Find the volume of tetradron with vertices $A_1(2,-1,2)$, $A_2(2,1,3)$, $A_3(1,0,1)$ and $A_4(6,4,5)$. Find its volume after linear transformation T defined by Matrix B

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{matrix}$$

Formula to use:

abs(|A|) (Volume of S)

$$A = \begin{bmatrix} 2 & 2 & 1 & 6 \\ -1 & 1 & 0 & 4 \\ 2 & 3 & 1 & 5 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$\det(A) = 11$

Volume of tetradron

$= \frac{1}{6}(\det(A)) = \frac{11}{6}$

$\det(B) = 7$

After linear transformation, abs(det(B)) (Volume of tetradron)

$= 7 \times \frac{11}{6} = \frac{77}{6}$

Prof Solution:

$$A_1A_2 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 2-2 \\ 1-(-1) \\ 3-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Formula:

$\vec{AB} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$

$$A_1A_3 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 0-(-1) \\ 1-2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$$

$$A_1A_4 = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 6-2 \\ 4-(-1) \\ 5-2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$$

$$\det \begin{bmatrix} 0 & -1 & 4 \\ 2 & 1 & 5 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 4 & 0 & -1 \\ 2 & 1 & 5 & 2 & 1 \\ 1 & -1 & 3 & 1 & -1 \end{bmatrix}$$

$= ((0 + (-5) + (-8)) - (4 + (-6)))$

$= -13 - (-2) = -11$

Vol after transformation

$= \text{abs}(|B|) \times \frac{11}{6}$

$= \frac{77}{6}$

abs(−11) = 11

Vol of tetrahedron = 11/6

Find volume of tetrahedron with vertices

a) $A_1(2,3,1)$, $A_2(4,1,-2)$, $A_3(6,3,7)$, $A_4(7,5,3)$

b) $A_1(-3,4,-7)$, $A_2(1,5,4)$, $A_3(-5,-2,0)$, $A_4(2,5,4)$

(a)

$$A_1A_2 = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}$$

$$A_1A_3 = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$$

$$A_1A_4 = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$$

$$\det \begin{bmatrix} 2 & 4 & 5 \\ -2 & 0 & 2 \\ -3 & 6 & 2 \end{bmatrix} \begin{matrix} 2 & 4 \\ -2 & 0 \\ -3 & 6 \end{matrix}$$

$$= (0 + (-24) + (-60)) - (0 + 24 + (-16))$$

$$= -84 - 8 = -92$$

$$\text{abs}(-92) = 92$$

$$\text{Vol} = \frac{1}{6} \times 92 = \frac{92}{6}$$

Prof ans : $\frac{140}{6}$

(b)

$$A_1A_2 = \begin{bmatrix} 4 \\ 1 \\ 11 \end{bmatrix}$$

$$A_1A_3 = \begin{bmatrix} 4 \\ 0 \\ 6 \end{bmatrix}$$

$$A_1A_4 = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}$$

$$\det \begin{bmatrix} 4 & 4 & 5 \\ 1 & 0 & 2 \\ 11 & 6 & 2 \end{bmatrix} \begin{matrix} 4 & 4 \\ 1 & 0 \\ 11 & 6 \end{matrix}$$

$$= (88 + 30) - (48 + 8)$$

$$= (118) - (56) = 62$$

$$\text{Vol} = \frac{1}{6} \times 62 = \frac{62}{6}$$

Prof ans : $\frac{151}{6}$

Find volume of parallelepiped with one vertex at the origin and adjacent vertices

c) $\begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \\ -5 \end{pmatrix}$

d) $\begin{pmatrix} -2 \\ 4 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ -4 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -5 \\ 4 \end{pmatrix}$

(c) $\det \begin{bmatrix} -3 & 1 & 1 \\ 1 & 2 & -1 \\ 2 & -2 & -5 \end{bmatrix} = 33$

(d) $\det \begin{bmatrix} -2 & 1 & -1 \\ 4 & -4 & -5 \\ 6 & -8 & 4 \end{bmatrix} = 74$

How volumes in a), b), c) and d) will change after transformation with matrix:

e) $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

f) $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 3 & 1 & 0 \end{pmatrix}$

(e) $\det \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = 1$

All volumes remain the same.

(f) $\det \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & 0 \\ 3 & 1 & 0 \end{pmatrix} = 2$

volumes will $\times 2$

if $r < c$,

max rank $= r$

if $r > c$,

max rank $= c$.

