29. Suppose γ is the reaction-rate coefficient for the reaction B + B → A. By the reaction, two B's react with each other to create one A. In other words, B decreases at the rate γb^2 and A increases at the rate $\gamma b^2/2$. The resulting system of the differential equations is

$$\frac{da}{dt} = k_1 - \alpha ab + \frac{\gamma b^2}{2}$$
$$\frac{db}{dt} = k_2 - \alpha ab - \gamma b^2.$$

30. The chance that two B's and an A molecule are close is proportional to ab^2, so

$$\frac{da}{dt} = k_1 - \alpha ab - \gamma ab^2$$
$$\frac{db}{dt} = k_2 - \alpha ab - 2\gamma ab^2,$$

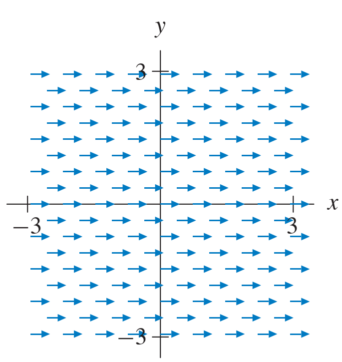
where γ is the reaction-rate parameter for the reaction that produces D from two B's and an A.

EXERCISES FOR SECTION 2.2

1. **(a)** $\mathbf{V}(x, y) = (1, 0)$

(b) See part (c).

(c)



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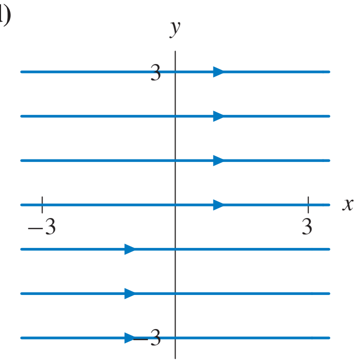
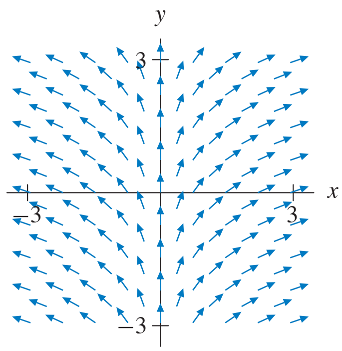


(e) As t increases, solutions move along horizontal lines toward the right.

2. **(a)** $\mathbf{V}(x, y) = (x, 1)$

(b) See part (c).

(c)



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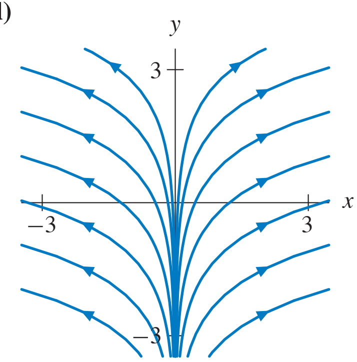
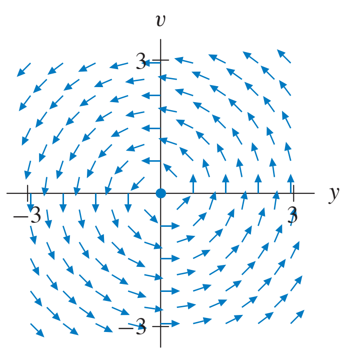


(e) As t increases, solutions move up and right if $x(0) > 0$, up and left if $x(0) < 0$.

3. **(a)** $\mathbf{V}(y, v) = (-v, y)$

(b) See part (c).

(c)



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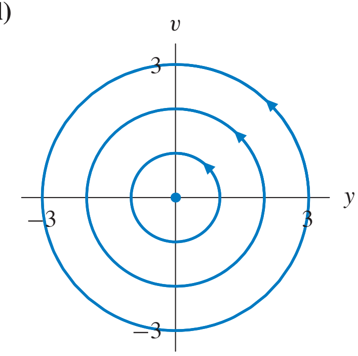
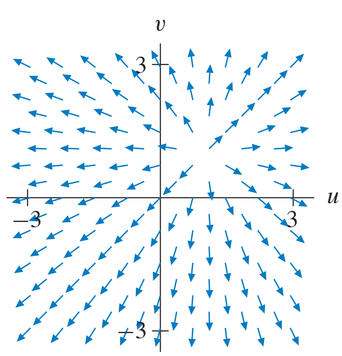


(e) As t increases, solutions move on circles around (0, 0) in the counter-clockwise direction.

