HW5

December 23, 2024

1 Homework 5 - Naiara Alonso Montes

1.1 Problem 1

1.1.1 How many errors can correct this code?

$n = 15$, $k = 9$, $q = 16$

$d - 1 = n - k \Rightarrow d = 15 - 9 + 1 = 7$

$d = 2t + 1 \Rightarrow t = \frac{d-1}{2} = \frac{7-1}{2} = 3$

This code can correct 3 errors

1.1.2 Find error positions

$s_1 = \alpha^{13}$

$s_2 = \alpha^4$

$s_3 = \alpha^8$

$s_4 = \alpha^2$

$s_5 = \alpha^3$

$s_6 = \alpha^8$

$s(x) = \alpha^{13} + \alpha^4 x + \alpha^8 x^2 + \alpha^2 x^3 + \alpha^3 x^4 + \alpha^8 x^5$

$\Lambda(x) = 1 + \alpha^3 x + \alpha^{11} x^2 + \alpha^9 x^3$

Error positions: $x_1 = \alpha$, $x_2 = \alpha^7$, $x_3 = \alpha^{13}$

1.1.3 Find the error magnitude polynomial

$\Omega(x) = s(x)\Lambda(x) = \alpha^{13} + x + \alpha^2 x^2 + \alpha^{14} x^6 + \alpha^6 x^7 + \alpha^2 x^8 \bmod x^6 = \alpha^{13} + x + \alpha^{14} + \alpha^6 x = \alpha^2 + \alpha^6 x$

1.1.4 Find the error values by Forney's algorithm

$y_i = \frac{x_i^{-1}\Omega(x_i^{-1})}{\Pi_{j\neq i}(1 - x_j x_i^{-1})}$

$y_1 = \frac{\alpha^{14}\cdot(\alpha^2+\alpha^5)}{1+\alpha^7\cdot\alpha^{13}\alpha^{14}} = \frac{\alpha^0}{\alpha} = \alpha^{14}$

$y_2 = \frac{\alpha^8\cdot(\alpha^2+\alpha^{14})}{1+\alpha\cdot\alpha^{13}\alpha^8} = \frac{\alpha^{13}}{\alpha^9} = \alpha^4$

$y_3 = \frac{\alpha^2 \cdot (\alpha^2 + \alpha^8)}{1 + \alpha \cdot \alpha^7 \alpha^2} = \frac{\alpha^2}{\alpha^5} = \alpha^{12}$

1.2 Problem 2

1.2.1 Encode the message 1 0 1 1 1 1 0 0 0 1 1 1 by the product code consisting of $[5, 4, 2]$ code C_1 and $[4, 3, 2]$ code C_2.

Display in a matrix based on C_1 and C_2 parameters.

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \tag{1}$$

For each row, encode using C_1 generator matrix G_1:

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{2}$$

- Row 1: 1011 -> G_1 Row 1 = 10111
- Row 2: 1100 -> G_1 Row 2 = 11000
- Row 3: 0111 -> G_1 Row 3 = 01111

Encoded message: 10111100001111

1.2.2 What are the paramenters of the product code?

$n = 20$, $k = 12$, $d \geq 4$

1.2.3 Write down a generator matrix of the product code

$$G_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{3}$$

$$G_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{4}$$

$$G_3 = \begin{pmatrix} \begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&0\end{pmatrix} \\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&0\end{pmatrix} \\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \\ \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} & \begin{pmatrix}1&0&0&1\\0&1&0&1\\0&0&1&0\end{pmatrix} & \begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \end{pmatrix} \tag{5}$$

