Math 2410-010/015 | **Exam II** | **April 7$^{\text{th}}$, 2017**

Name: KEY

Instructions:

- Answer each question to the best of your ability.
- All answers must be written clearly. Be sure to erase or cross out any work that you do not want graded. Partial credit can not be awarded unless there is legible work to assess.
- You may use a calculator, but you must show all your work in order to receive credit.

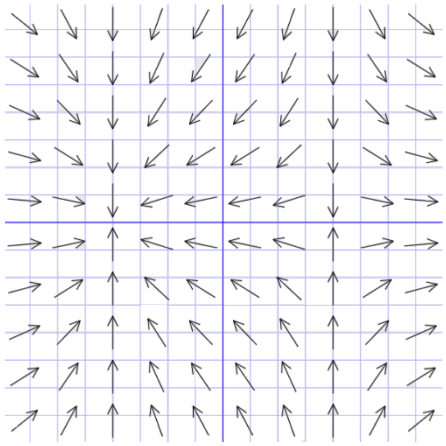
Academic Integrity Agreement

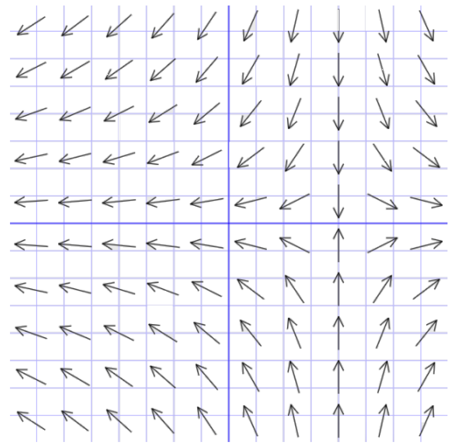
I certify that all work given in this examination is my own and that, to my knowledge, has not been used by anyone besides myself to their personal advantage. Further, I assert that this examination was taken in accordance with the academic integrity policies of the University of Connecticut.

Signed:________________________
(full name)

Questions:	1	2	3	4	5	6	7	**Total**
Points:	20	10	10	15	25	10	10	100
Score:								

1. (20 points) The direction fields of four systems of differential equations are given below. They are plotted for $-2 \le x \le 2$ and $-2 \le y \le 2$.

I


II


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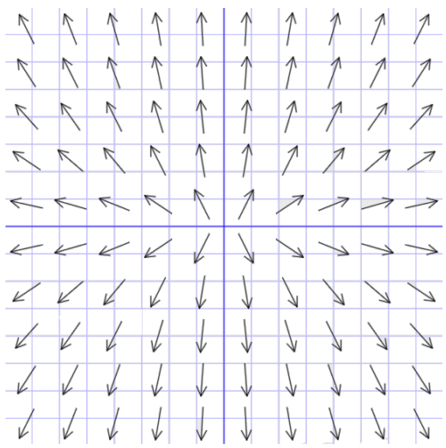

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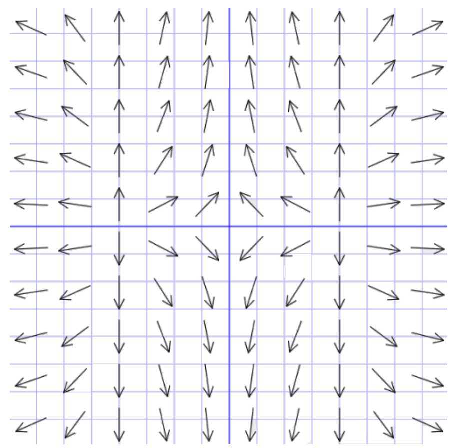


Each is generated by one of the following six systems of differential equations.

(A) $\dfrac{dx}{dt} = 2x$, $\dfrac{dy}{dt} = y$

(B) $\dfrac{dx}{dt} = x$, $\dfrac{dy}{dt} = 2y$

(C) $\dfrac{dx}{dt} = x - 1$, $\dfrac{dy}{dt} = -y$

(D) $\dfrac{dx}{dt} = x^2 - 1$, $\dfrac{dy}{dt} = y$

(E) $\dfrac{dx}{dt} = x^2 - 1$, $\dfrac{dy}{dt} = -y$

(F) $\dfrac{dx}{dt} = x(x-1)(x+1)$, $\dfrac{dy}{dt} = y$

(a) (5 points) Direction field I is generated by the system **E**.

