Homework 4 - Design and Analysis of Algorithms - Naiara Alonso Montes

Problem 1

Original problem

$$\min 2x_1 - 3x_2 + 4x_3 - x_4$$

$$\begin{aligned} s.t. \quad x_1 + 2x_2 - x_4 &\leq 3 \\ 2x_1 + x_2 - 3x_3 &\geq -1 \\ x_1 - x_2 + 3x_3 + 3x_4 &= 2 \\ x_1 &\geq 0 \\ x_4 &\geq 0 \end{aligned}$$

Process

1. From min to max

$$\max -2x_1 + 3x_2 - 4x_3 + x_4$$

$$\begin{aligned} s.t. \quad x_1 + 2x_2 - x_4 &\leq 3 \\ 2x_1 + x_2 - 3x_3 &\geq -1 \\ x_1 - x_2 + 3x_3 + 3x_4 &= 2 \\ x_1 &\geq 0 \\ x_4 &\geq 0 \end{aligned}$$

2. From equality to inequality

$$\max -2x_1 + 3x_2 - 4x_3 + x_4$$

$$\begin{aligned} s.t. \quad x_1 + 2x_2 - x_4 &\leq 3 \\ 2x_1 + x_2 - 3x_3 &\geq -1 \\ x_1 - x_2 + 3x_3 + 3x_4 &\leq 2 \\ x_1 - x_2 + 3x_3 + 3x_4 &\geq 2 \\ x_1 &\geq 0 \\ x_4 &\geq 0 \end{aligned}$$

3. From *greater than* to *lower than*

$$\max -2x_1 + 3x_2 - 4x_3 + x_4$$

$$\begin{aligned} s.t. \quad x_1 + 2x_2 - x_4 &\leq 3 \\ -2x_1 - x_2 + 3x_3 &\leq 1 \\ x_1 - x_2 + 3x_3 + 3x_4 &\leq 2 \\ -x_1 + x_2 - 3x_3 - 3x_4 &\leq 2 \\ x_1 &\geq 0 \\ x_4 &\geq 0 \end{aligned}$$

4. From *unbounded* to *bounded*

$$x_1;\ x_2 = z_2 - z_3;\ x_4 = z_4 - z_5;\ x_4 = z_6$$

$$\max -2z_1 + 3z_2 - 3z_3 - 4z_4 + 4z_5 + z_6$$

$$\begin{aligned} s.t. \quad z_1 + 2z_2 - 2z_3 - z_6 &\le 3 \\ -2z_1 - z_2 + z_3 + 3z_4 - 3z_5 &\le 1 \\ z_1 - z_2 + z_3 + 3z_4 - 3z_5 + 3z_6 &\le 2 \\ -z_1 + z_2 - z_3 - 3z_4 + 3z_5 - 3z_6 &\le 2 \\ z_1 &\ge 0 \\ z_2 &\ge 0 \\ z_3 &\ge 0 \\ z_4 &\ge 0 \\ z_5 &\ge 0 \\ z_6 &\ge 0 \end{aligned}$$

5. In matrix representation

$$b^T = \begin{pmatrix} -2 & 3 & -3 & -4 & 4 & 1 \end{pmatrix} \tag{1}$$

$$b^T = \begin{pmatrix} 3 & 1 & 2 & -2 \end{pmatrix} \tag{2}$$

$$A = \begin{pmatrix} 1 & 2 & -2 & 0 & 0 & -1 \\ -2 & -1 & 1 & 3 & -3 & 0 \\ 1 & -1 & 1 & 3 & -3 & 3 \\ -1 & 1 & -1 & -3 & 3 & -3 \end{pmatrix} \tag{3}$$