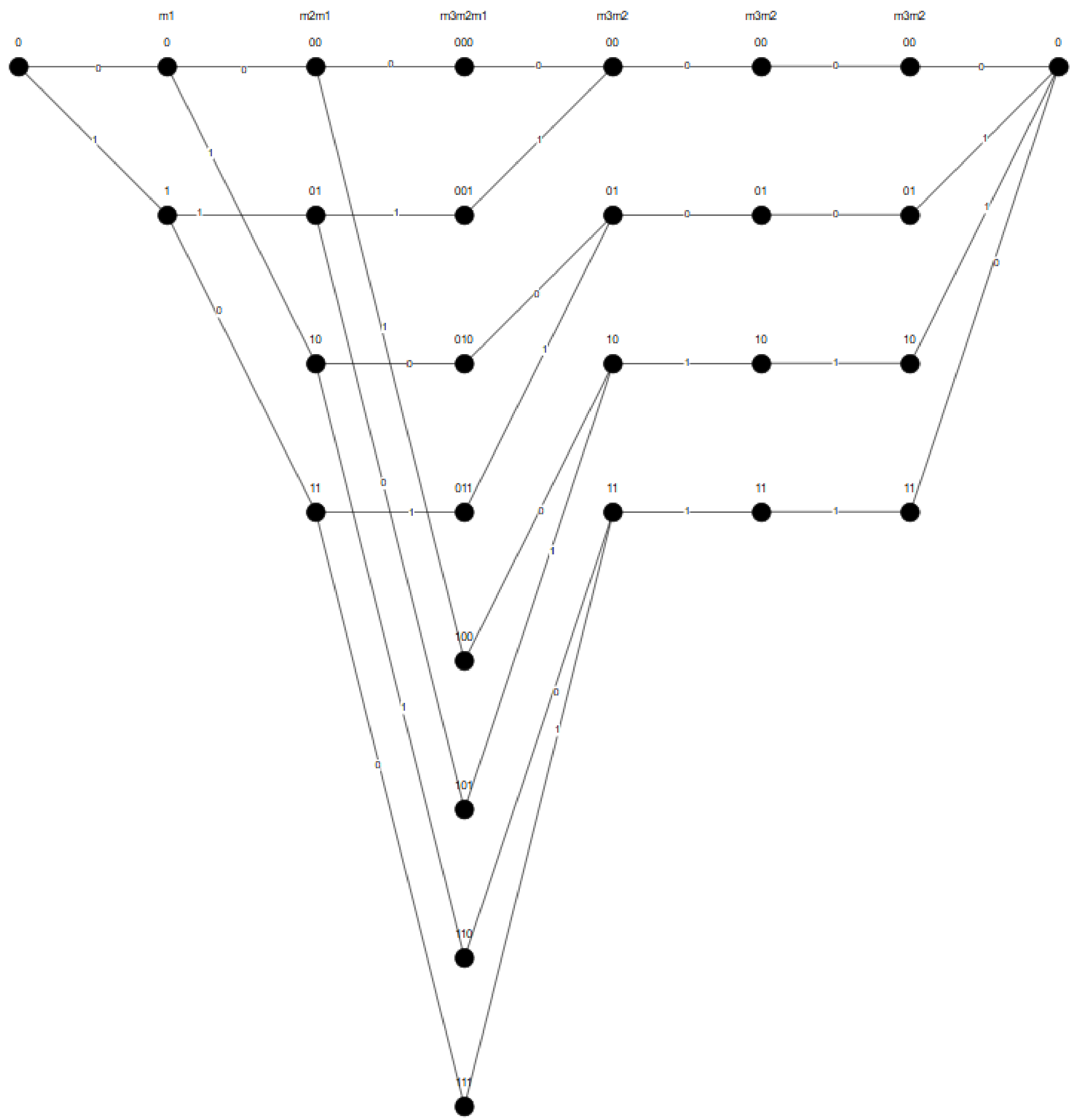
1.3 Problem 3

I tried to do this exercise a hundred million times but I still don't know how to get the Trellis from either generator matrix or parity check matrix. The only thing I did was going from generator to parity check matrix, and then I got stuck there.

1.3.1 Construct the minimal code trellis.

$$\begin{pmatrix}1&1&1&1&0&0&0\\0&1&0&1&1&0&1\\0&1&1&0&1&1&0\end{pmatrix} \equiv \begin{pmatrix}1&1&1&1&0&0&0\\0&1&0&1&1&0&1\\0&0&1&1&0&1&1\end{pmatrix} \tag{6}$$