Given the matrix A

a) $A = \begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 7 & 5 \end{pmatrix}_{3\times 2}$ b) $A = \begin{pmatrix} 3 & 5 & 8 \\ 4 & 1 & 5 \\ 1 & 7 & 8 \end{pmatrix}$ c) $A = \begin{pmatrix} 1 & 8 & 3 & 2 \\ 2 & 16 & 6 & 4 \end{pmatrix}$

d) $A = \begin{pmatrix} 3 & 1 & 5 & 7 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 6 & 5 \end{pmatrix}$

find RANK A, Col(A), N(A) (Null space), basis of Col A, basis of N(A)

(a) Rank $A = 2$, $\text{col}(A) = \text{Span}\left\{ \begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \right\}$, $N(A) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

Basis of Col (A) : $\begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$, basis $N(A)$: $\emptyset$

$$\begin{bmatrix} 3 & 1 \\ 2 & 1 \\ 7 & 5 \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1/3} \begin{bmatrix} 1 & \frac{1}{3} \\ 2 & 1 \\ 7 & 5 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & \frac{1}{3} \\ 7 & 5 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 7R_1} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & \frac{1}{3} \\ 0 & \frac{8}{3} \end{bmatrix}$$

$$\xrightarrow{R_2 = 3R_2} \begin{bmatrix} 1 & \frac{1}{3} \\ 0 & 1 \\ 0 & \frac{8}{3} \end{bmatrix} \xrightarrow{R_1 \leftarrow R_1 - \frac{R_2}{3}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & \frac{8}{3} \end{bmatrix} \xrightarrow{R_3 = R_3 - \frac{8R_2}{3}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

(b) $\begin{pmatrix} 3 & 5 & 8 \\ 4 & 1 & 5 \\ 1 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Rank $= 2$, $\text{col}(A) = \text{Span}\left\{ \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix} \right\}$,

find $N(A)$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{aligned} x + z &= 0 \\ y + z &= 0 \\ 0 &= 0 \end{aligned}$

$\Downarrow$

$x = -z$

$y = -z$

$z = z$

$\Downarrow$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} z$

$\therefore N(A) = \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\}$

Basis of col (A) $= \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ 1 \\ 7 \end{pmatrix}$

Basis of $N(A) = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$

$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

(c) $A = \begin{pmatrix} 1 & 8 & 3 & 2 \\ 2 & 16 & 6 & 4 \end{pmatrix} = \begin{bmatrix} 1 & 8 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Rank $= 1$; $\text{col}(A) = \text{span}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$,

$\text{Null}(A) = \text{Span}\left\{ \begin{pmatrix} -8 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ $x = -8y - 3z - 2a$

basis of $\text{col}(A) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{bmatrix} x \\ y \\ z \\ a \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

basis of $\text{null}(A) = \begin{pmatrix} -8 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}$

(d) $A = \begin{pmatrix} 3 & 1 & 5 & 7 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Rank $= 3$, $\text{col}(A) = \begin{pmatrix} 3 & & 1 & & 5 \\ 1 & & 1 & & 3 \\ 2 & , & 1 & , & 6 \end{pmatrix}$

$\text{Null}(A) = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, basis of $\text{col}(A) =$

basis of $\text{null}(A) = \}$

$a = -2d$

$b = -1d$

$c = 0$

$d = d$

$\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

why not reverse the signs

Given matrix $A = \begin{pmatrix} 1 & 0 & 2 \\ 5 & 2 & 4 \\ 3 & 1 & 3 \end{pmatrix}$ find Rank A, Col(A)
N(A), basis of Col (A), basis of N(A).

$$\begin{pmatrix} 1 & 0 & 2 \\ 5 & 2 & 4 \\ 3 & 1 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

Should have this

Rank : 2, $\quad Col(A) = Span\left\{ \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \cancel{\begin{pmatrix} 2 \\ 4 \\ 3 \end{pmatrix}} \right\},$

$a = -2c$

$b = 3c$

$c = c$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} c$$

$$Nul(A) = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

basis of Col(A) $= \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$

$$Nul(A) = \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

Given the matrix A

a) $A = \begin{pmatrix} 3 & 5 \\ 6 & 16 \end{pmatrix}$ b) $A = \begin{pmatrix} 4 & 1 & 2 \\ -4 & 0 & 3 \end{pmatrix}$ c) $A = \begin{pmatrix} 3 & 1 & 6 \\ -3 & 0 & -2 \\ 3 & 3 & 17 \end{pmatrix}$

d) $A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 10 \end{pmatrix}$ find it's LU factorization.

