4.1 Vector spaces

Definition. A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ and for all scalars c and d. $\vec{u}, \vec{v} \to V$

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$ is in V. $\vec{u}+\vec{v} \to V$
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$. → if no 0 vector, not a vector space
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$. → $\vec{u} \to V$, $c\vec{u} \to V$
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

Following simple facts can be proved from the axioms: For each $\mathbf{u}$ in V and scalar c,

- $0\mathbf{u}=\mathbf{0}$.
- $c\mathbf{0}=\mathbf{0}$.
- $-\mathbf{u}=(-1)\mathbf{u}$.

Examples of vector spaces

$\{y_k\} + \{z_k\} = (\{y_k + z_k\})$

y -1 + z -2
y -2 + z -1
y 0 + z 0
y 1 + z 1

1. $\mathbb{R}^n$ $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$ ← real number

→ -∞ to ∞

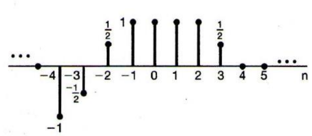
2. $\mathbb{S}$: space of all doubly infinite sequences of numbers

$$\{y_k\}=(\ldots,y_{-2},y_{-1},y_0,y_1,y_2,\ldots)$$

Define addition: If $\{z_k\}$ is another element in $\mathbb{S}$, then sum $\{y_k\}+\{z_k\}$ is the sequence $\{y_k+z_k\}$ formed by adding corresponding terms of $\{y_k\}$ and $\{z_k\}$.

Define scalar multiplication: The scalar multiple $c\{y_k\}$ is the sequence $\{cy_k\}$.

every element of sequence



Examples of vector spaces (contd.)

3. $\mathbb{P}_n$: polynomials of degree n, $(n>0)$

$$\mathbf{p}(t)=a_0+a_1t+a_2t^2+\cdots+a_nt^n$$

Degree : highest power of t in $\mathbf{p}$. If $\mathbf{p}(t)=a_0\neq 0$, degree of $\mathbf{p}$ is zero.

If all the coefficients are zero, $\mathbf{p}$ is called the *zero polynomial*. → $\vec{0}$ vector member of vector space

Define addition: If $\mathbf{q}(t)=b_0+b_1t+b_2t^2+\cdots+b_nt^n$, then the sum $\mathbf{p}+\mathbf{q}$ is defined by

$$\begin{aligned}(\mathbf{p}+\mathbf{q})(t) &= \mathbf{p}(t)+\mathbf{q}(t)\\ &= (a_0+b_0)+(a_1+b_1)t+\cdots+(a_n+b_n)t^n\end{aligned}$$

Define scalar multiplication: The scalar multiple $c\mathbf{p}$ is the polynomial

$$(c\mathbf{p})(t)=c\mathbf{p}(t)=ca_0+(ca_1)t+\cdots+(ca_n)t^n$$

→ multiplied by c

$p(t) + q(t) = r(t)$ → in $\mathbb{R}_n$

- Axioms 1 and 6: satisfied because $\mathbf{p}+\mathbf{q}$ and $c\mathbf{p}$ are polynomials of degree less than or equal to n
- Axioms 2, 3, and 7-10: satisfied because of properties of real numbers
- Axiom 4: zero polynomial acts as the zero vector
- Axiom 5: $(-1)\mathbf{p}$ acts as the negative of $\mathbf{p}$

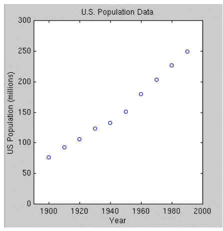
$7: c(\vec{u}+\vec{v}) = c\vec{u} + c\vec{v}$

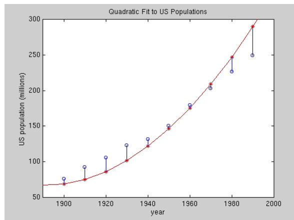
$c(p+q) = cp + cq$

$4: \vec{u} + \vec{0} = \vec{u}$

$p + 0 = p$

Non-linear curve fitting





vectors not limited to $[\vdots]$, can be {sequence}, polynomials & functions

Examples of vector spaces (contd.)

4. Let V be the set of all real-valued functions defined on a set $\mathbb{D}$

↳ hence V is a vector space

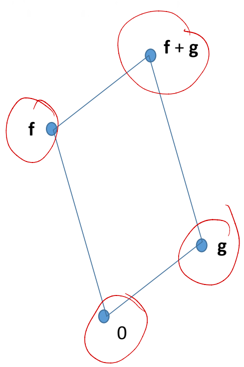
Define addition: $\mathbf{f}+\mathbf{g}$ is the function whose value at t in the domain $\mathbb{D}$ is $\mathbf{f}(t)+\mathbf{g}(t)$.

Define scalar multiplication: The scalar multiple $c\mathbf{f}$ is the function whose value at t is $c\mathbf{f}(t)$.

$c\vec{u}$ in V $\quad cf(t) \to D$ $\quad \Downarrow D$

- Axioms 1 and 6: obvious
- Axiom 4: zero vector is the function that is identically zero, $\mathbf{f}(t) = 0$
- Axiom 5: Negative of $\mathbf{f}$ is $(-1)\mathbf{f}$
- Rest of the axioms: satisfied because of properties of real numbers

Think of an element of a vector space as one "point" or vector in the vector space



4.2 Subspaces

$\vec{0} \in V$

↳ also in H

Definition. A **subspace** of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.

$V: \vec{u}, \vec{v}, \vec{u}+\vec{v}$

b. H is closed under vector addition, i.e., for each $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

$V: c\vec{u}$

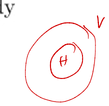
c. H is closed under scalar multiplication, i.e., for each $\mathbf{u}$ in H and each scalar c, the vector $c\mathbf{u}$ is in H.

In applications, subspaces of $\mathbb{R}^n$ arise

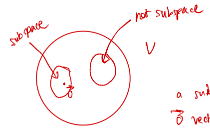
- as the set of all solutions to a system of homogeneous linear equations
- as the set of all linear combinations of certain specified vectors

Subspace H of V is itself a *vector space* under vector space operations already defined in V.

add , zero , scalar



- Axioms 1, 4 and 6: same as (a), (b) and (c)
- Axioms 2, 3, and 7-10: automatically true in H because they apply to all elements of V, including those in H
- Axiom 5: if $\mathbf{u}$ is in H, then $(-1)\mathbf{u}$ is in H [by property (c) and by $(-1)\mathbf{u} = -\mathbf{u}$]



a subspace with just the $\vec{0}$ vector is also a subspace

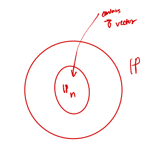
Examples of subspaces

1. The set consisting of only the zero vector in a vector space V : **zero subspace** written as $\{\mathbf{0}\}$.

Examples of subspaces (contd.)