4. **(a)** $\mathbf{V}(u, v) = (u - 1, v - 1)$

(b) See part (c).

(c)



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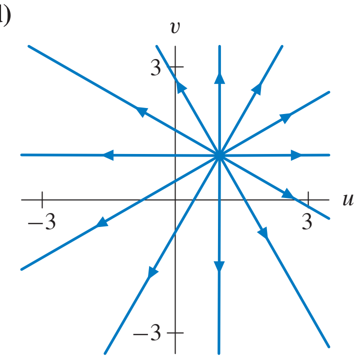
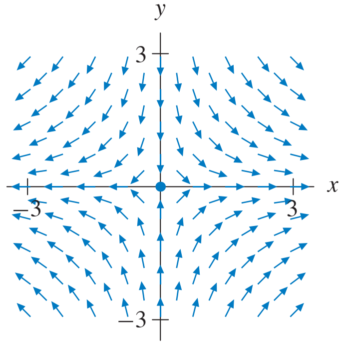


(e) As t increases, solutions move away from the equilibrium point at (1, 1).

5. **(a)** $\mathbf{V}(x, y) = (x, -y)$

(b) See part (c).

(c)



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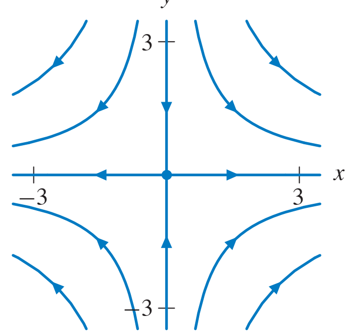
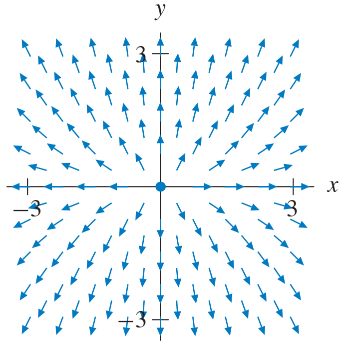


(e) As t increases, solutions move toward the x-axis in the y-direction and away from the y-axis in the x-direction.

6. **(a)** $\mathbf{V}(x, y) = (x, 2y)$

(b) See part (c).

(c)



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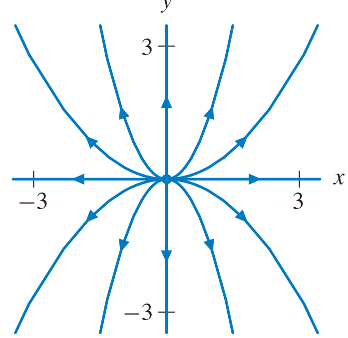


(e) As t increases, solutions move away from the equilibrium point at the origin.

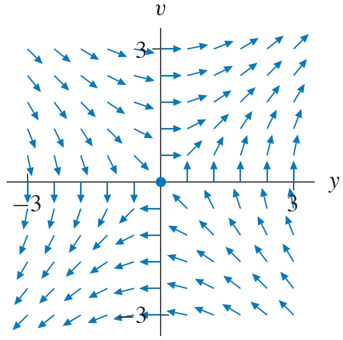
7. **(a)** Let $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = y.$$

Thus the associated vector field is $\mathbf{V}(y, v) = (v, y)$.

(b) See part (c).

