# Mechanics of Euler-Bernoulli beam in tension or compression

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## Contents

## 1. Torque, curvature and stress of an Euler-Bernoulli beam.

The Euler-Bernoulli beam model is a simplified description of beam bending in which *only the axial stress profile inside the beam is considered*. A general description would also include the effect of shear forces, corrugation and warping of the cross-section. The Euler-Bernoulli model is applicable to beams which are long compared to their thickness, and also to thin plates in the "plane-strain" limit.

Consider a straight elastic beam of length $L$ and arbitrary profile with cross sectional area $A(s)$, thickness ("height") $h(s) << L$ and width $w(s) << L$, where $s \epsilon (0..L)$ is the curvilinear coordinate along the beam. The beam profile can be a solid or hollow bar, or have a circular, oval or rectangular cross-section. Other examples are H-, I-, T- and U- profiles.

The analysis is based on the assumption that each cross section of a bending beam stays normal to the beam axis. Figure 1 shows a small section of a bending beam with a rectangular cross section, which is subjected to external torques $\tau_0$ and $\tau_L = -\tau_0$ at its ends. The section has initial length $\Delta L_0$, thickness $h$ in the bending plane, and (out-of-plane) width $w$. As the section is short, its bent shape is a circular extrusion of angle $\Delta\theta \ll 1$ with respect to some "center" point C. The beam's radius of curvature $R$ is defined as the distance from C to all circular arcs of length $\Delta L_0$. On these arcs, making up the so-called neutral plane (colored in blue) the axial stress equals 0. Apparently, beam arcs further away from the center are stretched, while arcs on the other side of the neutral line are compressed.

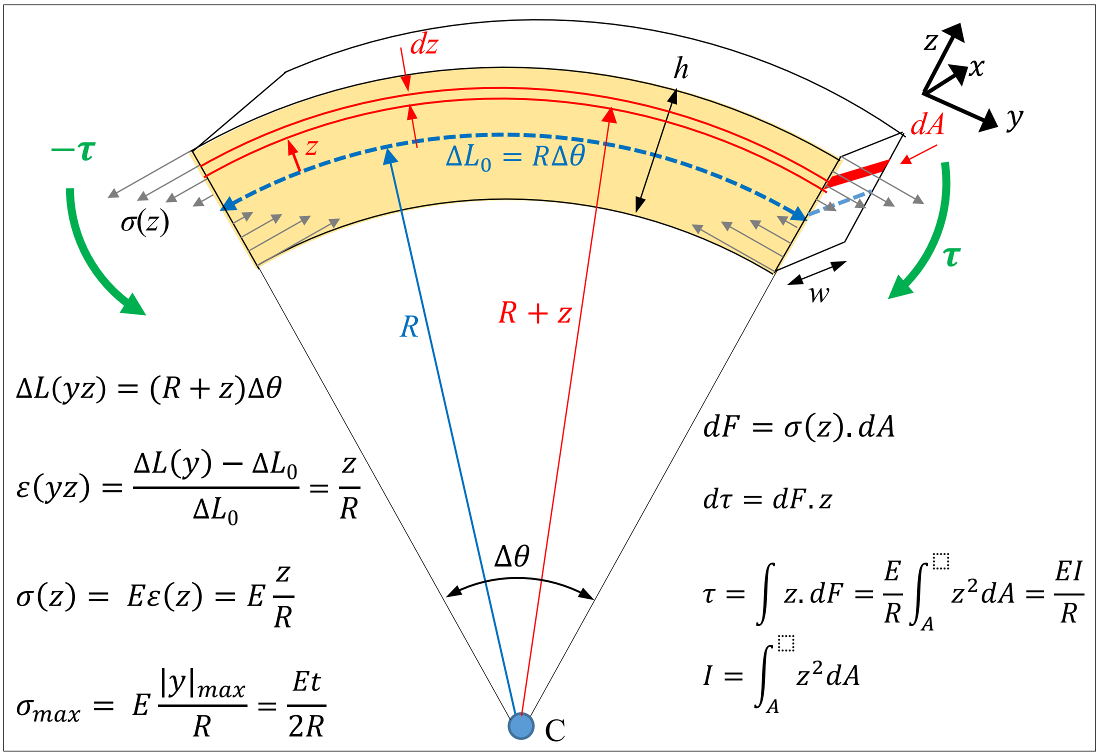



**Figure 1. Sketch of small bending section of an Euler-Bernoulli beam with rectangular cross section. A thin beam slice of thickness $dy$ at radial distance $y$ from the neutral line is shown in red. The equations show the elongation $\Delta L(y)$, axial strain $\varepsilon(y)$, axial stress $\sigma(y)$ and the axial force $dF$ of the slice, as well as its contribution to the torque $d\tau$.**

The geometrical and mechanical equations in Figure 1 hopefully make clear that the total torque on any beam cross section is related to its (local) radius of curvature according to :

(1) $$\tau(s) = \frac{EI(s)}{R(s)} = EI(s)\kappa(s), \qquad I(s) = \int_A z^2 dA$$

where $E$ is the axial elastic modulus. The product EI(s) is called the (local) *flexural rigidity* of the beam. $I(s)$ is called the *second moment of area* of the cross section. From now on the dependency of the coordinate $s$ is omitted, unless it is relevant. As shown in eq. (1), the torque can also be expressed in the parameter $\kappa(s) = 1/R(s)$, called *curvature*.

For the rectangular cross section in Figure 1 the calculation of this integral is straightforward, because the neutral plane is apparently situated exactly between the upper and lower bending surfaces. Dropping the In that case, using $dA = w.dz$ :

(2) $$I = \mathrm{w}\int_{-h/2}^{h/2} z^2 dz = \frac{1}{12}wh^3$$ (rectangular cross section with $w$ normal to bending plane)

The coordinate along the beam is dropped from now on.
Similarly, for a solid circular cross section of radius $r$ with $dA = rd\phi dr$ and $y = r\sin\phi$:

(3) $$I = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{r}(r\sin\phi)^2 . r\, d\phi dr = \frac{\pi}{4}r^4$$ (circular cross section)

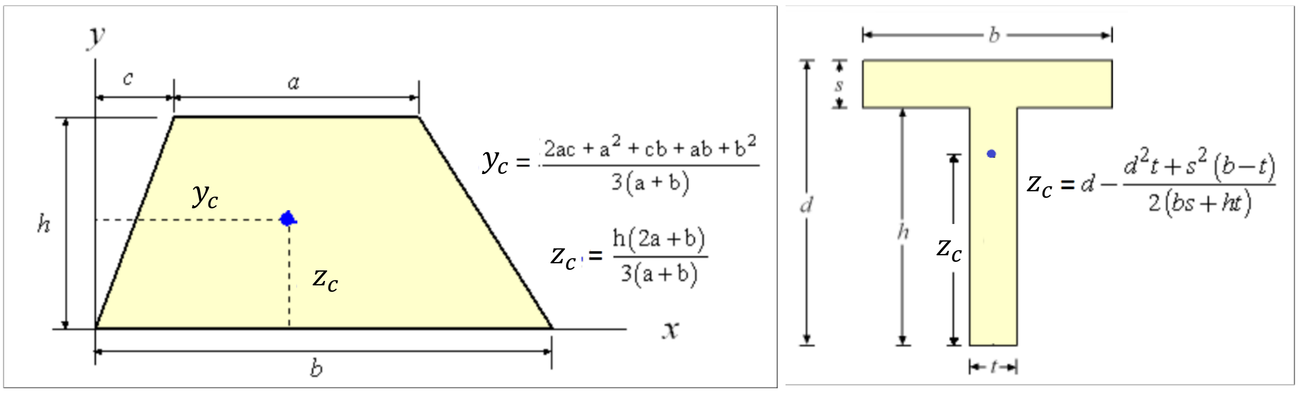
Calculation of the second moment of area of less symmetric cross sections require the location of the neutral plane. For bending in the $z$-$y$ and $x$-$y$ planes respectively they are defined as:

(4) $$I_x = \int (z - z_c)^2 dA$$
$$I_y = \int (x - x_c)^2 dA$$

$z_c, x_c$ are the coordinates of the so-called *centroid of the cross section*. If the mass density of the beam is constant, the centroid corresponds to the center of mass of the cross section, and is found by solving the *first moment of area* equations:

(5) $$0 = \int (z - z_c) dA$$
$$0 = \int (x - x_c) dA$$

**Figure 2. Examples of centroid expressions for trapezoidal and T cross sections. (www.efunda.com/designstandards/beams)**



## 2. Bending of a Euler Bernoulli beam subjected to external forces.

In this section we derive and solve the equation of motion of a straight, stiff wire, subjected to a static tension force *T*, a shear force *F* and a distributed shear force *f* (unit N/m). The *y*-axis is chosen along the initial beam axis, and the *z*-axis is chosen in the plane of bending. We restrict ourselves to *small angular displacements*: $\theta \ll 1$. In that case $\sin\theta \cong \theta$ and $\cos\theta \cong 1$, to first order, and the curvilinear coordinate along the beam (*s*) can be identified with the *y* coordinate. The beam displacement at any *y* is assumed to be normal to the wire and called *z*(*y*), as illustrated in Figure 3. The angular displacement $\theta$ and curvature $\kappa$ can (to first order) be approximated by:

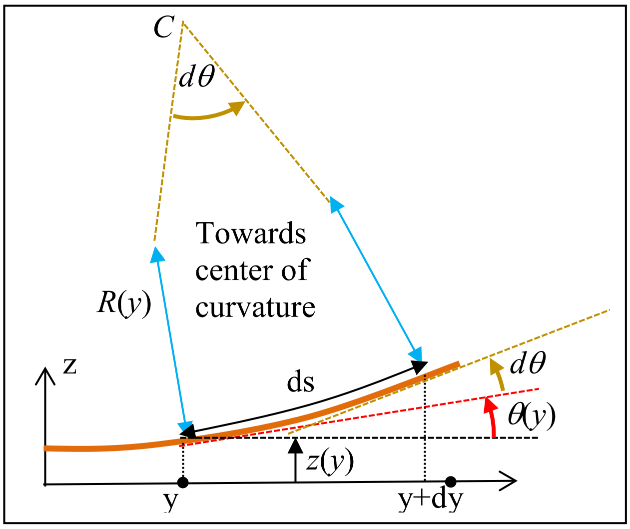


**Figure 3. wire angular and lateral displacement**

$$\theta(y) = arctan\left(\frac{dz}{dy}\right) \cong \frac{dz}{dy} \quad (6)$$

$$\kappa(y) = \frac{1}{R(y)} \cong \frac{d\theta}{dy} \cong \frac{d^2z}{dy^2} \quad (7)$$

Note that eq (6) has been substituted into eq (7). $R(y) = 1/\kappa$ is the local *radius of curvature*. The beam torque can be expressed in terms of z(y) using eq (1) as:

$$\tau(y) = EI(y)\kappa(y) = EI(y)\frac{d^2z}{dy^2} \quad (8)$$

### 2.1. Internal force and torque.

When a beam is loaded its material gets a state of stress invoked by the external forces and torques. For instance if a tension force *T* is applied on the right end, while the left end is fixed to a wall, the tension force is transmitted via the beam to the other end. Consider an arbitrary point $y_0$ on the beam (Figure 5). At this junction the right section (pink), exerts an axial force *T* on the left section (red). From the third law of Newton (*action = reaction*) it follows that an opposite directed force of the same size ($-T$) is acting on the right section. The same holds for the shear force *F* and the torque $\tau$. These quantities are called *internal* and are by convention defined as the forces and torque acting on the left section (i.e. $y<y_0$) for any junction $y_0$ considered.

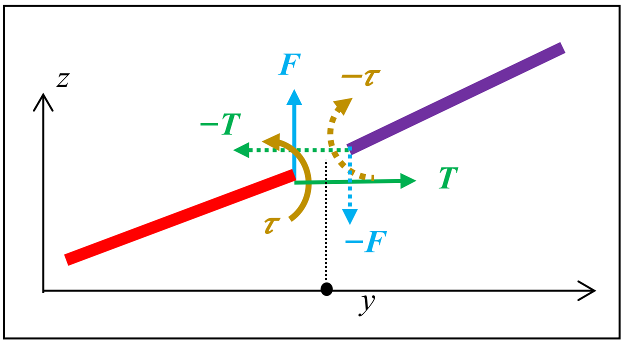


**Figure 4. Internal torque and forces in a beam**

**2.2. Beam equation in static equilibrium and general solutions.**

Consider an infinitesimal beam segment of length *dy* in a state of tension and bending (Figure 5). As the segment is small, the body force *f*(*y*) can be considered constant. Moreover, as the body force *f*(*y*) is only acting the *z* direction, the internal tension *T* in the beam is constant, i.e. not dependent on *y*. However, the internal shear force *F* and torque $\tau$ may vary along the segment. In equilibrium the total shear force and the torque acting on the segment *dy* must be 0:

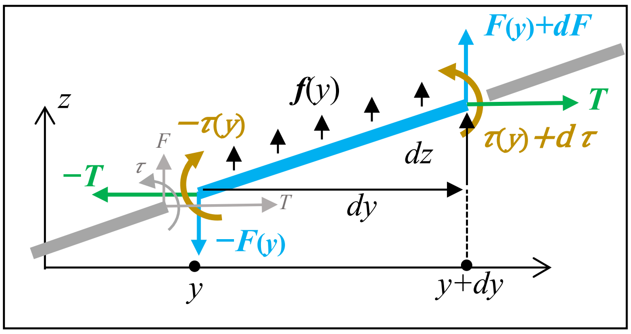



**Figure 5. Wire segment under tension, loaded by distributed force**

$$f(y)dy + dF = 0 \tag{9}$$

$$d\tau - T dz + F(y)dy + f(y)\int_0^{dy} \eta d\eta = 0 \tag{10}$$

The integral equals $dy^2/2$, which is second order in *dy*, so this term vanishes in the limit $dy \rightarrow 0$ (compared to the other terms), and these equations can be rewritten in differential way as:

$$\frac{dF}{dy} = -f(y) \tag{11}$$

$$\frac{d\tau}{dy} - T\frac{dz}{dy} + F(y) = 0 \tag{12}$$

Substitution of eq(11) into the derivative of eq (12) results into:

$$\frac{d^2\tau}{dy^2} - T\frac{d^2z}{dy^2} - f(y) = 0 \tag{13}$$

Finally, put eq (8) into eq (13) an obtain the differential equation for the displacement of a *beam in tension or compression* (*T*<*0*) *in static equilibrium*:

$$\boxed{\frac{d^2}{dy^2}\left(EI(y)\frac{d^2z}{dy^2}\right) - T\frac{d^2z}{dy^2} - f(y) = 0} \quad \text{(beam with axial load in static equilibrium)} \tag{14}$$

For beams of *uniform cross section and uniform distributed load* $f_0$ this simplifies to

The general solution of eq (14) is:

$$z(y) = C_1 + C_2 y - \frac{f_0}{2T}y^2 + C_3 e^{py} + C_4 e^{-py} \quad , p = \sqrt{\frac{T}{EI}} \quad \text{(uniform } EI \text{ and } f\text{)} \tag{15}$$

This can be proved by substitution it in eq (15). This solution can equivalently be written as:

$$z(y) = C_1 + C_2 y - \frac{f_0}{2T}y^2 + C_3 \sinh(py) + C_4 \cosh(py) \tag{16}$$

In case of a *compressive force* $P = -T$ it is often convenient to write eq (16) as[1]:

$$z(y) = C_1 + C_2 y + \frac{f_0}{2P}y^2 + C_3 \sin(ky) + C_4 \cos(ky),\ \ k = i.p = \sqrt{\mathrm{P/EI}}\ , i = \sqrt{-1} \tag{17}$$

For *T*=*P*=0 the general solution reads:

$$z(y) = C_1 + C_2 y + C_3 y^2 + C_4 y^3 + \frac{f_0}{24EI}y^4 \tag{18}$$

Note that this solution is not simply found by substituting *T*=0 in eq (16) or (17). Instead, this substitution is most easily done in eq(14). Straightforward integration delivers then eq (18) .

[1] Eq (17) is obtained using these identities: $\cosh Ix = \cos x$, $\sinh Ix = I \sin x$, $\cos Ix = \cos x$, $\sin Ix = I \sinh x$

**2.3. Beam equation of motion and general solutions.**

The analysis of the response of an elastic beam subjected to external dynamic forces (i.e. varying in time) requires to include the inertial forces invoked by the lateral acceleration $a(y)$ at each point of the beam[2]. This is simply done by adding a term $-dm.a(y)$ to eq(9), where $dm = \mu(y)dy$ is the mass of the segment *dy*:

$$(f(y) - \mu(y)a(y))dy + dF = 0 \tag{19}$$

$\mu(y)$=$\rho A$ is the beam linear mass density (unit: kg/m). The minus sign reflects the fact that the acceleration force invokes an equal but opposite recoil force on the beam itself (again: action = reaction). To put it otherwise: any force used to accelerate the beam is not contributing to the beam bending. The equation of motion is simply obtained by subtracting $\mu(y).a$ to $f(y)$ in eq (14):

$$\frac{\partial^2}{\partial y^2}\left(EI(y)\frac{\partial^2 z}{\partial y^2}\right) - T\frac{\partial^2 z}{\partial y^2} + \mu(y)\frac{\partial^2 z}{\partial t^2} - f(y,t) = 0 \quad \text{(elastic beam equation of motion)} \tag{20}$$

Unlike the static beam equation, this one is a linear partial differential equation with displacement *z* dependent on both the position on the beam, and the time: *z*(*y*,*t*).

In many applications it is convenient to write the equation of motion in the frequency domain by writing $z(y,t)$ and $f(y,t)$ as Fourier transforms:

$$z(y,t) = \int_{-\infty}^{\infty} \hat{z}(y,\omega)e^{i\omega t}d\omega \quad , \quad f(y,t) = \int_{-\infty}^{\infty} \hat{f}(y,\omega)e^{i\omega t}d\omega \tag{21}$$

Substitution into eq (20) results in an ordinary differential equation that should hold for all $\omega$:

$$\boxed{\frac{d^2}{dy^2}\left(EI(y)\frac{d^2\hat{z}}{dy^2}\right) - T\frac{d^2\hat{z}}{dy^2} - \omega^2\mu(y)\hat{z}(y) - \hat{f}(y,\omega) = 0} \tag{22}$$

For a beam with both uniform flexural rigidity $EI$ and body force amplitude $\hat{f}(\omega)$ this simplifies to:

$$EI\frac{d^4\hat{z}}{dy^4} - T\frac{d^2\hat{z}}{dy^2} - \omega^2\mu(y)\hat{z}(y,\omega) - \hat{f}(\omega) = 0 \tag{23}$$

As can be verified by back substitution, the general solution reads:

$$\hat{z}(y,\omega) = C_1e^{\alpha y} + C_2e^{-\alpha y} + C_3e^{i\beta y} + C_4e^{-i\beta y} - \frac{\hat{f}(\omega)}{q\omega^2}, \tag{24}$$

Or, equivalently:

$$\hat{z}(y,\omega) = C_1\sinh\alpha y + C_2\cosh\alpha y + C_3\sin\beta y + C_4\cos\beta y \tag{25}$$

with:

$$\begin{aligned} 2\alpha^2 &= \sqrt{p^4 + 4q\omega^2} + p^2 \\ 2\beta^2 &= \sqrt{p^4 + 4q\omega^2} - p^2 \end{aligned} \quad , \quad q = \frac{\mu}{EI} \; , p = \sqrt{\frac{T}{EI}} \tag{26}$$

Just like in the previous section (eq (17)), one might prefer to substitute $k = i.p$ in case of compression ($P = -T > 0$).