(b) (5 points) Direction field II is generated by the system **C**.

(c) (5 points) Direction field III is generated by the system **B**.

(d) (5 points) Direction field IV is generated by the system **F**.

2. (10 points) Consider the partially decoupled system

$$\frac{dx}{dt} = 2x - 8y^2$$
$$\frac{dy}{dt} = -3y$$

(a) (5 points) What are the equilibrium points for this system?

Solution: Equilibrium points (x_0, y_0) are such that the constant functions $x(t) = x_0$ and $y(t) = y_0$ are solutions to the system. That is, such that

$$2x_0 - 8y_0^2 = 0$$
$$-3y_0 = 0$$

We see the second equation implies that y_0 must be 0. Plugging this information into the first equation yields $2x_0 - 0 = 0$. So we see x_0 must be zero as well. We conclude the only equilibrium points is $(0, 0)$.

(b) (5 points) Find the general solution of this system.

Solution: To solve this partially decoupled system of differential equations we focus first on the second equation

$$\frac{dy}{dt} = -3y.$$

By separation of variables, we see that the general solution to this differential equation is $y(t) = k_1 e^{-3t}$. Knowing the form of $y(t)$ allows us to rewrite the first equation as

$$\frac{dx}{dt} = 2x - 8(k_1 e^{-3t})^2.$$

This is a first-order non-homogeneous linear differential equation. Thus, to solve we may use the method of integrating factors or undetermined coefficients. Why not both?

First, let's solve using an integrating factor. In this case, our integrating factor is $\mu(t) = e^{\int -2\,dt} = e^{-2t}$, so

$$\begin{aligned} x(t) = \frac{1}{\mu(t)} \int \mu(t) b(t)\, dt &= e^{2t} \int -8k_1^2 e^{-8t}\, dt \\ &= e^{2t}\left(k_1^2 e^{-8t} + k_2\right) = k_1^2 e^{-6t} + k_2 e^{2t}. \end{aligned}$$

Now, let's solve the same equation using the method of undetermined coefficients. Here we need to find a solution x_h to the associated homogeneous equation $dx/dt = 2x$ and a particusolution x_p to the differential equation itself. Finding x_h is easily done by separation of variables: $x_h = e^{2t}$ for example. To find x_p, we note that x_p should be such that

$$\frac{dx_p}{dt} - 2x_p = -8k_1^2 e^{-6t}.$$

It seems trying $x_p(t) = ae^{-6t}$ where a is an undetermined coefficient will be fruitful. Then, this function will be a solution if

$$\begin{aligned} \frac{dx_p}{dt} - 2x_p = -6ae^{-6t} - 2ae^{-6t} &= -8ae^{-6t} \\ &= -8k_1^2 e^{-6t}. \end{aligned}$$

That is, if $a = k_1^2$. So $x_p(t) = k_1^2 e^{-6t}$ and

$$x(t) = k_2 x_h + x_p = k_2 e^{2t} + k_1^2 e^{-6t}.$$

In either case, we found the same function for $x(t)$ and can conclude that

$$\begin{aligned} x(t) &= k_1^2 e^{-6t} + k_2 e^{2t} \\ y(t) &= k_1 e^{-3t} \end{aligned}$$

is the general solution to this system.

3. (10 points) Consider the following second-order differential equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0$$

(a) (5 points) Find two non-zero solutions that are not multiples of each other.

Solution: The function e^{st} will solve this differential equation if

$$\frac{d^2}{dt^2}e^{st} + 5\frac{d}{dt}e^{st} + 6e^{st} = 0.$$

We see this is true if s is such that

$$\frac{d^2}{dt^2}e^{st} + 5\frac{d}{dt}e^{st} + 6e^{st} = (s^2 + 5s + 6)e^{st} = 0.$$

This only holds if $s^2 + 5s + 6 = (s + 2)(s + 3) = 0$ because e^{st} is positive for every t, regardless of s. So $s = -2$ or $s = -3$. Thus $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-3t}$ are solutions of the system. Notice they are not multiplies of each other.

(b) (5 points) Convert this equation to a first-order system.

