EXERCISES FOR SECTION 2.4

1. To check that $dx/dt = 2x + 2y$, we compute both

$$\frac{dx}{dt} = 2e^t$$

and

$$2x + 2y = 4e^t - 2e^t = 2e^t.$$

To check that $dy/dt = x + 3y$, we compute both

$$\frac{dy}{dt} = -e^t,$$

and

$$x + 3y = 2e^t - 3e^t = -e^t.$$

Both equations are satisfied for all t. Hence $(x(t), y(t))$ is a solution.

2. To check that $dx/dt = 2x + 2y$, we compute both

$$\frac{dx}{dt} = 6e^{2t} + e^t$$

and

$$2x + 2y = 6e^{2t} + 2e^t - 2e^t + 2e^{4t} = 6e^{2t} + 2e^{4t}.$$

Since the results of these two calculations do not agree, the first equation in the system is not satisfied, and $(x(t), y(t))$ is not a solution.

3. To check that $dx/dt = 2x + 2y$, we compute both

$$\frac{dx}{dt} = 2e^t - 4e^{4t}$$

and

$$2x + 2y = 4e^t - 2e^{4t} - 2e^t + 2e^{4t} = 2e^t.$$

Since the results of these two calculations do not agree, the first equation in the system is not satisfied, and $(x(t), y(t))$ is not a solution.

4. To check that $dx/dt = 2x + 2y$, we compute both

$$\frac{dx}{dt} = 4e^t + 4e^{4t}$$

and

$$2x + 2y = 8e^t + 2e^{4t} - 4e^t + 2e^{4t} = 4e^t + 4e^t.$$

To check that $dy/dt = x + 3y$, we compute both

$$\frac{dy}{dt} = -2e^t + 4e^{4t},$$

and

$$x + 3y = 4e^t + e^{4t} - 6e^t + 3e^{4t} = -2e^t + 4e^{4t}.$$

Both equations are satisfied for all t. Hence $(x(t), y(t))$ is a solution.

5. The second equation in the system is $dy/dt = -y$, and from Section 1.1, we know that $y(t)$ must be a function of the form $y_0 e^{-t}$, where y_0 is the initial value.

6. Yes. You can always show that a given function is a solution by verifying the equations directly (as in Exercises 1–4).

To check that $dx/dt = 2x + y$, we compute both

$$\frac{dx}{dt} = 8e^{2t} + e^{-t}$$

and

$$2x + y = 8e^{2t} - 2e^{-t} + 3e^{-t} = 8e^{2t} + e^{t}.$$

To check that $dy/dt = -y$, we compute both

$$\frac{dy}{dt} = -3e^{-t},$$

and

$$-y = -3e^{-t}.$$

Both equations are satisfied for all t. Hence $(x(t), y(t))$ is a solution.

7. From the second equation, we know that $y(t) = k_1 e^{-t}$ for some constant k_1. Using this observation, the first equation in the system can be rewritten as

$$\frac{dx}{dt} = 2x + k_1 e^{-t}.$$

This equation is a first-order linear equation, and we can derive the general solution using the Extended Linearity Principle from Section 1.8 or integrating factors from Section 1.9.

Using the Extended Linearity Principle, we note that the general solution of the associated homogeneous equation is $x_h(t) = k_2 e^{2t}$.

To find one solution to the nonhomogeneous equation, we guess $x_p(t) = \alpha e^{-t}$. Then

$$\begin{aligned}\frac{dx_p}{dt} - 2x_p &= -\alpha e^{-t} - 2\alpha e^{-t} \\ &= -3\alpha e^{-t}.\end{aligned}$$

Therefore, $x_p(t)$ is a solution if $\alpha = -k_1/3$.

The general solution for $x(t)$ is

$$x(t) = k_2 e^{2t} - \frac{k_1}{3} e^{-t}.$$

8. **(a)** No. Given the general solution

$$\left(k_2 e^{2t} - \frac{k_1}{3} e^{-t}, k_1 e^{-t}\right),$$

the function $y(t) = 3e^{-t}$ implies that $k_1 = 3$. But this choice of k_1 implies that the coefficient of e^{-t} in the formula for $x(t)$ is -1 rather than $+1$.