Using LU factorization find solution for $Ax = b$

a) $b = \begin{pmatrix} 8 \\ 22 \end{pmatrix}$ b) $b = \begin{pmatrix} 8 \\ -1 \end{pmatrix}$ c) $b = \begin{pmatrix} 9 \\ -5 \\ 20 \end{pmatrix}$ d) $b = \begin{pmatrix} 5 \\ 4 \\ 4 \end{pmatrix}$

(a) $A = \begin{pmatrix} 3 & 5 \\ 6 & 16 \end{pmatrix} = \underbrace{\begin{pmatrix} \# & 0 \\ \# & \# \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} \# & \# \\ 0 & \# \end{pmatrix}}_{U}$

$$\begin{pmatrix} 1 & 0 \\ \boxed{-2} & 1 \end{pmatrix} \begin{pmatrix} 3 & 5 \\ 6 & 16 \end{pmatrix} \xrightarrow[r^2 \leftarrow r^2 - 2r^1]{} = \begin{pmatrix} \boxed{3} & \boxed{5} \\ 0 & \boxed{6} \end{pmatrix}$$

$$\begin{bmatrix} 3 & 5 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix}$$

(b) $A = \begin{pmatrix} 4 & 1 & 2 \\ -4 & 0 & 3 \end{pmatrix}$

$$\begin{pmatrix} 1 & 0 \\ \boxed{1} & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 \\ -4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} \boxed{4} & \boxed{1} & \boxed{2} \\ 0 & \boxed{1} & \boxed{5} \end{pmatrix}$$

$\rightsquigarrow$

$r2 \leftarrow r2 + r1$

$$\begin{pmatrix} 4 & 1 & 2 \\ -4 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 4 & 1 & 2 \\ 0 & 1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 & 2 \\ 0 & 1 & 5 \end{pmatrix}$$

(c) $A = \begin{pmatrix} 3 & 1 & 6 \\ -3 & 0 & -2 \\ 3 & 3 & 17 \end{pmatrix}$

always subtract multiples of higher rows

$$\begin{pmatrix} 1 & 0 & 0 \\ \boxed{1} & 1 & 0 \\ \boxed{-1} & \boxed{-2} & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 6 \\ -3 & 0 & -2 \\ 3 & 3 & 17 \end{pmatrix} = \begin{pmatrix} \boxed{3} & \boxed{1} & \boxed{6} \\ 0 & \boxed{} & \boxed{} \\ 0 & 0 & \boxed{} \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 6 \\ -3 & 0 & -2 \\ 3 & 3 & 17 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 + R_1]{(-1)} \begin{pmatrix} 3 & 1 & 6 \\ 0 & 1 & 4 \\ 3 & 3 & 17 \end{pmatrix}$$

$$\xrightarrow[R_3 \leftarrow R_3 - R_1]{(1)} \begin{pmatrix} 3 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & 2 & 11 \end{pmatrix} \xrightarrow[R_3 \leftarrow R_3 - 2R_2]{(2)} \begin{pmatrix} 3 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 6 \\ -3 & 0 & -2 \\ 3 & 3 & 17 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 3 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

??

$$= \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 1 & 6 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix}4&1&2\\-4&0&3\end{pmatrix} = \begin{pmatrix}1&0\\-1&1\end{pmatrix}\begin{pmatrix}4&1&2\\0&1&5\end{pmatrix}$$

$$b = \begin{pmatrix}8\\-1\end{pmatrix}$$

$$Ly=b \;:\; \begin{pmatrix}1&0&8\\-1&1&-1\end{pmatrix} \xrightarrow{R_2 \leftarrow R_2+R_1} \begin{pmatrix}1&0&8\\0&1&7\end{pmatrix}$$

$$Ux=y \;:\; \begin{pmatrix}4&1&2&8\\0&1&5&7\end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{1}{4}R_1} \begin{pmatrix}1&\frac{1}{4}&\frac{1}{2}&2\\0&1&5&7\end{pmatrix}$$

$$\begin{bmatrix}1&0&-\frac{3}{4}&\frac{1}{4}\\0&1&5&7\end{bmatrix}$$

$$x_1 + \frac{1}{4}x_2 + \frac{1}{2}x_3 = 2$$

$$x_2 + 5x_3 = 7$$

$$x_1 - \frac{3}{4}x_3 = \frac{1}{4}$$

$$x_2 + 5x_3 = 7$$

$$x_1 = 2 - \frac{1}{4}x_2 - \frac{1}{2}x_3$$

$$x_2 = 7 - 5x_3$$

$$x_1 = \frac{1}{4} + \frac{3}{4}x_3$$

$$x_2 = 7 - 5x_3$$

$$x_3 = x_3$$

$$x_3 = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix} \\ t\end{bmatrix}x_3$$

$$\begin{pmatrix}\frac{1}{4}+\frac{3}{4}t\\ 7 - 5t\\ t\end{pmatrix}$$

$$\underset{L}{\begin{pmatrix}1&0\\2&1\end{pmatrix}}\underset{U}{\begin{pmatrix}3&5\\0&6\end{pmatrix}} \qquad b=\begin{pmatrix}8\\22\end{pmatrix}$$

$$Ly=b: \quad \begin{pmatrix}1&0&8\\2&1&22\end{pmatrix} \xrightarrow{R_2\leftarrow R_2-2R_1} \begin{pmatrix}1&0&8\\0&1&6\end{pmatrix}$$

$$Ux=y: \quad \begin{pmatrix}3&5&8\\0&6&6\end{pmatrix} \longrightarrow \begin{pmatrix}1&0&\boxed{\begin{matrix}1\\1\end{matrix}}\\0&1&\end{pmatrix}$$

