Robust Model Predictive Control: A Survey

Alberto Bemporad and Manfred Morari

Automatic Control Laboratory, Swiss Federal Institute of Technology (ETH),
Physikstrasse 3, CH-8092 Zürich, Switzerland,
bemporad,morari@aut.ee.ethz.ch, http://control.ethz.ch/

Abstract. This paper gives an overview of robustness in *Model Predictive Control* (MPC). After reviewing the basic concepts of MPC, we survey the uncertainty descriptions considered in the MPC literature, and the techniques proposed for robust constraint handling, stability, and performance. The key concept of "closed-loop prediction" is discussed at length. The paper concludes with some comments on future research directions.

1 Introduction

Model Predictive Control (MPC), also referred to as *Receding Horizon Control* and *Moving Horizon Optimal Control*, has been widely adopted in industry as an effective means to deal with multivariable constrained control problems [42,57]. The ideas of receding horizon control and model predictive control can be traced back to the 1960s [28], but interest in this field started to surge only in the 1980s after publication of the first papers on IDCOM [59] and *Dynamic Matrix Control* (DMC) [25,26], and the first comprehensive exposition of *Generalized Predictive Control* (GPC) [22,23]. Although at first sight the ideas underlying the DMC and GPC are similar, DMC was conceived for multivariable constrained control, while GPC is primarily suited for single variable, and possibly adaptive control.

The conceptual structure of MPC is depicted in Fig. 1. The name MPC stems from the idea of employing an explicit *model* of the plant to be controlled which is used to *predict* the future output behavior. This prediction capability allows solving optimal control problems on line, where tracking error, namely the difference between the predicted output and the desired reference, is minimized over a future horizon, possibly subject to constraints on the manipulated inputs and outputs. When the model is linear, then the optimization problem is quadratic if the performance index is expressed through the ℓ_2-norm, or linear if expressed through the ℓ_1/ℓ_∞-norm. The result of the optimization is applied according to a *receding horizon* philosophy: At time t only the first input of the optimal command sequence is

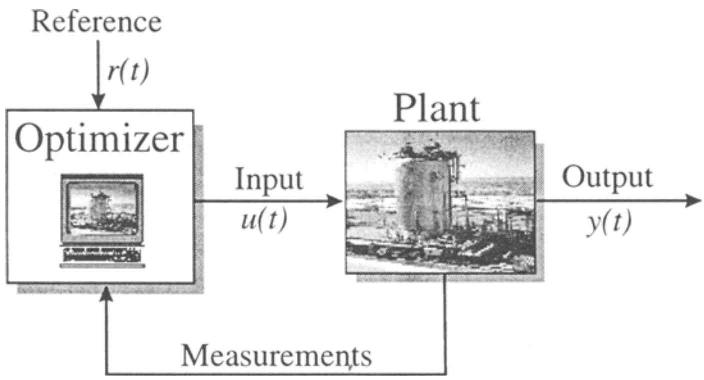

Fig. 1. Basic structure of Model Predictive Control

actually applied to the plant. The remaining optimal inputs are discarded, and a new optimal control problem is solved at time $t+1$. This idea is illustrated in Fig. 2. As new measurements are collected from the plant at each time t, the receding horizon mechanism provides the controller with the desired feedback characteristics.

The issues of feasibility of the on-line optimization, stability and performance are largely understood for systems described by linear models, as testified by several books [14,65,46,24,12,17] and hundreds of papers [36][1]. Much progress has been made on these issues for nonlinear systems [48], but for practical applications many questions remain, including the reliability and efficiency of the on-line computation scheme. Recently, application of MPC to hybrid systems integrating dynamic equations, switching, discrete variables, logic conditions, heuristic descriptions, and constraint prioritizations have been addressed in [9]. They expanded the problem formulation to include integer variables, yielding a Mixed-Integer Quadratic or Linear Program for which efficient solution techniques are becoming available.

A fundamental question about MPC is its *robustness* to model uncertainty and noise. When we say that a control system is robust we mean that stability is maintained and that the performance specifications are met for a specified range of model variations and a class of noise signals (uncertainty range). To be meaningful, any statement about "robustness" of a particular control algorithm must make reference to a specific uncertainty range as well as specific stability and performance criteria. Although a rich theory has been developed for the robust control of linear systems, very little is known about the robust control of linear systems with constraints. Recently, this type of problem has been addressed in the context of MPC. This paper will give an overview of these attempts to endow MPC with some robustness

[1] Morari [51] reports that a simple database search for "predictive control" generated 128 references for the years 1991-1993. A similar search for the years 1991-1998 generated 2802 references.