(c)



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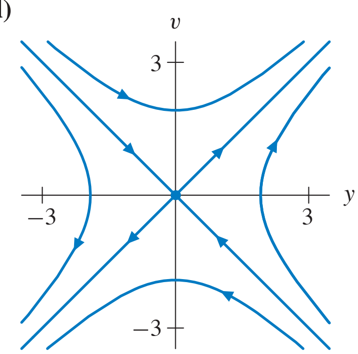


(e) As t increases, solutions in the 2nd and 4th quadrants move toward the origin and away from the line $y = -v$. Solutions in the 1st and 3rd quadrants move away from the origin and toward the line $y = v$.

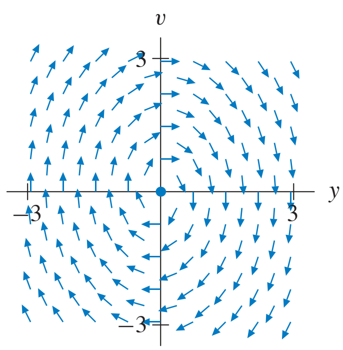
8. **(a)** Let $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -2y.$$

Thus the associated vector field is $\mathbf{V}(y, v) = (v, -2y)$.

(b) See part (c).

(c)



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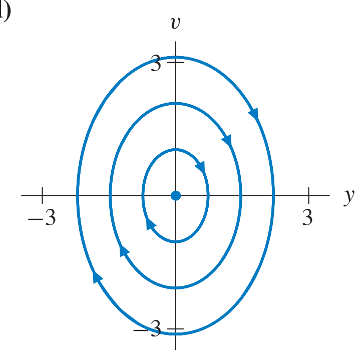
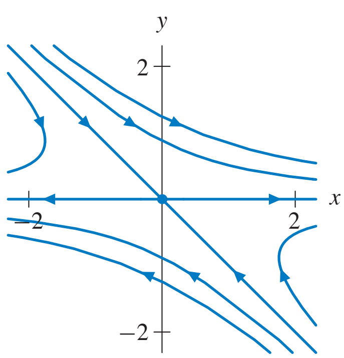


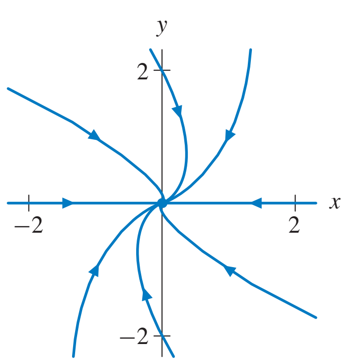
(e) As t increases, solutions move around the origin on ovals in the clockwise direction.

9. **(a)**



(b) The solution tends to the origin along the line $y = -x$ in the xy-phase plane. Therefore both $x(t)$ and $y(t)$ tend to zero as $t \to \infty$.

10. **(a)**



(b) The solution enters the first quadrant and tends to the origin tangent to the positive x-axis. Therefore $x(t)$ initially increases, reaches a maximum value, and then tends to zero as $t \to \infty$. It remains positive for all positive values of t. The function $y(t)$ decreases toward zero as $t \to \infty$.

11. **(a)** There are equilibrium points at $(\pm 1, 0)$, so only systems (ii) and (vii) are possible. Since the direction field points toward the x-axis if $y \neq 0$, the equation $dy/dt = y$ does not match this field. Therefore, system (vii) is the system that generated this direction field.

(b) The origin is the only equilibrium point, so the possible systems are (iii), (iv), (v), and (viii). The direction field is not tangent to the y-axis, so it does not match either system (iv) or (v). Vectors point toward the origin on the line $y = x$, so $dy/dt = dx/dt$ if $y = x$. This condition is not satisfied by system (iii). Consequently, this direction field corresponds to system (viii).

(c) The origin is the only equilibrium point, so the possible systems are (iii), (iv), (v), and (viii). Vectors point directly away from the origin on the y-axis, so this direction field does not correspond to systems (iii) and (viii). Along the line $y = x$, the vectors are more vertical than horizontal. Therefore, this direction field corresponds to system (v) rather than system (iv).

(d) The only equilibrium point is $(1, 0)$, so the direction field must correspond to system (vi).