Gram - Schmid process can only orthogonalise linearly independent set of vectors.

rank(A) < n, can find least square soln.

given 2 vector v & w

find the set where it spans $\mathbb{R}^3$

(v & w is LI)

(w & w & u is NOT LI)

↓

u useless

① find vector outside u span

② apply projection on it $(y - \hat{y})$

Hence guaranteed orthogonal

* cross product of 2 vectors

zero vector inside a set is not orthogonal

↓

must be non zero

* linearly dependent set ~~can~~ cannot be orthogonal

↳ ON Basis?

if b is in Col(A), least square soln of $Ax = b$
is exact soln.

QR factorisation
→ must be linearly independent (col vector of A)

$A = QR$

$R = Q^T A$

columns of A are $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

⇒ least square soln of $Ax = b$ NOT unique

⇒ cols linearly dependent

12. Suppose **z** is in $\mathbb{R}^3$ & {**u**,**v**,**w**} is an orthonormal basis of $\mathbb{R}^3$. We know **z** = a**u** + b**v** + c**w** where a,b,c are scalars. **Explain if we can or can't compute b if we only know v & not u & w.**

can compute.

$\|z\|^2 = a^2 + b^2 + c^2$ → True

14. Find the **subspace of** $\mathbb{R}^3$ that is **perpendicular** to the **vector (1,2,3)**

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 0 \rightarrow x + 2y + 3z = 0$

A in subspace

15. Find the vector in subspace W=span{(1,0,0), (1,9,0)} that is closest to (1,1,1).

$\to (1,1,0)$

span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 9 \\ 0 \end{pmatrix} \right\}$ ← not orthogonal basis

Proj ⇒ need to be orthogonal basis

→ maybe use gramschmidt

18. Given {**u**,**v**,**w**} is in $\mathbb{R}^3$, where {**u**,**v**} is independent but {**u**,**v**,**w**} is not. **Outline briefly an algorithm** on how one can still compute an orthogonal basis for $\mathbb{R}^3$ containing {**u**}.

only u & v valid vectors

2 vectors cannot span $\mathbb{R}^3$

need to cross product u & v to get 3rd vector

perform Gramschmidt on u & v

Then set w = u × v

↓

{u, w, v} orthogonal

or projection ← perpendicular vector →

19. What happens if we perform Gram-Schmid orthogonalisation on {u,v,w} in Q16? Explain.

w is in the span {u, v}

→ gonna get zero vector

20. Prove/disprove: If {u, v} is orthogonal, then {(u.v)u, v} is linearly independent.

16. Let W =Span{$(-1,1,0)^t$, $(2,0,1)^t$}. Find the projection of $(5,6,7)^t$ on W. (BASIS NOT ORTHOGONAL! CANT USE PROJ FORMULA DIRECTLY!

17. Compute the shortest distance from above point (5,6,7) to W (ref Q16).

Q1 – use Gram sch to orthogonalise the proj.

16. Ans: $\begin{pmatrix} 5.5 \\ 6.5 \\ 6 \end{pmatrix}$

17. $\left\| \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} - \begin{pmatrix} 5.5 \\ 6.5 \\ 6 \end{pmatrix} \right\| = \sqrt{\frac{3}{2}}$

$$\begin{bmatrix} -0.7 \\ 0.7 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.6 \\ 0.6 \\ 0.6 \end{bmatrix}$$

$$\text{proj} = \frac{\begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}\begin{bmatrix} -0.7 \\ 0.7 \\ 0.7 \end{bmatrix}}{\begin{bmatrix} -0.7 \\ 0.7 \\ 0.7 \end{bmatrix}\begin{bmatrix} -0.7 \\ 0.7 \\ 0.7 \end{bmatrix}} \begin{bmatrix} -0.7 \\ 0.7 \\ 0.7 \end{bmatrix} +$$

$\frac{5.6}{1.47}$ $\frac{10.8}{1.08}$

$$\begin{matrix} -2.67 \\ 2.67 \\ 2.67 \end{matrix} + \begin{matrix} 6 \\ 6 \\ 6 \end{matrix} \quad \cancel{10.8}$$