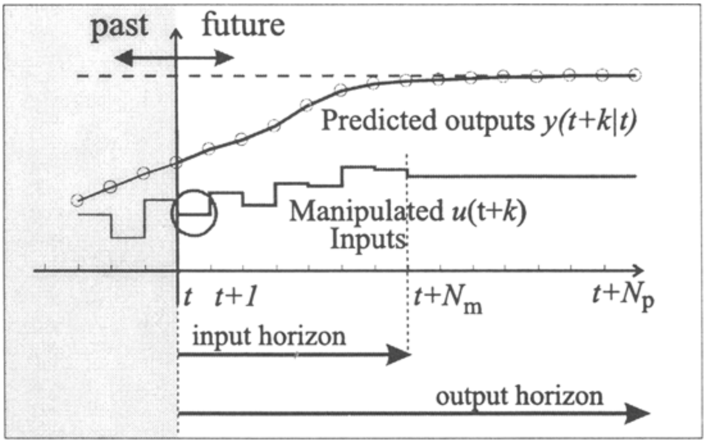

Fig. 2. Receding horizon strategy: only the first one of the computed moves $u(t)$ is implemented

guarantees. The discussion is limited to linear time invariant (LTI) systems with constraints. While the use of MPC has also been proposed for LTI systems without constraints, MPC does not have any practical advantage in this case. Many other methods are available which are at least equally suitable.

2 MPC Formulation

In the research literature MPC is formulated almost always in the state space. Let the model Σ of the plant to be controlled be described by the linear discrete-time difference equations

$$\Sigma : \begin{cases} x(t+1) = Ax(t) + Bu(t), & x(0) = x_0, \\ \quad y(t) = Cx(t) \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ denote the state, control input, and output respectively. Let $x(t+k, x(t), \Sigma)$ or, in short, $x(t+k|t)$ denote the prediction obtained by iterating model (1) k times from the current state $x(t)$.

A receding horizon implementation is typically based on the solution of the following open-loop optimization problem:

$$\min_{\mathbf{U} \triangleq \{u(t+k|t)\}_{k=t}^{t+N_m-1}} J(\mathbf{U}, x(t), N_p, N_m) = x^T(N_p)P_0x(N_p) + \sum_{k=0}^{N_p-1} x'(t+k|t)Qx(t+k|t) + \sum_{k=0}^{N_m-1} u'(t+k|t)Ru(t+k|t) \tag{2a}$$

$$\text{subject to} \qquad \begin{aligned} F_1 u(t+k|t) &\le G_1 \\ E_2 x(t+k|t) + F_2 u(t+k|t) &\le G_2 \end{aligned} \tag{2b}$$

$$\text{and} \qquad \text{“stability constraints”} \tag{2c}$$

where, as shown in Fig. 2, N_p denotes the length of the *prediction horizon* or *output horizon*, and N_m denotes the length of the *control horizon* or *input horizon* ($N_m \le N_p$). When $N_p = \infty$, we refer to this as the *infinite horizon problem*, and similarly, when N_p is finite, as a *finite horizon* problem. For the problem to be meaningful we assume that the polyhedron $\{(x,u) : F_1 u \le G_1,\ E_2 x + F_2 u \le G_2\}$ contains the origin ($x = 0, u = 0$). The constraints (2c) are inserted in the optimization problem in order to guarantee closed-loop stability, and will be discussed in the sequel.

The basic MPC law is described by the following algorithm:

Algorithm 1:

1. Get the new state $x(t)$
2. Solve the optimization problem (2)
3. Apply only $u(t) = u(t+0|t)$
4. $t \leftarrow t+1$. Go to 1.

2.1 Some Important Issues

Feasibility Feasibility of the optimization problem (2) at each time t must be ensured. Typically one assumes feasibility at time $t = 0$ and chooses the cost function (2a) and the stability constraints (2c) such that feasibility is preserved at the following time steps. This can be done, for instance, by ensuring that the shifted optimal sequence $\{u(t+1|t), \ldots, u(t+N_p|t), 0\}$ is feasible at time $t+1$. Also, typically the constraints in (2b) which involve state components are treated as *soft* constraints, for instance by adding the slack variable ϵ

$$E_2 x + F_2 u \le G_2 + \epsilon \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{3}$$

while pure input constraints $F_1 u \le G_1$ are maintained as *hard*. Relaxing the state constraints removes the feasibility problem at least for stable systems. Keeping the state constraints tight does not make sense from a practical point of view because of the presence of noise, disturbances, and numerical errors. As the inputs are generated by the optimization procedure, the input constraints can always be regarded as hard.

Stability In the MPC formulation (2) we have not specified the stability constraints (2c). Below we review some of the popular techniques used in the literature to "enforce" stability. They can be divided into two main classes. The first uses the value $V(t) = J(\mathbf{U}^*, x(t), N_p, N_m)$ attained for the minimizer $\mathbf{U}^* \triangleq \{u^*(t+1|t), \ldots, u^*(t+N_m|t)\}$ of (2) at each time t as a Lyapunov function. The second explicitly requires that the state $x(t)$ is shrinking in some norm.

