13. If $R - 5F > 0$, the number of predators increases and, if $R - 5F < 0$, the number of predators decreases. Since the condition on prey is same, we modify only the predator part of the system. the modified rate of change of the predator population is

$$\frac{dF}{dt} = -F + 0.9RF + k(R - 5F)$$

where $k > 0$ is the immigration parameter for the predator population.

14. In both cases the rate of change of population of prey decreases by a factor of kF. Hence we have

(i) $dR/dt = 2R - 1.2RF - kF$

(ii) $dR/dt = 2R - R^2 - 1.2RF - kF$

15. Suppose $y = 1$. If we can find a value of x such that $dy/dt = 0$, then for this x and $y = 1$ the predator population is constant. (This point may not be an equilibrium point because we do not know if $dx/dt = 0$.) The required value of x is $x = 0.05$ in system (i) and $x = 20$ in system (ii). Survival for one unit of predators requires 0.05 units of prey in (i) and 20 units of prey in (ii). Therefore, (i) is a system of inefficient predators and (ii) is a system of efficient predators.

16. At first, the number of rabbits decreases while the number of foxes increases. Then the foxes have too little food, so their numbers begin to decrease. Eventually there are so few foxes that the rabbits begin to multiply. Finally, the foxes become extinct and the rabbit population tends to the constant population $R = 3$.

17. **(a)** For the initial condition close to zero, the pest population increases much more rapidly than the predator. After a sufficient increase in the predator population, the pest population starts to decrease while the predator population keeps increasing. After a sufficient decrease in the pest population, the predator population starts to decrease. Then, the population comes back to the initial point.

(b) After applying the pest control, you may see the increase of the pest population due to the absence of the predator. So in the short run, this sort of pesticide can cause an explosion in the pest population.

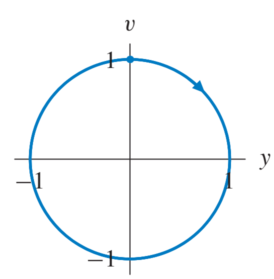
18. One way to consider this type of predator-prey interaction is to raise the growth rate of the prey population. If only weak or sick prey are removed, the remaining population may be assumed to be able to reproduce at a higher rate.

19. **(a)** Substituting $y(t) = \sin t$ into the left-hand side of the differential equation gives

$$\begin{aligned}\frac{d^2y}{dt^2} + y &= \frac{d^2(\sin t)}{dt^2} + \sin t \\ &= -\sin t + \sin t \\ &= 0,\end{aligned}$$

so the left-hand side equals the right-hand side for all t.

(b)



(c) These two solutions trace the same curve in the yv-plane—the unit circle.

(d) The difference in the two solution curves is in how they are parameterized. The solution in this problem is at (0, 1) at time $t = 0$ and hence it lags behind the solution in the section by $\pi/2$. This information cannot be observed solely by looking at the solution curve in the phase plane.

20. **(a)** If we substitute $y(t) = \cos\beta t$ into the left-hand side of the equation, we obtain

$$\begin{aligned}\frac{d^2y}{dt^2} + \frac{k}{m}y &= \frac{d^2(\cos\beta t)}{dt^2} + \frac{k}{m}\cos\beta t\\ &= -\beta^2\cos\beta t + \frac{k}{m}\cos\beta t\\ &= \left(\frac{k}{m} - \beta^2\right)\cos\beta t\end{aligned}$$

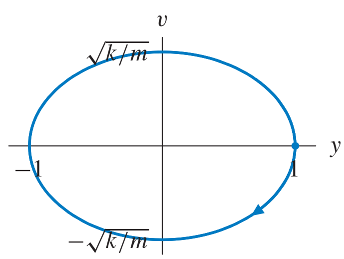
Hence, in order for $y(t) = \cos\beta t$ to be a solution we must have $k/m - \beta^2 = 0$. Thus,

$$\beta = \sqrt{\frac{k}{m}}.$$

(b) Substituting $t = 0$ into $y(t) = \cos\beta t$ and $v(t) = y'(t) = -\beta\sin\beta t$ we obtain the initial conditions $y(0) = 1$, $v(0) = 0$.