Find cross product of

$\vec{a} = (7, 3, -4)$ & $\vec{b} = (1, 0, 6)$

$$\vec{a} \times \vec{b}$$

$$\begin{matrix} 7 & 3 & -4 & 7 & 3 & -4 \\ 1 & 0 & 6 & 1 & 0 & 6 \end{matrix}$$

$$\begin{pmatrix} 18 - 0 & -4 - 42 & 0 - 3 \\ 18 \; , & -46 \; , & -3 \end{pmatrix}$$

Part 2

$A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $u = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $v = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

$$Au = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 6 \\ -5 \end{bmatrix} = \begin{bmatrix} 6+(-30) \\ 30+(-10) \end{bmatrix} = \begin{bmatrix} -24 \\ 20 \end{bmatrix}$$
$$= -4\begin{bmatrix} 6 \\ -5 \end{bmatrix}$$
$$= -4u$$

$$Av = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3+(-12) \\ 15+(-4) \end{bmatrix}$$
$$= \begin{bmatrix} -9 \\ 11 \end{bmatrix} \neq \lambda \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

u is an eigen vector with eigenvalue (-4)

Av not multiple of v $\therefore$ not eigen vector

$Ax = 7x$

$(A - 7I)x = 0$

$$A - 7I = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix} - \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} -6 & 6 \\ 5 & -5 \end{bmatrix}$$

linearly dependent

$\Downarrow$

has non trivial soln

$\therefore 7$ is eigenvalue of A

To find eigen vector, row operations:

$$\begin{bmatrix} -6 & 6 & 0 \\ 5 & -5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_2 \; \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_2$$

$\rightarrow$ vector this form with $x_2 \neq 0$ is an eigen vector corresponding to $\lambda = 7$

To find eigenvalues:

$(A - 7I)x = 0$

eigenvectors:

RREF with 0 vector

solve for free variable

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}, \quad \text{eigen value} = 2$$

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

RR ~~for~~ for $(A - 2I)x = 0$

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -1 & 6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

equation $(A - 2I)x = 0$ has free variables

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} x_3$$

basis $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$\begin{bmatrix} 6 & -3 & 1 \\ 3 & 0 & 5 \\ 2 & 2 & 6 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & -3 & 1 \\ 3 & -5 & 5 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 3 & -5 & 5 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

all linearly independent $\therefore$ not eigenvalue

$$A^2 x = A(\lambda x) = \lambda Ax = \lambda^2 x$$

$$(A - \lambda I)x = 0$$

$$\det\left(\begin{bmatrix} 1-\lambda & -4 \\ 4 & 2-\lambda \end{bmatrix}\right) = (1-\lambda)(2-\lambda) - (-4)(4)$$

$$= 2 - \lambda - 2\lambda + \lambda^2 + 16$$

$$ax^2 + bx + c = 0$$

$$= \lambda^2 - 3\lambda + 18$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{-63}}{2}$$

Diagonalization

① find eigen values of A

$\rightarrow \det(A - \lambda I) = 0$

② find 3 linearly independent eigenvectors of A (find eigen vectors)

③ Construct P from vectors in ②

$P = [v_1 \ v_2 \ v_3]$

④ Construct D for corresponding eigen values

EXAMPLE 5 Determine if the following matrix is diagonalizable.

$$A = \begin{bmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{bmatrix}$$

SOLUTION This is easy! Since the matrix is triangular, its eigenvalues are obviously 5, 0, and -2. Since A is a 3×3 matrix with three distinct eigenvalues, A is diagonalizable. ■

PRACTICE PROBLEMS

1. Compute A^8, where $A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}$.

2. Let $A = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Suppose you are told that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A. Use this information to diagonalize A.

3. Let A be a 4×4 matrix with eigenvalues 5, 3, and -2, and suppose you know that the eigenspace for $\lambda = 3$ is two-dimensional. Do you have enough information to determine if A is diagonalizable?

WEB

SOLUTIONS TO PRACTICE PROBLEMS

1. $\det(A - \lambda I) = \lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1)$. The eigenvalues are 2 and 1, and the corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Next, form

$$P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

Since $A = PDP^{-1}$,

$$\begin{aligned} A^8 = PD^8P^{-1} &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2^8 & 0 \\ 0 & 1^8 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 256 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix} \end{aligned}$$

2. Compute $A\mathbf{v}_1 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = 1 \cdot \mathbf{v}_1$, and

$$A\mathbf{v}_2 = \begin{bmatrix} -3 & 12 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \cdot \mathbf{v}_2$$

So, $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors for the eigenvalues 1 and 3, respectively. Thus

$$A = PDP^{-1}, \quad \text{where} \quad P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