- *End (Terminal) Constraint* [38,39]. The stability constraint (2c) is

$$x(t+N_p|t) = 0 \tag{4}$$

This renders the sequence $\mathbf{U}_1 \triangleq \{u^*(t+1|t), \ldots, u^*(t+N_m|t), 0\}$ feasible at time $t+1$, and therefore $V(t+1) \leq J(\mathbf{U}_1, x(t+1), N_p, N_m) \leq J(\mathbf{U}^*, x(t), N_p, N_m) = V(t)$ is a Lyapunov function of the system [34,10]. The main drawback of using terminal constraints is that the control effort required to steer the state to the origin can be large, especially for short N_p, and therefore feasibility is more critical because of (2b). The domain of attraction of the closed-loop (MPC+plant) is limited to the set of initial states x_0 that can be steered to 0 in N_p steps while satisfying (2b), which can be considerably smaller then the set of initial states steerable to the origin in an arbitrary number of steps. Also, performance can be negatively affected because of the artificial terminal constraint. A variation of the terminal constraint idea has been proposed where only the unstable modes are forced to zero at the end of the horizon [58]. This mitigates some of the mentioned problems.
- *Infinite Output Prediction Horizon* [34,58,71]. For asymptotically stable systems, no stability constraint is required if $N_p = +\infty$. The proof is again based on a similar Lyapunov argument.
- *Terminal Weighting Matrix* [37,40]. By choosing the terminal weighting matrix P_0 in (2a) as the solution of a Riccati inequality, stability can be guaranteed without the addition of stability constraints.
- *Invariant terminal set* [64]. The idea is to relax the terminal constraint (4) into the set-membership constraint

$$x(t+N_p|t) \in \Omega \tag{5}$$

and set $u(t+k|t) = F_{LQ}x(t+k|t)$, $\forall k \geq N_m$, where F_{LQ} is the LQ feedback gain. The set Ω is invariant under LQ regulation and such that the constraints are fulfilled inside Ω. Again, stability can be proved via Lyapunov arguments.
- *Contraction Constraint* [53,73]. Rather then relying on the optimal cost $V(t)$ as a Lyapunov function, the idea is to require explicitly that the state $x(t)$ is decreasing in some norm

$$\|x(t+1|t)\| \leq \alpha \|x(t)\|, \ \alpha < 1 \tag{6}$$

Following this idea, Bemporad [4] proposed a technique where stability is guaranteed by synthesizing a quadratic Lyapunov function for the system, and by requiring that the terminal state lies within a level set of the Lyapunov function, similar to (5).

Computation The complexity of the solver for the optimization problem (2) depends on the choice of the performance index and the stability constraint (2c). When $N_p = +\infty$, or the stability constraint has the form (4), or the form (5) and Ω is a polytope, the optimization problem (2) is a *Quadratic Program* (QP). Alternatively, one obtains a *Linear Program* (LP) by formulating the performance index (2a) in $\|\cdot\|_1$ or $\|\cdot\|_\infty$ [19]. The constraint (6) is convex, and is quadratic or linear depending if $\|\cdot\|_2$ or $\|\cdot\|_1/\|\cdot\|_\infty$ is chosen. When $\|\cdot\|_2$ is used, second-order cone programming algorithms [44] can be adopted conveniently.

3 Robust MPC — Problem Definition

The basic MPC algorithm described in the previous section assumes that the plant Σ_0 to be controlled and the model Σ used for prediction and optimization are the same, and no unmeasured disturbance is acting on the system. In order to talk about robustness issues, we have to relax these hypotheses and assume that (i) the true plant $\Sigma_0 \in \mathcal{S}$, where $\mathcal{S}$ is a given family of LTI systems, and/or (ii) an unmeasured noise $w(k)$ enters the system, namely

$$\Sigma : \begin{cases} x(t+1) = Ax(t) + Bu(t) + Hw(t), \quad x(0) = x_0, \\ \quad\;\; y(t) = Cx(t) + Kw(t) \end{cases} \tag{7}$$

where $w(t) \in \mathcal{W}$ and $\mathcal{W}$ is a given set (usually a polytope).

We will refer to ***robust stability***, ***robust constraint fulfillment***, and ***robust performance*** of the MPC law if the respective property is guaranteed for all possible $\Sigma_0 \in \mathcal{S}$, $w(t) \in \mathcal{W}$.