Solution: To convert this second-order differential equation into a system of first-order equations we introduce a new variable to represent the derivative of y. Let

$$x = \frac{dy}{dt}.$$

This will be one of the two equations in our system. Note $dx/dt = d^2y/dt^2$ so the equation in question becomes

$$\frac{dx}{dt} + 5x + 6y = 0.$$

Solving for $\frac{dx}{dt}$ gives us the other equation of the desired system. Thus, we have

$$\begin{aligned} \frac{dx}{dt} &= -5x - 6y \\ \frac{dy}{dt} &= x \end{aligned}$$

4. (15 points) Consider the linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \mathbf{Y}.$$

(a) (5 points) The two functions

$$\mathbf{Y}_1(t) = e^{-2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} \text{ and } \mathbf{Y}_2(t) = e^{-2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix}$$

are solutions to the system. Verify this fact for $\mathbf{Y}_1(t)$.

Solution: To show $\mathbf{Y}_1(t)$ is a solution, we simply need show that $\frac{d\mathbf{Y}_1}{dt} = \mathbf{A}\mathbf{Y}_1$ where $A = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix}$. That that end, note

$$\begin{aligned} \frac{d\mathbf{Y}_1}{dt} = \frac{d}{dt} e^{-2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} &= -2e^{-2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + e^{-2t} \begin{pmatrix} -3\sin 3t \\ 3\cos 3t \end{pmatrix} \\ &= \begin{pmatrix} -2e^{-2t}\cos 3t - 3e^{-2t}\sin 3t \\ -2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{A}\mathbf{Y}_1 = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} e^{-2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} &= e^{-2t} \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} \\ &= e^{-2t} \left[\begin{pmatrix} -2\cos 3t \\ 3\cos 3t \end{pmatrix} + \begin{pmatrix} -3\sin 3t \\ -2\sin 3t \end{pmatrix} \right] \\ &= \begin{pmatrix} -2e^{-2t}\cos 3t - 3e^{-2t}\sin 3t \\ -2e^{-2t}\sin 3t + 3e^{-2t}\cos 3t \end{pmatrix}. \end{aligned}$$

As these two expressions agree, we conclude that $\mathbf{Y}_1(t)$ is a solution.

(b) (5 points) Show $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are linearly independent.

Solution: To show these two solutions are linearly independent, we need verify that their initial conditions are linearly independent. The initial conditions are

$$\mathbf{Y}_1(0) = e^0 \begin{pmatrix} \cos 0 \\ \sin 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \mathbf{Y}_1(0) = e^0 \begin{pmatrix} -\sin 0 \\ \cos 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Note

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

As this determinant is nonzero, we conclude the initial conditions are linearly independent vectors and hence, $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are linearly independent solutions.

(c) (5 points) Find the particular solution to the system with initial value $\mathbf{Y}(0) = (2, 3)$.

Solution: Because $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are linearly independent solutions, we know the general solution to the system is

$$\mathbf{Y}(t) = k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}_2(t) = k_1 e^{-2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix}$$

for some constants k_1 and k_2. Thus, the function that solves this system with initial condition $\mathbf{Y}(0) = (2, 3)$ is of the form

$$\mathbf{Y}(t) = k_1 e^{-2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix}$$

with k_1 and k_2 such that

$$\mathbf{Y}(0) = k_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}.$$

Clearly, we must have $k_1 = 2$ and $k_2 = 3$. Thus the solution we desire is

$$\mathbf{Y}(t) = 2e^{-2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix} + 3e^{-2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix}.$$

5. (25 points) Consider the following system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & 2 \\ 0 & -2 \end{pmatrix} \mathbf{Y}.$$

(a) (5 points) Find the eigenvalues of the coefficient matrix. Find an eigenvector for each eigenvalue.

Solution: An eigenvalue of the coefficient matrix is a number λ such that

$$\det \begin{pmatrix} 3-\lambda & 2 \\ 0 & -2-\lambda \end{pmatrix} = (3-\lambda)(-2-\lambda) = 0.$$

We see the eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = -2$. To find an eigenvector for λ_1, we seek nontrivial (x_0, y_0) such that

$$\begin{pmatrix} 3-\lambda_1 & 2 \\ 0 & -2-\lambda_1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \lambda \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

This is equivalent to finding a nonzero solution to

$$\begin{cases} (3-3)x_0 + 2y_0 & = 0 \\ \quad (-2-3)y_0 & = 0 \end{cases} \Longrightarrow \begin{cases} 2y_0 = 0 \\ -5y_0 = 0 \end{cases}$$

So an eigenvector is any (x_0, y_0) such that $y_0 = 0$ and $x_0 \neq 0$. We use $(1, 0)$ for our eigenvector with eigenvalue $\lambda_1 = 3$. To find an eigenvector with eigenvalue $\lambda_2 = -2$, we proceed similarly:

$$\begin{cases} (3+2)x_0 + 2y_0 & = 0 \\ \quad (-2+2)y_0 & = 0 \end{cases}$$

has nonzero solutions (x_0, y_0) where $x_0 = (-2/5)y_0$. We use $(-2, 5)$ as our eigenvector with eigenvalue $\lambda_2 = -2$.