$f(t) + g(t)$

$p(t) + q(t)$



2. $\mathbb{P}$: set of all polynomials with real coefficients, with operations in $\mathbb{P}$ defined as for functions.
For each $n \geq 0$, $\mathbb{P}_n$ is a subspace of $\mathbb{P}$. [properties (a), (b) and (c)]

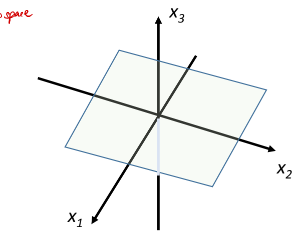
$\mathbb{R}^2$ not even a subset of $\mathbb{R}^3$ $\therefore$ not a subspace

3. Is $\mathbb{R}^2$ a subspace of $\mathbb{R}^3$? $\mathbb{R}^2$ $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbb{R}^3$, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

No

Is the set $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} : s \text{ and } t \text{ are real} \right\}$ a subset of $\mathbb{R}^3$?

Yes



H is a subspace of $\mathbb{R}^3$

4.3 Subspace spanned by a set

Recall: Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is the set of all vectors that can be written as linear combinations of $\mathbf{v}_1, \cdots, \mathbf{v}_p$.

$a_1 v_1 + a_2 v_2$

Exercise 4.3.1

Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V, let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that H is a subspace of V.

V, span $\{\vec{v}\}$ ← subspace

Theorem 4.1. *If* $\mathbf{v}_1, \cdots, \mathbf{v}_p$ *are in a vector space* V, *then* Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ *is a subspace of* V.

Span$\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ is called **the subspace spanned** by $\mathbf{v}_1, \cdots, \mathbf{v}_p$.
Given any subspace H of V, a **spanning set** for H is a set $\mathbf{v}_1, \cdots, \mathbf{v}_p$ in H such that $H = \text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$.

conditions to check it is subspace $\begin{cases} \text{(a) zero vector} \\ \text{(b) closed under addition} \\ \text{(c) closed under multiplication} \end{cases}$

(a) $\vec{0} = \vec{0} v_1 + \vec{0} v_2 \Rightarrow$ zero vector is in H.

(b) closed under addition : $\vec{u} = s_1 \vec{v}_1 + s_2 \vec{v}_2$, $\vec{w} = t_1 \vec{v}_1 + t_2 \vec{v}_2$

Axioms 2, 3, 8 for V, $\vec{u} + \vec{w} = (s_1 \vec{v}_1 + s_2 \vec{v}_2) + (t_1 \vec{v}_1 + t_2 \vec{v}_2) = (s_1 + t_1) \vec{v}_1 + (s_2 + t_2) \vec{v}_2$

$\therefore \vec{u} + \vec{w}$ is in H

Closed under scalar multiplication :

If c is any scalar, by axioms 7 and 9,

$c\vec{u} = c(s_1 \vec{v}_1 + s_2 \vec{v}_2) = c s_1 \vec{v}_1 + c s_2 \vec{v}_2$

$\Rightarrow c\vec{u}$ is in H

(a), (b), (c) satisfied : H is a subspace of V

← $\boxed{\text{Refer to theorem 4.1}}$

R'

Exercise 4.3.2

Let H be the set of all vectors of the form $(a-3b, b-a, a, b)$ where a and b are in $\mathbb{R}$. Show that H is a subspace of $\mathbb{R}^4$.

real numbers

Solution:

$$H = \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{v}_1} + b\underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{v}_2}.$$

$H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

$\therefore H$ is a subspace of $\mathbb{R}^4$.

H is a linear combination of v_1 & v_2 $\Rightarrow$ span of v_1, v_2

4.4 The Null Space of a Matrix

NO PRE LEC

System of homogeneous equations

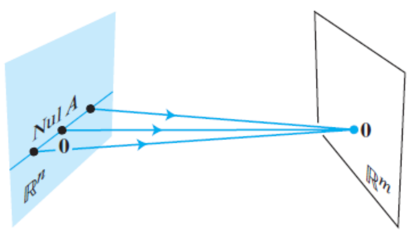
$$\begin{aligned} x_1 - 3x_2 - 2x_3 &= 0 \\ -5x_1 + 9x_2 + x_3 &= 0 \end{aligned}$$

Rewritten as $A\mathbf{x} = \mathbf{0}$, where $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$

Null space: the set of $\mathbf{x}$ that satisfy $A\mathbf{x} = \mathbf{0}$

Definition. The **null space** of an $m \times n$ matrix A, written as $\mathbf{N}(A)$, is the set of all soutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$, i.e.,

$$\mathbf{N}(A) = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}$$



All $\mathbf{x}$ in $\mathbb{R}^n$ mapped into the zero vector in $\mathbb{R}^m$ via the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

$Au = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore$ u belongs to N(A)

Exercise 4.4.1

Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Determine if $\mathbf{u}$ belongs to $\mathbf{N}(A)$.

Exercise 4.4.2

Describe the null space of $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$.

Solution

Apply elimination on $A\mathbf{x} = \mathbf{0}$.

$$\begin{aligned} ①\quad x_1 + 2x_2 &= 0 \\ ②\quad 3x_1 + 6x_2 &= 0 \end{aligned} \rightarrow \begin{aligned} x_1 + 2x_2 &= 0 \\ \underbrace{0 = 0}_{② - 3①} \end{aligned}$$

The line $x_1 + 2x_2 = 0$ is $\mathbf{N}(A)$. It contains all solutions (x_1, x_2).

$x_1 = -2x_2$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2t \\ t \end{pmatrix}$

Set free variable x_2 to some value, say, 1. Then $x_1 = -2$.

$\mathbf{N}(A)$ contains all multiples of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Theorem 4.2. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to the system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.*

Proof. □

$Ax = 0 \qquad x = 0$

$A(x_1 + x_2) = \underbrace{Ax_1}_{0} + \underbrace{Ax_2}_{0} = 0$

$Acx = c\underbrace{Ax}_{0} = 0$

closed under summation , multiplication

Exercise 4.4.3

Find a spanning set for the null space of the matrix

$\begin{bmatrix} -3 \\ 1 \\ \end{bmatrix}$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Solution

Apply elimination on $A\mathbf{x} = \mathbf{0}$ to obtain reduced row echelon form of augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$.