(c) The solution is $y(t) = \cos((\sqrt{k/m})t)$ and the period of this function is $2\pi/(\sqrt{k/m})$, which simplifies to $2\pi\sqrt{m}/\sqrt{k}$.

(d)



21. Hooke's law tells us that the restoring force exerted by a spring is linearly proportional to the spring's displacement from its rest position. In this case, the displacement is 3 in. while the restoring force is 12 lbs. Therefore, 12 lbs. $= k \cdot 3$ in. or $k = 4$ lbs. per in. $= 48$ lbs. per ft.

22. **(a)** First, we need to determine the spring constant k. Using Hooke's law, we have 4 lbs $= k \cdot 4$ in. Thus, $k = 1$ lbs/in $= 12$ lbs/ft. We will measure distance in feet since the mass is extended 1 foot.

To determine the mass of a 4 lb object, we use the fact that the force due to gravity is mg where $g = 32$ ft/sec^2. Thus, $m = 4/32 = 1/8$.

Using the model

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0,$$

for the undamped harmonic oscillator, we obtain

$$\frac{d^2y}{dt^2} + 96y = 0, \quad y(0) = 1, \quad y'(0) = 0$$

as our initial-value problem.

(e) As t increases, solutions in the 2nd and 4th quadrants move toward the origin and away from the line $y = -v$. Solutions in the 1st and 3rd quadrants move away from the origin and toward the line $y = v$.

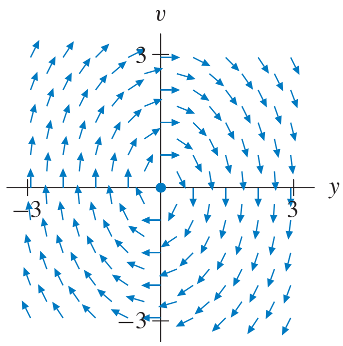
8. **(a)** Let $v = dy/dt$. Then

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = -2y.$$

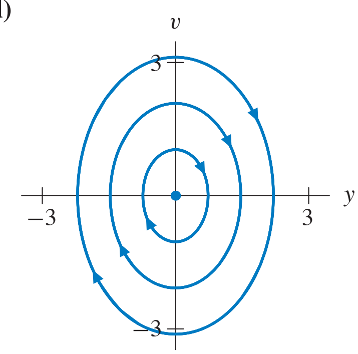
Thus the associated vector field is $\mathbf{V}(y, v) = (v, -2y)$.

(b) See part (c).

(c)

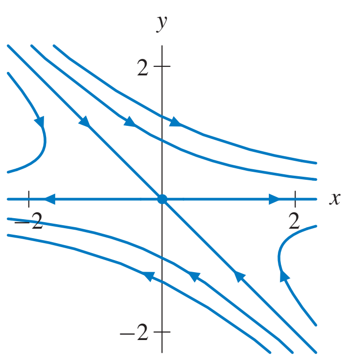


(d)



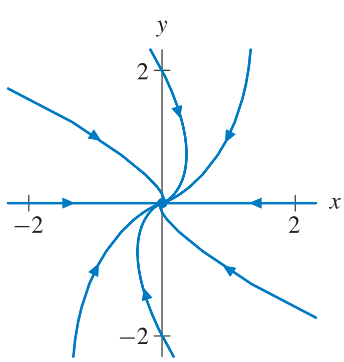
(e) As t increases, solutions move around the origin on ovals in the clockwise direction.

9. **(a)**



(b) The solution tends to the origin along the line $y = -x$ in the xy-phase plane. Therefore both $x(t)$ and $y(t)$ tend to zero as $t \to \infty$.

