Ray tracing with bilinear interpolation

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March 2024

We can approximate the slowness $s(x,z)$ at any arbitrary position within a pixel using bilinear interpolation as

$$s(x,z) \approx \omega_{11}s(x_1,z_1) + \omega_{12}s(x_1,z_2) + \omega_{21}s(x_2,z_1) + \omega_{22}s(x_2,z_2) = \sum_{i,j=1}^{2} \omega_{ij}s_{ij}, \tag{1}$$

with

$$\begin{aligned} \omega_{11} &= \tfrac{(x_2-x)(z_2-z)}{\Delta x \Delta z} \\ \omega_{12} &= \tfrac{(x_2-x)(z-z_1)}{\Delta x \Delta z} \\ \omega_{21} &= \tfrac{(x-x_1)(z_2-z)}{\Delta x \Delta z} \\ \omega_{22} &= \tfrac{(x-x_1)(z-z_1)}{\Delta x \Delta z}, \end{aligned} \tag{2}$$

where the subscripts 1 or 2 refer to the positions of the vertices of a pixel and are defined in Fig. 1.

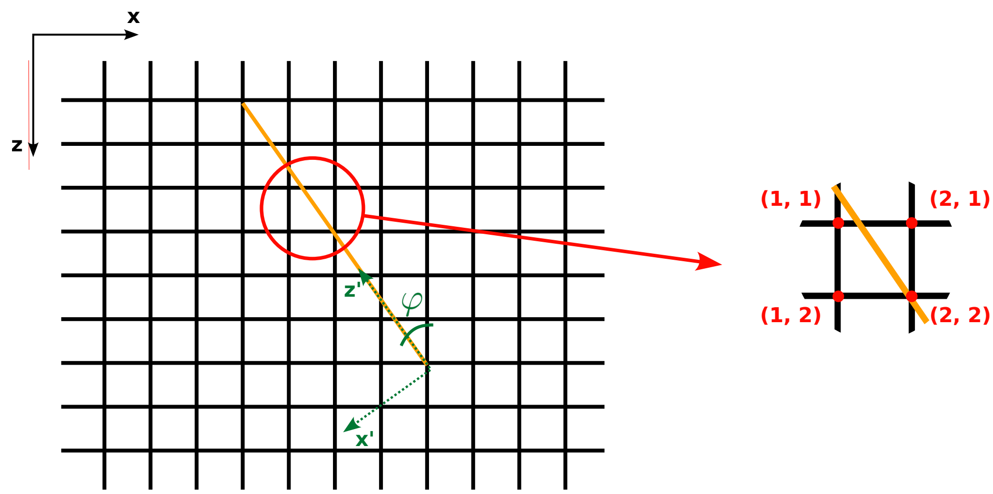

Figure 1:

For speed-of-sound tomography, the time-of-flight t along the ray path $L(\varphi)$ is computed solving the line integral

$$t = \int_{L(\varphi)} s(x,z)dl, \tag{3}$$

where dl is the differential arc length along the ray, and φ indicates the propagation direction of the ray. To simplify the computations, we rotate the coordinate system in order to align it with the ray path (see x',z'-coordinate system in Fig. 1), with the transformation defined as

$$\begin{aligned} x &= -\cos\varphi x' - \sin\varphi z' + x_0 \\ z &= \sin\varphi x' - \cos\varphi z' + z_0, \end{aligned} \tag{4}$$

where the position (x_0, z_0) refers to the origin of the ray where the new coordinate system is located.

Let us focus on an arbitrary pixel crossed by the ray. By using Eqs. (1), (2), and (4), the integral in Eq. (3) reduces to

$$\int_{l_0}^{l_1} s(0,z')dz' = \sum_{i,j=1}^{2} s_{ij} \int_{l_0}^{l_1} \omega_{ij}dz'. \tag{5}$$