As part of the modelling effort it is necessary to arrive at an appropriate description of the uncertainty, i.e. the sets $\mathcal{S}$ and $\mathcal{W}$. This is difficult because there is very little experience and no systematic procedures are available. On one hand, the uncertainty description should be "tight", i.e. it should not include "extra" plants which do not exist in the real situation. On the other hand, there is a trade-off between realism and the resulting computational complexity of the analysis and controller synthesis. In other words, the uncertainty description should lead to a simple (non-conservative) ***analysis*** procedure to determine if a particular system with controller is stable and meets the performance requirements in the presence of the specified uncertainty. Alternatively, a computationally tractable *synthesis* procedure should exist to design a controller which is robustly stable and satisfies the robust performance specifications.

At present all the proposed uncertainty descriptions and associated analysis/synthesis procedures do little more than provide different handle to the engineer to detect and avoid sensitivity problems. They do not address the trade-off alluded to above, in a systematic manner.

For example, for simplicity some procedures consider only the uncertainty introduced by the set of unmeasured bounded inputs. There is the implicit assumptions that the other model uncertainty is in some way covered in this manner. There has been no rigorous analysis, however, to determine the exact relationship between the input set $\mathcal{W}$ and the covered set $\mathcal{S}$ — if such a relationship does indeed exist.

In the remaining part of the paper we will describe the different uncertainty descriptions which have been used in robust MPC, comment on the robustness analysis of standard (no uncertainty description) MPC, and give an overview of the problems associated with the synthesis of robust MPC control laws.

4 Uncertainty Descriptions

Different uncertainty sets $\mathcal{S}$, $\mathcal{W}$ have been proposed in the literature in the context of MPC, and are mostly based on *time-domain* representations. Frequency-domain descriptions of uncertainty are not suitable for the formulation of robust MPC because MPC is primarily a time-domain technique.

4.1 Impulse/Step-Response

Uncertainties on the impulse-response or step-response coefficients provide a practical description in many applications, as they can be easily determined from experimental tests, and allow a reasonably simple way to compute robust predictions. Uncertainty is described as range intervals over the coefficients of the impulse- and/or step-response. In the simplest SISO (single-input single-output) case, this corresponds to set

$$\Sigma : \; y(t) = \sum_{k=0}^{N} h(t)u(t-k) \tag{8}$$

and

$$\mathcal{S} = \{\Sigma : \; h_t^- \leq h(t) \leq h_t^+\}, \; t = 0, \ldots, N \tag{9}$$

where $[h_t^-, h_t^+]$ are given intervals. For $N < \infty$, $\mathcal{S}$ is a set of FIR models.

A similar type of description can be used for step-response models

$$y(t) = \sum_{k=0}^{N} s(t)[u(t-k) - u(t-k-1)], \; s(t) \in [s_t^-, s_t^+] \tag{10}$$

Impulse- and step-response descriptions are only equivalent when there is no uncertainty. If there is uncertainty they behave rather differently [8]. In order to arrive at a tight uncertainty description both may have to be used simultaneously and further constraints may have to be imposed on the coefficient variations as we will explain.

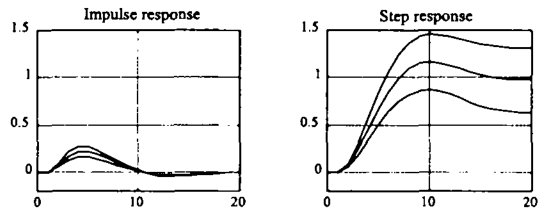

Fig. 3. Step-response interval ranges (right) arising from an impulse-response description (left)

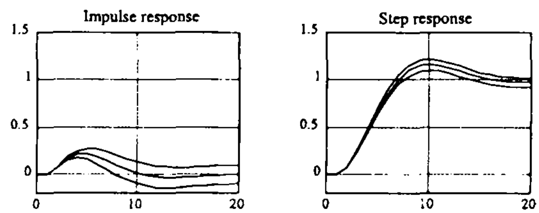

Fig. 4. Impulse-response interval ranges (left) arising from a step-response description (right)

Consider Fig. 3, which depicts perturbations expressed only in terms of the impulse response. The resulting step-response uncertainty is very large as $t \to \infty$. This may not be a good description of the real situation. Conversely, as depicted in Fig. 4, uncertainty expressed only in terms of the step response could lead to nonzero impulse-response samples at large values of t, for instance because the DC-gain from u to y is uncertain. Hence any a priori information about asymptotic stability properties would not be exploited.

Also, the proposed bounds would allow the step response to be highly oscillatory, though the process may be known to be overdamped. Similar comments apply to the impulse response. Thus this description may introduce high frequency model uncertainty artificially and may lead to a conservative design. This deficiency can be alleviated by imposing a correlation between neighboring uncertain coefficients as proposed by Zheng [73].