(b) (5 points) For each eigenvalue, specify a corresponding straight-line solution.

Solution: For λ_1 we have

$$\mathbf{Y}_1(t) = e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

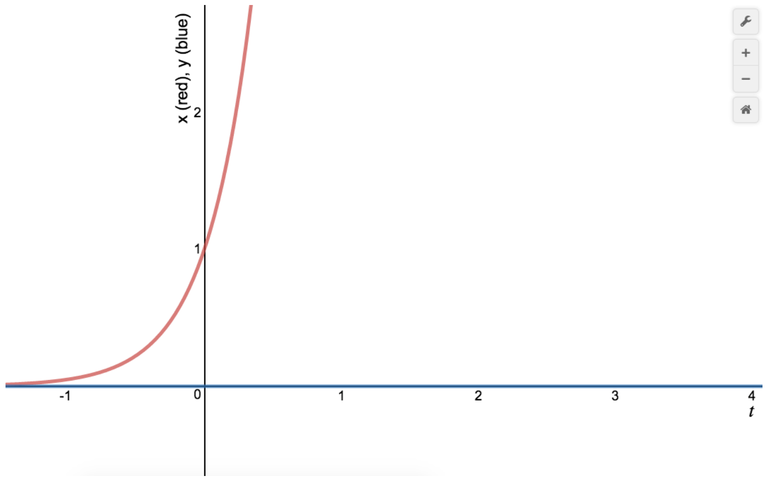
and for λ_2 we have

$$\mathbf{Y}_2(t) = e^{-2t} \begin{pmatrix} -2 \\ 5 \end{pmatrix}.$$

(c) (5 points) Sketch the $x(t)$ and $y(t)$ graphs for each straight-line solution.

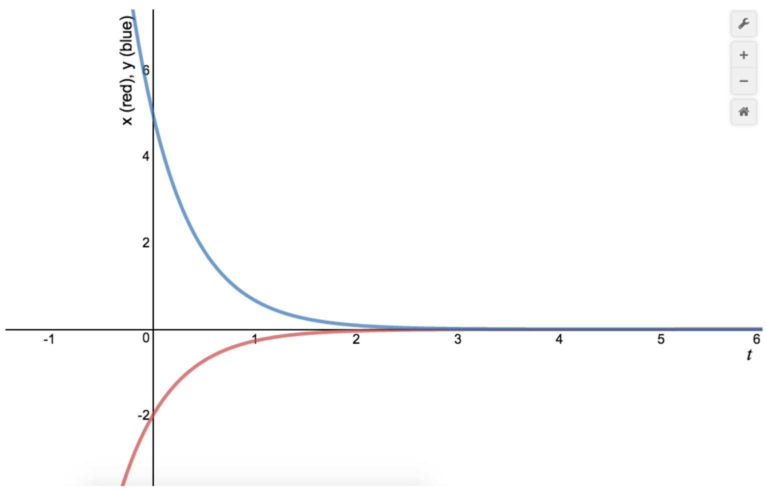
Solution: For $\mathbf{Y}_1(t)$ we have

$$x(t) = e^{3t} \text{ and } y(t) = 0.$$



For $\mathbf{Y}_2(t)$ we have

$$x(t) = -2e^{-2t} \text{ and } y(t) = 5e^{-2t}.$$



(d) (5 points) Give the general solution.