Note that for $\omega = 0$ eq, the parameters $\alpha$ and $\beta$ reduce to $\alpha$=$p$ and $\beta$=*0,* and eq(22) reduces to the static beam equation (14), as expected.

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[2] We neglect the effect of the inertial torque invoked by the angular acceleration $\ddot{\theta}(y)$ of the beam, which only plays a role at very high frequencies, when the wavelength approaches the beam thickness *h*. So we assume $\beta h \ll 1$

**2.4. Travelling transversal waves.**
An interesting set of special solutions of eq (23) in absence of a distributed force are harmonic transversal waves: $z(y,t) = Ae^{i(\beta y - \omega t)}$ travelling at wave speed $c(\omega) = \omega/\beta(\omega)$. The wave number $\beta = 2\pi/\lambda$ is apparently dependent on the frequency (eq (27)), and so is the wave velocity. In other words: the wave medium shows dispersion.

$$(27) \qquad c(\omega) = \frac{\omega}{\beta} = \frac{\sqrt{2}\,.\omega}{\sqrt{\sqrt{p^4+4q\omega^2}-p^2}} \quad = \sqrt{\frac{1}{\sqrt{(T/F_{\mathrm{ch}})^2+1}\ -2T/F_{\mathrm{ch}}}}\, c_{\mathrm{ch}}$$

In the last expression $c(\omega)$ is written in terms of a characteristic force and velocity, defined as:

$$(28) \qquad F_{\mathrm{ch}}(\omega) = 4\omega\sqrt{EI\mu}\,, \quad c_{\mathrm{ch}}(\omega) = \frac{\omega^2 EI}{\mu}$$

For compressed beams eq (27) is of course only meaningful at compression forces ($P = -T$) below the buckling force: $P \ll P_{buckle}$.

The velocities at large tension, large compression, and at zero tension or compression simplify to:

$$(29) \qquad c_{T\gg F_{ch}} \cong \sqrt{T/\mu}, \quad c_{P_{buckle}\gg P\gg F_{ch}} \cong \omega\sqrt{EI/P}, \qquad c_0 = \sqrt{\omega}\,\sqrt[4]{\frac{\mathrm{EI}}{\mu}} \ (T{=}P{=}0)$$

As expected, at large tension the velocity approaches the well-known “violin string” thumb rule.

**Dispersion relation.**
At zero tension or compression the dispersion relation reads, after solving $\omega$ from $\beta c_0 = \omega$

$$(30) \qquad \omega(\beta) = \beta c(\omega) = \beta^2\sqrt{\mu/EI}$$

**2.5. Calculation of the internal force and torque.**

The internal force and torque are already defined and shown in section 2.1 (eq (8) and (12)). For convenience they are recalled here in terms of the beam deflection function $z(y)$:

(31) $$\tau(y) = EI(y)\frac{d^2z}{dy^2} \quad ,$$

(32) $$F(y) = T\frac{dz}{dy} - EI(y)\frac{d^3z}{dy^3}$$

These expressions are quite useful, for instance to express proper boundary conditions (section 2.6), or to express a stiffness or transfer matrix (section 2.7)

**2.6. Overview of boundary condition equations for beam ends.**

Specific solutions of statically or dynamically bending beams require the specification of the amplitude constants $C_1$, $C_2$,…. $C_4$ in a general solution, like eq (15)(18) and eq (25). These constants can be solved from a same number (4) of boundary conditions, which can be any geometrical constraint or prescribed external force or torque on one of the end points. These can be expressed in the solution $z(y)$ and its derivatives, as shown in the previous section. From the resulting set of *linear equations* the amplitudes can be solved.

This holds for both statically and dynamically bending beams. In the last case the boundary equations refer to amplitudes of force $\hat{F}$, torque $\hat{\tau}$, displacement $\hat{z}(y)$ and its derivatives.

Another useful type of boundary condition allow to add springs and dashpots to the beam ends. These also result into relations between $z(y)$ and its derivatives.

- Fixed displacement → $z(y) = z_e$
- Fixed rotation → $\frac{dz}{dy} = \theta_e$
- Fixed force → $T\frac{dz}{dy} - EI\frac{d^3z}{dy^3} = F_e$
- Fixed torque → $\frac{d^2z}{dy^2} = \tau_e$
- Linear spring $F = -k_z.z$ → $T\frac{dz}{dy} - EI\frac{d^3z}{dy^3} = -k_z z(y)$
- Angular spring $\tau = -k_\theta.\theta$ → $EI\frac{d^2z}{dy^2} = -k_\theta\frac{dz}{dy}$
- Linear dashpot $F = -c_z.\frac{\partial z}{\partial t}$ → $T\frac{d\hat{z}}{dy} - EI\frac{d^3\hat{z}}{dy^3} = -c_z.I\omega.\hat{z}(y)$
- Angular dashpot $\tau = -c_\theta.\frac{\partial\theta}{\partial t}$ → $EI\frac{d^2\hat{z}}{dy^2} = -c_\theta.I\omega.\frac{d\hat{z}}{dy}$

# 3. Beams in static equilibrium: displacement functions and matrix relations

## 3.1. Typical static bending deformation of axially loaded beams.

Suspension beams/wires and flexes of inverted pendula are examples of bending beams in tension and compression respectively. In this section we consider them in a typical bending state, expressed by the following geometric boundary conditions:

$$(33)\qquad z(0)=0 \qquad \theta(0)=\frac{dz}{dy}(0)=0$$
$$z(L)=z_L \qquad \theta(L)=\frac{dz}{dy}(L)=0$$

In other words, the beam ends stay parallel, but the end point gets a displacement $z_L$, such that the beam adopts an "S" shape. The corresponding displacement function $z(y)$ is obtained by substituting these 4 equations into a general solution such as eq. (16) in section 2.2, and solve the 4 amplitudes $C_1 \dots . C_4$. The result is (see Appendix A):

$$(34)\qquad z(y)=z_L\frac{\sinh pL(py-\sinh py)+(\cosh pL-1)(\cosh py-1)}{pL\sinh pL-2(\cosh pL-1)} \quad \text{(tension, } p=\sqrt{T/EI}\text{)}$$

$$(35)\qquad z(y)=z_L\frac{\sin kL(ky-\sin ky)-(\cos kL-1)(\cos ky-1)}{kL\sin kL+2(\cos kL-1)} \quad \text{(compression, } k=\sqrt{P/EI}\text{)}$$

Figure 6 shows the normalized displacement curves for beams in tension in compression and without axial load, all in absence of an external distributed load. The corresponding expressions are obtained after some simplifications, also shown in Appendix A.

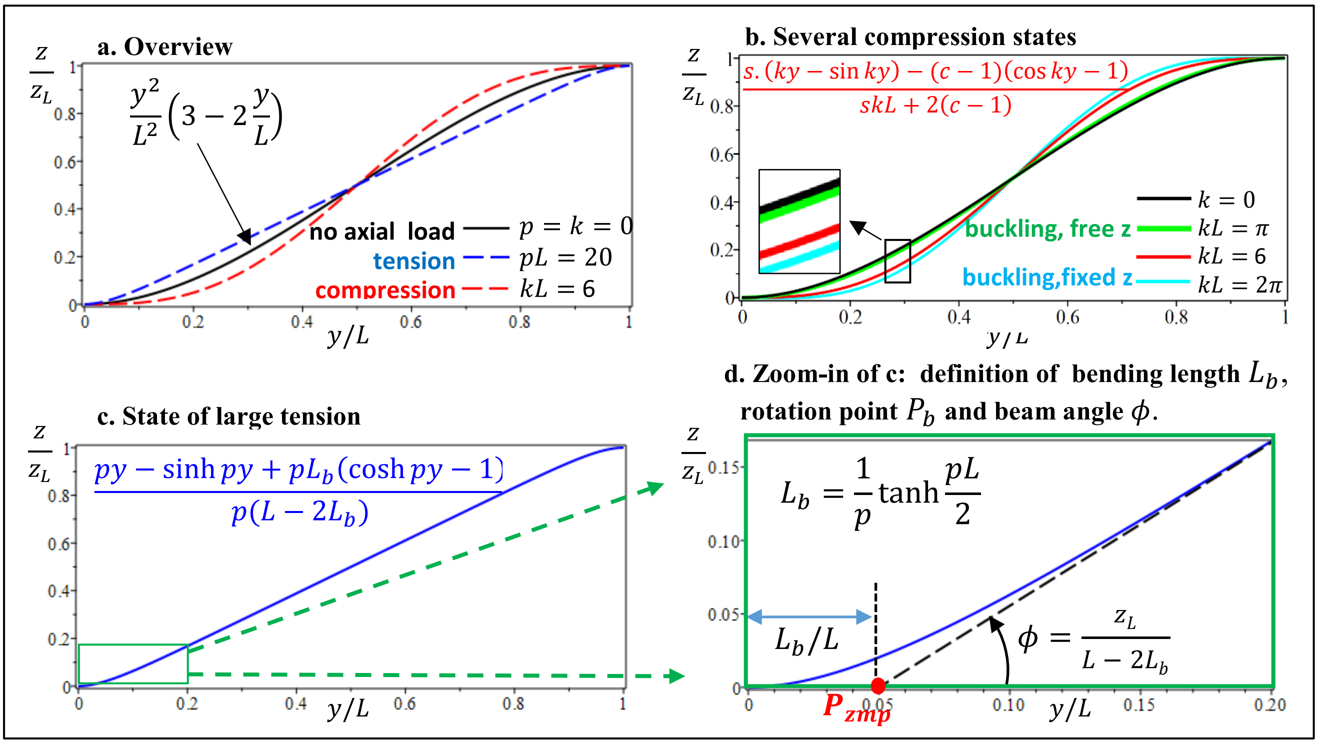



**Figure 6. Displacement functions $z(y)$ of a beam with $z(0)=\theta(0)=\theta(L)=0$ and fixed $z(L)=z_L$.**

***Discussion of Figure 6:***

**No axial load (*T*=*P*=0),** Figure 6a, black line.
In this case both eq (34)and (35) simplify to:

$$z(y) = \frac{y^2}{L^2}\left(3 - 2\frac{y}{L}\right). \tag{36}$$

This is the well-known third order expression, most easily obtained after substituting (33) into eq. (18).