10. **(a)**



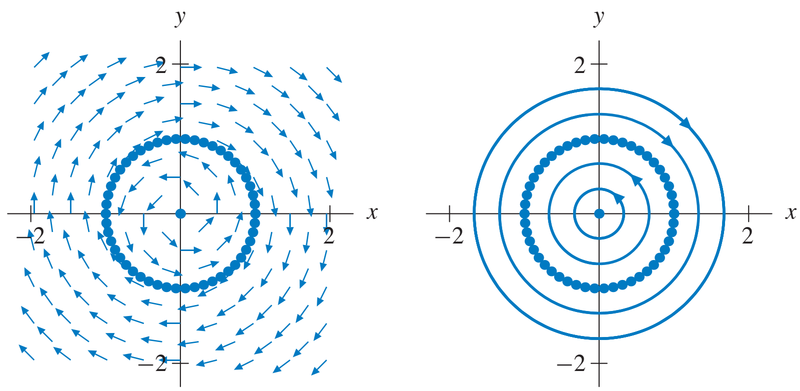
(b) The solution enters the first quadrant and tends to the origin tangent to the positive x-axis. Therefore $x(t)$ initially increases, reaches a maximum value, and then tends to zero as $t \to \infty$. It remains positive for all positive values of t. The function $y(t)$ decreases toward zero as $t \to \infty$.

18. **(a)** To find the equilibrium points, we solve the system of equations

$$\begin{cases} y(x^2 + y^2 - 1) = 0 \\ -x(x^2 + y^2 - 1) = 0. \end{cases}$$

If $x^2 + y^2 = 1$, then both equations are satisfied. Hence, any point on the unit circle centered at the origin is an equilibrium point. If $x^2 + y^2 \neq 1$, then the first equations implies $y = 0$ and the second equation implies $x = 0$. Hence, the origin is the only other equilibrium point.

(b)



(c) As t increases, typical solutions move on a circle around the origin, either counter-clockwise inside the unit circle, which consists entirely of equilibrium points, or clockwise outside the unit circle.

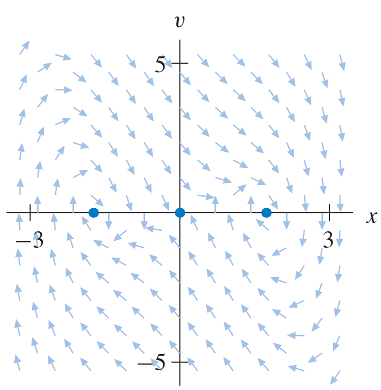
19. **(a)** Let $v = dx/dt$. Then

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = 3x - x^3 - 2v.$$

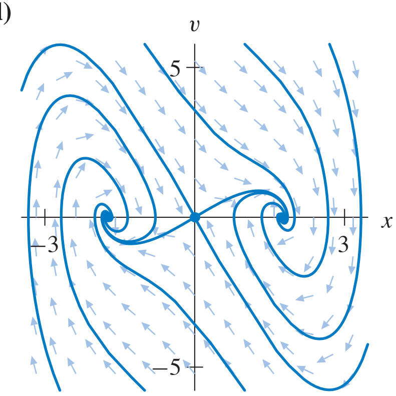
Thus the associated vector field is $\mathbf{V}(x, v) = (v, 3x - x^3 - 2v)$.

(b) Setting $\mathbf{V}(x, v) = (0, 0)$ and solving for (x, v), we get $v = 0$ and $3x - x^3 = 0$. Hence, the equilibria are $(x, v) = (0, 0)$ and $(x, v) = (\pm\sqrt{3}, 0)$.

(c)



(d)



(e) As t increases, almost all solutions spiral to one of the two equilibria $(\pm\sqrt{3}, 0)$. There is a curve of initial conditions that divides these two phenomena. It consists of those initial conditions for which the corresponding solutions tend to the equilibrium point at $(0, 0)$.

EXERCISES FOR SECTION 2.3

1. **(a)** See part (c).