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{array}{rcrcrcrcrcl} x_1 & - & 2x_2 & & & - & x_4 & + & 3x_5 & = & 0 \\ & & & & x_3 & + & 2x_4 & - & 2x_5 & = & 0 \\ & & & & & & & & 0 & = & 0 \end{array}$$

General solution : $x_1 = 2x_2 + x_4 - 3x_5$ and $x_3 = -2x_4 + x_5$, with x_2, x_4, and x_5 as free variables.

Decompose the vector giving the general solution into a linear combination of vectors where *the weights are the free variables.*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = x_2\mathbf{u} + x_4\mathbf{v} + x_5\mathbf{w}$$

Every linear combination of $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ is an element of $\mathbf{N}(A)$ and vice versa. Thus $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a spanning set for $\mathbf{N}(A)$.

4.5 The Column Space of a Matrix

Definition. The **column space** of an $m \times n$ matrix A, written as $\mathbf{C}(A)$, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$, then $\mathbf{C}(A) = \operatorname{Span}\{\mathbf{a}_1 \cdots \mathbf{a}_n\}$, i.e.,

each is a column

$$\mathbf{C}(A) = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$$

$A = [\vec{a}_1, \vec{a}_2]$

linear combination : $x_1\vec{a} + x_2\vec{a}_2 = \vec{b} \Leftarrow Ax$

$[\vec{a}_1, \vec{a}_2]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2$
$= \vec{b}$

Theorem 4.3. *The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.*

Proof. $\operatorname{Span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ is a subspace [Theorem 4.1]

Columns of A are in $\mathbb{R}^m$. □

m entries

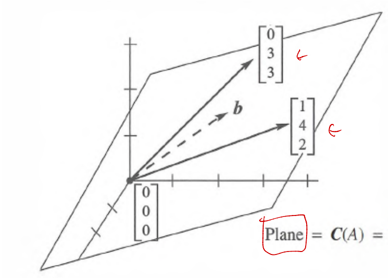
Recall (from Chapter 1, slide 29)): The columns of A span $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$.

$x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b} \Leftarrow Ac$

find a x_1 & x_2 for every b in $\mathbb{R}^m$ $\Rightarrow$ a_1 & a_2 span all of $\mathbb{R}^m$ $\Rightarrow$ becomes column space of matrix A

Restating the above:
The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

$m = 3$, $n = 2$



$$A = \begin{array}{c} \begin{array}{cc} \vec{a}_1 & \vec{a}_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix} \end{array}_{3\times 2}$$

$C(A)$ is 3 A

$$b = .4\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + .3\begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$A\vec{x}$: $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$

$Ax = b$ has $x = \begin{bmatrix} .4 \\ .3 \end{bmatrix}$

Plane $= C(A) =$ all vectors Ax

Exercise 4.5.1

Let $A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$. 3×4

$m = 3$, $n = 4$

(a) $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + x_4\vec{a}_4 = [\vec{b}_1]_{3\times 1} \in \mathbb{R}^3$

(b) Null space: $A\vec{x} = \vec{0}$ $[\ ]_{3\times 4} \begin{bmatrix} \vec{x} \end{bmatrix}_{4\times 1} = \vec{0}$ $\rightarrow \vec{x} \in \mathbb{R}^4$

a. If the column space of A is a subspace of $\mathbb{R}^k$, what is k? $k = 3$

b. If the null space of A is a subspace of $\mathbb{R}^k$, what is k? $k = 4$

c. Find a nonzero vector in $\mathbf{C}(A)$ and a nonzero vector in $\mathbf{N}(A)$.

(c) $C(A) = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

$\vec{x}$ in $N(A) = \begin{bmatrix} 9 \\ 5 \\ -1 \\ 0 \end{bmatrix}$

$x_1 + 9x_3 = 0$
$x_2 - 5x_3 = 0$
$x_4 = 0$

$x_1 = -9x_3$
$x_2 = 5x_3$
$x_4 = 0$

solve for $Ax = 0$

$\Rightarrow x_3$ is the free variable, choose $x_3 = -1$

[Augmented] [0]

$$\rightarrow \begin{bmatrix} 1 & 0 & 9 & 0 & 0 \\ 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

d. If $\mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$

(i) $A\vec{u} = \vec{0}$? $\begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i. Determine if $\mathbf{u}$ is in $\mathbf{N}(A)$. No. Could $\mathbf{u}$ be in $\mathbf{C}(A)$? No

ii. Determine if $\mathbf{v}$ is in $\mathbf{C}(A)$. Yes. Could $\mathbf{v}$ be in $\mathbf{N}(A)$? No

Column space of A should have only 3 entries, but u has 4 entries

$\therefore$ u cannot be in $C(A)$ $\Rightarrow$ $C(A)$ is a subspace of $\mathbb{R}^3$

(ii) $A\vec{x} = \vec{v}$ is consistent or not?

$[A\ \vec{v}] \rightarrow \begin{bmatrix} 2 & 4 & -2 & 1 & 3 \\ 0 & 1 & -5 & -4 & -2 \\ 0 & 0 & 0 & 17 & 1 \end{bmatrix} \Rightarrow$ REF

$\therefore$ consistent $\Rightarrow \vec{v}$ is in $C(A)$

$N(A)$: $A\vec{v} = \vec{0}$, 3×4, $\vec{v}_{4\times 1}$ $\leftarrow$ $N(A)$ is a subspace of $\mathbb{R}^4$

v in $\mathbb{R}^3$

Nul A $\equiv \mathbf{N}(A)$

Col A $\equiv \mathbf{C}(A)$

N(A)	C(A)
1. Nul A is a subspace of $\mathbb{R}^n$.	1. Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	2. Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	3. It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	4. There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	5. A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	6. Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	7. Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	8. Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

4.6 Kernel and Range of a Linear Transformation

$A\vec{x} = \vec{b}$, $T(\vec{x})$

$\Rightarrow x$ got transformed

- Generalize definition of linear transformation to include vector spaces

Definition. A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W, such that

1. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}$, $\mathbf{v}$ in V, and
2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u}$ in V and all scalars c.

$A\vec{x} = \vec{0}$

$T(\vec{u}) = \vec{0}$

forms kernel of transformation

Definition. The **kernel**(or **null space**) of such a T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$ (the zero vector in W).

