Math 2410-10/15 **Quiz #3** **February 6^{th}, 2017**

Note: Partial credit can not be awarded unless there is legible work to assess.

1. Suppose we have the differential equation

$$\frac{dy}{dt} = f(t, y).$$

Assume the function f satisfies the hypotheses of the Existence and Uniqueness Theorem in the entire ty-plane. Also assume that $y_1(t) = t+2$ and $y_2(t) = -t^2$ are solutions for all t. What can you conclude about a solution $y(t)$ defined for all t with the initial condition $y(0) = 1$? Specifically, give any bounds on the values of $y(t)$ and discuss it's limiting behavior when t approaches negative infinity and when t approaches positive infinity.

Solution: The simplest conclusion we can make is that such a solution indeed exists. Now, because $y_1(0) = 2$ and $y_2(0) = 0$ we have that

$$0 = y_2(0) < 1 = y(0) < 2 = y_1(0).$$

So, by Uniqueness, $y(t)$ can never equal or exceed $y_1(t)$ as this would imply that $y(t) = y_1(t)$. Similarly, $y(t)$ can never equal or be below $y_2(t)$. Thus, for all t, we have the following bounds on $y(t)$:

$$-t^2 < y(t) < t + 2.$$

Now, with regard to the limiting behavior of $y(t)$ we see this bounds ensure that $y(t) \to -\infty$ as $t \to -\infty$ because both $\lim_{t\to-\infty} y_1(t)$ and $\lim_{t\to-\infty} y_2(t)$ diverge to negative infinity. In the case of the limiting behavior of $y(t)$ as $t \to \infty$, we can say considerably less: in fact, becuase $t + 2 \to \infty$ and $-t^2 \to -\infty$ as $t \to \infty$, we can not conclude anything about the limiting behavior of $y(t)$ as $t \to \infty$. It could diverge or converge to any value.

2. Give the phase line of

$$\frac{dy}{dt} = 3y^3 - 12y^2$$

and sketch possible solutions in the ty-plane relative to the equilibrium solutions.
Bonus: (1 point) Identify any equilibrium points as sinks, sources, or nodes.

Solution: To draw the phase line we first identify equilibrium points and then determine the sign of dy/dt between these. Notice

$$3y^3 - 12y^2 = 3y^2(y-4) = 0$$

if and only if $y = 0$ or $y = 4$. These are our equilibrium points. For $y < 0$, $3y^2(y-4) < 0$. For $0 < y < 4$, $3y^2(y-4) < 0$. For $y > 4$, $3y^2(y-4) > 0$. This information is all we need to draw the phase line (right).

To sketch possible solutions in the ty-plane relative to the equilibrium solutions, we first plot the equilibrium solutions at $y = 0$ and $y = 4$ (red) and then sketch solutions around them ensuring:

- that any solutions with initial condition above 4 (orange) are always increasing away from $y = 4$;
- that any solutions with initial condition between 0 and 4 (cyan) are always decreasing toward $y = 0$ and never equal $y = 4$;
- and that any solutions with initial condition below 0 (green) are always decreasing away from $y = 0$.