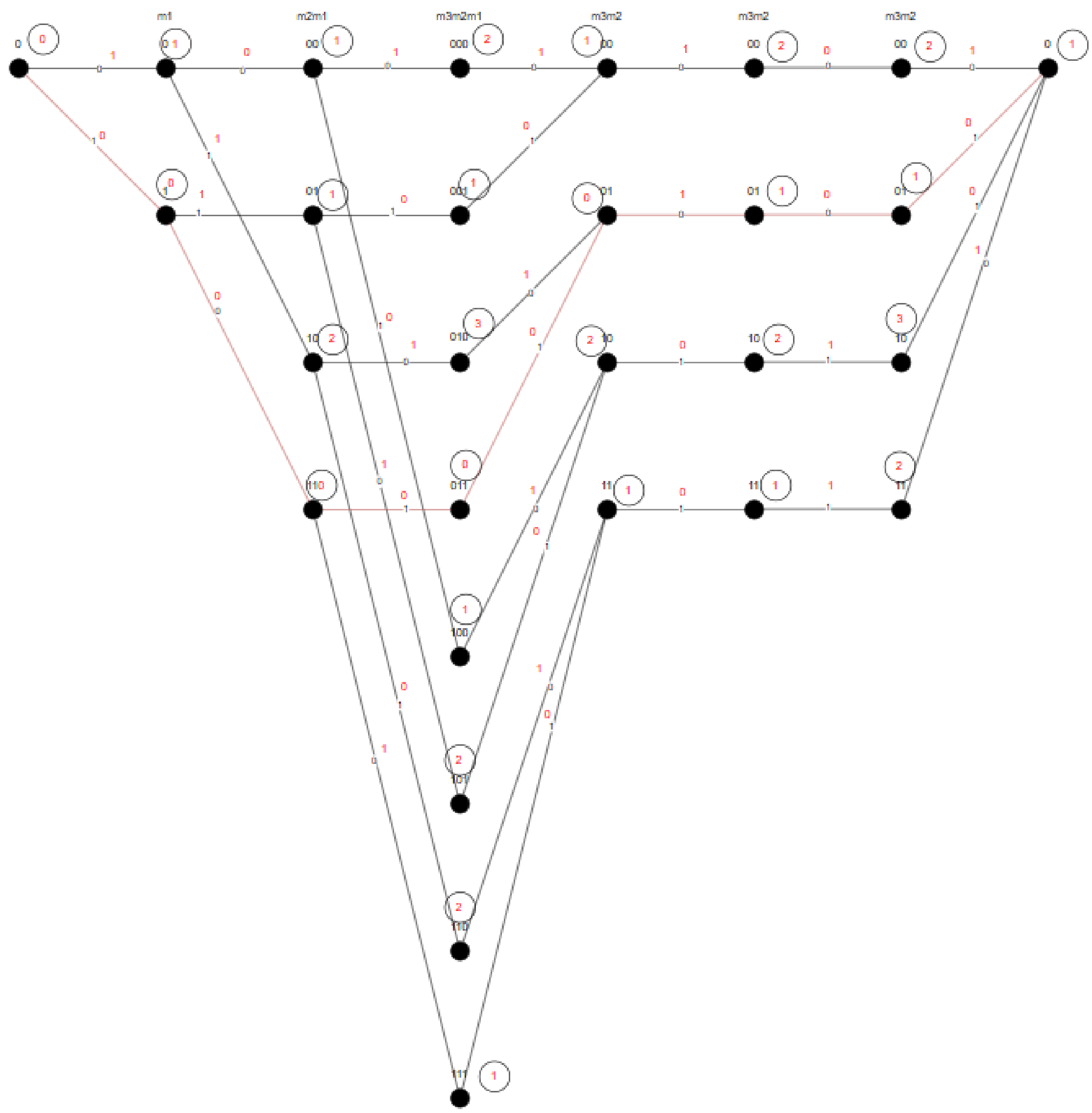
- Active row 1, span 3
- Active row 2, span 5
- Active row 3, span 4