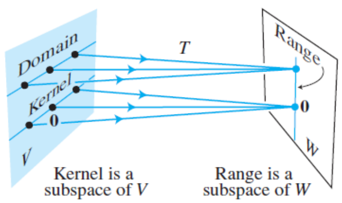
Definition. The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

$A\vec{x} = \vec{0}$

$A\vec{x}$

If $T(\mathbf{x}) = A\mathbf{x}$, then

- Kernel $= \mathbf{N}(A)$
- Range $= \mathbf{C}(A)$



4.7 Bases

- Linear independence (again! introduced in Sec 1.9 for $\mathbb{R}^n$)

Definition. An indexed set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in V is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$$

$c_1\vec{v}_1 + c_2\vec{v}_2 = 0$

$c_1\vec{v}_1 = -c_2\vec{v}_2$

$v_2 = -\frac{c_1}{c_2}\vec{v}_1$ → v_2 as a multiple of v_1

$c_1 \neq 0$

$c_2 \neq 0$

$c_1\vec{v}_1 = \vec{0}$, $\vec{v}_1 \neq 0$

only has trivial solution

- A set containing a single vector $\mathbf{v}$ is linearly independent if and only if $\mathbf{v} \neq \mathbf{0}$.
- A set of two vectors is linearly dependent if and only if one of the vectors is a multiple of other.
- Any set containing the zero vector is linearly dependent. $c_1\vec{0} + c_2\vec{v}_2 = \vec{0}$

Theorem 4.4. *An indexed set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.*

(Skip proof)

$\vec{v}_1, \vec{v}_2, \vec{v}_3$

$\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2$

⇒ linearly dependent

$p_3(t) = 4p_1(t) + p_2(t)$

linearly dependent

⇒ based previous theorem

Examples: Linearly independent or not?

- $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t$, and $\mathbf{p}_3(t) = 4 - t$. (polynomial)
- The set $\{\sin t, \cos t\}$ in $\underbrace{C[0,1]}_{\text{vector space}}$, the space of all continuous functions on $0 \le t \le 1$

  sin t & cos t not multiples of each other in vectorspace C[0,1]
  ⇒ not linear independent
- The set $\{\sin t \cos t, \sin 2t\}$ in $C[0,1]$, the space of all continuous functions on $0 \le t \le 1$

$\sin 2t = 2 \sin t \cos t$ ⇒ multiple of each other

⇒ linearly dependent

Basis : "minimum" set of vector that span the subspace
↓
no redundant vectors

Definition. Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a **basis** for H if

(i) $\mathcal{B}$ is a linearly independent set, and

(ii) the subspace spanned by $\mathcal{B}$ coincides with H, i.e.,

$$H = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

- Also true when $H = V$ since every vector space is a subspace of itself. $\Rightarrow$ a basis of V is a linearly independent set that spans V.

Examples:

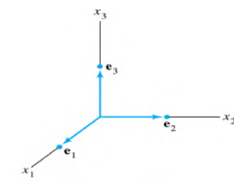
- Invertible matrix $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$

  Columns of A form a basis for $\mathbb{R}^n$

  $\left.\begin{array}{l} \text{– they are linearly independent} \\ \text{– they span } \mathbb{R}^n \end{array}\right\}$ Invertible matrix theorem (Theorem 2.4)

- Columns of I_n

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$



The standard basis for $\mathbb{R}^3$.

The set $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ is called the **standard basis** for $\mathbb{R}^n$

Exam-like question

Exercise 4.7.1:

Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

check for linearly independence :

$$\begin{vmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 7 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 6$$

↓
check it invertible

Since det $\neq 0$, it is linearly independent.

Exercise 4.7.2:

Let $S = \{1, t, t^2, \dots, t^n\}$. Verify that S is a basis for $\mathbb{P}_n$. This basis is called the **standard basis** for $\mathbb{P}_n$.

4.8 The Spanning Set Theorem

$c_1\vec{v_1} + c_2\vec{v_2} = c_1\vec{v_1} + c_2\vec{v_2} + 0\vec{v_3}$

$\Rightarrow$ every vector in span $\{\vec{v_1}, \vec{v_2}\}$ (LHS) belongs to H (RHS)

Exercise 4.8.1:

Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$, and H = Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. subspace

Note that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$. Show that Span$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ = Span$\{\mathbf{v}_1, \mathbf{v}_2\}$. Then, find a basis for the subspace H.

The set v_1, v_2, v_3 is not linearly independent, cause v_3 is a linear combination of v_1 & v_2

$\therefore$ set not linearly independent

Let $\vec{x}$ be any vector in H

since $\vec{x}$ is in the span $\{v_1, v_2, v_3\}$, it can be written as

$\vec{x} = c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3}$

$= c_1\vec{v_1} + c_2\vec{v_2} + c_3(5\vec{v_1} + 3\vec{v_2})$

$= c_1\vec{v_1} + c_2\vec{v_2} + 5c_3\vec{v_1} + 3c_3\vec{v_2}$

$= c_1\vec{v_1} + 5c_3\vec{v_1} + c_2\vec{v_2} + 3c_3\vec{v_2}$

$= (c_1 + 5c_3)\vec{v_1} + (c_2 + 3c_3)\vec{v_2}$

$\Rightarrow \vec{x}$ belongs to span $\{\vec{v_1}, \vec{v_2}\}$ $\rightarrow$ as we have written $\vec{x}$ as a linear combination of v_1 and v_2

$\therefore$ H & span $\{\vec{v_1}, \vec{v_2}\}$ are the same set of vectors

Since $\{\vec{v_1}, \vec{v_2}\}$ are linearly independent and they span H, they form a basis for H

Theorem 4.5. *Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a set in V, and let H = Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.*

a. If one of the vectors in S - say, $\mathbf{v}_k$ - is a linear combination of the remaining vectors in S, then the set formed from S by removing $\mathbf{v}_k$ still spans H.

b. If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

(Skip proof)

span $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ = span $\{\vec{v_1}, \vec{v_2}\}$

Subset of this: $\{\vec{v_1}, \vec{v_2}\}$ basis for H

remove v_3 $\downarrow$ make it a linear combination of v_1 & v_2

4.9 Bases for $\mathbf{N}(A)$ and $\mathbf{C}(A)$

if there is a set of vectors in a subspace and if they are:

- linearly dependent
- span subspace

$\Rightarrow$ they are the basis for that subspace

$\mathbf{N}(A)$ null space

Recall from Exercise 4.4.3:

- found vectors that span $\mathbf{N}(A)$
- vectors were linearly independent (when $\mathbf{N}(A)$ contains non-zero vectors)

So, that method produces a *basis* for $\mathbf{N}(A)$.