Another subtle point is that the uncertain FIR model (8) is usually unsuitable if the coefficients must be assumed to be time varying in the analysis or synthesis. In this case, the model would predict output variations even when the input is constant, which is usually undesirable. Writing the model in the form

$$\Sigma : \; y(t) = y(t-1) + \sum_{k=0}^{N} h(t)[u(t-k) - u(t-k-1)] \tag{11}$$

removes this problem.

In conclusion, simply allowing the step- or impulse-response coefficients to vary within intervals is rarely a useful description of model uncertainty

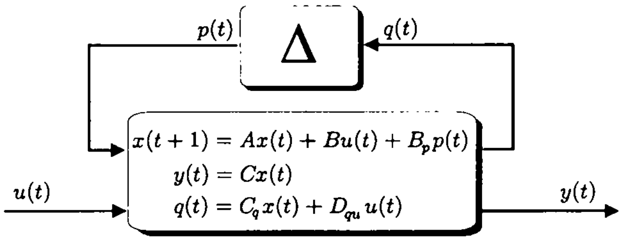

Fig. 5. Structured feedback uncertainty

unless additional precautions are taken. Nevertheless, compared to other descriptions, it leads to computationally simpler algorithms when adopted in robust MPC design, as will be discussed in Sect. 9

4.2 Structured Feedback Uncertainty

A common paradigm for robust control consists of a linear time-invariant system with uncertainties in the feedback loop, as depicted in Fig. 5 [35]. The operator Δ is block-diagonal, $\Delta = \text{diag}\{\Delta_1, \dots, \Delta_r\}$, where each block Δ_i represents either a memoryless time-varying matrix with $\|\Delta_i(t)\|_2 = \bar{\sigma}(\Delta_i(t)) \leq 1$, $\forall i = 1, \dots, r$, $t \geq 0$; or a convolution operator (e.g. a stable LTI system) with the operator norm induced by the truncated ℓ_2-norm less than 1, namely $\sum_{j=0}^{t} p'(j)p(j) \leq \sum_{j=0}^{t} q'(j)q(j)$, $\forall t \geq 0$. When Δ_i are stable LTI systems, this corresponds to the frequency domain specification on the z-transform $\hat{\Delta}_i(z)$ $\|\hat{\Delta}(z)\|_{\mathcal{H}_\infty} < 1$.

4.3 Multi-Plant

We refer to a *multi-plant* description when model uncertainty is parameterized by a finite list of possible plants [3]

$$\Sigma \in \{\Sigma_1, \dots, \Sigma_n\} \tag{12}$$

When we allow the real system to vary within the convex hull defined by the list of possible plants we obtain the so called polytopic uncertainty.

4.4 Polytopic Uncertainty

The set of models $\mathcal{S}$ is described as

$$\begin{aligned} x(t+1) &= A(t)x(t) + B(t)u(t) \\ y(t) &= Cx(t) \\ [A(t)\ B(t)] &\in \Omega \end{aligned}$$

and $\Omega = \text{Co}\{[A_1\ B_1], \dots, [A_M\ B_M]\}$, the convex hull of the "extreme" models $[A_i\ B_i]$ is a polytope. As remarked by Kothare *et al.* [35], polytopic uncertainty is a conservative approach to model a nonlinear system $x(t+1) = f(x(k), u(k), k)$ when the Jacobian $[\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial u}]$ is known to lie in the polytope Ω.

4.5 Bounded Input Disturbances

The uncertainty is limited to the unknown disturbance $w \in \mathcal{W}$ in (7), the plant Σ_0 is assumed to be known ($\mathcal{S} = \{\Sigma_0\}$). Also, one assumes that bounds on the disturbance are known, i.e. $\mathcal{W}$ is a given set. Although the assumption of knowing model Σ_0 might seem restrictive, the description of uncertainty by additive terms $w(t)$ that are known to be bounded in some norm is a reasonable choice, as shown in the recent literature on robust control and identification [50,45].

5 Robustness Analysis

We distinguish robustness *analysis*, i.e. analysis of the robustness properties of *standard* MPC designed for a nominal model without taking into account uncertainty, and *synthesis* of MPC algorithms which are robust by construction.

The robustness analysis of MPC control loops is more difficult than the synthesis, where the controller is designed in such a way that *it is* robustly stabilizing. This is not unlike the situation in the nominal case where the stability analysis of a closed loop MIMO system with multiple constraints is essentially impossible. On the other hand, the MPC technology leads naturally to a controller such that the closed loop system is guaranteed to be stable. There is a need for analysis tools, however, because standard MPC algorithms typically require less on-line computations, which is desirable for implementation.

Indeed, there are very few analysis methods discussed in the literature. By using a contraction mapping theorem, Zafiriou [68] derives a set of sufficient conditions for nominal and robust stability of MPC. Because the conditions are difficult to check he also states some necessary conditions for these sufficient conditions.

