Math 2410-10/15 **Quiz #5** **March 6^{th}, 2017**

Note: Partial credit can not be awarded unless there is legible work to assess.

1. Find the general solution of the following system.

$$\frac{dx}{dt} = 3x$$
$$\frac{dy}{dt} = x + y$$

You may report your answer in vector notation $\mathbf{Y}(t)$ or as two functions $x(t)$ and $y(t)$.

Solution: This is a partially decoupled system. Hence, we solve first for $x(t)$ and then use this to solve for $y(t)$. We see $x(t) = k_1e^{3t}$ for some constant k_1 by applying separation of variables to $dx/dt = 3x$. Thus, substituting $x(t)$ into the second differential equation yields

$$\frac{dy}{dt} = k_1e^{3t} + y,$$

a non-homogeneous linear differential equation. To solve this, we can apply the method of undetermined coefficients or use an integrating factor. Let's do both:

Int. factor: Here $a(t) = 1$, so $g(t) = -1$ and $b(t) = k_1e^{3t}$. Since $g(t) = -1$, our integrating factor is $\mu(t) = e^{\int -1\,dt} = e^{-t}$. Thus

$$y(t) = \frac{1}{\mu(t)}\int \mu(t)b(t)\,dt = e^t\int k_1e^{2t}\,dt = e^t\left(\frac{k_1}{2}e^{2t} + k_2\right) = k_2e^t + \frac{k_1}{2}e^{3t}$$

where k_2 is a new arbitrary constant introduced by integration.

Undet. coeff.: To solve

$$\frac{dy}{dt} = k_1e^{3t} + y$$

by undetermined coefficients we first observe that a solution to the associated homogeneous equation is $y_h(t) = e^t$. Next, we seek a function $y_p(t)$ such that

$$\frac{dy_p}{dt} - y_p = k_1e^{3t}.$$

That is, we seek a particular solution y_p to the differential equation. To that end, try $y_p(t) = ae^{3t}$ for some undetermined coefficient a. Note y_p is a particular solution if

$$\frac{dy_p}{dt} - y_p = 3ae^{3t} - ae^{3t} = 2ae^{3t} = k_1e^{3t}.$$

That is, if $a = k_1/2$. Hence $y_p(t) = (k_1/2)e^{3t}$ is our particular solution. Thus, the general solution to the differential equation is

$$y(t) = k_2y_h(t) + y_p(t) = k_2e^t + \frac{k_1}{2}e^{3t}$$

for an arbitrary constant k_2.

In either case we see that $y(t) = k_2 e^t + (k_1/2)e^{3t}$. Reporting our solution as two functions, we have

$$\begin{aligned} x(t) &= k_1 e^{3t} \\ y(t) &= k_2 e^t + (k_1/2)e^{3t} \end{aligned}$$

form the general solution of the system. Reporting our solution in vector notation, we have

$$\mathbf{Y}(t) = \begin{pmatrix} k_1 e^{3t} \\ k_2 e^t + \dfrac{k_1}{2} e^{3t} \end{pmatrix}$$

is the general solution to the system.

2. Considering the following second-order differential equation.

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} - 7x = 0 \tag{1}$$

(i) Convert (1) into a system of first-order differential equations by letting $\dfrac{dx}{dt} = y$.

(ii) Give two solutions to (1).

Solution: To solve (i), if $dx/dt = y$, then

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} - 7x = \frac{dy}{dt} + 6y - 7x = 0.$$

Thus, solving for dy/dt yields the following first-order system:

$$\begin{aligned} \frac{dx}{dt} &= y \\ \frac{dy}{dt} &= 7x - 6y \end{aligned}$$

To solve (ii), we use "the method of the lucky guess" by finding values of s such that $x(t) = e^{st}$ solves (1). Note e^{st} is a solution if

$$\begin{aligned} \frac{d^2x}{dt^2} + 6\frac{dx}{dt} - 7x &= \frac{d^2}{dt^2}e^{st} + 6\frac{d}{dt}e^{st} - 7e^{st} \\ &= s^2 e^{st} + 6se^{st} - 7e^{st} \\ &= (s^2 + 6s - 7)e^{st} \\ &= (s-1)(s+7)e^{st} = 0. \end{aligned}$$

And as $e^{st} > 0$ for all s and t, this implies $(s-1)(s+7) = 0$. Thus if $s = 1$ or $s = -7$, then e^{st} is a solution. We conclude $x_1(t) = e^t$ and $x_2(t) = e^{-7t}$ are two solutions to (1).