12. The equilibrium solutions are those solutions for which $dR/dt = 0$ and $dF/dt = 0$ simultaneously. To find the equilibrium points, we must solve the system of equations

$$\begin{cases} 2R\left(1 - \dfrac{R}{2}\right) - 1.2RF = 0 \\ -F + 0.9RF = 0. \end{cases}$$

The second equation is satisfied if $F = 0$ or if $R = 10/9$, and we consider each case independently. If $F = 0$, then the first equation is satisfied if and only if $R = 0$ or $R = 2$. Thus two equilibrium solutions are $(R, F) = (0, 0)$ and $(R, F) = (2, 0)$.

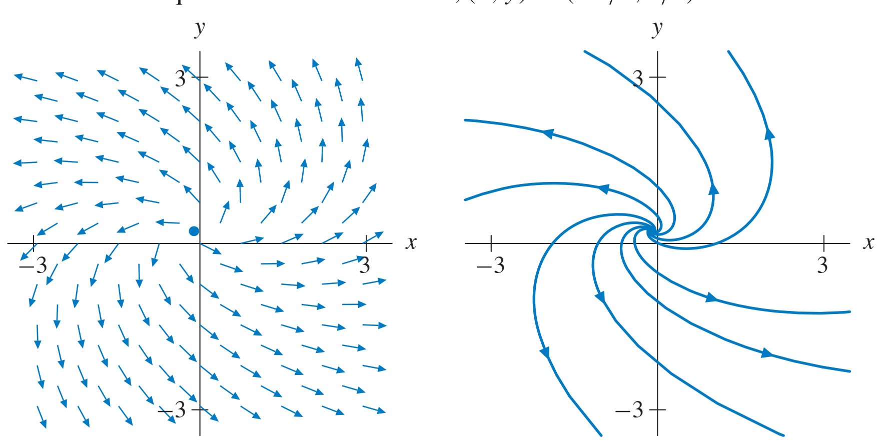
If $R = 10/9$, we substitute this value into the first equation and obtain $F = 20/27$.

13. **(a)** To find the equilibrium points, we solve the system of equations

$$\begin{cases} 4x - 7y + 2 = 0 \\ 3x + 6y - 1 = 0. \end{cases}$$

These simultaneous equations have one solution, $(x, y) = (-1/9, 2/9)$.

(b)

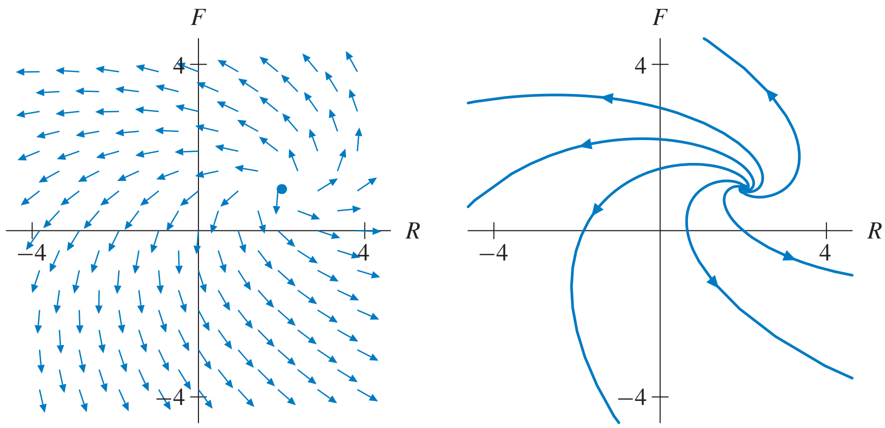


(c) As t increases, typical solutions spiral away from the origin in the counter-clockwise direction.

14. **(a)** To find the equilibrium points, we solve the system of equations

$$\begin{cases} 4R - 7F - 1 = 0 \\ 3R + 6F - 12 = 0. \end{cases}$$

These simultaneous equations have one solution, $(R, F) = (2, 1)$.