$\mathbf{C}(A)$

Exercise 4.9.1

Find a basis for $\mathbf{C}(B)$, where

Reduced row echelon form

$$B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_5 \end{bmatrix} = \begin{bmatrix} \boxed{1} & 4 & 0 & 2 & 0 \\ 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

non-pivot columns

Solution

$$\begin{array}{cc} & p \\ n & \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix} \end{array}$$

There will be columns that don't contain pivots

If $p > n$, the columns are linearly dependent.

Each nonpivot column of B is a linear combination of the pivot columns.

$\mathbf{b}_2 = 4\mathbf{b}_1, \mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$

spanning set theorem: span $\{v_1, v_2, v_3\}$ = span $\{v_1, v_2\}$ ⇒ can "throw away" what was dependent on the others

By Theorem 4.5, discard $\mathbf{b}_2$ and $\mathbf{b}_4$; $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ will still span $\mathbf{C}(B)$. Let

dependent

By inspection, can see that vectors in S are linearly independent

$$S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$ ← b_1, b_3, b_5

check if linearly independent

By Theorem 4.4, since $\mathbf{b}_1 \neq \vec{0}$ and no vector in S is a linear combination of the vectors that precede it, S is linearly independent.

$\therefore S$ is a basis for $\mathbf{C}(B)$.

$A\vec{x} = \vec{0}$ can be written as: $x_1\vec{a_1} + x_2\vec{a_2} = \vec{0}$
⇒ depending on solution, can say if $\vec{a_1}$ & $\vec{a_2}$ linearly independent or not
if $x_1 = x_2 = 0$ ⇒ linearly independent

- For a matrix A that is *not* in reduced row echelon form

Recall: Any linear dependence relationship among the columns of A can be expressed in the form $A\mathbf{x} = \mathbf{0}$.

→ stop at row echelon form

If A is row reduced to B, $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have exactly the same set of solutions.

no change in value of x

If $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$, then

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{0} \text{ and } x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n = \mathbf{0}$$ $B\vec{x} = \vec{0}$

have the same set of solutions.

Columns of A have *exactly the same linear dependence relationships* as the columns of B.

$A \xrightarrow{\text{EROs}} B$ ⇒ what relationship between columns of A, same goes for columns of B

Exercise 4.9.2

It can be shown that

① swap col.
② sum / subtract
③ mult. by c

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}$$

is row equivalent to the matrix B in Exercise 4.9.1. Find a basis for $\mathbf{C}(A)$.

↳ was in the RREF

Solution

In Exercise 4.9.1, $\mathbf{b}_2 = 4\mathbf{b}_1, \mathbf{b}_4 = 2\mathbf{b}_1 - \mathbf{b}_3$.

So, $\mathbf{a}_2 = 4\mathbf{a}_1, \mathbf{a}_4 = 2\mathbf{a}_1 - \mathbf{a}_3$ (Check!)

linearly dependent on a_1 & a_3

Discard $\mathbf{a}_2$ and $\mathbf{a}_4$. For $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ to be linearly independent, $\{\mathbf{b}_1, \mathbf{b}_3, \mathbf{b}_5\}$ should also be linearly independent, which is true.

Therefore, $\{\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5\}$ is a basis for $\mathbf{C}(A)$.

linearly dependent on pivot columns

form basis

Theorem 4.6. *The pivot columns of a matrix A form a basis for* $\mathbf{C}(A)$.

Proof. Let B be the reduced echelon form of A. $A \xrightarrow{\text{EROs}} B$

The set of pivot columns of B are linearly independent.

Since A is row equivalent to B, the pivot columns of A are linearly independent, AND

Every nonpivot column of A is a linear combination of the pivot columns of A.

So, the nonpivot columns of A can be discarded from the spanning set for $\mathbf{C}(A)$ [Spanning Set theorem].

This leaves the pivot columns of A as the basis for $\mathbf{C}(A)$. □

4.10 Coordinate Systems

$\mathbb{R}^2 \rightarrow$ basis vectors $\vec{i}$ & $\vec{j}$

Let $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \cdots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

$\vec{x} = a\vec{i} + b\vec{j}$

$[x]_B = \begin{bmatrix} a \\ b \end{bmatrix}$

Definition. Suppose $\mathcal{B} = \{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ is a basis for V and $\mathbf{x}$ is in V. The **coordinates of x relative to the basis** $\mathcal{B}$ (or the $\mathcal{B}-$**coordinates of x**) are the weights $c_1, \cdots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Co-ordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$, or the $\mathcal{B}-$**coordinate vector of x**

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

Exercise 4.10.1

Consider a basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2,\}$ for $\mathbb{R}^2$, where $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Suppose an $\mathbf{x}$ in $\mathbb{R}^2$ has the coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\mathbf{x}$.

$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$

$[x]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Solution

$$\mathbf{x} = (-2)\mathbf{b}_1 + 3\mathbf{b}_2 = -2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 6 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\vec{x} = c_1\vec{e}_1 + c_2\vec{e}_2$

Example

Consider the *standard basis* $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2,\}$. If $[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$, $\rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

then $\mathbf{x} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$, i.e., $[\mathbf{x}]_{\mathcal{E}} = \mathbf{x}$.

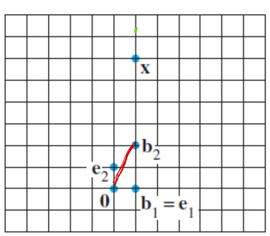
$\vec{x} = -2\vec{b}_1 + 3\vec{b}_2$

$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

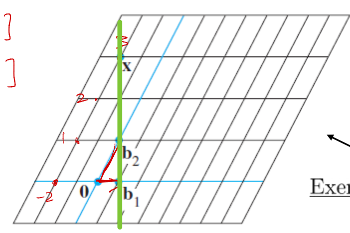
$\vec{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$

x, b_1, b_2 on a straight line



Standard graph paper



$\mathcal{B}-$graph paper

Exercise 4.10.1

solve for c_1, c_2:

$P_B^{-1}x = [x]_B$

$[x]_B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$= \frac{1}{2-(-1)}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$= \begin{bmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}+\frac{5}{3} \\ -\frac{4}{3}+\frac{10}{3} \end{bmatrix} = \begin{bmatrix} \frac{9}{3} \\ \frac{6}{3} \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Exercise 4.10.2

Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2,\}$. Find the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ of $\mathbf{x}$ relative to $\mathcal{B}$.