(b) We guess that there are solutions of the form $y(t) = e^{st}$ for some choice of the constant s. To determine these values of s, we substitute $y(t) = e^{st}$ into the left-hand side of the differential equation, obtaining

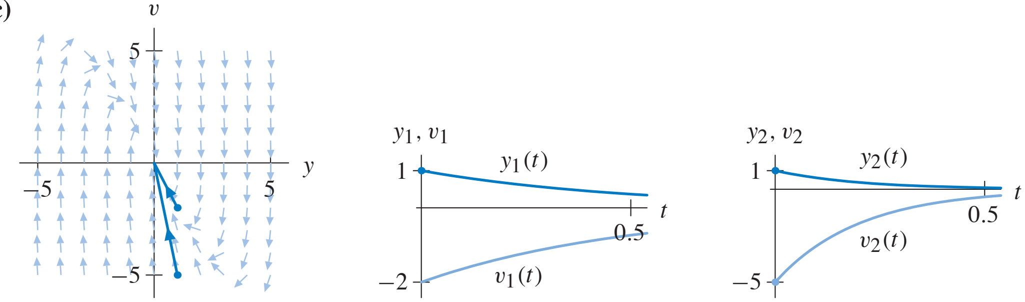
$$\begin{aligned}\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y &= \frac{d^2(e^{st})}{dt^2} + 7\frac{d(e^{st})}{dt} + 10(e^{st}) \\ &= s^2e^{st} + 7se^{st} + 10e^{st} \\ &= (s^2 + 7s + 10)e^{st}\end{aligned}$$

In order for $y(t) = e^{st}$ to be a solution, this expression must be 0 for all t. In other words,

$$s^2 + 7s + 10 = 0.$$

This equation is satisfied only if $s = -2$ or $s = -5$. We obtain two solutions, $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-5t}$, of this equation.

(c)



2. **(a)** See part (c).

(b) We guess that there are solutions of the form $y(t) = e^{st}$ for some choice of the constant s. To determine these values of s, we substitute $y(t) = e^{st}$ into the left-hand side of the differential equation, obtaining

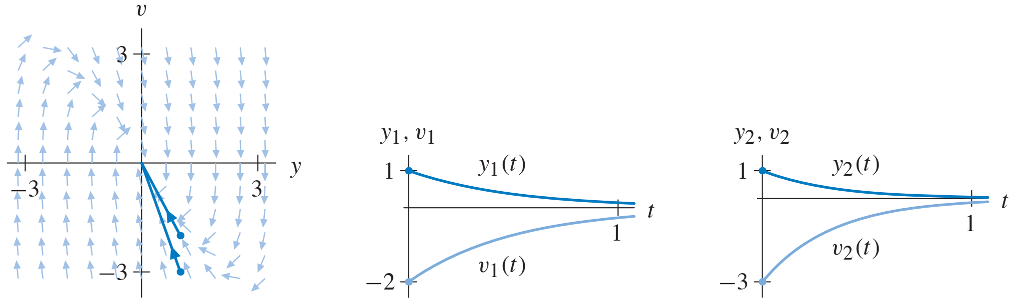
$$\begin{aligned}\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y &= \frac{d^2(e^{st})}{dt^2} + 5\frac{d(e^{st})}{dt} + 6(e^{st}) \\ &= s^2e^{st} + 5se^{st} + 6e^{st} \\ &= (s^2 + 5s + 6)e^{st}\end{aligned}$$

In order for $y(t) = e^{st}$ to be a solution, this expression must be 0 for all t. In other words,

$$s^2 + 5s + 6 = 0.$$

This equation is satisfied only if $s = -3$ or $s = -2$. We obtain two solutions, $y_1(t) = e^{-3t}$ and $y_2(t) = e^{-2t}$, of this equation.

(c)



3. **(a)** See part (c).