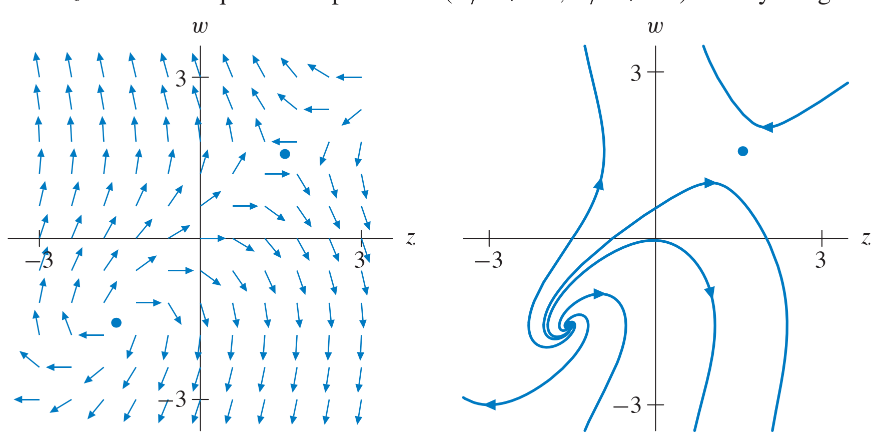
(b) As t increases, typical solutions spiral away from the equilibrium point at (2, 1)

15. **(a)** To find the equilibrium points, we solve the system of equations

$$\begin{cases} \cos w = 0 \\ -z + w = 0. \end{cases}$$

The first equation implies that $w = \pi/2 + k\pi$ where k is any integer, and the second equation implies that $z = w$. The equilibrium points are $(\pi/2 + k\pi, \pi/2 + k\pi)$ for any integer k.

(b)



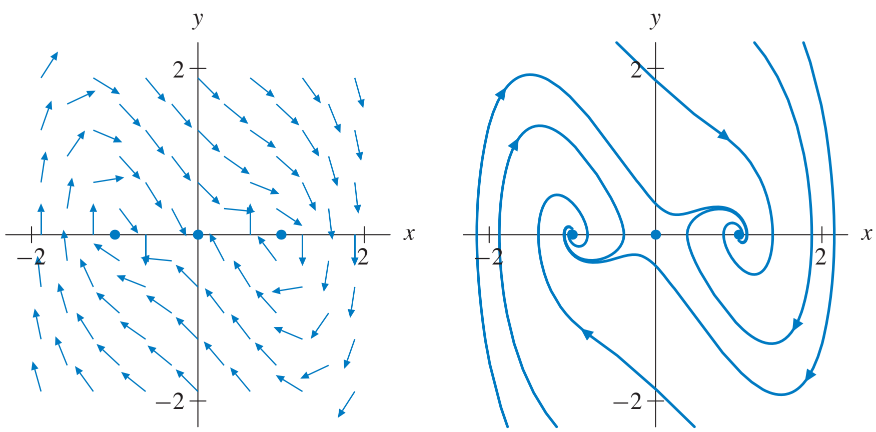
(c) As t increases, typical solutions move away from the line $z = w$, which contains the equilibrium points. The value of w is either increasing or decreasing without bound depending on the initial condition.

16. **(a)** To find the equilibrium points, we solve the system of equations

$$\begin{cases} x - x^3 - y = 0 \\ y = 0. \end{cases}$$

Since $y = 0$, we have $x^3 - x = 0$. If we factor $x - x^3$ into $x(x-1)(x+1)$, we see that there are three equilibrium points, $(0,0)$, $(1,0)$, and $(-1,0)$.