3. Yes, A is diagonalizable. There is a basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for the eigenspace corresponding to $\lambda = 3$. In addition, there will be at least one eigenvector for $\lambda = 5$ and one for $\lambda = -2$. Call them $\mathbf{v}_3$ and $\mathbf{v}_4$. Then $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent by Theorem 2 and Practice Problem 3 in Section 5.1. There can be no additional eigenvectors that are linearly independent from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$, because the vectors are all in $\mathbb{R}^4$. Hence the eigenspaces for $\lambda = 5$ and $\lambda = -2$ are both one-dimensional. It follows that A is diagonalizable by Theorem 7(b).

envalue
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$$\begin{Bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{Bmatrix}$$

$= (4-\lambda)(-1-\lambda) - (-3)(2)$

$= -4 - 4\lambda + \lambda + \lambda^2 + 6$

$= \lambda^2 - 3\lambda + 2$

$\therefore$ eigen 2 & 1

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix}$$

$4x - 3y = x$
$2x - y = y$

$\rightarrow$ $x = \frac{2y}{2}$
$x = y$

$x = \frac{1}{1}y$

$4x - 3y = 2x$
$2x - y = 2y$

$x = \frac{3}{2}y$

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, \quad \text{find } A^8.$$

① find eigenvalues

$$(A - \lambda I)x = \begin{bmatrix} 4-\lambda & -3 \\ 2 & -1-\lambda \end{bmatrix} = 0$$

$$\begin{aligned} \det &= (4-\lambda)(-1-\lambda) - (-3)(2) \\ &= -4 - 4\lambda + \lambda + \lambda^2 + 6 \\ &= \lambda^2 - 3\lambda + 2 \\ &= (\lambda - 1)(\lambda - 2) \end{aligned}$$

eigen values are 2, 1.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{3^2 - 4(1)(2)}}{2}$$

$$= \frac{3 \pm \sqrt{1}}{2}$$

$$= \frac{3-1}{2} = 1$$

$$= \frac{3+1}{2} = 2$$

② find eigen vectors

$$(A - 2I)x = \begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 2 & -1 \end{bmatrix}$$

$2x_1 - 3x_2 = 0$

$2x_1 = 3x_2 \Rightarrow x_1 = \frac{3}{2}x_2$

choose $x_2 = 1$, $x_1 = \frac{3}{2}$

$$\therefore \begin{bmatrix} 3/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -3 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 2 & 0 \end{bmatrix}$$

$3x_1 - 3x_2 = 0$

$x_1 = x_2$

Chose $x_2 = 1$

$$\therefore \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

③ Construct P with vectors in ②

$$P = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

④ Construct D with eigenvalues as the diagonals

(Diagonals need to correspond with vectors in P)

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

⑤ $P^{-1} D P = A$

$$P^{-1} = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$$

$$A^8 = \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 2^8 & 0 \\ 0 & 1^8 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 766 & -765 \\ 510 & -509 \end{bmatrix}$$

PRACTICE PROBLEMS

1. Find $T(a_0 + a_1 t + a_2 t^2)$, if T is the linear transformation from $\mathbb{P}_2$ to $\mathbb{P}_2$ whose matrix relative to $\mathcal{B} = \{1, t, t^2\}$ is

$$[T]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}$$

2. Let A, B, and C be $n \times n$ matrices. The text has shown that if A is similar to B, then B is similar to A. This property, together with the statements below, shows that "similar to" is an *equivalence relation*. (Row equivalence is another example of an equivalence relation.) Verify parts (a) and (b).
   a. A is similar to A.
   b. If A is similar to B and B is similar to C, then A is similar to C.

RCISES

$\mathbf{b}_2, \mathbf{b}_3\}$ and $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$ be bases for vector
d W, respectively. Let $T : V \to W$ be a linear
n with the property that

$-5\mathbf{d}_2$, $T(\mathbf{b}_2) = -\mathbf{d}_1 + 6\mathbf{d}_2$, $T(\mathbf{b}_3) = 4\mathbf{d}_2$

ix for T relative to $\mathcal{B}$ and $\mathcal{D}$.

$\mathbf{d}_2\}$ and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ be bases for vector spaces
pectively. Let $T : V \to W$ be a linear transfor-
he property that

Find the matrix for T relative to $\mathcal{B}$ and the standard basis for $\mathbb{R}^2$.

5. Let $T : \mathbb{P}_2 \to \mathbb{P}_3$ be the transformation that maps a polynomial $\mathbf{p}(t)$ into the polynomial $(t + 5)\mathbf{p}(t)$.
   a. Find the image of $\mathbf{p}(t) = 2 - t + t^2$.
   b. Show that T is a linear transformation.
   c. Find the matrix for T relative to the bases $\{1, t, t^2\}$ and $\{1, t, t^2, t^3\}$.