Solution

$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$

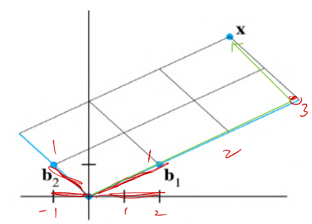
$[x]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Let $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$c_1\underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{b_1} + c_2\underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{b_2} = \underbrace{\begin{bmatrix} 4 \\ 5 \end{bmatrix}}_{x}$$

or $\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ or $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

solve for c_1 & c_2



The $\mathcal{B}$-coordinate vector of $\mathbf{x}$ is $(3, 2)$.

$A\vec{x} = \vec{b}$

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ P_{\mathcal{B}} & [\mathbf{x}]_{\mathcal{B}} & \mathbf{x} \end{matrix}$$

The matrix $P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$ changes the $\mathcal{B}-$coordinates of a vector $\mathbf{x}$ into the standard coordinates of $\mathbf{x}$.

$$\mathbf{x} = P_{\mathcal{B}} \, [\mathbf{x}]_{\mathcal{B}}$$

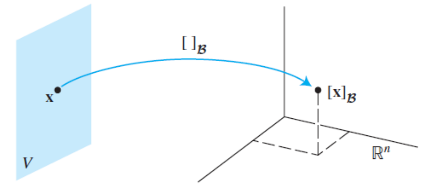
Since the columns of $P_{\mathcal{B}}$ form a basis for $\mathbb{R}^2$, $P_{\mathcal{B}}$ is invertible (by the Invertible Matrix Theorem). Therefore,

$\Leftarrow$ columns linearly independent & span $\mathbb{R}^2$

$$P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$$

$P_{\mathcal{B}}^{-1}\mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$ can be seen as a one-to-one linear transformation $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$

Extending to $\mathbb{R}^n$:



If $\mathbf{u}_1, \ldots, \mathbf{u}_p$ are in V and if $c_1, \ldots, c_p$ are scalars, then

coordinate vector

$$[c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p]_{\mathcal{B}} = c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_p[\mathbf{u}_p]_{\mathcal{B}}$$

The $\mathcal{B}-$coordinate vector of a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ is the *same* linear combinatin of their coordinate vectors.

4.11 Dimension of a Vector Space

Definition. If V is a vector space, the **dimension** of V, written as dim V, is the number of vectors in a basis for V. The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be zero.

e.g. standard basis for $\mathbb{R}^3$ $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Examples

- Standard basis for $\mathbb{R}^n$ contains n vectors $\Rightarrow$ dim $\mathbb{R}^n = n$
- Standard polynomial basis $\{1, t, t^2\} \Rightarrow$ dim $\mathbb{P}_2 = 3$

$\Rightarrow$ dimension of $\mathbb{P}_n$ is $n+1$

Exercise 4.11.1 Find the dimension of the subspace

$$H = \left\{ \begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$

① find a basis for H

$$\begin{bmatrix} a - 3b + 6c \\ 5a + 4d \\ b - 2c - d \\ 5d \end{bmatrix} = a\begin{pmatrix} 1 \\ 5 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 6 \\ 0 \\ -2 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 4 \\ -1 \\ 5 \end{pmatrix}$$

② Are they linearly independant?

$$\begin{bmatrix} 1 & -3 & 6 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \xrightarrow{\text{ERO}_s} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ not all columns have pivot positions

$\therefore$ not linearly independent

column 3 not linearly independent

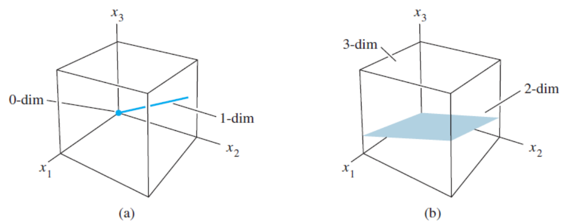
$\therefore$ basis for H is

$$\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \right\}$$

$\therefore$ dimension of H is 3

Example

- Subspaces of $\mathbb{R}^3$



Quiz 1 tested till HERE

Consider $\mathbb{R}^3$

$\Rightarrow$ Any set of > 3 vectors in $\mathbb{R}^3$ is linearly dependent

$\Rightarrow$ Any set of < 3 vector in $\mathbb{R}^3$ does not span $\mathbb{R}^3$

$\Rightarrow$ Any basis for $\mathbb{R}^3$ must have exactly 3 vectors.

4.12 Dimensions of $\mathbf{C}(A)$ and $\mathbf{N}(A)$

$\mathbf{C}(A)$

Pivot columns of A form a basis for $\mathbf{C}(A)$.

So, dim $\mathbf{C}(A)$ = number of pivot columns of A.

$\mathbf{N}(A)$

For $m \times n$ matrix A, suppose $A\mathbf{x} = \mathbf{0}$ has k free variables.

Spanning set for $\mathbf{N}(A)$ will have k linearly independent vectors - one for each free variable.

So, dim $\mathbf{N}(A)$ = number of free variables, k.

Exercise 4.12.1

Find the dimension of the null space and column space of

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}_{3\times 5}$$

$A\vec{x} = \vec{0}$
↳ cause we are interested in the null space

Solution

Echelon form of augmented matrix $\begin{bmatrix} A & \mathbf{0} \end{bmatrix}$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- 3 free variables $(x_2, x_4, x_5) \Rightarrow \boxed{\text{dim } \mathbf{N}(A) = 3}$
- A has two pivot columns $\Rightarrow$ dim $\boxed{\mathbf{C}(A) = 2}$

4.13 The Row Space

$m \times n$

$\Rightarrow$ each row vector is a dimension of n

The set of all linear combinations of the rows (row vectors) of a matrix A is called the **row space** of A.

Since rows of A are columns of A^T, row space is denoted by $\mathbf{C}(A^T)$.

use this to denote row space

For an $m \times n$ matrix, each row has n entries $\Rightarrow \mathbf{C}(A^T)$ is a subspae of $\mathbb{R}^n$.