**Beam in tension** ($pL = 20$)**,** Figure 6a, c. d, in blue:
At large tension ($pL \gg 1$) the beam is almost a straight line with inclination $\phi$, crossing $z = 0$ and $z = z_L$ at the so-called *zero moment points* $P_{zmp}$ at distance $L_b$ from the beam ends, see Figure 6d. Most of the curvature is restricted to the sections close to the beam ends. $L_b$ is called the *bending length*. It can be shown that[3]:

$$L_b = \frac{1}{p}\frac{\cosh pL - 1}{\sinh(pL)} = \frac{1}{p}\tanh\frac{pL}{2}, \tag{37}$$

and

$$\phi = \left.\frac{dz}{dy}\right|_{y=L/2} = \frac{z_L}{L - 2L_b} \qquad \text{(for } pL \gg 1\text{)} \tag{38}$$

Using the bending length, eq. (34) simplifies to (Appendix A):

$$z(y) = z_L\frac{py - \sinh py + pL_b(\cosh py - 1)}{p(L - 2L_b)} \tag{39}$$

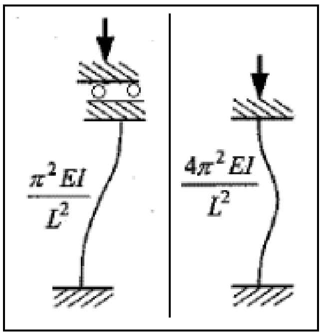
**Beam in compression,** Figure 6a, b, in red:
In this case a similar expression as eq (39) can be obtained using $L_b$. However, this is not always recommended, not only because singularities in eq (39) can frustrate its numerical calculation, but also because there is no meaningful physical interpretation of $L_b$ in the compression case. If preferred, you may shorten eq (35) simply to:

$$z(y) = z_L\frac{s.(ky - \sin ky) - (c - 1)(\cos ky - 1)}{skL + 2(c - 1)} \quad , \quad \text{with: } s = \sin kL,\ c = \cos kL \tag{40}$$

Further simplifications are possible at the buckling loads for **free** and **fixed** $z(L)$, In those cases the displacement functions turn out to be sinusoidal (Appendix A):

$$P = P_{buckle,free} = \frac{\pi^2 EI}{L^2} \rightarrow kL = \pi \rightarrow z(y) = \frac{1}{2}\left(1 - \cos\frac{\pi y}{L}\right) \tag{41}$$

$$P = P_{buckle,fixed} = \frac{4\pi^2 EI}{L^2} \rightarrow kL = 2\pi \rightarrow z(y) = \frac{y}{L} - \frac{1}{2\pi}\sin\frac{2\pi y}{L} \tag{42}$$




They are shown in Figure 6b, in green and light blue, together with the case *kL*=0 (eq (36)).

[3] In eq. (37) this identity is applied: $tanh\frac{x}{2} = \frac{\cosh x - 1}{\sinh x}$.

**3.2. Beam end-to-end transfer matrix in static equilibrium.**
Consider a uniform beam bending in the $y$-$z$ plane, of which, at the left end ($y = 0$), all linear and angular displacements and forces are known. This state can be represented by a vector that we label "$y = 0$". This state can only be realized for certain displacements and forces on the other end ($y = L$), that we label "$y = L$". The fact that all equations so far are linear is suggesting that the elements of these two vectors are linearly related to each other. In other words, their relation can be written as a matrix equation:

$$\begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=L} = [M] \begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=0}$$

**Beam in compression.**
Let us check this, for instance for a beam in compression, without distributed load and in static equilibrium. We can apply eq (17), and rewrite it, to our convenience (with $f_0 = 0$):

(43) $$z(y) = C_1 + C_2 y + C_3 \sin(ky) + C_4 \cos(ky), \quad k = \sqrt{P/EI}$$

This general solution can be used to calculate all elements of these two vectors. Expressions for $\theta$, $F$ and $\tau$ on both ends are directly obtained from sections 2.5 and 2.6:

(44) $$\theta_{0,L} = \left.\frac{dz}{dy}\right|_{0,L}, \qquad F_{0,L} = -P\left.\frac{dz}{dy}\right|_{0,L} - EI\left.\frac{d^3z}{dy^3}\right|_{0,L}, \qquad \tau_{0,L} = \left.\frac{d^2z}{dy^2}\right|_{0,L}$$

It is tedious but straight forward to substitute $z(y)$ and its 1st, 2nd and 3rd derivatives into the above equations. My Maple/Mathematica can do it easily (see Maple listing in appendix C). Result:

(45) $$\begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=L} = \begin{bmatrix} 1 & \frac{s}{k} & \frac{s/k-L}{P} & \frac{c-1}{P} \\ 0 & c & \frac{c-1}{P} & \frac{sk}{P} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{sP}{k} & -\frac{s}{k} & c \end{bmatrix} \begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=0} \quad \text{with} \quad \begin{aligned} c &= \cos kL \\ s &= \sin kL \\ k &= \sqrt{P/EI} \\ P &= \text{compr. force} \end{aligned}$$

**Beam in tension**.
Similarly, for a beam in tension, the end-to-end relation reads (appendix C):

(46) $$\begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=L} = \begin{bmatrix} 1 & \frac{sh}{p} & -\frac{sh/p-L}{T} & \frac{ch-1}{T} \\ 0 & ch & \frac{1-ch}{T} & \frac{sh.p}{T} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{sh.T}{p} & -\frac{sh}{p} & ch \end{bmatrix} \begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=0} \quad \text{with} \quad \begin{aligned} ch &= \cosh pL \\ sh &= \sinh pL \\ p &= \sqrt{T/EI} \\ T &= \text{tension force} \end{aligned}$$

These matrices are quite similar: the matrix of Eq (46) can also directly obtained from the matrix of Eq (45) with these substitutions:

$$T = -P, \; s = I.sh, \; c = ch, \; k = I.p, \quad \text{with } I = \sqrt{-1}$$

Note that all $I$'s are cancelled, except for the last element of the second row (indicated in yellow), in which the product $I^2 = -1$ appears.

**Beam at zero axial load.**
In the limit *T,P*→0 both Eq (45) and (46) reduce to (Appendix D):

$$(47)\qquad \begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=L} = \begin{bmatrix} 1 & L & -\frac{L^3}{6EI} & \frac{L^2}{2EI} \\ 0 & 1 & -\frac{L^2}{2EI} & \frac{L}{EI} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -L & 1 \end{bmatrix} \begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=0}$$

**Applications of end-to-end matrices.**
Mechanic end-to-end matrices are very useful for systems consisting of a series of *N* connected beams of different lengths, flexural rigidity and axial load. The end-to-end matrix for such a system with total length *L* simply reads:

$$(48)\qquad \begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=L} = [M_N][M_{N-1}]\ldots.[M_2][M_1] \begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=0} \quad \text{matrix of chain of beam sections}$$

An example of a series of beams with different axial loads is a vertical suspension chain with masses $m_i$ (i=1..*N*). The tension in the *N* wires is then $T_i = \sum_{i=N}^{1} m_{N-i+1} g$, if the upper wire has index 1.

It is also possible to include massless rigid bodies in a static chain. In that case you just take the limit $EI \rightarrow \infty$. The end-to-end matrix of a static rigid body of length *L* is:

$$(49)\qquad \begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=L} = \begin{bmatrix} 1 & L & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -LT & 1 \end{bmatrix} \begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{y=0} \quad \text{rigid body, tension } T \text{ or compression } T=-P$$

Similarly springs can added in the chain For instance in case of a linear and an angular spring ($k_z$ and $k_\theta$), both connecting points *a* and *b*, the corresponding matrix reads:

$$(50)\qquad \begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{b} = \begin{bmatrix} 1 & 0 & 1/k_z & 0 \\ 0 & 1 & 0 & 1/k_\theta \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} z \\ \theta \\ F \\ \tau \end{bmatrix}_{a}$$

In chapter 4 the end-to-end matrix of a beam in harmonic motion is discussed. In that case the inertial forces have to be taken into account, see also section 2.3. The elements of its matrix become then a function of frequency. In the assembled chain transfer matrix the beam's "rigid body", "violin" and/or "banana" modes may show up as peaks in the displacement spectrum of the suspended body at the end of the chain (typically a mirror).

In the next section the end-to-end matrices presented so far serve as a basis to construct stiffness matrices.

### 3.3. Stiffness matrix in static equilibrium.

The stiffness matrix of a bending beam is defined as the relation between displacements and rotations of the end points to the forces and torques acting on the end points. This can be done by shuffling the matrix equation (45) and solve the forces and torques at both ends. The result is a symmetric matrix.