SOLUTIONS TO PRACTICE PROBLEMS

1. Let $\mathbf{p}(t) = a_0 + a_1 t + a_2 t^2$ and compute

$$[T(\mathbf{p})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\mathbf{p}]_{\mathcal{B}} = \begin{bmatrix} 3 & 4 & 0 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 3a_0 + 4a_1 \\ 5a_1 - a_2 \\ a_0 - 2a_1 + 7a_2 \end{bmatrix}$$

So $T(\mathbf{p}) = (3a_0 + 4a_1) + (5a_1 - a_2)t + (a_0 - 2a_1 + 7a_2)t^2$.

2. a. $A = (I)^{-1}AI$, so A is similar to A.
   b. By hypothesis, there exist invertible matrices P and Q with the property that $B = P^{-1}AP$ and $C = Q^{-1}BQ$. Substitute the formula for B into the formula for C, and use a fact about the inverse of a product:

$$C = Q^{-1}BQ = Q^{-1}(P^{-1}AP)Q = (PQ)^{-1}A(PQ)$$

This equation has the proper form to show that A is similar to C.

COMPLEX EIGENVALUES

Since the characteristic equation of an $n \times n$ matrix involves a polynomial of degree n, the equation always has exactly n roots, counting multiplicities, *provided that possibly complex roots are included*. This section shows that if the characteristic equation of a real matrix A has some complex roots, then these roots provide critical information about A. The key is to let A act on the space $\mathbb{C}^n$ of n-tuples of complex numbers.[1]

PRACTICE PROBLEMS

1. Let $\mathbf{a} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$. Compute $\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}$ and $\left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}\right)\mathbf{a}$.

2. Let $\mathbf{c} = \begin{bmatrix} 4/3 \\ -1 \\ 2/3 \end{bmatrix}$ and $\mathbf{d} = \begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix}$.
   a. Find a unit vector $\mathbf{u}$ in the direction of $\mathbf{c}$.
   b. Show that $\mathbf{d}$ is orthogonal to $\mathbf{c}$.
   c. Use the results of (a) and (b) to explain why $\mathbf{d}$ must be orthogonal to the unit vector $\mathbf{u}$.

3. Let W be a subspace of $\mathbb{R}^n$. Exercise 30 establishes that $W^\perp$ is also a subspace of $\mathbb{R}^n$. Prove that $\dim W + \dim W^\perp = n$.

① $\begin{bmatrix} -2 \\ 1 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = (-2)(-3) + (1)(1) = 6+1 = 7$

$\begin{bmatrix} -2 \\ 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = (-2)(-2) + (1)(1) = 4+1 = 5$

$\frac{7}{5}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -14/5 \\ 7/5 \end{pmatrix}$

② $||c|| = \sqrt{\left(\frac{4}{3}\right)^2 + (-1)^2 + \left(\frac{2}{3}\right)^2}$ ← scale first

$\dfrac{[4/3 \;\; -1 \;\; 2/3]}{||c||}$

$\begin{bmatrix} 4/3 \\ -1 \\ 2/3 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \\ -1 \end{bmatrix} = 20/3 - 6 - \frac{2}{3}$

$= \frac{20}{3} - \frac{18}{3} - \frac{2}{3}$

$= 0 \quad \therefore d \perp c$

Orthogonal projection of y onto u:

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u$$

component of y orthogonal to u:

$$y - \hat{y}$$

$$y = \hat{y} + (y - \hat{y})$$

Span $\{u_1, u_2\}$

$$y = \frac{y \cdot u_1}{y \cdot u_1} u_1 + \frac{y \cdot u_2}{y \cdot u_2} u_2$$

orthonormal : dot product $= 0$ & dot product of self $= 1$

$v_1 v_2 = 0$ $\qquad$ $v_1 v_1 = 1$

PRACTICE PROBLEMS

1. Let $\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. Show that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for $\mathbb{R}^2$.
2. Let $\mathbf{y}$ and L be as in Example 3 and Figure 3. Compute the orthogonal projection $\hat{\mathbf{y}}$ of $\mathbf{y}$ onto L using $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ instead of the $\mathbf{u}$ in Example 3.
3. Let U and $\mathbf{x}$ be as in Example 6, and let $\mathbf{y} = \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix}$. Verify that $U\mathbf{x} \cdot U\mathbf{y} = \mathbf{x} \cdot \mathbf{y}$.
4. Let U be an $n \times n$ matrix with orthonormal columns. Show that $\det U = \pm 1$.

SOLUTIONS TO PRACTICE PROBLEMS

1. The vectors are orthogonal because

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = -2/5 + 2/5 = 0$$

They are unit vectors because

$$\|\mathbf{u}_1\|^2 = (-1/\sqrt{5})^2 + (2/\sqrt{5})^2 = 1/5 + 4/5 = 1$$
$$\|\mathbf{u}_2\|^2 = (2/\sqrt{5})^2 + (1/\sqrt{5})^2 = 4/5 + 1/5 = 1$$

In particular, the set $\{\mathbf{u}_1, \mathbf{u}_2\}$ is linearly independent, and hence is a basis for $\mathbb{R}^2$ since there are two vectors in the set.