1.3.2 What is the state complexity of the constructed trellis?

$min = \{2^k, 2^{n-k}\} = \{2^3, 2^{7-3}\} = 2^3 = 8$

1.3.3 Decode the received vector $y = (1011101)$ by using the Viterby algorithm.



The decoded codeword is: 1011001

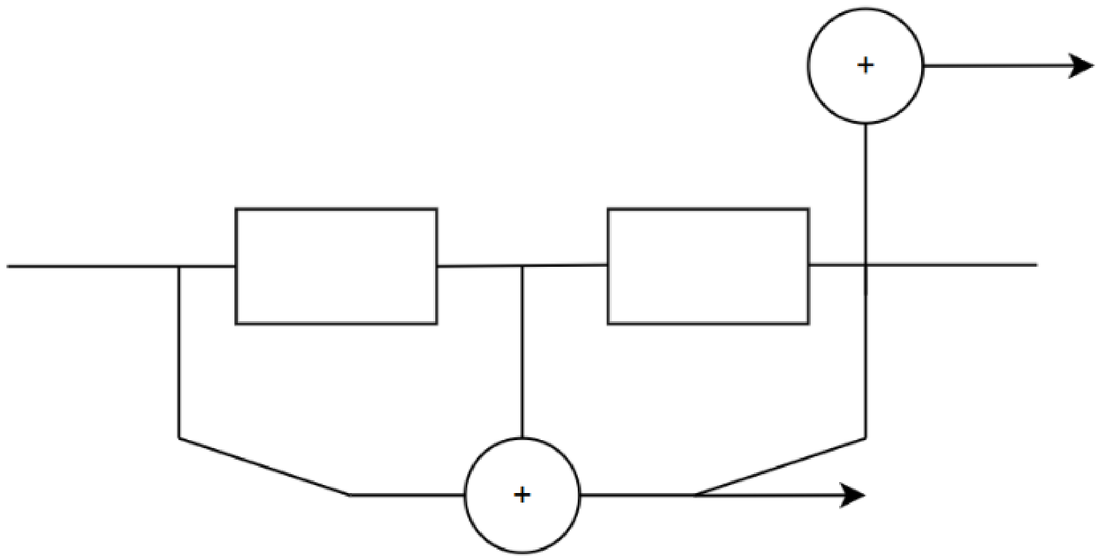
1.4 Problem 4

1.4.1 Write down a generator matrix of the code in polynomial and binary forms.

$$G(D) = \begin{pmatrix} D^2 & 1 + D + D^2 \end{pmatrix} \tag{7}$$

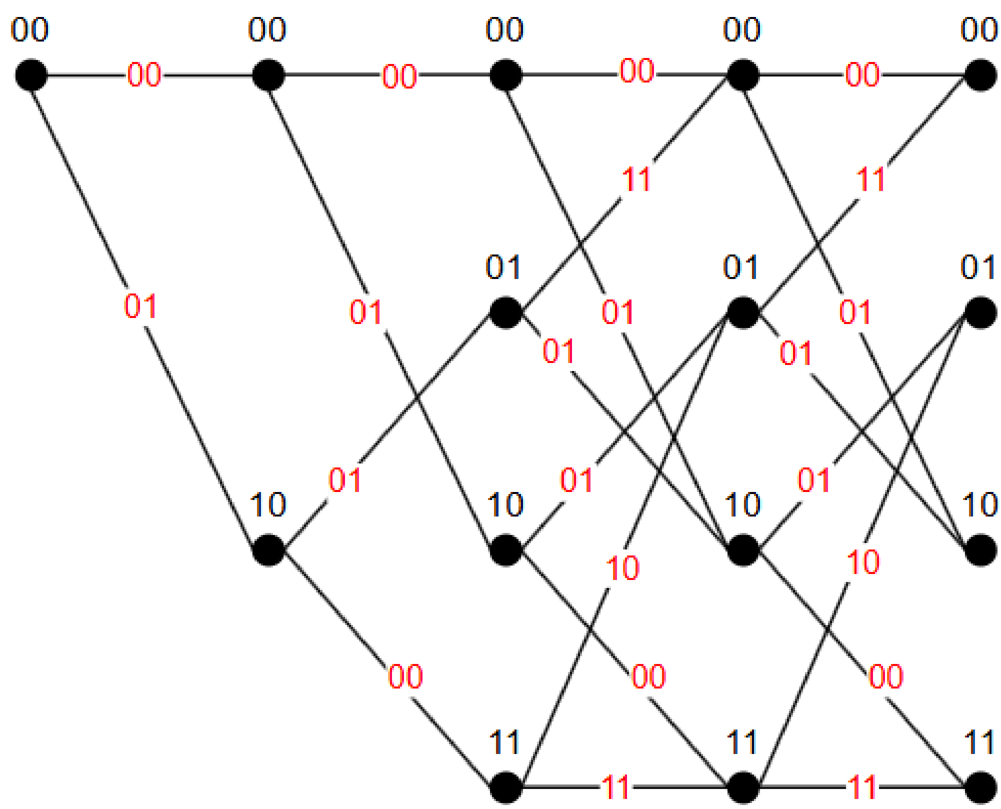
$$G(D) = \begin{pmatrix} (01) & (01) & (11) & ... & ... & ... & (00) & ... \\ (00) & (01) & (01) & (11) & ... & ... & (00) & ... \\ (00) & (00) & (01) & (01) & (11) & ... & (00) & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \\ ... & ... & ... & ... & ... & ... & ... & ... \end{pmatrix} \tag{8}$$