**Beam in tension.**

The stiffness matrix equation reads, using the bending length $L_b$ (eq (37)), see Appendix C:

$$(51)\qquad \begin{bmatrix} F_{0,\mathrm{r}} \\ \tau_{0,\mathrm{r}} \\ F_L \\ \tau_L \end{bmatrix} = \frac{T}{L-2L_b}\begin{bmatrix} 1 & L_b & -1 & L_b \\ & \frac{L.ch}{p.sh}-\frac{1}{\mathrm{p}^2} & -L_b & \frac{1}{\mathrm{p}^2}-\frac{L}{p.sh} \\ & & 1 & -L_b \\ sym & & & \frac{L.ch}{p.sh}-\frac{1}{\mathrm{p}^2} \end{bmatrix}\begin{bmatrix} z_0 \\ \theta_0 \\ z_L \\ \theta_L \end{bmatrix},\quad \begin{matrix} ch = \cosh pL \\ sh = \sinh pL \\ p = \sqrt{T/EI} \\ L_b = \frac{1}{p}\tanh\frac{pL}{2} \end{matrix}$$

**Beam in compression.**

Similarly, the stiffness matrix equation reads, see Appendix B:

$$(52)\quad \begin{bmatrix} F_{0,\mathrm{r}} \\ \tau_{0,\mathrm{r}} \\ F_L \\ \tau_L \end{bmatrix} = \begin{bmatrix} -F_0 \\ -\tau_0 \\ F_L \\ \tau_L \end{bmatrix} = \frac{P}{L-2L_b}\begin{bmatrix} -1 & -L_b & 1 & -L_b \\ & \frac{L.c}{k.s}-\frac{1}{k^2} & L_b & \frac{1}{k^2}-\frac{L}{k.s} \\ & & -1 & L_b \\ sym & & & \frac{L.c}{ks}-\frac{1}{k^2} \end{bmatrix}\begin{bmatrix} z_0 \\ \theta_0 \\ z_L \\ \theta_L \end{bmatrix},\quad \begin{matrix} c = \cos kL \\ s = \sin kL \\ k = \sqrt{P/EI} \\ L_b = \frac{1}{k}\frac{1-\cos kL}{\sin(kL)} = \frac{1}{k}\tan\frac{kL}{2} \end{matrix}$$

- Note the minus signs in front of $F_0$ and $\tau_0$. This is done to convert then from *internal force & torque* (which by definition act on the left side of $y = 0$, see section 2.1) to their *reaction force & torque* $F_{0,\mathrm{r}}$ and $\tau_{0,\mathrm{r}}$ acting on the beam itself.
- The convention to define stiffness matrices based on the forces & torques *acting on the beam ends* always leads to a *symmetric matrix*.
- The first column and third columns differ only in sign, as you may have noticed. This means that this *stiffness matrix is singular and cannot be inverted*.
- Note that the definitions of $L_b$ in eq (51) and (52) are equivalent, which justifies usage of the same symbol:

$$T = -P \rightarrow \quad p = Ik \ \rightarrow L_b = \frac{1}{p}\tanh\frac{pL}{2} = \frac{1}{Ik}\tanh\frac{IkL}{2} = \frac{1}{k}\tan\frac{kL}{2} \quad \text{q.e.d.}$$

- The "length" $L_b$ is just a placeholder. There is no physical interpretation of $L_b$ in case of compression. Unlike the tension case, $L_b$ can have negative values (in intervals $kL = (2n-1)\pi \ldots n2\pi$) and even reach infinity ($kL = n\pi$). This might frustrate numerical evaluation of the forces in (52). Just take care…

**Beam at zero axial load.**

In the limit *T,P*→0 both Eq (51) and (52) reduce to (Appendix D):

$$(53)\qquad \begin{bmatrix} F_{0,\mathrm{r}} \\ \tau_{0,\mathrm{r}} \\ F_L \\ \tau_L \end{bmatrix} = \frac{12EI}{L^3}\begin{bmatrix} 1 & L/2 & -1 & L/2 \\ & 1 & -\mathrm{L}/2 & L^2/6 \\ & & 1 & -L/2 \\ \mathrm{sym} & & & L^2/3 \end{bmatrix}\begin{bmatrix} z_0 \\ \theta_0 \\ z_L \\ \theta_L \end{bmatrix}$$

**3.4. Stiffness and compliance matrices of single-sided clamped beams.**
If a beam is completely fixed at one end, for example at *y=0*, it makes sense to define the other end's stiffness matrix, which connects force and torque at (in this example) *y=L* to the displacement and rotation *of the same end*. This results in a 2x2 matrix, not only symmetric, but also non-singular, i.e. invertible, unlike the matrices in the previous section. The stiffness matrix at *y=L*, $[\mathrm{K}_L]$, and its reciprocal called *compliance matrix* $[\mathrm{C}_L]$ are defined as:

(54) $$\begin{bmatrix} F_L \\ \tau_L \end{bmatrix} = [\mathrm{K}_L]\begin{bmatrix} z_L \\ \theta_L \end{bmatrix}, \qquad \begin{bmatrix} z_L \\ \theta_L \end{bmatrix} = [\mathrm{C}_L]\begin{bmatrix} F_L \\ \tau_L \end{bmatrix}, \quad \text{with } [\mathrm{C}_L] = [\mathrm{K}_L]^{-1}$$

**Beam in tension.**
Setting $z_0 = \theta_0 = 0$ in Eq (51), we get this 2x2 symmetric sub-matrix of the matrix in Eq (51):

(55) $$[\mathrm{K}_L] = \frac{T}{L-2L_b}\begin{bmatrix} 1 & -L_b \\ -L_b & \frac{L}{p.\tanh(pL)} - \frac{1}{\mathrm{p}^2} \end{bmatrix} \quad , \quad L_b = \frac{1}{p}\tanh\frac{pL}{2}$$

The compliance matrix turns out to be:

(56) $$[\mathrm{C}_L] = \frac{\tanh(pL)}{T}\begin{bmatrix} \frac{L}{\tanh(pL)} - \frac{1}{p} & pL_b \\ pL_b & p \end{bmatrix}$$

$[C_L] = [\mathrm{K}_L]^{-1}$ is called the *compliance matrix*.

**Beam in compression.**
The same approach, either using eq (52) or substituting $T = -P$ and $p = Ik$ into eq (55)(56) and (56) renders:

(57) $$[\mathrm{K}_L] = \frac{P}{L-2L_b}\begin{bmatrix} -1 & L_b \\ L_b & \frac{L}{k.\tan(kL)} - \frac{1}{\mathrm{k}^2} \end{bmatrix} \quad , \quad L_b = \frac{1}{k}\tan\frac{kL}{2}$$

(58) $$[C_L] = \frac{\tan(k\,L)}{P}\begin{bmatrix} \frac{1}{k} - \frac{L}{\tan(kL)} & kL_b \\ kL_b & k \end{bmatrix}$$

**Beam at zero axial load (*T*=*P*=0).**
In the limit for $P \rightarrow 0$ eq (57) and (58) reduce to these well-known expressions:

(59) $$[\mathrm{K}_L] = EI\begin{bmatrix} 12/L^3 & -6/L^2 \\ -6/L^2 & 4/\mathrm{L} \end{bmatrix}$$

(60) $$[\mathrm{C}_L] = \frac{1}{EI}\begin{bmatrix} L^3/3 & L^2/2 \\ L^2/2 & \mathrm{L} \end{bmatrix}$$

### 3.5. Examples cases of statically loaded beams or series of beams.

In the examples in which a single beam is discussed, the results are expressed in dimensionless parameters, after defining the so-called *reference force*, composed of "real" beam properties:

(61) force: $F_{ref} = \frac{EI}{L^2}$

Using $F_{ref}$ and the beam length $L$, the various matrix elements can be expressed and plotted in terms of more general dimensionless parameters $t$ (tension), $c$ (compression), $\kappa_z$ and $\kappa_\theta$ (linear and angular stiffness) defined as:

$$(62)\quad \begin{cases} \text{tension:} & T = t\,F_{ref}, \quad p = \sqrt{\frac{T}{EI}} = \frac{1}{L}\sqrt{t}\ , \quad pL = \sqrt{t}\ , \quad L_b = \frac{L}{\sqrt{t}}\tanh\frac{\sqrt{t}}{2} \\ \text{compression}^4\text{:} & P = ç\,F_{ref}, \quad k = \sqrt{\frac{P}{EI}} = \frac{1}{L}\sqrt{ç}\ , \quad kL = \sqrt{ç}\ , \quad L_b = \frac{L}{\sqrt{ç}}\tan\frac{\sqrt{ç}}{2} \\ \text{stiffnesses:} & K_z = \kappa_z\frac{F_{ref}}{L}, \quad K_\theta = \kappa_\theta\,F_{ref}L \end{cases}$$

**Angular stiffness of single-sided clamped beam in compression, loaded by a torque.**
This is obtained from element $\mathbf{C}_{2,2}$ in eq (58):

$$(63)\quad K_\theta = \frac{1}{C_{22}} = \frac{P}{k\tan(kL)} = \frac{(ç.F_{ref})}{\left(\frac{1}{L}\sqrt{ç}\right)\tan\sqrt{c}} = \frac{\sqrt{ç}}{\tan\sqrt{ç}}.F_{ref}L \rightarrow \kappa_\theta(ç) = \frac{\sqrt{ç}}{\tan\sqrt{ç}}$$

The function $\kappa_\theta(ç)$ is plotted in Figure 7. Note that, using the relations (62) again:

- At zero axial load: $\kappa_\theta(0) = 1 \rightarrow$ stiffness is $K_\theta = (1)\,F_{ref}L = EI/L$, just as expected.
- $\kappa_\theta(\pi^2/4) = 0 \rightarrow$ the compression force is $P = (\pi^2/4)EI/L^2$. That is, no surprise, the beam's buckling force, shown in the insert. Above that force the stiffness is negative, and the system becomes unstable.