(b)



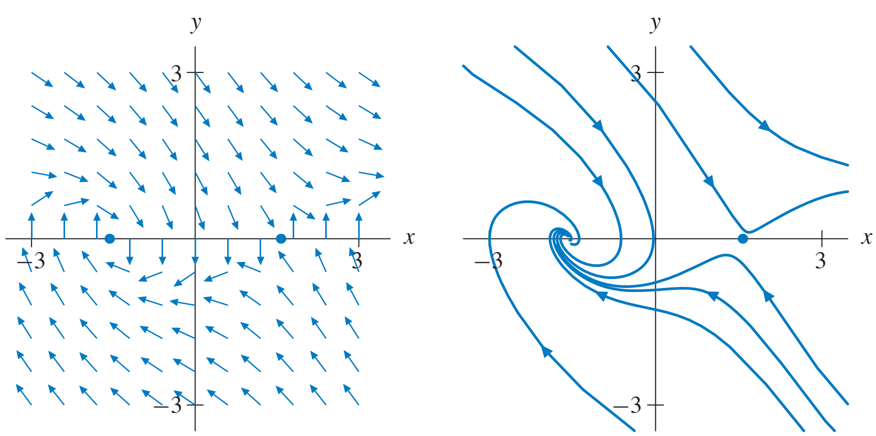
(c) As t increases, typical solutions spiral toward either $(1,0)$ or $(-1,0)$ depending on the initial condition.

17. **(a)** To find the equilibrium points, we solve the system of equations

$$\begin{cases} y = 0 \\ -\cos x - y = 0. \end{cases}$$

We see that $y = 0$, and thus $\cos x = 0$. The equilibrium points are $(\pi/2 + k\pi, 0)$ for any integer k.

(b)



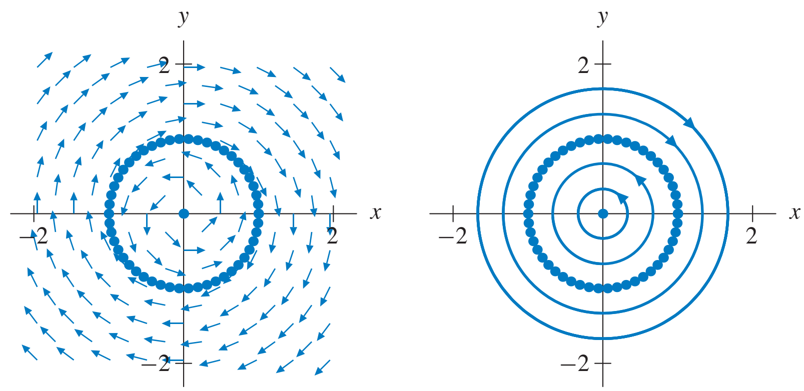
(c) As t increases, typical solutions spiral toward one of the equilibria on the x-axis. Which equilibrium point the solution approaches depends on the initial condition.

18. **(a)** To find the equilibrium points, we solve the system of equations

$$\begin{cases} y(x^2 + y^2 - 1) = 0 \\ -x(x^2 + y^2 - 1) = 0. \end{cases}$$

If $x^2 + y^2 = 1$, then both equations are satisfied. Hence, any point on the unit circle centered at the origin is an equilibrium point. If $x^2 + y^2 \neq 1$, then the first equations implies $y = 0$ and the second equation implies $x = 0$. Hence, the origin is the only other equilibrium point.

(b)



(c) As t increases, typical solutions move on a circle around the origin, either counter-clockwise inside the unit circle, which consists entirely of equilibrium points, or clockwise outside the unit circle.

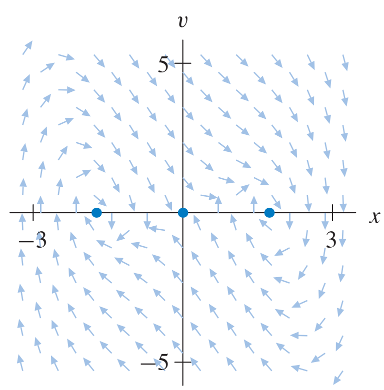
19. **(a)** Let $v = dx/dt$. Then

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = 3x - x^3 - 2v.$$

Thus the associated vector field is $\mathbf{V}(x, v) = (v, 3x - x^3 - 2v)$.