6. Dual problem

$$\min 3y_1 + y_2 + 2y_3 - 2y_4$$

$$\begin{aligned} s.t. \quad y_1 - 2y_2 + y_2 - y_4 &\ge -2 \\ 2y_1 - y_2 - y_3 + y_4 &\ge 3 \\ -2y_1 + y_2 + y_3 - y_4 &\ge -3 \\ 3y_2 + 3y_3 - 3y_4 &\ge -4 \\ -3y_2 - 3y_3 + 3y_4 &\ge 4 \\ -y_1 + 3y_3 - 3y_4 &\ge 1 \\ y_1 &\ge 0 \\ y_2 &\ge 0 \\ y_3 &\ge 0 \\ y_4 &\ge 0 \end{aligned}$$

Problem 2

(a)

$$
\begin{aligned}
& \max f_{s,a} + f_{s,b} - f_{c,s} \\
s.t. \quad & -f_{s,a} + f_{a,c} + f_{a,t} \le 0 \\
& f_{s,a} - f_{a,c} - f_{a,t} \le 0 \\
& -f_{s,b} + f_{b,c} + f_{b,t} \le 0 \\
& f_{s,b} - f_{b,c} - f_{b,t} \le 0 \\
& -f_{a,c} - f_{b,c} + f_{c,s} + f_{c,t} \le 0 \\
& f_{a,c} + f_{b,c} - f_{c,s} - f_{c,t} \le 0 \\
& f_{s,a} \le 8 \\
& f_{s,b} \le 7 \\
& f_{a,c} \le 4 \\
& f_{b,c} \le 3 \\
& f_{c,s} \le 2 \\
& f_{c,t} \le 2 \\
& f_{a,t} \le 3 \\
& f_{b,t} \le 5 \\
& f_{s,a} \ge 0 \\
& f_{s,b} \ge 0 \\
& f_{a,c} \ge 0 \\
& f_{b,c} \ge 0 \\
& f_{c,s} \ge 0 \\
& f_{c,t} \ge 0 \\
& f_{a,t} \ge 0 \\
& f_{b,t} \ge 0
\end{aligned}
$$

$$c^T = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

$$b^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 8 & 7 & 4 & 3 & 2 & 2 & 3 & 5 \end{pmatrix} \tag{5}$$

$$A = \begin{pmatrix}
-1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \tag{6}$$

(b)

$$
\begin{aligned}
\min 8y_7 + 7y_8 + 4y_9 + 3y_{10} + 2y_{11} + 2y_{12} + 3_{13} + 5y_{14} \\
s.t. \quad -y_1 + y_2 + y_7 &\geq 1 \\
-y_3 + y_4 + y_8 &\geq 1 \\
y_1 - y_2 - y_5 + y_6 + y_9 &\geq 0 \\
y_3 - y_4 - y_5 + y_6 + y_{10} &\geq 0 \\
y_5 - y_6 + y_{11} &\geq -1 \\
y_5 - y_6 + y_{12} &\geq 0 \\
y_1 - y_2 + y_{13} &\geq 0 \\
y_3 - y_4 + y_{14} &\geq 0 \\
y_1 &\geq 0 \\
y_2 &\geq 0 \\
y_3 &\geq 0 \\
y_4 &\geq 0 \\
y_5 &\geq 0 \\
y_6 &\geq 0 \\
y_7 &\geq 0 \\
y_8 &\geq 0 \\
y_9 &\geq 0 \\
y_{10} &\geq 0 \\
y_{11} &\geq 0 \\
y_{12} &\geq 0 \\
y_{13} &\geq 0 \\
y_{14} &\geq 0
\end{aligned}
$$

(c)

The dual of the general form is the LP for the min cut problem. Using paths we can create the general form. In this LP problem, E is the set of edges and V is the set of vertex.

- Definition of variables

$$y_e = \{\{u, v\}, \ldots, \{n, m\}\} \in E$$

$$x_u = \{v_a, v_b, \ldots, v_x\} \in V$$

$$x_v = \{v_a, v_b, \ldots, v_x\} \in V$$

$$x_u \neq x_v$$

$$P \text{ is the set of paths from } s \text{ to } t$$

In this case, y_e will be equal to 1 if the edge is in a given cut, else if 0.

$$
\begin{aligned}
\min \sum\nolimits_{\{x_u, x_v\}} c(\{x_u, x_v\}) \cdot y_e(\{x_u, x_v\}) \\
s.t. \quad \sum_{\{x_u, x_v\}} y_e(\{x_u, x_v\}) \geq 1 \text{ for all paths in } P \text{ from } s \text{ to } t \\
y_e(\{x_u, x_v\}) \in [0, 1]
\end{aligned}
$$

The result of the LP problem will provide the value of the cut.