1.4.2 Draw the encoder scheme



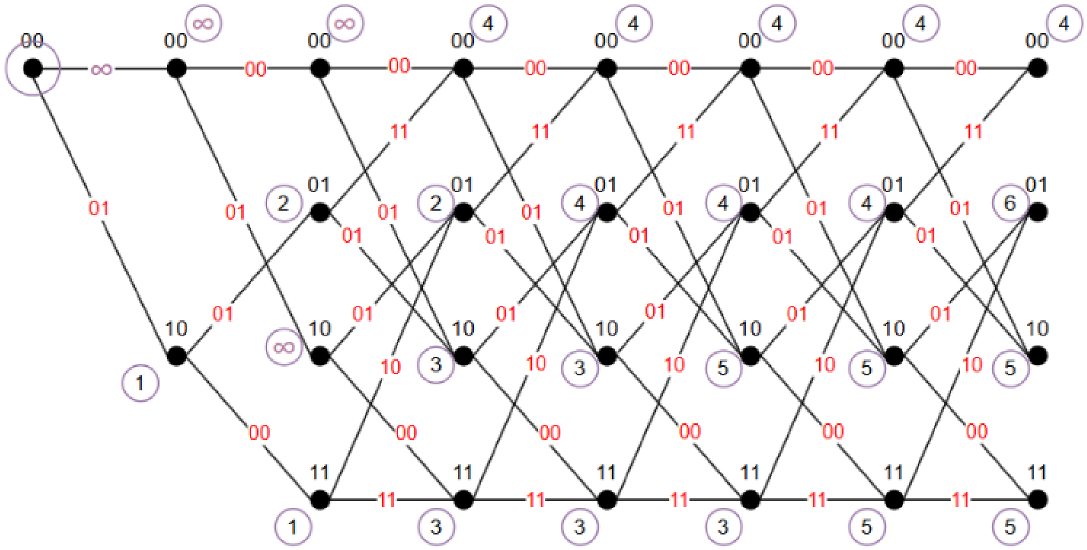
1.4.3 Draw the trellis diagram

Based on this, I constructed the trellis diagram:



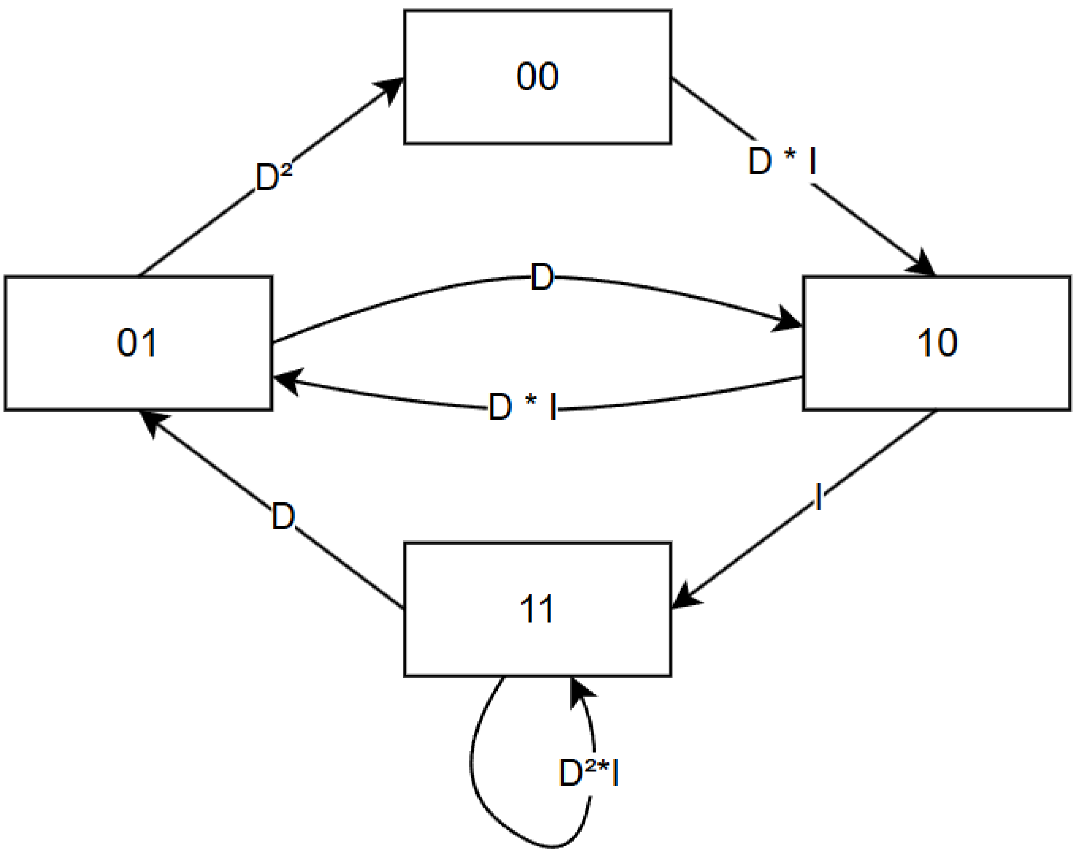
The diagram continues...