Genceli and Nikolaou [29] give sufficient conditions for robust closed-loop stability and investigate robust performance of dynamic matrix control (DMC) systems with hard input/soft output constraints. The authors consider an ℓ_1-norm performance index, a terminal state condition as a stability constraint, an impulse-response model with bounds on the variations of the coefficients. They derive a robustness test in terms of simple inequalities to be satisfied. This simplicity is largely lost in the extension to the MIMO case.

Primbs and Nevistíc [56] provide an off-line robustness analysis test of constrained finite receding horizon control which requires the solution of a set of linear matrix inequalities (LMIs). The test is based on the so called S-procedure and provides a (conservative) sufficient condition for $V(t)$ to be decreasing for all $\Sigma \in \mathcal{S}$, $\forall w(t) \in \mathcal{W}$. Both polytopic and structured uncertainty descriptions are considered. The authors also extend the idea to develop a robust synthesis method. It requires the solution of bilinear matrix inequalities (BMIs) and is computationally demanding.

More recently, Primbs [55] presented a new formulation of the analysis technique which is less conservative. The idea is to express the (optimal) input $u(t)$ obtained by the MPC law through the Lagrangian multipliers λ associated with the optimization problem (2a), and then to write the S-procedure in the $[x, u, \lambda]$-space.

6 Robust MPC Synthesis

In light of the discussion in Section 2.1, one has the following alternatives when synthesizing robust MPC laws:

1. Optimize performance of the nominal model or robust performance ?
2. Enforce state constraints on the nominal model or robustly ?
3. Adopt an *open-loop* or a *closed-loop prediction* scheme ?
4. How to guarantee robust stability ?

In the remaining part of the section we will discuss these questions.

6.1 Nominal vs. Robust Performance

The performance index (2a) depends on one particular model Σ and disturbance realization $w(t)$. In an uncertainty framework, two strategies are possible: (i) define a nominal model $\hat{\Sigma}$ and nominal disturbance $\hat{w}(t) = 0$, and optimize nominal performance; or (ii) solve the min-max problem to optimize robust performance

$$\min_{\mathbf{U}} \max_{\substack{\Sigma \in \mathcal{S} \\ \{w(k+t)\}_{k=0}^{N_p-1} \subseteq \mathcal{W}}} J(\mathbf{U}, x(t), \Sigma, w(\cdot)) \tag{13}$$

Min-max robust MPC was first proposed by Campo and Morari [18], and further developed in [2] and [69] for SISO FIR plants. Kothare *et al.* [35] optimize robust performance for polytopic/multi-model and structured feedback uncertainty, Scokaert and Mayne [63] for input disturbances only, and Lee and Yu [41] for linear time-varying and time-invariant state-space models depending on a vector of parameters $\theta \in \Theta$, where Θ is either an ellipsoid or a polyhedron. However it has two possible drawbacks. The first one is computational: Solving the problem (13) is computationally much more demanding than solving (2a) for a nominal model $\hat{\Sigma}$, $w(t) = 0$. However, under slightly restrict assumptions on the uncertainty, quite efficient algorithms are possible [73]. The second one is that the control action may be excessively conservative.

6.2 Input and State Constraints

In the presence of uncertainty, the constraints on the states variables (2b) can be enforced for all plant $\Sigma \in \mathcal{S}$ (robust constraint fulfillment) or for a nominal system $\hat{\Sigma}$ only. One also has to distinguish between hard and soft state constraints, although the latter are preferable for the reasons discussed in Section 2.1. As command inputs are directly generated by the optimizer, input constraints do not present any additional difficulty relative to the nominal MPC case.

For uncertainty described in terms of $w(t) \in \mathcal{W}$ only, when the set $\mathcal{W}$ is a polyhedron, state constraints can be tackled through the theory of *maximal output admissible sets* MOAS developed in [31,32]. The theory provides tools to enforce hard constraints on states despite the presence of input disturbances, by computing the minimum output prediction horizon N_p which guarantees robust constraint fulfillment.

In [47,63] tools from MOAS theory are used to synthesize robust minimum-time control on line. The technique is based on the computation of the level sets of the value function, and deals with hard input/state constraints.

Bemporad and Garulli [7] also consider the effect of the worst input disturbance over the prediction horizon, and enforce constraint fulfillment for all possible disturbance realizations (output prediction horizons are again computed through algorithms inspired by MOAS theory). In addition, the authors consider the case when full state information is not available. They use the so-called *set-membership* (SM) state estimation [62,13], through recursive algorithms based on parallelotopic approximation of the state uncertainty set [66,21].

When impulse-response descriptions are adopted, output constraints can be easily related to the uncertainty intervals of the impulse-response coefficients. For embedding input and state constraint into LMIs, the reader is referred to [35].