(b) Setting $\mathbf{V}(x, v) = (0, 0)$ and solving for (x, v), we get $v = 0$ and $3x - x^3 = 0$. Hence, the equilibria are $(x, v) = (0, 0)$ and $(x, v) = (\pm\sqrt{3}, 0)$.

(c)



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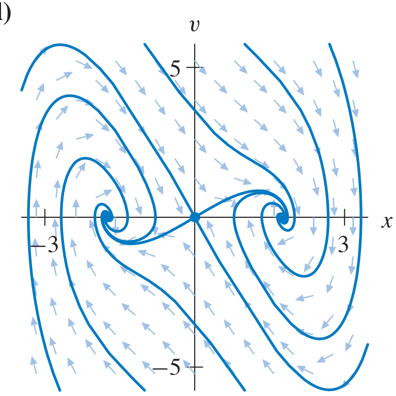


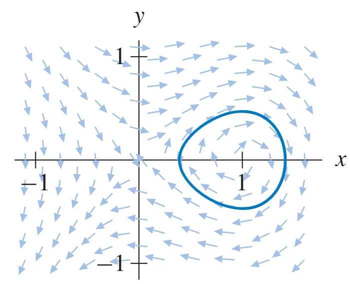
(e) As t increases, almost all solutions spiral to one of the two equilibria $(\pm\sqrt{3}, 0)$. There is a curve of initial conditions that divides these two phenomena. It consists of those initial conditions for which the corresponding solutions tend to the equilibrium point at $(0, 0)$.

20. Consider a point (y, v) on the circle $y^2 + v^2 = r^2$. We can consider this point to be a radius vector—one that starts at the origin and ends at the point (y, v). If we compute the dot product of this vector with the vector field $\mathbf{F}(y, v)$, we obtain

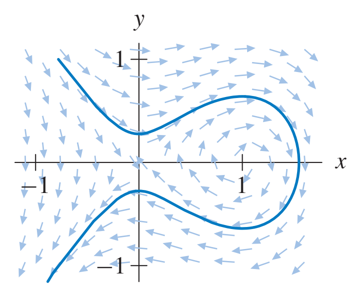
$$(y, v) \cdot \mathbf{F}(y, v) = (y, v) \cdot (v, -y) = yv - vy = 0.$$

Since the dot product of these two vectors is 0, the two vectors are perpendicular. Moreover, we know that any vector that is perpendicular to the radius vector of a circle must be tangent to that circle.

21. **(a)** The $x(t)$- and $y(t)$-graphs are periodic, so they correspond to a solution curve that returns to its initial condition in the phase plane. In other words, its solution curve is a closed curve. Since the amplitude of the oscillation of $x(t)$ is relatively large, these graphs must correspond to the outermost closed solution curve.

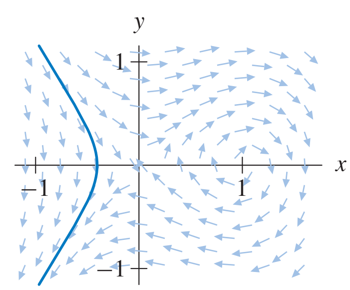


(b) The graphs are not periodic, so they cannot correspond to the two closed solution curves in the phase portrait. Both graphs cross the t-axis. The value of $x(t)$ is initially negative, then becomes positive and reaches a maximum, and finally becomes negative again. Therefore, the corresponding solution curve is the one that starts in the second quadrant, then travels through the first and fourth quadrants, and finally enters the third quadrant.



(c) The graphs are not periodic, so they cannot correspond to the two closed solution curves in the phase portrait. Only one graph crosses the t-axis. The other graph remains negative for all time. Note that the two graphs cross.

The corresponding solution curve is the one that starts in the second quadrant and crosses the x-axis and the line $y = x$ as it moves through the third quadrant.



(d) The $x(t)$- and $y(t)$-graphs are periodic, so they correspond to a solution curve that returns to its initial condition in the phase plane. In other words, its solution curve is a closed curve. Since the amplitude of the oscillation of $x(t)$ is relatively small, these graphs must correspond to the innermost closed solution curve.