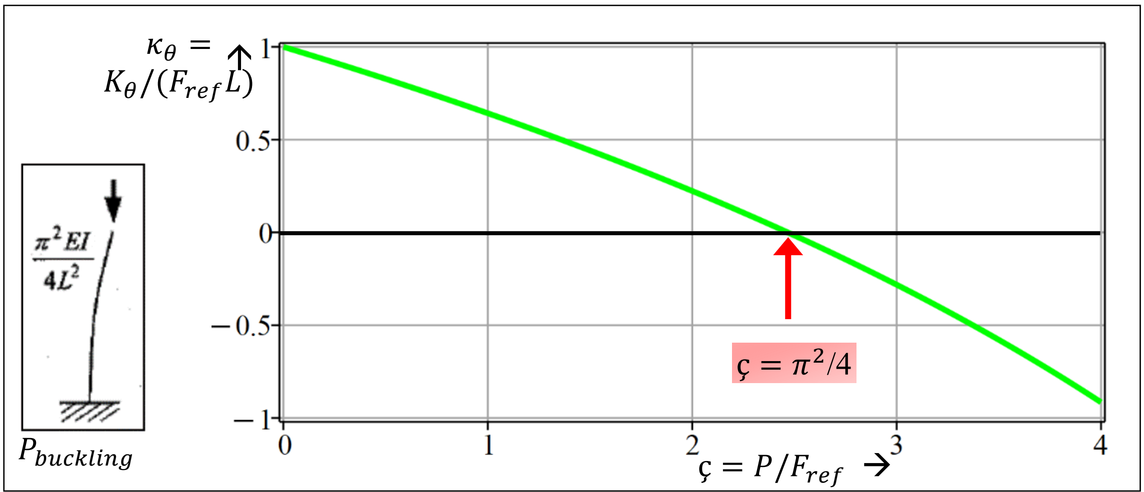



**Figure 7. Angular stiffness of a single-sided beam versus the compressive force.**

[4] The symbol ç for dimensionless compression is chosen to distinguish it from "$c$" as symbol for "cos" in other sections.

**Linear stiffness of single-sided clamped beam with $\boldsymbol{\theta(L) = 0}$.**
For this beam in tension, the linear stiffness is obtained from element $\mathbf{K}_{1,1}$ in eq (55)(56).

$$K_z = \mathbf{K}_{1,1} = \frac{T}{L-L_b} = \frac{T}{L-\frac{1}{p}\tanh\frac{pL}{2}} = \frac{t}{1-\frac{1}{\sqrt{t}}\tanh\frac{\sqrt{t}}{2}}\frac{F_{ref}}{L} \tag{64}$$

$$\rightarrow \kappa_z(t) = \frac{t}{1-\frac{1}{\sqrt{t}}\tanh\frac{\sqrt{t}}{2}} = 12 + \frac{5}{6}t + O(t^2)$$

$$\lim_{t\to\infty} \frac{d\kappa_z}{dt} = 1$$

$\kappa_z(t)$ is shown in Figure 8. Linear stiffness of guided double-clamped beam Figure 8, while its derivative (in tension) is shown in Figure 9. Note that, using the relations (62):

- At zero axial load: $\kappa_z(0) = 12 \;\rightarrow K_z = \frac{12EI}{L^2}$, just as expected.
  $\frac{d\kappa_z}{dt}(0) = \frac{6}{5} \;\rightarrow \frac{dK_z}{dT}(0) = \frac{6}{5L}$. Compare to “naïve” estimation: $\frac{1}{L}$.
- $\kappa_z(\pi^2) = 0 \;\rightarrow$ force is $P = (\pi^2/4)EI/L^2$ the stiffness becomes 0. That is, no surprise, the beam’s buckling force under the given constraints, shown in the insert. Above the buckling force the stiffness is negative, and the system becomes unstable.
- At (very) large tension $d\kappa_z/dt$ approaches the naïve value indeed.

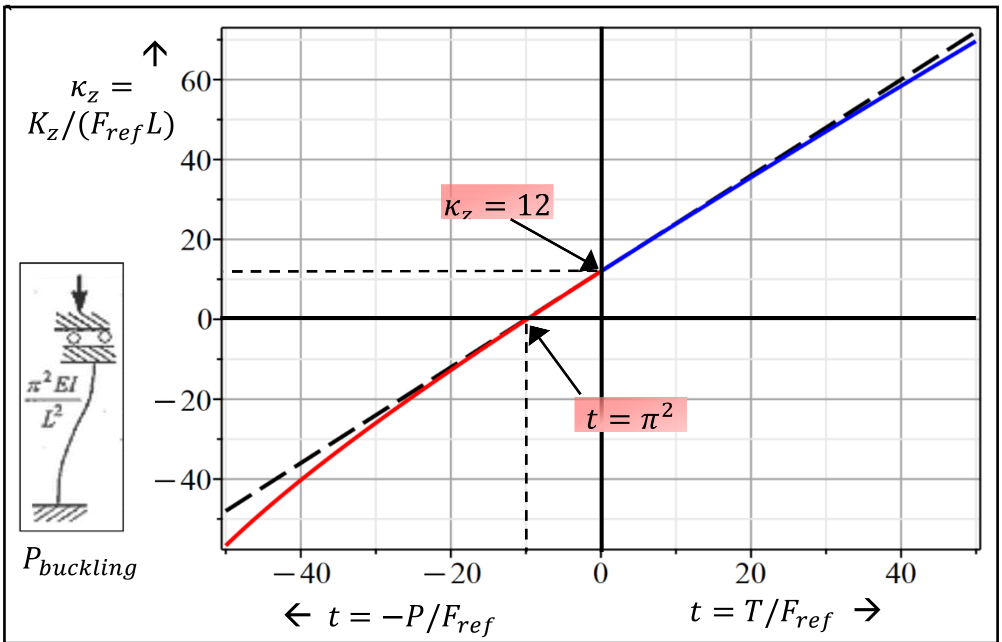



**Figure 9. Linear stiffness of guided double-clamped beam in tension (blue) and compression (bred). The dashed line is the first order expansion of $\kappa_z(t)$, having slope 5/6. The insert shows the buckling state.**

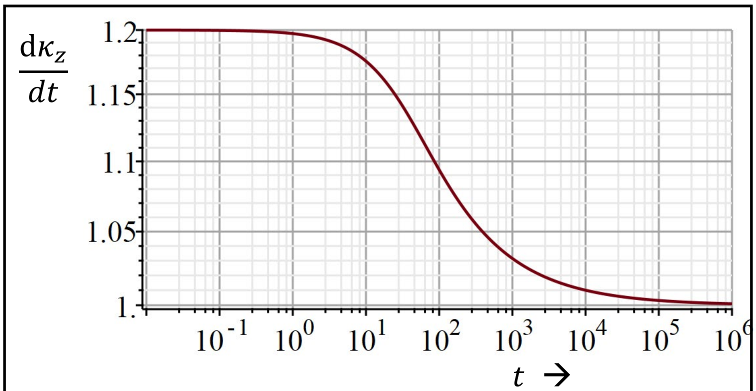



**Figure 8. Derivative of stiffness vs tension**

**Transfer matrix of a flex-leg-flex suspension beam (TBD)**
This is the design concept presently discussed in the ET mirror suspension group.

## 4. Beams in dynamic state: modal shapes, transfer matrices (TBD)

This will result in violin modes. It should also reveal the percussion effect, and the effect of internal mode showing up in mirror vibrations.

As a check I'll derive and check use Ruggi's end-to-end transfer matrix of a suspension wire of length L, density rho, modulus E, cross-sectional area S, flexural rigidity J and tension T:

$$\begin{pmatrix} z_{out} \\ \theta x_{out} \\ Fz_{out} \\ M\theta x_{out} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_1^2 c_3 + \lambda_3^2 c_1}{\beta} & \frac{L}{\beta}\left(\lambda_1 s_1 + \lambda_3 s_3\right) & \frac{L^3}{EJ\gamma^2\beta}\left(\lambda_3 s_1 - \lambda_1 s_3\right) & \frac{L^2}{EJ\beta}\left(c_3 - c_1\right) \\ \frac{\gamma^2}{L\beta}\left(\lambda_3 s_1 - \lambda_1 s_3\right) & \frac{\lambda_1^2 c_1 + \lambda_3^2 c_3}{\beta} & \frac{L^2}{EJ\beta}\left(c_1 - c_3\right) & -\frac{L}{EJ\beta}\left(\lambda_1 s_1 + \lambda_3 s_3\right) \\ \frac{EJ\gamma^2}{L^3\beta}\left(\lambda_1^3 s_3 + \lambda_3^3 s_1\right) & \frac{EJ\gamma^4}{L^2\beta}\left(c_1 - c_3\right) & \frac{\lambda_1^2 c_3 + \lambda_3^2 c_1}{\beta} & \frac{\gamma^2}{L\beta}\left(\lambda_1 s_3 - \lambda_3 s_1\right) \\ \frac{EJ\gamma^4}{L^2\beta}\left(c_3 - c_1\right) & \frac{EJ}{L\beta}\left(\lambda_3^3 s_3 - \lambda_1^3 s_1\right) & -\frac{L}{\beta}\left(\lambda_1 s_1 + \lambda_3 s_3\right) & \frac{\lambda_1^2 c_1 + \lambda_3^2 c_3}{\beta} \end{pmatrix} \begin{pmatrix} z_{in} \\ \theta x_{in} \\ Fz_{in} \\ M\theta x_{in} \end{pmatrix}$$

$$\gamma^4 = \frac{\omega^2 \rho S L^4}{EJ} \quad \lambda = \sqrt{\frac{EJ}{T}} \quad \tau = \left(\frac{L}{\lambda}\right)^2 \quad \beta = \sqrt{4\gamma^4 + \tau^2} \quad \lambda_1 = \sqrt{(\beta + \tau)/2} \quad \lambda_3 = \sqrt{(\beta - \tau)/2}$$

$$s_1 = \sinh(\lambda_1) \quad s_3 = \mathrm{sin}\cancel{\mathrm{h}}(\lambda_3) \quad c_1 = \cosh(\lambda_1) \quad c_3 = \mathrm{cos}\cancel{\mathrm{h}}(\lambda_3)$$