Robust fulfillment of state constraints can result in a very conservative behavior. Such an undesirable effect can be mitigated by using closed-loop prediction (see Sect. 8). Alternatively, when violations of the constraints are allowed, it can be more convenient to impose constraint satisfaction on the nominal plant $\hat{\Sigma}$ only.

Although unconstrained MPC for uncertain systems has been investigated, we do not review this literature here, because many superior linear robust control techniques are available.

7 Robust Stability

The minimum closed-loop requirement is robust stability, i.e., stability in the presence of uncertainty. In MPC the various design procedures achieve robust stability in two different ways: indirectly by specifying the performance objective and uncertainty description in such a way that the optimal control computations lead to robust stability; or directly by enforcing a type of robust contraction constraint which guarantees that the state will shrink for all plants in the uncertainty set.

7.1 Min-max performance optimization

While the generalization (13) of nominal MPC to the robust case appears natural, it is not without pitfalls. The min-max formulation as proposed in [18] alone does not guarantee robust stability as was demonstrated by Zheng [73] through a counterexample. To ensure robust stability the uncertainty must be assumed to be time varying. This added conservativeness may be prohibitive for demanding applications.

7.2 Robust contraction constraint

For stable plants, Zheng [73] introduces the stability constraint

$$\|x(t+1|t)\|_P \leq \lambda \|x(t)\|_P, \lambda < 1. \tag{14}$$

which forces the state to contract. When $P \succ 0$ is chosen as the solution of the Lyapunov equation $A'PA - P = -Q$, $Q \succ 0$, then this constraint can always be met for some u ($u(t+k) = 0$ satisfies this constraint and any constraint on u). Zheng [73] achieves robust stability by requiring the state to contract for all plants in $\mathcal{S}$. For the uncertain case constraint (14) is generalized by maximizing $\|x(t+1|t)\|_P$ over $\Sigma \in \mathcal{S}$.

For the multi-plant description, Badgwell [3] proposes a robust MPC algorithm for stable, constrained, linear plants that is a direct generalization of the nominally stabilizing regulator presented by Rawlings and Muske [58]. By using Lyapunov arguments, robust stability can be proved when the following stability constraint is imposed for each plant in the set.

$$J(\mathbf{U}, x(t), \Sigma_i) \leq J(\mathbf{U}_1^*, x(t), \Sigma_i) \tag{15}$$

This can be seen as a special case of the contraction constraint, where $J(\mathbf{U}, x(t), \Sigma_i)$ is the cost associated with the prediction model Σ_i for a fixed pair (N_p, N_m), and $\mathbf{U}_1 \triangleq \{u^*(t|t-1), \ldots, u^*(t-1+N_m|t-1), 0\}$ is the shifted optimal sequence computed at time $t-1$. Note that the stability constraints (15) are quadratic.

7.3 Robustly Invariant Terminal Sets

Invariant ellipsoidal terminal sets have been proposed recently in the nominal context as relaxations of the terminal equality constraint mentioned in Section 2.1 (see for instance [4] and references therein). Such techniques can be extended to robust MPC formulations, for instance by using the LMI techniques developed in [35]. Invariant terminal ellipsoid inevitably lead to Quadratically Constrained Quadratic Programs (QCQP), which can be solved through interior-point methods [44]. Alternatively, one can determine polyhedral robustly terminal invariant sets [16], which would lead to linear constraints, and therefore quadratic programming (QP), which is computationally cheaper than QCQP, at least for small/medium size problems.

8 Closed-Loop Prediction

Let us consider the design of a predictive controller which guarantees that hard state constraints are met in the presence of input disturbances $w(t)$. In order to achieve this task for every possible disturbance realization $w(t) \in \mathcal{W}$, the control action must be chosen safe enough to cope with the effect of the worst disturbance realization [30]. This effect is typically evaluated by predicting the *open-loop* evolution of the system driven by such a worst-case disturbance. This can be very conservative because in actual operation the disturbance effect is mitigated by feedback.

Lee and Yu [41] show that this problem can be addressed rigorously via Bellman's principle of optimality but that this is impractical for all but the simplest cases. As a remedy they introduce the concept of *closed-loop prediction*. For closed-loop prediction a feedback term $F_k x(t+k|t)$ is included in the expression for $u(t+k|t)$,

$$u(t+k|t) = F_k x(t+k|t) + v(k), \tag{16}$$

and the MPC controller optimizes with respect to both F_k and $v(k)$.

The benefit of this feedback formulation is discussed in [5] and is briefly reviewed here. In open-loop prediction the disturbance effect is passively suffered, while *closed-loop prediction* attempts to reduce the effect of disturbances. In open-loop schemes the uncertainty produced by the disturbances grows over the prediction horizon.