(b) We guess that there are solutions of the form $y(t) = e^{st}$ for some choice of the constant s. To determine these values of s, we substitute $y(t) = e^{st}$ into the left-hand side of the differential equation, obtaining

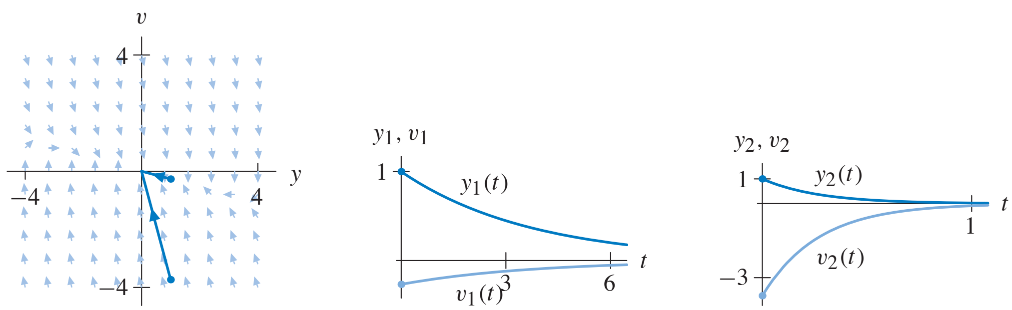
$$\begin{aligned}\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y &= \frac{d^2(e^{st})}{dt^2} + 4\frac{d(e^{st})}{dt} + e^{st} \\ &= s^2e^{st} + 4se^{st} + e^{st} \\ &= (s^2 + 4s + 1)e^{st}\end{aligned}$$

In order for $y(t) = e^{st}$ to be a solution, this expression must be 0 for all t. In other words,

$$s^2 + 4s + 1 = 0.$$

Applying the quadratic formula, we obtain the roots $s = -2 \pm \sqrt{3}$ and the two solutions, $y_1(t) = e^{(-2-\sqrt{3})t}$ and $y_2(t) = e^{(-2+\sqrt{3})t}$, of this equation.

(c)



4. **(a)** See part (c).

(b) We guess that there are solutions of the form $y(t) = e^{st}$ for some choice of the constant s. To determine these values of s, we substitute $y(t) = e^{st}$ into the left-hand side of the differential equation, obtaining

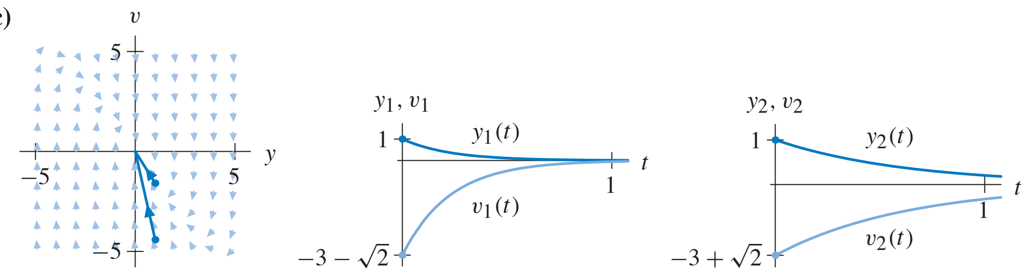
$$\begin{aligned}\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 7y &= \frac{d^2(e^{st})}{dt^2} + 6\frac{d(e^{st})}{dt} + 7e^{st} \\ &= s^2e^{st} + 6se^{st} + 7e^{st} \\ &= (s^2 + 6s + 7)e^{st}\end{aligned}$$

In order for $y(t) = e^{st}$ to be a solution, this expression must be 0 for all t. In other words,

$$s^2 + 6s + 7 = 0.$$

Applying the quadratic formula, we obtain the roots $s = -3 \pm \sqrt{2}$ and the two solutions, $y_1(t) = e^{(-3-\sqrt{2})t}$ and $y_2(t) = e^{(-3+\sqrt{2})t}$, of this equation.

(c)



5. **(a)** See part (c).