# Appendix A. S-shaped beam deflection function at various axial loads (Maple listing)

(Maple listing)

*restart* :  **# general solution**

$z := y \rightarrow A \cdot \sinh(p\,y) + B \cdot \cosh(p\,y) + C \cdot y + \mathrm{D}$ : # eq 16

$\tau(y) := EI \dfrac{d^2}{dy^2}(z(y))$ :  # eq. 31

$F(y) := T \dfrac{d}{dy}(z(y)) \; - EI \cdot \dfrac{d^3}{dy^3}(z(y))$ :  # eq. 32

$\theta(y) := \dfrac{d}{dy}(z(y))$ :

$eqns := subs\Big(EI = \dfrac{T}{p^2},$

$[\quad F0 = simplify(subs(y=0,\; F(y))),$  # forces, torques,

$\tau0 = \; simplify(subs(y=0,\; \tau \cdot (y))),$  # displacemenst and angles

$FL = simplify(subs(y=L,\; F(y))),$  # at beam ends

$\tau L = \; simplify(subs(y=L,\; \tau(y))),$

$z0 = \; simplify(subs(y=0,\; z(y)\;)),$

$\theta 0 = \; simplify(subs(y=0,\; \theta(y))),$

$zL = simplify(subs(y=L, z(y)))\;,$

$\theta L = \; simplify(subs(y=L,\; \theta(y)))$

$]\Big)$ :

$vars := [A, B, C, \mathrm{D}, F0, FL, \tau0, \tau L]$ :  # variables to be solved

$sol := solve(eqns, vars)[1]$ :  # solutions

**#Displacement function of clamped − clamped guided beam, unit end displacement $z(L)$**

$subses := \theta 0 = 0, \theta L = 0, z0 = 0, zL = 1$ :  # S-beam boundary conditions

$zsol := simplify(subs(sol, subses, z(y)));$  # general solution of S − beam

$z\_Tension := simplify\big(subs\big(\cosh(pL) = 1 + \; \sinh(pL) \cdot p \cdot L_b, zsol\big)\big);$ # case tension $T$ : introduce bending length

$zP := simplify\Big(subs\Big(p = \dfrac{k}{I}, zsol\Big)\Big)$ :  # next lines: just to simplify expression

$subs(\cos(kL) = c, c = cp + 1, \sin(kL) = s, zP)$ : $subs(cp = c - 1, collect(\%, [cp, s]))$ :

$z\_Compression := subs(c = \cos(kL), s = \sin(kL), \%);$

$z\_No\_axial\_force := expand(limit(zsol, p = 0));$  # no axial load (P=T=0)

$$zsol := \frac{(\cosh(py) - 1)\cosh(pL) + (py - \sinh(py))\sinh(pL) - \cosh(py) + 1}{2 + pL\sinh(pL) - 2\cosh(pL)}$$

$$z\_Tension := \frac{\cosh(py)\,pL_b - \sinh(py) + (y - L_b)\,p}{p\,(L - 2L_b)}$$

$$z\_Compression := \frac{(-\cos(ky) + 1)(\cos(kL) - 1) + (ky - \sin(ky))\sin(kL)}{kL\sin(kL) + 2\cos(kL) - 2}$$

$$z\_No\_axial\_force := \frac{3y^2}{L^2} - \frac{2y^3}{L^3}$$

**# deflection at buckling loads :**

$$limit\left(simplify\left(subs\left(k=\sqrt{\frac{P}{EI}},P=\frac{\pi^2 EI}{L^2}-\alpha,EI=1,zP\right)\right),\alpha=0\right)\text{ assuming } positive:$$

$$zy\left(p_{buckle,free}\right)=limit\left(simplify\left(subs\left(k=\sqrt{\frac{P}{EI}},P=\frac{4\pi^2 EI}{L^2}-\alpha,EI=1,zP\right)\right),\alpha=0\right)\text{ assuming } positive;$$

$$zy\left(p_{buckle,fixed}\right)=subs\left(p2=2\pi,expand\left(subs\left(\pi=\frac{p2}{2},\%\right)\right)\right);$$

$$zy\left(p_{buckle,fixed}\right)=-\frac{\cos\left(\frac{\pi y}{L}\right)}{2}+\frac{1}{2}$$

$$zy\left(p_{buckle,free}\right)=\frac{-\sin\left(\frac{2\pi y}{L}\right)L+2\pi y}{2L\pi} \qquad (2)$$

**# beam in tension: inclination angle $\phi$**

$$\phi(p)=simplify\left(subs\left(y=\frac{L}{2},z_L\cdot\frac{\mathrm{d}}{\mathrm{d}y}zT\right)\right); \quad \#\ definition\ of\ \phi$$

$$denominator := denom(rhs(\%)):$$

$$enumerator := subs\left(L_b=\frac{1}{p}\,\frac{\cosh(pL)-1}{\sinh(pL)},numer(rhs(\%\%))\right):$$

$$expand(subs(p=2q,enumerator)):$$

$$\phi(p)=\frac{subs\left(q=\frac{p}{2},\%\right)}{denominator}:collect\left(\%,z_L\right);$$

$$\phi(p\to inf)=limit(rhs(\%),p=\infty)\text{ assuming } L>0;$$

$$\phi(p)=\frac{z_L\left(\sinh\left(\frac{pL}{2}\right)pL_b-\cosh\left(\frac{pL}{2}\right)+1\right)}{L-2L_b}$$

$$\phi(p)=\frac{\left(-\frac{1}{\cosh\left(\frac{pL}{2}\right)}+1\right)z_L}{L-2L_b}$$

$$\phi(p\mapsto inf)=\frac{z_L}{L-2L_b}$$

## Appendix B. Static beam in compression: calculation of matrices (Maple)

**Basic equations**

```
restart;  # Flex in compression
z := A·sin(k y) + B·cos(k y) + C·y + D :
F := −P d/dy (z)  − EI· d^3/dy^3 (z) :
τ := EI d^2/dy^2 (z) :
θ := d/dy (z) :
eqns := subs( EI = P/k^2 ,
[      F0 = simplify(subs(y = 0, F)),
       τ0 = simplify(subs(y = 0, τ)),
       FL = simplify(subs(y = L, F)),
       τL = simplify(subs(y = L, τ)),
       z0 = simplify(subs(y = 0, z)),
       θ0 = simplify(subs(y = 0, θ)),
       zL = simplify(subs(y = L, z)) ,
       θL = simplify(subs(y = L, θ))
]) :
```

**End-to-end matrix**

```
vars := [A, B, C, D, zL, θL, FL, τL] :
sol := solve(eqns, vars)[1] :            # solve amplitudes and  y=L components
sys := [seq( sol[i], i = 5 ..8)] :       # select solved L components
var := [z0, θ0, F0, τ0] :                # create y=0 vector
V0 := Vector(var) :
                       #convert equations "sys" to Matrix M+ vector VL
with(LinearAlgebra) : M, VL := eval(−GenerateMatrix(sys, var)) :
VL = M, V0;
M := simplify(subs(sin(k L) = s, cos(k L) = c, M)) :
VL = M, V0;                                # display in most compact way
```

$$\begin{bmatrix} zL \\ \theta L \\ FL \\ \tau L \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sin(kL)}{k} & -\frac{kL-\sin(kL)}{Pk} & -\frac{\cos(kL)\,k-k}{Pk} \\ 0 & \cos(kL) & \frac{\cos(kL)-1}{P} & \frac{\sin(kL)\,k}{P} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{\sin(kL)\,P}{k} & -\frac{\sin(kL)}{k} & \cos(kL) \end{bmatrix}, \begin{bmatrix} z0 \\ \theta 0 \\ F0 \\ \tau 0 \end{bmatrix}$$

$$\begin{bmatrix} zL \\ \theta L \\ FL \\ \tau L \end{bmatrix} = \begin{bmatrix} 1 & \frac{s}{k} & \frac{-kL+s}{Pk} & \frac{-c+1}{P} \\ 0 & c & \frac{c-1}{P} & \frac{s\,k}{P} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{s\,P}{k} & -\frac{s}{k} & c \end{bmatrix}, \begin{bmatrix} z0 \\ \theta 0 \\ F0 \\ \tau 0 \end{bmatrix}$$