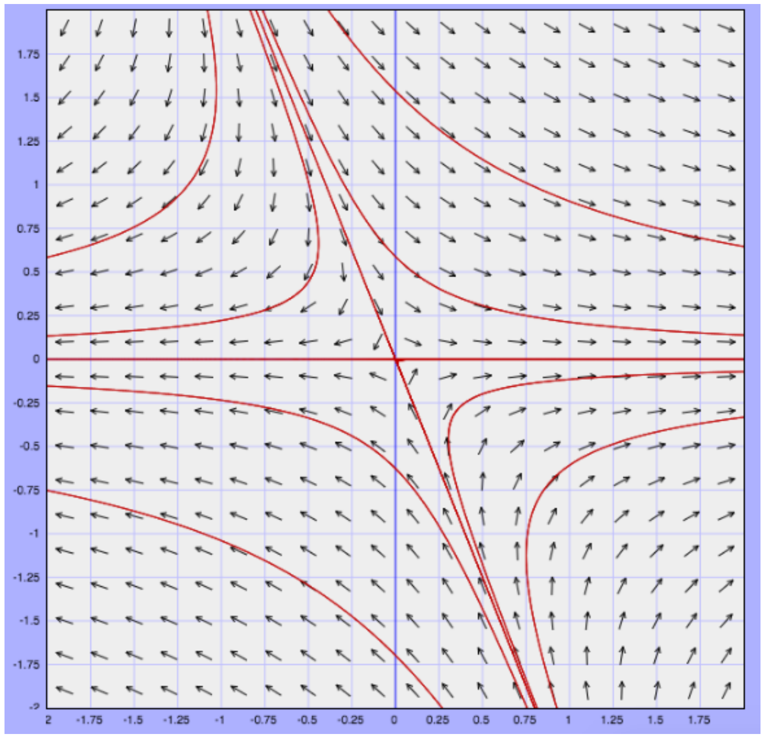
Solution: Since $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ are straight-line solutions with respect to different eigenvalues, we know they are linearly independent. Hence, the general solution is

$$\mathbf{Y}(t) = \mathbf{Y}_1(t) + \mathbf{Y}_2(t) = k_1 e^{3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + k_2 e^{-2t} \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

for arbitrary constants k_1 and k_2.

(e) (5 points) Sketch the phase-portrait of the system.

Solution: As this system has two real eigenvalues, one of which is negative and one of which is positive we know the origin will be a saddle. The straight-line solutions lie on the lines $x = 0$ and $y = (-5/2)x$ for λ_1 and λ_2 respectively. Since $\lambda_1 > 0$ the straight-line solutions leave the origin along $x = 0$. Since $\lambda_2 < 0$ the straight-line solutions approach the origin along $y = (-5/2)x$. The other curves can be found by either using the general solution or recognizing the general behavior of saddles.



6. (10 points) The linear system

$$\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -3 & -5 \\ 3 & 1 \end{pmatrix} \mathbf{Y}$$

has complex eigenvalues $\lambda_1 = -1 + i\sqrt{11}$ and $\lambda_2 = -1 - i\sqrt{11}$.

(a) (1 point) Determine if the origin is a spiral sink, spiral source, or a center.

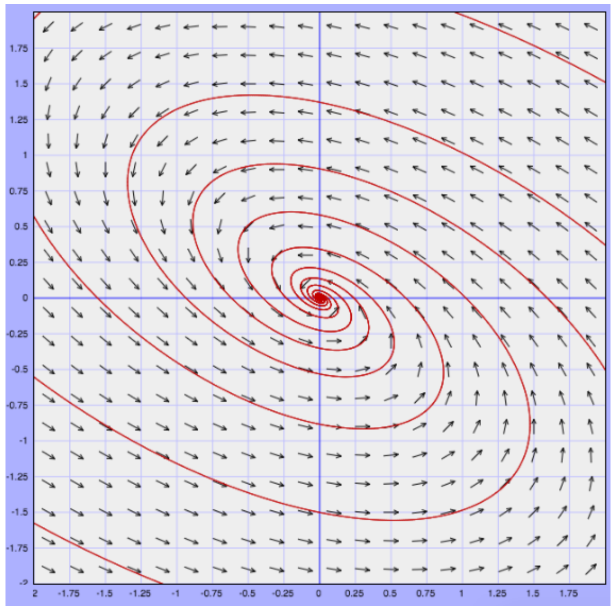
Solution: To determine if the origin is a spiral sink, spiral source, or a center we investigate the real part of the eigenvalues. In this case, it is -1. As $-1 < 0$ we conclude that the origin is a spiral sink.

(b) (4 points) Determine if solution curves travel clockwise or counterclockwise around the origin.

Solution: To determine this, we investigate the vector field at a specific point and see which directions solutions are traveling. The vector field at $(1, 0)$ is

$$\begin{pmatrix} -3 & -5 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \end{pmatrix}.$$

The direction of this vector is up and to the left with it's base at $(1, 0)$ on the horizontal axis. This indicates that solutions travel counterclockwise around the origin. The phase portrait is given below for reference.