Theorem 4.7. *If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.*

(Skip proof)

non-zero row will form a basis for b as well as for a

Exercise 4.13.1

Find bases for the row space, column space and null space of

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

4×5

Solution

Reduce A to echelon form:

$$A \sim B = \begin{bmatrix} \boxed{1} & 3 & -5 & 1 & 5 \\ 0 & \boxed{1} & -2 & 2 & -7 \\ 0 & 0 & 0 & \boxed{-4} & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

forms basis for row space of A

Row space $\mathbf{C}(A^T)$

From Theorem 4.7,

Basis for $\mathbf{C}(A^T) : \{(1, 3, -5, 1, 5), (0, 1, -2, 2, -7), (0, 0, 0, -4, 20)\}$

Take from $A \sim B$

Column space $\mathbf{C}(A)$

Pivots are in columns 1, 2 and 4. Hence, columns 1, 2, and 4 of A (not B!) form the basis:

$$\text{Basis for } \mathbf{C}(A) : \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$$

Take from A

Null space $\mathbf{N}(A)$

Need reduced row echelon form of A:

$$A \sim B \sim C = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_3 - x_5$

$x_2 = 2x_3 - 3x_5$

$x_4 = 5x_5$

$A\mathbf{x} = \mathbf{0}$ is equivalent to $C\mathbf{x} = \mathbf{0}$

$$\begin{array}{rcrcrcrcl} x_1 & + & & & x_3 & & & + & x_5 & = & 0 \\ & & x_2 & - & 2x_3 & & & + & 3x_5 & = & 0 \\ & & & & & & x_4 & - & 5x_5 & = & 0 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

Similar to Exercise 4.4.3 (slide 21),

$$\text{Basis for } \mathbf{N}(A) : \left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$$

4.14 Rank

→ of matrix

E.g. $\begin{bmatrix} | & | & | & | \end{bmatrix}$ basis, 3 $A\vec{x} \Rightarrow$ Span 3D space

→ find basis for C(A)

Definition. The **rank** of A is the dimension of the column space of A.

Recall: $A\mathbf{x} = \mathbf{b}$ is consistent when $\mathbf{b}$ is in $\mathbf{C}(A)$.

Rank of A is the dimension of the set of $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ is consistent.

Suppose $A \in \mathbb{R}^{n\times n}$ has rank p. This means that if we take all vectors $\mathbf{x} \in \mathbb{R}^{n\times 1}$, then $A\mathbf{x}$ spans p dimensional space $\Rightarrow$ no unique solution. $0, \cancel{1}, \infty$

Example

any vector in column space, multiple of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

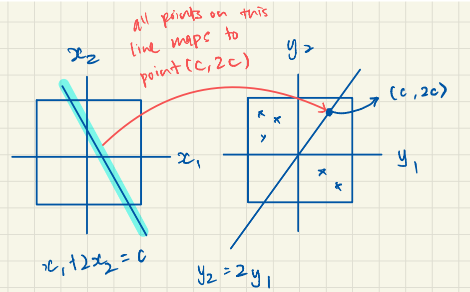
$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = A\mathbf{x} = \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 4x_2 \end{bmatrix}$$

dim of A = 1

∴ Rank of $A = 1 \Rightarrow A\mathbf{x}$ spans a 1 dimensional space, i.e., a line $y_2 = 2y_1$ passing through the origin.

Points on $x_1 - x_2$ plane are mapped onto a line $y_2 = 2y_1$.

(Also see Exercise 4.13.1) to find if have 0 or ∞ solutions, need to find domain & codomain (refer to next slide)



Question: what happens to points not on line $y_2 = 2y_1$ (indicated by x's)

$\Rightarrow A\underline{\underline{x}} = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$ no x

↓

Ax spans ONLY $y_2 = 2y_1$ (no solution)

Points on $x_1 - x_2$ plane are mapped onto a line $y_2 = 2y_1$

$\Rightarrow A x = \begin{bmatrix} 2 \\ 2c \end{bmatrix}$

↓

∞ solutions

∴ Depending on where y_1 & y_2 points are or where y vector is, you can have either 0 or ∞ solutions

known as rank deficient matrix, not full rank

(if $n \times n$ matrix has rank n, it is full rank)

→ where $p < n$

- The dimensions of the column space and the row space of an $m \times n$ matrix A are equal.

Example

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}_{4\times 5} \text{ Echelon form of } A\text{: } A \sim B = \begin{bmatrix} \boxed{1} & 3 & -5 & 1 & 5 \\ 0 & \boxed{1} & -2 & 2 & -7 \\ 0 & 0 & 0 & \boxed{-4} & 20 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns form a basis for C(A)

By Theorem 4.6 (slide 42), rank of A is the number of pivot columns in A, i.e., the number of pivot positions in B. → cause A & B row equivalent

Rows with pivots in B form a basis for the row space of A.

Row rank = column rank

Therefore, rank of A is also the dimension of the row space of A.

- Rank of A = dim $\mathbf{C}(A)$ = the number of columns with pivots
- dim $\mathbf{N}(A)$ = the number of free variables (in $A\mathbf{x} = \mathbf{0}$) = the number of columns without pivots
- Obviously,

$$\underbrace{\#(\text{columns with pivots})}_{\text{column space of } A \,\to\, (\text{rank of } A)} + \underbrace{\#(\text{columns without pivots})}_{(\dim N(A))} = \underbrace{\#(\text{columns})}_{\text{total number of}}$$

Theorem 4.8. *The Rank nullity Theorem*

If A is a matrix with n columns then,

$$\text{rank of } A + \underbrace{\dim \mathbf{N}(A)}_{\text{nullity}} = n.$$

- Gives a relationship between the solution set of $A\mathbf{x} = \mathbf{0}$, i.e., $\mathbf{N}(A)$ and the set of vectors $\mathbf{b}$ that make $A\mathbf{x} = \mathbf{b}$ consistent, i.e., $\mathbf{C}(A)$

a linear system can have either 0, 1, ∞ solutions

Exercise 4.14.1

2 of ∞ solutions

A scientist has found two solutions to a homogeneous system of 40 equations in 42 variables. The solutions are linearly independent and span $\mathbf{N}(A)$. $A\vec{x}=\vec{0}$

Can the scientist be *certain* that an associated nonhomogeneous system (with the same coefficients) has a solution? if above condition is satisfied $A\vec{x}=\vec{b}$

Solution $[A]_{40\times 42}\,[\vec{x}]_{42\times 1} = [\vec{b}]_{40\times 1}$ ⇒ we can say that: $\vec{x}\in\mathbb{R}^{42}$, $\vec{b}\in\mathbb{R}^{40}$

A: 40×42 coefficient matrix.