**Stiffness matrix**

```
# Stiffness matrix K.
eqnsK := subs(F0 = -F0r, τ0 = -τ0r, eqns) : #switch sign of internal forces at y=0
varsK := [A, B, C, D, F0r, τ0r, FL, τL] :
solK := solve(eqnsK, varsK)[1] :                # solve amplitudes, forces, torques
sysK := [seq( solK[i], i = 5..8)] :             #select solved L components
var := [z0, θ0, zL, θL] :                       #create  displacement  vector
Vdisp := Vector(var) :
                          #convert equations "sys" to Matrix K + vector VF
with(LinearAlgebra) : K, VF := eval(-GenerateMatrix(sysK, var)) :
K := simplify(K) :
K := (subs(sin(k L) = s, cos(k L) = c, K)) :
fac1 := P/(L s k + 2 c - 2) :  fac2 := fac1·k·s;        # common factors
VF = fac1, K/fac1, Vdisp;
simplify(subs(L c k = L_c_l, c = 1 - s·k·L_b, L_c_l = L c k, [VF = fac2, K/fac2, Vdisp])), L_b = (1/k) tan(k L/2)
```

$$
\begin{bmatrix} F0r \\ \tau 0r \\ FL \\ \tau L \end{bmatrix} = \frac{P}{L s k + 2 c - 2}, \begin{bmatrix} -s k & c-1 & s k & c-1 \\ c-1 & \frac{L c k - s}{k} & -c+1 & -\frac{k L - s}{k} \\ s k & -c+1 & -s k & -c+1 \\ c-1 & -\frac{k L - s}{k} & -c+1 & \frac{L c k - s}{k} \end{bmatrix}, \begin{bmatrix} z0 \\ \theta 0 \\ zL \\ \theta L \end{bmatrix}
$$

$$
\begin{bmatrix} F0r \\ \tau 0r \\ FL \\ \tau L \end{bmatrix} = \frac{P}{L - 2 L_b}, \begin{bmatrix} -1 & -L_b & 1 & -L_b \\ -L_b & \frac{L c k - s}{k^2 s} & L_b & \frac{-k L + s}{k^2 s} \\ 1 & L_b & -1 & L_b \\ -L_b & \frac{-k L + s}{k^2 s} & L_b & \frac{L c k - s}{k^2 s} \end{bmatrix}, \begin{bmatrix} z0 \\ \theta 0 \\ zL \\ \theta L \end{bmatrix}, \quad L_b = \frac{\tan\left(\frac{k L}{2}\right)}{k}
$$

# Appendix C. Static beam in tension: calculation of matrices (Maple)

**Basic equations.**
Indicated in green: differences with Appendix B.

*restart;* # *Flex in tension*

$$z := A\cdot\sinh(p\,y) + B\cosh(p\,y) + C\cdot y + D:$$

$$F := T\frac{d}{dy}(z) \; - EI\cdot\frac{d^3}{dy^3}(z):$$

$$\tau := EI\frac{d^2}{dy^2}(z):$$

$$\theta := \frac{d}{dy}(z):$$

$$\begin{aligned}
eqns := subs\Big(EI = \frac{T}{p^2},\ [\ & F0 = simplify(subs(y=0,\ F)),\\
& \tau 0 = simplify(subs(y=0,\ \tau)),\\
& FL = simplify(subs(y=L,\ F)),\\
& \tau L = simplify(subs(y=L,\ \tau)),\\
& z0 = simplify(subs(y=0,\ z)),\\
& \theta 0 = simplify(subs(y=0,\ \theta)),\\
& zL = simplify(subs(y=L,z)),\\
& \theta L = simplify(subs(y=L,\ \theta))\\
]\Big):&
\end{aligned}$$

$vars := [A, B, C, D, zL, \theta L, FL, \tau L]:$
$sol := solve(eqns, vars)[1]:$ **# solve amplitudes and y=L components**
$sys := [seq(\ sol[i], i=5..8)]:$ **# select solved L components**
$var := [z0, \theta 0, F0, \tau 0]:$ **# create y=0 vector**
$V0 := Vector(var):$
**#convert equations "sys" to Matrix M + vector VL**
$with(LinearAlgebra) : M, VL := eval(-GenerateMatrix(sys, var)):$
$VL = M, V0;$
$M := simplify(subs(\sinh(p\,L) = sh, \cosh(p\,L) = ch, M)):$
$VL = M, V0;$ **# display in most compact way**

**End-to-end matrix**

$vars := [A, B, C, D, zL, \theta L, FL, \tau L]:$
$sol := solve(eqns, vars)[1]:$ **# solve amplitudes and y=L components**
$sys := [seq(\ sol[i], i=5..8)]:$ **# select solved L components**
$var := [z0, \theta 0, F0, \tau 0]:$ **# create y=0 vector**
$V0 := Vector(var):$
**#convert equations "sys" to Matrix A + vector VL**
$with(LinearAlgebra) : M, VL := eval(-GenerateMatrix(sys, var)):$
$VL = M, V0;$
$M := simplify(subs(\sinh(p\,L) = sh, \cosh(p\,L) = ch, M)):$
$VL = M, V0;$ **# display in most compact way**

$$\begin{bmatrix} zL \\ \theta L \\ FL \\ \tau L \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sinh(p\,L)}{p} & \frac{p\,L - \sinh(p\,L)}{T\,p} & \frac{\cosh(p\,L)\,p - p}{T\,p} \\ 0 & \cosh(p\,L) & -\frac{\cosh(p\,L) - 1}{T} & \frac{\sinh(p\,L)\,p}{T} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\sinh(p\,L)\,T}{p} & -\frac{\sinh(p\,L)}{p} & \cosh(p\,L) \end{bmatrix}, \begin{bmatrix} z0 \\ \theta 0 \\ F0 \\ \tau 0 \end{bmatrix}$$

$$\begin{bmatrix} zL \\ \theta L \\ FL \\ \tau L \end{bmatrix} = \begin{bmatrix} 1 & \frac{sh}{p} & \frac{p\,L - sh}{T\,p} & \frac{ch - 1}{T} \\ 0 & ch & \frac{-ch + 1}{T} & \frac{sh\,p}{T} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{sh\,T}{p} & -\frac{sh}{p} & ch \end{bmatrix}, \begin{bmatrix} z0 \\ \theta 0 \\ F0 \\ \tau 0 \end{bmatrix}$$

## Stiffness matrix

```
# Stiffnes matrix K.
eqnsK := subs(F0 =−F0r, τ0 =−τ0r, eqns) : # switch sign of internal forces at y=0
varsK := [A, B, C, D, F0r, τ0r, FL, τL] :
solK := solve(eqnsK, varsK)[1] :          # solve amplitudes, forces, torques
sysK := [seq( solK[i], i = 5 ..8)] :       # select solved L components
var := [z0, θ0, zL, θL] :                  # create displacement vector
Vdisp := Vector(var) :

with(LinearAlgebra) : K, VF := −GenerateMatrix(sysK, var) : #convert equations "sys" to Matrix K+ vector VF
K := simplify(K) :
K := (subs(sinh(p L) = sh, cosh(p L) = ch, K)) :  # short notation
```

$fac1 := \dfrac{T}{p\,L\,sh - 2\,ch + 2}$ : $fac2 := fac1 \cdot p \cdot sh$ : # common factors

$VF = fac1, \dfrac{K}{fac1}, Vdisp;$

$simplify\left(subs\left(L\,ch\,p = L\_ch\_p, \mathbf{ch = 1 + sh \cdot p \cdot L_b}, L\_ch\_p = L\,ch\,p, \left[VF = fac2, \dfrac{K}{fac2}, Vdisp\right]\right)\right) \quad , L_b = \dfrac{1}{p}\tanh\left(\dfrac{pL}{2}\right);$

$$\begin{bmatrix} F0r \\ \tau 0r \\ FL \\ \tau L \end{bmatrix} = \frac{T}{p\,L\,sh - 2\,ch + 2}, \begin{bmatrix} sh\,p & ch - 1 & -sh\,p & ch - 1 \\ ch - 1 & \dfrac{L\,ch\,p - sh}{p} & -ch + 1 & -\dfrac{p\,L - sh}{p} \\ -sh\,p & -ch + 1 & sh\,p & -ch + 1 \\ ch - 1 & -\dfrac{p\,L - sh}{p} & -ch + 1 & \dfrac{L\,ch\,p - sh}{p} \end{bmatrix}, \begin{bmatrix} z0 \\ \theta 0 \\ zL \\ \theta L \end{bmatrix}$$

$$\begin{bmatrix} F0r \\ \tau 0r \\ FL \\ \tau L \end{bmatrix} = \frac{T}{L - 2\,L_b}, \begin{bmatrix} 1 & L_b & -1 & L_b \\ L_b & \dfrac{L\,ch\,p - sh}{p^2\,sh} & -L_b & \dfrac{-p\,L + sh}{p^2\,sh} \\ -1 & -L_b & 1 & -L_b \\ L_b & \dfrac{-p\,L + sh}{p^2\,sh} & -L_b & \dfrac{L\,ch\,p - sh}{p^2\,sh} \end{bmatrix}, \begin{bmatrix} z0 \\ \theta 0 \\ zL \\ \theta L \end{bmatrix}, L_b = \frac{\tanh\left(\dfrac{pL}{2}\right)}{p}$$

## Appendix D. Static beam, no tension or compression: calculation of matrices (Maple)

Basic equations: see Appendix B: beam in tension
same results obtained when starting from basic equations in Appendix C (beam in tension)

**End-to-end matrix**

```
# Given end-to-end matrix M for compressed beam: take limit P→0:
subs(s = sin(kL), c = cos(kL), k = sqrt(P/EI), M) :
MTM[limit](%, P, 0);
```

$$\begin{bmatrix} 1 & L & -\frac{L^3}{6\,EI} & \frac{L^2}{2\,EI} \\ 0 & 1 & -\frac{L^2}{2\,EI} & \frac{L}{EI} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -L & 1 \end{bmatrix}$$

**Stiffness matrix**

```
# Given end−to−end stiffness matrix K for compressed beam: take limit P→0:
KK := subs(s = sin(kL), c = cos(kL), k = sqrt(P/EI), K) :
fac := 12 EI / L^3 :
fac, MTM[limit](KK, P, 0) / fac;
```

$$\frac{12\,EI}{L^3}\begin{bmatrix} 1 & \frac{L}{2} & -1 & \frac{L}{2} \\ \frac{L}{2} & \frac{L^2}{3} & -\frac{L}{2} & \frac{L^2}{6} \\ -1 & -\frac{L}{2} & 1 & -\frac{L}{2} \\ \frac{L}{2} & \frac{L^2}{6} & -\frac{L}{2} & \frac{L^2}{3} \end{bmatrix}$$