The integrals on the right-hand side provide the weights for each grid point for tracing the rays. Thus, we focus on deriving their values in the following. The first integral is

$$
\begin{aligned}
\int_{l_0}^{l_1} \omega_{11} dz' &= \underbrace{\frac{1}{\Delta x \Delta z}\int_{l_0}^{l_1} x_2 z_2 dz'}_{1} - \underbrace{\frac{1}{\Delta x \Delta z}\int_{l_0}^{l_1} (-\sin\varphi z' + x_0) z_2 dz'}_{2} \\
&- \underbrace{\frac{1}{\Delta x \Delta z}\int_{l_0}^{l_1} x_2(-\cos\varphi z' + z_0) dz'}_{3} \\
&+ \underbrace{\frac{1}{\Delta x \Delta z}\int_{l_0}^{l_1} (-\sin\varphi z' + x_0)(-\cos\varphi z' + z_0) dz'}_{4},
\end{aligned}
\tag{6}
$$

where each term is reduced to

$$
1 = \tfrac{x_2 z_2 (l_1 - l_0)}{\Delta x \Delta z}, \tag{7}
$$

$$
2 = -\tfrac{\sin\varphi z_2}{2\Delta x \Delta z}(l_1^2 - l_0^2) + \tfrac{x_0 z_2 (l_1 - l_0)}{\Delta x \Delta z}, \tag{8}
$$

$$
3 = -\tfrac{\cos\varphi x_2}{2\Delta x \Delta z}(l_1^2 - l_0^2) + \tfrac{z_0 x_2 (l_1 - l_0)}{\Delta x \Delta z}, \tag{9}
$$

$$
4 = \tfrac{\sin\varphi\cos\varphi}{3\Delta x \Delta z}(l_1^3 - l_0^3) - \tfrac{\cos\varphi x_0 + \sin\varphi z_0}{2\Delta x \Delta z}(l_1^2 - l_0^2) + \tfrac{x_0 z_0 (l_1 - l_0)}{\Delta x \Delta z}. \tag{10}
$$

By combining all the terms, we find

$$
\begin{aligned}
\textstyle\int_{l_0}^{l_1} \omega_{11} dz' &= \tfrac{1}{\Delta x \Delta z}(z_2 - z_0)(x_2 - x_0)(l_1 - l_0) \\
&+ \tfrac{1}{2\Delta x \Delta z}\left(\sin\varphi(z_2 - z_0) + \cos\varphi(x_2 - x_0)\right)(l_1^2 - l_0^2) \\
&+ \tfrac{1}{3\Delta x \Delta z}\sin\varphi\cos\varphi(l_1^3 - l_0^3).
\end{aligned}
\tag{11}
$$

Similarly, the rest of the integrals in Eq. (5) become:

$$
\begin{aligned}
\textstyle\int_{l_0}^{l_1} \omega_{12} dz' &= \tfrac{1}{\Delta x \Delta z}(z_0 - z_1)(x_2 - x_0)(l_1 - l_0) \\
&+ \tfrac{1}{2\Delta x \Delta z}\left(\sin\varphi(z_0 - z_1) - \cos\varphi(x_2 - x_0)\right)(l_1^2 - l_0^2) \\
&- \tfrac{1}{3\Delta x \Delta z}\sin\varphi\cos\varphi(l_1^3 - l_0^3).
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\textstyle\int_{l_0}^{l_1} \omega_{21} dz' &= \tfrac{1}{\Delta x \Delta z}(z_2 - z_0)(x_0 - x_1)(l_1 - l_0) \\
&+ \tfrac{1}{2\Delta x \Delta z}\left(-\sin\varphi(z_2 - z_0) + \cos\varphi(x_0 - x_1)\right)(l_1^2 - l_0^2) \\
&- \tfrac{1}{3\Delta x \Delta z}\sin\varphi\cos\varphi(l_1^3 - l_0^3).
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\textstyle\int_{l_0}^{l_1} \omega_{22} dz' &= \tfrac{1}{\Delta x \Delta z}(z_0 - z_1)(x_0 - x_1)(l_1 - l_0) \\
&- \tfrac{1}{2\Delta x \Delta z}\left(\sin\varphi(z_0 - z_1) + \cos\varphi(x_0 - x_1)\right)(l_1^2 - l_0^2) \\
&+ \tfrac{1}{3\Delta x \Delta z}\sin\varphi\cos\varphi(l_1^3 - l_0^3).
\end{aligned}
\tag{14}
$$