(b) To determine that $\mathbf{Y}(t)$ is not a solution without reference to the general solution, we check the equation $dx/dt = 2x + y$. We compute both

$$\frac{dx}{dt} = -e^{-t}$$

and

$$2x + y = 2e^{-t} + 3e^{-t}.$$

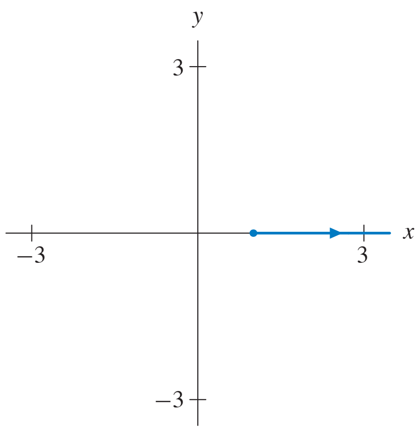
Since these two functions are not equal, $\mathbf{Y}(t)$ is not a solution.

9. **(a)** Given the general solution

$$\left(k_2 e^{2t} - \frac{k_1}{3}e^{-t}, k_1 e^{-t}\right),$$

we see that $k_1 = 0$, and therefore $k_2 = 1$. We obtain $\mathbf{Y}(t) = (x(t), y(t)) = (e^{2t}, 0)$.

(b)



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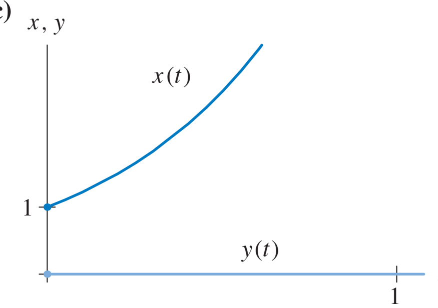
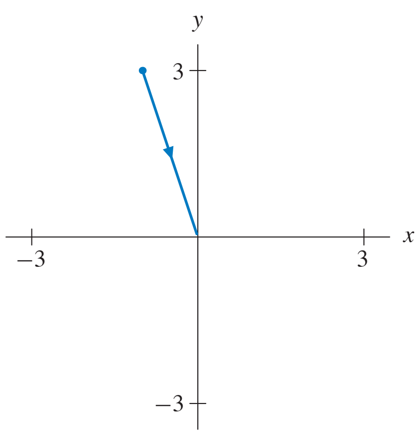


10. **(a)** Given the general solution

$$\left(k_2 e^{2t} - \frac{k_1}{3}e^{-t}, k_1 e^{-t}\right),$$

we see that $k_1 = 3$, and therefore $k_2 = 0$. We obtain $\mathbf{Y}(t) = (x(t), y(t)) = (-e^{-t}, 3e^{-t})$.

(b)



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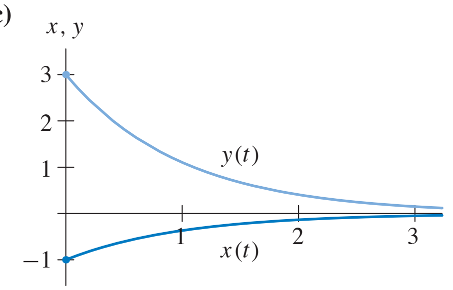


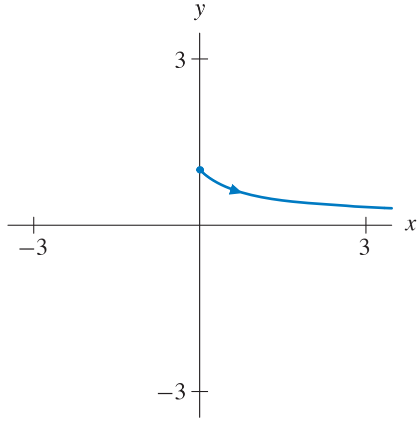
11. **(a)** Given the general solution

$$\left(k_2 e^{2t} - \frac{k_1}{3} e^{-t}, k_1 e^{-t}\right),$$

we see that $k_1 = 1$, and therefore $k_2 = 1/3$. We obtain

$$\mathbf{Y}(t) = (x(t), y(t)) = \left(\tfrac{1}{3} e^{2t} - \tfrac{1}{3} e^{-t}, e^{-t}\right).$$

