by one of the following six systems of differential equations.

(A) $$\frac{dx}{dt} = 2x \qquad \frac{dy}{dt} = y$$

(B) $$\frac{dx}{dt} = x \qquad \frac{dy}{dt} = 2y$$

(C) $$\frac{dx}{dt} = x - 1 \qquad \frac{dy}{dt} = -y$$

(D) $$\frac{dx}{dt} = x^2 - 1 \qquad \frac{dy}{dt} = y$$

(E) $$\frac{dx}{dt} = x^2 - 1 \qquad \frac{dy}{dt} = -y$$

(F) $$\frac{dx}{dt} = x(x-1)(x+1) \qquad \frac{dy}{dt} = y$$

(a) (5 points) Direction field I is generated by the system _________.

(b) (5 points) Direction field II is generated by the system _________.

(c) (5 points) Direction field III is generated by the system _________.

(d) (5 points) Direction field IV is generated by the system _________.

2. (10 points) Consider the partially decoupled system

$$\begin{aligned}\frac{dx}{dt} &= 2x - 8y^2 \\ \frac{dy}{dt} &= -3y\end{aligned}$$

(a) (5 points) What are the equilibrium points for this system?

(b) (5 points) Find the general solution of this system.

3. (10 points) Consider the following second-order differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

(a) (5 points) Find two non-zero solutions that are not multiples of each other.

(b) (5 points) Convert this equation to a first-order system.

4. (15 points) Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \mathbf{Y}.$$

(a) (5 points) The two functions

$$\mathbf{Y}_1(t) = e^{-2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} \text{ and } \mathbf{Y}_2(t) = e^{-2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix}$$

are solutions to the system. Verify this fact for $\mathbf{Y}_1(t)$.

(b) (5 points) Show $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are linearly independent.

(c) (5 points) Find the particular solution to the system with initial value $\mathbf{Y}(0) = (2, 3)$.

5. (25 points) Consider the following system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix} \mathbf{Y}.$$

(a) (5 points) Find the eigenvalues of the coefficient matrix. Find an eigenvector for each eigenvalue.

(b) (5 points) For each eigenvalue, specify a corresponding straight-line solution.

(c) (5 points) Sketch the $x(t)$ and $y(t)$ graphs for each straight-line solution.

(d) (5 points) Give the general solution.

(e) (5 points) Sketch the phase-portrait of the system.

6. (10 points) The linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & -5 \\ 3 & 1 \end{pmatrix} \mathbf{Y}$$

has complex eigenvalues $\lambda_1 = -1 + i\sqrt{11}$ and $\lambda_2 = -1 - i\sqrt{11}$.

(a) (1 point) Determine if the origin is a spiral sink, spiral source, or a center.

(b) (4 points) Determine if solution curves travel clockwise or counterclockwise around the origin.

(c) (5 points) Given that $\begin{pmatrix} 5 \\ -2 - i\sqrt{11} \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = -1 + i\sqrt{11}$, find the general solution of the system.

7. (10 points) Verify the linearly principle for linear systems of differential equations. That is, suppose $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are solutions to the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

and show

(a) (5 points) that for any constant k, the function $k\mathbf{Y}_1$ is a solution as well;

(b) (5 points) and that the sum $\mathbf{Y}_1 + \mathbf{Y}_2$ is also a solution to the system.

You may use any facts about matrix arithmetic without justification.

8. (3 points (bonus)) Give any three items from the table of contents for the lecture on section 3.1.

9. (2 points (bonus)) During the "primer on complex numbers," I mentioned a consequence of Euler's formula that some have called "the most beautiful identity in mathematics." What is that identity?

10. (5 points (bonus)) Suppose that a matrix $\mathbf{A}$ with real entries has the complex eigenvalues $\lambda = a \pm bi$ with $b \neq 0$. Suppose also that $\mathbf{Y}_0 = (x_1 + iy_1, x_2 + iy_2)$ is an eigenvector with eigenvalue $\lambda = a + bi$. Show that $(x_1 - iy_1, x_2 - iy_2)$ is an eigenvector with eigenvalue $\overline{\lambda} = a - bi$. In other words, show that the complex conjugate of an eigenvector with eigenvalue λ, is an eigenvector with eigenvalue $\overline{\lambda}$, the complex conjugate of λ.