(b) We guess that there are solutions of the form $y(t) = e^{st}$ for some choice of the constant s. To determine these values of s, we substitute $y(t) = e^{st}$ into the left-hand side of the differential equation, obtaining

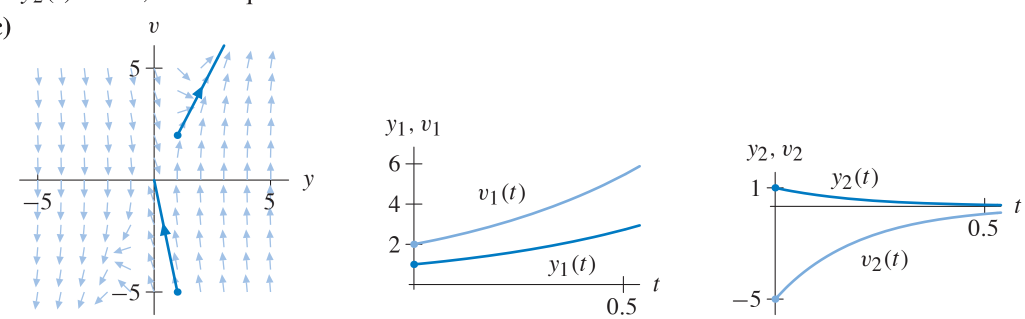
$$\begin{aligned}\frac{d^2y}{dt^2} + 3\frac{dy}{dt} - 10y &= \frac{d^2(e^{st})}{dt^2} + 3\frac{d(e^{st})}{dt} - 10(e^{st}) \\ &= s^2e^{st} + 3se^{st} - 10e^{st} \\ &= (s^2 + 3s - 10)e^{st}\end{aligned}$$

In order for $y(t) = e^{st}$ to be a solution, this expression must be 0 for all t. In other words,

$$s^2 + 3s - 10 = 0.$$

This equation is satisfied only if $s = -5$ or $s = 2$. We obtain two solutions, $y_1(t) = e^{-5t}$ and $y_2(t) = e^{2t}$, of this equation.

(c)



6. **(a)** See part (c).

(b) We guess that there are solutions of the form $y(t) = e^{st}$ for some choice of the constant s. To determine these values of s, we substitute $y(t) = e^{st}$ into the left-hand side of the differential equation, obtaining

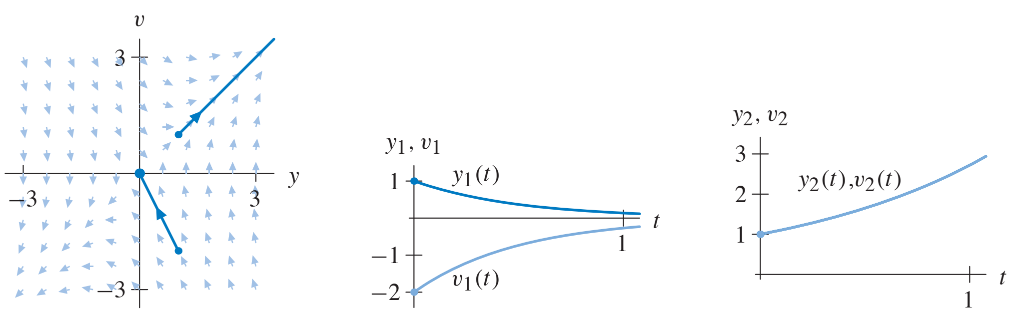
$$\begin{aligned}\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y &= \frac{d^2(e^{st})}{dt^2} + \frac{d(e^{st})}{dt} - 2(e^{st}) \\ &= s^2e^{st} + se^{st} - 2e^{st} \\ &= (s^2 + s - 2)e^{st}\end{aligned}$$

In order for $y(t) = e^{st}$ to be a solution, this expression must be 0 for all t. In other words,

$$s^2 + s - 2 = 0.$$

This equation is satisfied only if $s = -2$ or $s = 1$. We obtain two solutions, $y_1(t) = e^{-2t}$ and $y_2(t) = e^t$, of this equation.

