Homogeneous linear second-order differential equations
Example: solving an IVP

Math 2410-010/015

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Abstract

This note provides a quick summary of the methods we developed for finding the general solution of any homogeneous linear second-order differential equation with constant coefficients in order to demonstrate the solving of an IVP involving a second-order equation. It does not explain why these methods work: to understand that consult your notes from the lecture.

In section 3.6, our chief concern is to find the general solution to a differential equation of the form

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0. \tag{1}$$

Being able to do this allows us to, of course, solve any initial value problem. We will provide an example of solving such an initial value problem in section 2. Before we do that, let us recap how to find the general solution of equations like 1. In section 2.3, we saw that a function e^{st} solves 1 if s is such that

$$s^2 + ps + q = 0.$$

So, finding a solution to the differential equation is as simple as finding a root of a quadratic polynomial.[1] Now, based on our knowledge of solving first-order linear systems, we have seen that we can derive the general equation from these e^{st} equations. We summarize below:

for a differential equation

$$\frac{d^2y}{dt^2} + p\frac{dy}{dt} + qy = 0.$$

the function e^{st} is a solution if s is such that $s^2 + ps + q = 0$. Furthermore based on the roots of this polynomial, we can find the general solution as follows

[1] You should understand *why* this method works for finding a solution of the form e^{st}. See page 185 of the text.

Roots	General solution
$s = a$ and $s = b$	$y(t) = k_1 e^{at} + k_2 e^{bt}$
Only $s = a$	$y(t) = k_1 e^{at} + k_2 t e^{at}$
$s = a \pm bi$	$y(t) = k_1 e^{at}\cos(bt) + k_2 e^{at}\sin(bt)$

where k_1 and k_2 are constants.

1 Finding general solutions

Here are a few examples of how we can find the general solution to equations of the form 1.

Example. Find the general solution of $\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 12y = 0$.

Solution. The general solution follows from the solutions e^{st} where s is such that $s^2 - 7s + 12 = 0$. This is equivalent to stating $(s-3)(s-4) = 0$. So $s = 3$ or $s = 4$. Hence, the general solution to this differential equation is $y(t) = k_1 e^{3t} + k_2 e^{4t}$.

Example. Find the general solution of $\frac{d^2y}{dt^2} + \sqrt{8}\frac{dy}{dt} + y = 0$.

Solution. The general solution follows from the solutions e^{st} where s is such that $s^2 - 2s + 2 = 0$. Thus

$$s = \frac{2 \pm \sqrt{4 - 4(2)}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i.$$

Hence, the general solution to this differential equation is $y(t) = k_1 e^{t}\cos(t) + k_2 e^{t}\sin(t)$.

Example. Find the general solution of $\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 36y = 0$.

Solution. The general solution follows from the solutions e^{st} where s is such that $s^2 + 12s + 36 = 0$. This is equivalent to stating $(s+6)^2 = 0$. So $s = -6$. Hence, the general solution to this differential equation is $y(t) = k_1 e^{-6t} + k_2 t e^{-6t}$.

2 An initial value problem

We now turn our attention to the topic of this short note: an initial value problem with a homogeneous linear second-order differential equation. We hope to solve the following initial value problem

$$3\frac{d^2y}{dt^2} + 36\frac{dy}{dt} + 108y = 0 \quad y(0) = 1,\ y'(0) = 0.$$

Before we proceed, note that *two* initial conditions (one for y and one for y') are specified. This is to ensure that we are indeed seeking *one* particular solution, as opposed to a family of solutions. This is exactly analgous to providing two initial conditions (one for y and one for v) in a initial value problem involving the corresponding linear system

$$\begin{aligned} \frac{dy}{dt} &= v \\ \frac{dv}{dt} &= -36y - 12v \end{aligned}$$

Now, onto the formal statement of the problem and its solution/

Example. Find a function $y(t)$ such that

$$3\frac{d^2y}{dt^2} + 36\frac{dy}{dt} + 108y = 0 \quad y(0) = 1,\ y'(0) = 0. \tag{2}$$

In other words, solve this IVP.

Solution. The first step in solving any IVP is to find the general solution to the differential equation, and then specify constants that make that solution satisfy the initial condition. To that end, we seek the general solution of 2. Notice the differential equation in 2 is seen to be equal to

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 36y = 0$$

by dividing through by 3. This is exactly the differential equation we solved in the previous example. Thus, we know its general solution is $y(t) = k_1e^{-6t} + k_2te^{-6t}$. To solve the initial value problem, we need to determine values of k_1 and k_2 such that $y(0) = 1$ and $y'(0) = 0$. Observe

$$y'(t) = -6k_1e^{-6t} + k_2e^{-6t} - 6k_2te^{-6t} = (-6k_1 + k_2)e^{-6t} + k_2te^{-6t}.$$

Thus we see

$$\begin{aligned} y(0) &= k_1 &&= 1 \\ y'(0) &= -6k_1 + k_2 &&= 0 \end{aligned}$$

if and only if $k_1 = 1$ and $k_2 = 6$. Hence the solution to initial value problem is the particular solution $e^{-6t} + 6te^{-6t}$.

3 Conclusion

We see that solving an initial value problem for a second-order differential equation is different from the previous IVPs we solved in that an initial condition is given both for the function and its derivative. Outside of that, the usual strategy of finding the general solution and then using the initial condtion to determine values of constants still applies.