(b)



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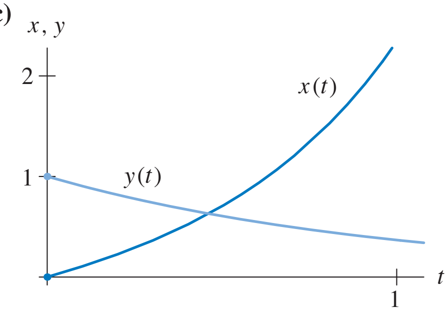


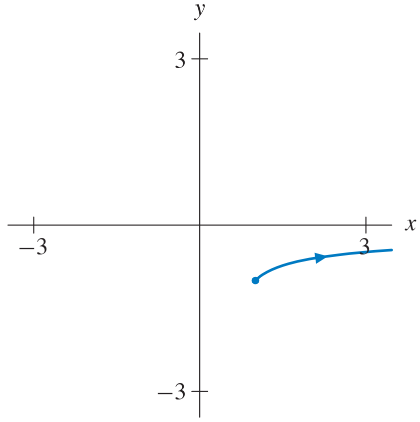
12. **(a)** Given the general solution

$$\left(k_2 e^{2t} - \frac{k_1}{3} e^{-t}, k_1 e^{-t}\right),$$

we see that $k_1 = -1$, and therefore $k_2 = 2/3$. We obtain

$$\mathbf{Y}(t) = (x(t), y(t)) = \left(\tfrac{2}{3} e^{2t} + \tfrac{1}{3} e^{-t}, -e^{-t}\right).$$

(b)



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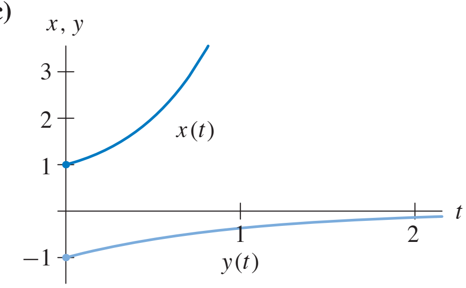

13. **(a)** For this system, we note that the equation for dy/dt is a homogeneous linear equation. Its general solution is

$$y(t) = k_2 e^{-3t}.$$

Substituting $y = k_2 e^{-3t}$ into the equation for dx/dt, we have

$$\begin{aligned}\frac{dx}{dt} &= 2x - 8(k_2 e^{-3t})^2 \\ &= 2x - 8k_2^2 e^{-6t}\end{aligned}$$

This equation is a linear and nonhomogeneous. The general solution of the associated homogeneous equation is $x_h(t) = k_1 e^{2t}$. To find one particular solution of the nonhomogeneous equation, we guess

$$x_p(t) = \alpha e^{-6t}.$$

With this guess, we have

$$\begin{aligned}\frac{dx_p}{dt} - 2x_p &= -6\alpha e^{-6t} - 2\alpha e^{-6t} \\ &= -8\alpha e^{-6t}.\end{aligned}$$

Therefore, $x_p(t)$ is a solution if $\alpha = k_2^2$. The general solution for $x(t)$ is $k_1 e^{2t} + k_2^2 e^{-6t}$, and the general solution for the system is

$$(x(t), y(t)) = (k_1 e^{2t} + k_2^2 e^{-6t}, k_2 e^{-3t}).$$

(b) Setting $dy/dt = 0$, we obtain $y = 0$. From $dx/dt = 2x - 8y^2 = 0$, we see that $x = 0$ as well. Therefore, this system has exactly one equilibrium point, $(x, y) = (0, 0)$.

(c) If $(x(0), y(0)) = (0, 1)$, then $k_2 = 1$. We evaluate the expression for $x(t)$ at $t = 0$ and obtain $k_1 + 1 = 0$. Consequently, $k_1 = -1$, and the solution to the initial-value problem is

$$(x(t), y(t)) = (e^{-6t} - e^{2t}, e^{-3t}).$$

(d)