(c) (5 points) Given that $\begin{pmatrix} 5 \\ -2 - i\sqrt{11} \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda_1 = -1 + i\sqrt{11}$, find the general solution of the system.

Solution: Since we have an eigenvalue and eigenvector, we can construct a complex valued solution to the system:

$$\mathbf{Y}_1(t) = e^{(-1+i\sqrt{11})t} \begin{pmatrix} 5 \\ -2 - i\sqrt{11} \end{pmatrix}.$$

To find the general solution to this system, we decompose $\mathbf{Y}_1$ into real and imaginary parts:

$$\begin{aligned}
\mathbf{Y}_1(t) &= e^{(-1+i\sqrt{11})t} \begin{pmatrix} 5 \\ -2 - i\sqrt{11} \end{pmatrix} && (1)\\
&= e^{-t+i\sqrt{11}t} \begin{pmatrix} 5 \\ -2 - i\sqrt{11} \end{pmatrix} && (2)\\
&= e^{t} e^{i\sqrt{11}t} \begin{pmatrix} 5 \\ -2 - i\sqrt{11} \end{pmatrix} && (3)\\
&= e^{-t}(\cos(\sqrt{11}t) + i\sin(\sqrt{11}t)) \begin{pmatrix} 5 \\ -2 - i\sqrt{11} \end{pmatrix} && (4)\\
&= e^{-t} \begin{pmatrix} 5\cos(\sqrt{11}t) + 5i\sin(\sqrt{11}t) \\ (-2 - i\sqrt{11})\cos(\sqrt{11}t) + i(-2 - i\sqrt{11})\sin(\sqrt{11}t) \end{pmatrix} && (5)\\
&= e^{-t} \begin{pmatrix} 5\cos(\sqrt{11}t) + i(5\sin(\sqrt{11}t)) \\ (-2\cos(\sqrt{11}t) + \sqrt{11}\sin(\sqrt{11}t)) + i(-\sqrt{11}\cos(\sqrt{11}t) - 2\sin(\sqrt{11}t)) \end{pmatrix} && (6)\\
&= e^{-t} \left[\begin{pmatrix} 5\cos(\sqrt{11}t) \\ -2\cos(\sqrt{11}t) + \sqrt{11}\sin(\sqrt{11}t) \end{pmatrix} + i \begin{pmatrix} 5\sin(\sqrt{11}t) \\ -\sqrt{11}\cos(\sqrt{11}t) - 2\sin(\sqrt{11}t) \end{pmatrix} \right] && (7)\\
&= \underbrace{e^{-t} \begin{pmatrix} 5\cos(\sqrt{11}t) \\ -2\cos(\sqrt{11}t) + \sqrt{11}\sin(\sqrt{11}t) \end{pmatrix}}_{\mathbf{Y}_{re}(t)} + i \underbrace{\left[e^{-t} \begin{pmatrix} 5\sin(\sqrt{11}t) \\ -\sqrt{11}\cos(\sqrt{11}t) - 2\sin(\sqrt{11}t) \end{pmatrix} \right]}_{\mathbf{Y}_{im}(t)} && (8)
\end{aligned}$$

Note **??** is where we applied Euler's formula. The real and imaginary parts, $\mathbf{Y}_{re}$ and $\mathbf{Y}_{im}$, are linearly independent solutions to the system. Thus, we have that the general solution to the system is

$$
\begin{aligned}
\mathbf{Y}(t) &= k_1\mathbf{Y}_1(t) + k_2\mathbf{Y}(t) \\
&= k_1 e^{-t}\begin{pmatrix} 5\cos(\sqrt{11}t) \\ -2\cos(\sqrt{11}t) + \sqrt{11}\sin(\sqrt{11}t) \end{pmatrix} + k_2 e^{-t}\begin{pmatrix} 5\sin(\sqrt{11}t) \\ -\sqrt{11}\cos(\sqrt{11}t) - 2\sin(\sqrt{11}t) \end{pmatrix}
\end{aligned}
$$

for constants k_1 and k_2.