(c)



7. **(a)** Let $y_p(t)$ be any solution of the damped harmonic oscillator equation and $y_g(t) = \alpha y_p(t)$ where α is a constant. We substitute $y_g(t)$ into the left-hand side of the damped harmonic oscillator equation, obtaining

$$\begin{aligned}m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky &= m\frac{d^2y_g}{dt^2} + b\frac{dy_g}{dt} + ky_g \\ &= m\alpha\frac{d^2y_p}{dt^2} + b\alpha\frac{dy_p}{dt} + \alpha k y_p \\ &= \alpha\left(m\frac{d^2y_p}{dt^2} + b\frac{dy_p}{dt} + ky_p\right)\end{aligned}$$

Since $y_p(t)$ is a solution, we know that the expression in the parentheses is zero. Therefore, $y_g(t) = \alpha y_p(t)$ is a solution of the damped harmonic oscillator equation.

(b) Substituting $y(t) = \alpha e^{-t}$ into the left-hand side of the damped harmonic oscillator equation, we obtain

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{d^2(\alpha e^{-t})}{dt^2} + 3\frac{d(\alpha e^{-t})}{dt} + 2(\alpha e^{-t})$$

$$= \alpha e^{-t} - 3\alpha e^{-t} + 2\alpha e^{-t}$$

$$= (\alpha - 3\alpha + 2\alpha)e^{-t}$$

$$= 0.$$

We also get zero if we substitute $y(t) = \alpha e^{-2t}$ into the equation.

(c) If we obtain one nonzero solution to the equation with the guess-and-test method, then we obtain an infinite number of solutions because there are infinitely many constants α.

8. **(a)** Let $y_1(t)$ and $y_2(t)$ be any two solutions of the damped harmonic oscillator equation. We substitute $y_1(t) + y_2(t)$ into the left-hand side of the equation, obtaining

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = m\frac{d^2(y_1 + y_2)}{dt^2} + b\frac{d(y_1 + y_2)}{dt} + k(y_1 + y_2)$$

$$= \left(m\frac{d^2y_1}{dt^2} + b\frac{dy_1}{dt} + ky_1\right) + \left(m\frac{d^2y_2}{dt^2} + b\frac{dy_2}{dt} + ky_2\right)$$

$$= 0 + 0 = 0$$

because $y_1(t)$ and $y_2(t)$ are solutions.

(b) In the section, we saw that $y_1(t) = e^{-t}$ and $y_2(t) = e^{-2t}$ are two solutions to this differential equation. Note that the $y_1(0) + y_2(0) = 2$ and $v_1(0) + v_2(0) = -3$. Consequently, $y(t) = y_1(t) + y_2(t)$, that is, $y(t) = e^{-t} + e^{-2t}$, is the solution of the initial-value problem.

(c) If we combine the result of part (a) of Exercise 7 with the result in part (a) of this exercise, we see that any function of the form

$$y(t) = \alpha e^{-t} + \beta e^{-2t}$$

is a solution if α and β are constants. Evaluating $y(t)$ and $v(t) = y'(t)$ at $t = 0$ yields the two equations

$$\alpha + \beta = 3$$

$$-\alpha - 2\beta = -5.$$

We obtain $\alpha = 1$ and $\beta = 2$. The desired solution is $y(t) = e^{-t} + 2e^{-2t}$.

(d) Given that any constant multiple of a solution yields another solution and that the sum of any two solutions yields another solution, we see that all functions of the form

$$y(t) = \alpha e^{-t} + \beta e^{-2t}$$

where α and β are constants are solutions. Therefore, we obtain an infinite number of solutions to this equation.

9. We choose the left wall to be the position $x = 0$ with $x > 0$ indicating positions to the right. Each spring exerts a force on the mass. If the position of the mass is x, then the left spring is stretched by the amount $x - L_1$. Therefore, the force F_1 exerted by this spring is

$$F_1 = k_1 (L_1 - x).$$