1.4.4 Find the free distance of the code by using the Viterby algorithm.



1.5 Problem 5

1.5.1 For the convolutional code determined by the generator polynomials $(4,7)$ (octal form) find weight enumerator $T(D)$. Determine free distance of the code by using $T(D)$. Compare with free distance found in Problem 4



$g_0 = g_1(D)D^2$

$g_1 = g_2(D)D + g_3(D)D$

$g_2 = D + g_1(D)D$

$g_3 = g_2(D) + g_3(D)D^2$

$g_1 = D^2 + g_1 D^2 + g_3 D$

$g_3 = D + g_1 D + g_3 D^2$

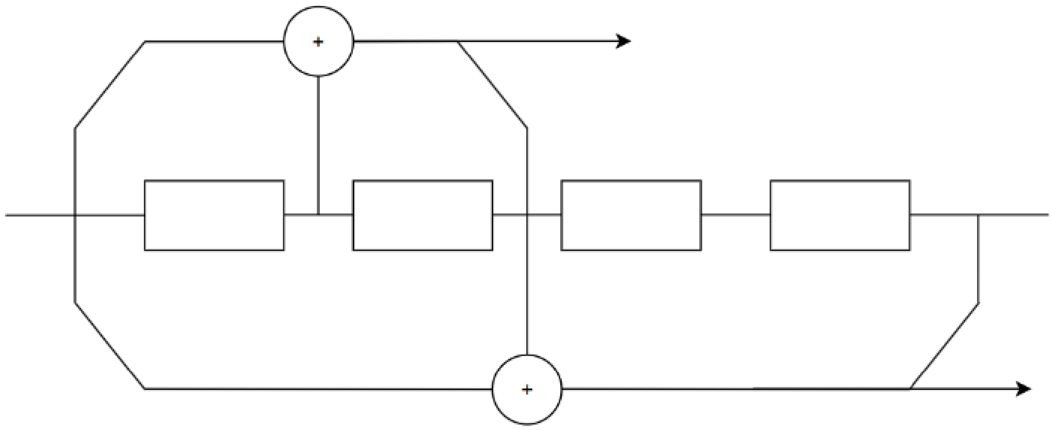
$g_1(1 - D^2) - g_3 D = D^2$

$-g_1 + g_3(1 - D^2) = D$

$g_1(D) = \frac{D^2 - D^4 - D}{1 + D^4 - D^2}$

$g_0(D) = \frac{D^4 - D^6 - D^3}{1 + D^4 - D2} = T(D)$

1.5.2 Check if the polynomials $(1+D+D^2)$, $(1+D^2+D^4)$ define a catastrophic encoder of a rate $1/2$ concolutional code. Explain your answer.



$GCD((1+D+D^2),(1+D^2+D^4)) = 1$

$$
\begin{array}{l|l}
\cancel{D^4} + \cancel{D^2} + 1 & D^2 + D + 1 \\
\hline
+\cancel{D^4}\ D^3 + \cancel{D^2} & D^2 + D + 1 \\
\quad \cancel{D^3} + 1 & \\
\cancel{D^3} + D^2 + D & \\
\quad \cancel{D^2} + \cancel{D} + \cancel{1} & \\
\quad \cancel{D^2} + \cancel{D} + \cancel{1} & \\
\qquad 0 &
\end{array}
$$

As the GCD is 1, the encoder cannot be catastrophic.

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[12]: !jupyter nbconvert --to pdf --embed-images HW5.ipynb
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[NbConvertApp] Converting notebook HW5.ipynb to pdf
[NbConvertApp] Writing 30400 bytes to notebook.tex
[NbConvertApp] Building PDF
[NbConvertApp] Running xelatex 3 times: ['xelatex', 'notebook.tex', '-quiet']
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[NbConvertApp] Running bibtex 1 time: ['bibtex', 'notebook']
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