(d)

By the constraint from the LP problem of the min-cut provided in the previous exercise, if an edge in a path belongs to the a given cut, the result of the constraint must be at least equal to 1, as all edges y_e were asigned with value 1.

Problem 3

(a) and (b)

$$x_1 = \text{type I};\ x_1 = \text{type II}$$

$$\max 100x_1 + 120x_2$$

$$\begin{aligned} s.t. \quad 4x_1 + 6x_2 &\leq 90 \\ 3x_1 + 3x_2 &\leq 60 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

I choose to maximize x_1

$$x_1 = 0;\ x_2 = 0$$

$$y_1 = ?;\ y_2 = x_2$$

$$y_1 = 60 - 3x_1 + 3y_2$$

$$x_1 = -\frac{1}{3}y_1 + 20 - y_2;\ x_2 = y_2$$

$$100 \cdot (-\frac{1}{3}y_1 + 20 - y_2) + 120y_2 = -\frac{100}{3}y_1 + 2000 - 100y_2 + 120y_2 = -\frac{100}{3}y_1 +$$

$$-\frac{4}{3}y_1 + 80 - 4y_2 + 6y_2 \leq 90 = -\frac{4}{3}y_1 + 2y_2 \leq 10$$

$$\begin{aligned} \max_{y_1, y_2} \quad & 1 - \frac{100}{3}y_1 + 20y_2 + 2000 \\ s.t. \quad & -\frac{4}{3}y_1 + 2y_2 \leq 10 \\ & -y_1 + 6y_2 \leq 0 \\ & -\frac{1}{3}y_1 - y_2 \geq -20 \\ & y_2 \geq 0 \end{aligned}$$

I choose to maximize y_2

$$y_1 = 0;\ y_2 = 20$$

$$z_1 = y_1;\ z_2 = 10 + \frac{4}{3}y_1 - 2y_2$$

$$y_1 = z_1;\ y_2 = 5 + \frac{2}{3}z_1 - \frac{1}{2}z_2$$

$$-\frac{100}{3}z_1 + 20 \cdot (5 + \frac{2}{3}z_1 - \frac{1}{2}z_2) + 2000 = -20z_1 - 10z_2 + 100$$

It is not possible to maximize anymore

$$z_1 = 1;\ z_2 = 0$$

$$y_1 = z_1 = 0;\ y_2 = 5 + \frac{2}{3} \cdot 0 - \frac{1}{2} \cdot 0 = 5$$

$$x_1 = -\frac{1}{3} \cdot 0 + 20 - 5 = 15;\ x_2 = y_2 = 5$$

Solution

$x_1 = 15$ and $x_2 = 5$. The constrains are met.

(c) and (d)

The dual repesentation is:

$$\min 90y_1 + 60y_2$$

$$\begin{aligned} s.t. \quad 4y_1 + 3y_2 &\geq 100 \\ 6y_1 + 3y_2 &\geq 120 \\ y_1 &\geq 0 \\ y_2 &\geq 0 \end{aligned}$$

It can be solved using a system of 2 linear equations, and then checking if the constraints are met, in this case the result of the min objective function must be equal to the result of the max function.

$$\begin{aligned} &4y_1 + 3y_2 = 100 \\ &6y_1 + 3y_2 = 120 \\ &y_2 = 40 - 2y_1 \\ &4y_1 + 3(40 - 2y_1) = 100 \\ &y_1 = 10 \\ &y_2 = 40 - 2(10) \\ &y_2 = 20 \end{aligned}$$

Both min and max results match.

In [ ]:

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!jupyter nbconvert --to html HW4.ipynb
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