As an example, consider a hard output constraint $y_{\min} \le y(t) \le y_{\max}$. The output evolution due to (16) from initial state $x(t)$ for $F_k \equiv F$ is

$$\begin{aligned} y(t+k|t) = C(A+BF)^k x(t) + \sum_{k=0}^{t-1} C(A+BF)^k v(k) + \\ + \sum_{k=0}^{t-1} C(A+BF)^k H w(t-1-k) + K w(t) \end{aligned} \tag{17}$$

It is clear that F offers some degrees of freedom to counteract the effect of $w(t)$ by modifying the multiplicative term $(A+BF)^k$. For instance, if F renders $(A+BF)$ nilpotent, $y(t+k|t)$ is only affected by the last n disturbance inputs $w(k-n+1), \ldots, w(k)$, and consequently no uncertainty accumulation occurs. On the other hand, if F is set to 0 (open-loop prediction) and A has eigenvalues close to the unit circle, the disturbance action leads to

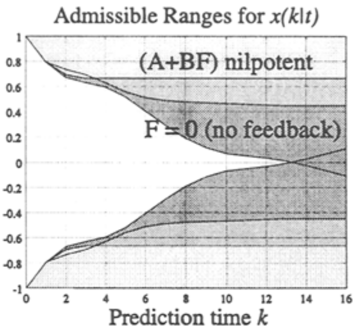

Fig. 6. Benefits of closed-loop prediction: Admissible ranges for the output $y(t+k|t)$ for different feedback LQ gains F (input weight $\rho = 0, 1, +\infty$)

very conservative constraints, and consequently to poor performance. Fig. 6 shows this effect for different gains F, selected by solving LQ problems with unit output weight and input weights $\rho = 0$, $\rho = 1$, and $\rho = +\infty$. The last one corresponds to open-loop prediction ($F = 0$).

For a wide range of uncertainty models, Kothare *et al.* [35] design, at each time step, a state-feedback control law that minimizes a 'worst-case' infinite horizon objective function, subject to input and output constraints. The authors transform the problem of minimizing an upper bound on the worst-case objective function to a convex optimization involving linear matrix inequalities (LMIs). A robustly stable MPC algorithm results. On one hand the closed-loop formulation reduces the conservativeness. On the other hand, the algorithm requires the uncertainty to be time-varying which may be conservative for some applications.

9 Computation

In the previous sections we discussed the formulation of various robust MPC algorithms, which differed with respect to the uncertainty descriptions, the performance criteria, and the type of stability constraints. In practice the choice is often dictated by computational considerations.

Uncertainty descriptions involving impulse/step-response coefficients or bounded input disturbances are easier to deal with, as the optimization problem can often be recast as an LP.

In [35] the authors solve optimal control problems with state-space uncertainty descriptions through LMIs. For the technique proposed in [33], where a worst case quadratic performance criterion is minimized over a finite set of models subject to input/state constraints, the authors report that problems with more than 1000 variables and 5000 constraints can be solved in a few minutes on a workstation by using interior-point methods.

For impulse and step response uncertainty, Bemporad and Mosca [8] propose a computationally efficient approach based on the *reference gover-*

nor [32,6]. The main idea is to separate the stabilization problem from the robust constraint fulfillment problem. The first is left to a conventional linear robust controller. Constraints are enforced by manipulating the desired set-points at a higher level (basically the reference trajectory is smoothed out when abrupt set-point changes would lead to constraint violations). The advantages of this scheme are that typically only one scalar degree of freedom suffices, as reported in [8], where the on-line optimization is reduced to a small number of LPs.

10 Conclusions and Research Directions

While this review is not complete it reflects the state of the art. It is apparent that none of the methods presented is suitable for use in industry except maybe in very special situations. The techniques are hardly an alternative to ad hoc MPC tuning based on exhaustive simulations for ranges of operating conditions. Choosing the right robust MPC technique for a particular application is an art and much experience is necessary to make it work — even on a simulation case study. Much research remains to be done, but the problems are difficult. Some topics for investigation are suggested next.

Contraction constraints have been shown to be successful tools to get stability guarantees, but typically performance suffers. By forcing the state to decrease in a somewhat arbitrary manner, the evolution is driven away from optimality as measured by the performance index. The contraction constraints which are in effect Lyapunov functions are only sufficient for stability. In principle, less restrictive criteria could be found. ***Integral Quadratic Constraints*** [49] could be embedded in robust MPC in order to deviate as little as possible from optimal performance but still guarantee robust stability.

Robustly invariant terminal sets can be adopted as an alternative to contraction constraints. As mentioned in Sect. 7, ellipsoids and polyhedra can be determined off-line, by utilizing tools from robustly invariant set theories [16].

The benefits of closed-loop prediction were addressed in Sect. 8. However very little research has been done toward the development of computationally efficient MPC algorithms.

Finally, the algorithms should be linked to appropriate identification procedures for obtaining the models and the associated uncertainty descriptions.

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