The two solutions are a basis for $\mathbf{N}(A)$. → linearly independent & span $N(A)$ means linearly independent & spanning

Therefore, the dim $\mathbf{N}(A)=2 \Rightarrow$ dim $\mathbf{C}(A)=42-2=40$. → 40 basis vectors for $C(A)$ → span all of $\mathbb{R}^{40}$

n; Null space of A

Rank Theorem

Since $\mathbb{R}^{40}$ is the only subspace of $\mathbb{R}^{40}$ whose dimension is 40, $\mathbf{C}(A)$ must be all of $\mathbb{R}^{40}$

→ any $\vec{b}\in\mathbb{R}^{40}$ can be written as a linear combination of the 40 vectors in A that form the basis for $C(A)$

for all b

⇒ Every homogeneous equation $A\mathbf{x}=\mathbf{b}$ has a solution.

Theorem 4.9. *The Invertible Matrix Theorem [Theorem 2.4](continued)*

Let A by an $n\times n$ matrix. Then the statement 'A is invertible' is equivalent to the following statements:

11. *The columns of A form a basis for $\mathbb{R}^n$*
12. $\mathbf{C}(A)=\mathbb{R}^n$
13. *dim* $\mathbf{C}(A)=n$
14. *Rank of* $A=n$
15. $\mathbf{N}(A)=\{\mathbf{0}\}$
16. *dim* $\mathbf{N}(A)=0$

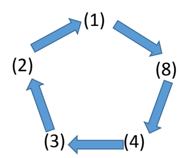
(6) ⇒ (12) ⇒ (13) ⇒ (14) ⇒ (16) ⇒ (15) ⇒ (4)

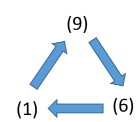
6. $Ax=b$ has at least one solution for each b in $\mathbb{R}^n$.

4. $Ax=0$ has only the trivial solution.

Theorem 2.4. *The Invertible Matrix Theorem*
Let A be a square $n\times n$ matrix. Then the following statements are equivalent, i.e., for a given A, the statements are either all true or all false.

1. *A is an invertible matrix.*
2. *A is row equivalent to I_n.*
3. *A has n pivot positions.*
4. *$A\mathbf{x}=\mathbf{0}$ has only the trivial solution.*
5. *The columns of A form a linearly independent set.*
6. *$A\mathbf{x}=\mathbf{b}$ has at least one solution for each $\mathbf{b}$ in $\mathbb{R}^n$.*
7. *The columns of A span $\mathbb{R}^n$.*
8. *There is an $n\times n$ matrix C such that $CA=I$.*
9. *There is an $n\times n$ matrix D such that $AD=I$.*
10. *A^T is an invertible matrix.*





(6) ⟺ (7)

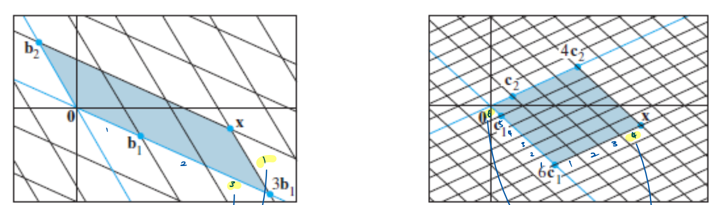
(4) ⟺ (5)

(1) ⟺ (10)

4.15 Change of basis

Recall: $\vec{x}$ = standard basis $e_1, e_2, \dots$ ⇒ can be represented in some other basis $[\vec{x}]_B$ using transformation matrix

- A problem could be more easily solved by changing the basis from $\mathcal{B}$ to $\mathcal{C}$
- How $[\mathbf{x}]_\mathcal{B}$ and $[\mathbf{x}]_\mathcal{C}$ are related for each $\mathbf{x}$ in V.



(a) (b)

Two co-ordinate systems for the same vector space

$[\mathbf{x}]_\mathcal{B}=\begin{bmatrix}3\\1\end{bmatrix}$ ⟷ Relation ? ⟷ $[\mathbf{x}]_\mathcal{C}=\begin{bmatrix}6\\4\end{bmatrix}$

Exercise 4.15.1

Consider two bases $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$ for a vector space V, such that

$$\mathbf{b}_1 = 4\mathbf{c}_1 + \mathbf{c}_2 \text{ and } \mathbf{b}_2 = -6\mathbf{c}_1 + \mathbf{c}_2$$

$\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -6 \\ 1 \end{bmatrix}$

If $\mathbf{x} = 3\mathbf{b}_1 + \mathbf{b}_2$, i.e., $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, find $[\mathbf{x}]_{\mathcal{C}}$.

$* \; T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})$

Solution

Coordinate mapping is a linear transformation (slide 48)

$$\begin{aligned} [\mathbf{x}]_{\mathcal{C}} &= [3\mathbf{b}_1 + \mathbf{b}_2]_{\mathcal{C}} \\ &= 3[\mathbf{b}_1]_{\mathcal{C}} + [\mathbf{b}_2]_{\mathcal{C}} \; * \end{aligned}$$

$$[\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \text{ and } [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$$

$$[\mathbf{x}]_{\mathcal{C}} = \underbrace{\begin{bmatrix} 4 & -6 \\ 1 & 1 \end{bmatrix}}_{\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}} \underbrace{\begin{bmatrix} 3 \\ 1 \end{bmatrix}}_{[\mathbf{x}]_{\mathcal{B}}} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

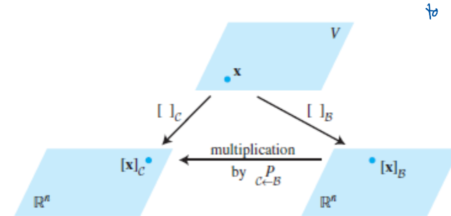
Theorem 4.9. *Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ such that*

$$[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}$$

The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are the $\mathcal{C}$-coordinate vectors of the vectors in the basis B, i.e.,

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [\mathbf{b}_1]_{\mathcal{C}} & [\mathbf{b}_2]_{\mathcal{C}} & \cdots & [\mathbf{b}_n]_{\mathcal{C}} \end{bmatrix}$$

← need to know this in order to find $[x]_C$



Two coordinate systems for *V*

***** END OF CHAPTER *****

Requirements for Subspace :

(i) zero 'vector' is the 3×3 zero vector $\Rightarrow$ Symmetric & in S.

(ii) Let A & B in S, with $A = A^T$ and $B = B^T$

$(A + B)^T = A^T + B^T = A + B$ $\therefore$ $A + B$ is symmetric & in S

(iii) Let A be in S and c a scalar. $(cA)^T = cA^T = cA$

$\therefore$ cA is also Symmetric & in S

Exam - like question

Exercise 4.7.1:

Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$.

check for linearly independence :

$$\begin{vmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ 0 & 7 & 5 \end{vmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 6$$

check if invertible

Since $\det \neq 0$, it is linearly independent.