7. (10 points) Verify the linearly principle for linear systems of differential equations. That is, suppose $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are solutions to the linear system

$$\frac{d\mathbf{Y}}{dt} = \mathbf{A}\mathbf{Y}$$

and show

(a) (5 points) that for any constant k, the function $k\mathbf{Y}_1$ is a solution as well;

(b) (5 points) and that the sum $\mathbf{Y}_1 + \mathbf{Y}_2$ is also a solution to the system.

You may use any facts about matrix arithmetic without justification.

Solution: For part (a), we need show that $d(k\mathbf{Y}_1)/dt = \mathbf{A}(k\mathbf{Y}_1)$. To that end note

$$\frac{d(k\mathbf{Y}_1)}{dt} = k\frac{d\mathbf{Y}_1}{dt} = k(\mathbf{A}\mathbf{Y}_1) = \mathbf{A}(k\mathbf{Y}_1)$$

where the middle step follows from the fact that $\mathbf{Y}_1$. So $k\mathbf{Y}_1(t)$ is a solution.

For part (b), similarly, we need show that $d(\mathbf{Y}_1 + \mathbf{Y}_2)/dt = \mathbf{A}(\mathbf{Y}_1 + \mathbf{Y}_2)$. Notice

$$\frac{d(\mathbf{Y}_1 + \mathbf{Y}_2)}{dt} = \frac{d\mathbf{Y}_1}{dt} + \frac{d\mathbf{Y}_2}{dt} = \mathbf{A}\mathbf{Y}_1 + \mathbf{A}\mathbf{Y}_2 = \mathbf{A}(\mathbf{Y}_1 + \mathbf{Y}_2).$$

So $(\mathbf{Y}_1 + \mathbf{Y}_2)$ is a solution as well.

This completes the problem.

8. (3 points (bonus)) Give any three items from the table of contents for the lecture on section 3.1.

Solution: The table of contents was:

Part 1: Linear systems and matrix notation

Part 2: Equilibrium solutions and the determinant

Part 3: The linearity principle for systems

Part 4: Initial value problems and linear independence

Part 5: The general solution

9. (2 points (bonus)) During the "primer on complex numbers," I mentioned a consequence of Euler's formula that some have called "the most beautiful identity in mathematics." What is that identity?

Solution: Euler's formula is $e^{i\theta} = \cos\theta + i\sin\theta$. If we substitute π for θ, we obtain

$$e^{i\pi} = \cos\pi + i\sin\pi = -1.$$

Rewriting this, we obtain *Euler's identity*:

$$e^{i\pi} + 1 = 0.$$

10. (5 points (bonus)) Suppose that a matrix $\mathbf{A}$ with real entries has the complex eigenvalues $\lambda = a \pm bi$ with $b \neq 0$. Suppose also that $\mathbf{Y}_0 = (x_1 + iy_1, x_2 + iy_2)$ is an eigenvector with eigenvalue $\lambda = a + bi$. Show that $(x_1 - iy_1, x_2 - iy_2)$ is an eigenvector with eigenvalue $\overline{\lambda} = a - bi$. In other words, show that the complex conjugate of an eigenvector with eigenvalue λ, is an eigenvector with eigenvalue $\overline{\lambda}$, the complex conjugate of λ.

Solution: Since $\mathbf{Y}_0 = (x_1 + iy_1, x_2 + iy_2)$ is an eigenvector with eigenvalue $\lambda = a + bi$ we have $\mathbf{AY}_0 = (a + bi)\mathbf{Y}_0$. In other words,

$$\mathbf{AY}_0 = \mathbf{A}\begin{pmatrix} x_1 + iy_1 \\ x_2 + iy_2 \end{pmatrix} = \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + i\mathbf{A}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = a\mathbf{Y}_0 + ib\mathbf{Y}_0 = (a + bi)\mathbf{Y}_0$$

Thus $\mathbf{A}(x_1, x_2) = a(x_1, x_2)$ and $\mathbf{A}(y_1, y_2) = b(y_1, y_2)$. Now we consider $\mathbf{A}(x_1 - iy_1, x_2 - iy_2)$:

$$\mathbf{A}\begin{pmatrix} x_1 - iy_1 \\ x_2 - iy_2 \end{pmatrix} = \mathbf{A}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - i\mathbf{A}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = a\mathbf{Y}_0 - ib\mathbf{Y}_0 = (a - bi)\mathbf{Y}_0.$$

Thus we see $(x_1 - iy_1, x_2 - iy_2)$ is an eigenvector with eigenvalue $\overline{\lambda} = a - bi$.