2. When $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$,

$$\hat{\mathbf{y}} = \frac{\mathbf{y} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} = \frac{20}{5} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

This is the same $\hat{\mathbf{y}}$ found in Example 3. The orthogonal projection does not seem to depend on the $\mathbf{u}$ chosen on the line. See Exercise 31.

3. $U\mathbf{y} = \begin{bmatrix} 1/\sqrt{2} & 2/3 \\ 1/\sqrt{2} & -2/3 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} -3\sqrt{2} \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -7 \\ 2 \end{bmatrix}$

Also, from Example 6, $\mathbf{x} = \begin{bmatrix} \sqrt{2} \\ 3 \end{bmatrix}$ and $U\mathbf{x} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$. Hence

$$U\mathbf{x} \cdot U\mathbf{y} = 3 + 7 + 2 = 12, \quad \text{and} \quad \mathbf{x} \cdot \mathbf{y} = -6 + 18 = 12$$

4. Since U is an $n \times n$ matrix with orthonormal columns, by Theorem 6, $U^T U = I$. Taking the determinant of the left side of this equation, and applying Theorems 5 and 6 from Section 3.2 results in $\det U^T U = (\det U^T)(\det U) = (\det U)(\det U) = (\det U)^2$. Recall $\det I = 1$. Putting the two sides of the equation back together results in $(\det U)^2 = 1$ and hence $\det U = \pm 1$.

PRACTICE PROBLEMS

1. Let $\mathbf{u}_1 = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix}$, and $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Use the fact that $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal to compute $\text{proj}_W \mathbf{y}$.
2. Let W be a subspace of $\mathbb{R}^n$. Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$ and let $\mathbf{z} = \mathbf{x} + \mathbf{y}$. If $\mathbf{u}$ is the projection of $\mathbf{x}$ onto W and $\mathbf{v}$ is the projection of $\mathbf{y}$ onto W, show that $\mathbf{u} + \mathbf{v}$ is the projection of $\mathbf{z}$ onto W.

SOLUTION TO PRACTICE PROBLEMS

1. Compute

$$\text{proj}_W \mathbf{y} = \frac{\mathbf{y}\cdot\mathbf{u}_1}{\mathbf{u}_1\cdot\mathbf{u}_1}\mathbf{u}_1 + \frac{\mathbf{y}\cdot\mathbf{u}_2}{\mathbf{u}_2\cdot\mathbf{u}_2}\mathbf{u}_2 = \frac{88}{66}\mathbf{u}_1 + \frac{-2}{6}\mathbf{u}_2$$

$$= \frac{4}{3}\begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix} - \frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -9 \\ 1 \\ 6 \end{bmatrix} = \mathbf{y}$$

In this case, $\mathbf{y}$ happens to be a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$, so $\mathbf{y}$ is in W. The closest point in W to $\mathbf{y}$ is $\mathbf{y}$ itself.

2. Using Theorem 10, let U be a matrix whose columns consist of an orthonormal basis for W. Then $\text{proj}_W\ \mathbf{z} = UU^T\ \mathbf{z} = UU^T\ (\mathbf{x} + \mathbf{y}) = UU^T\mathbf{x} + UU^T\mathbf{y} = \text{proj}_W\mathbf{x} + \text{proj}_W\mathbf{y} = \mathbf{u} + \mathbf{v}$.

The Gram–Schmidt Process

Given a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \frac{\mathbf{x}_2\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \frac{\mathbf{x}_3\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 \\
&\ \ \vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \frac{\mathbf{x}_p\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot\mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_p\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 - \cdots - \frac{\mathbf{x}_p\cdot\mathbf{v}_{p-1}}{\mathbf{v}_{p-1}\cdot\mathbf{v}_{p-1}}\mathbf{v}_{p-1}
\end{aligned}$$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W. In addition

$$\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \quad \text{for } 1 \le k \le p \qquad (1)$$

EXAMPLE 4 Find a QR factorization of $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

SOLUTION The columns of A are the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ in Example 2. An orthogonal basis for Col $A = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ was found in that example:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2' = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \\ 1/3 \end{bmatrix}$$

To simplify the arithmetic that follows, scale $\mathbf{v}_3$ by letting $\mathbf{v}_3' = 3\mathbf{v}_3$. Then normalize the three vectors to obtain $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, and use these vectors as the columns of Q:

$$Q = \begin{bmatrix} 1/2 & -3/\sqrt{12} & 0 \\ 1/2 & 1/\sqrt{12} & -2/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \\ 1/2 & 1/\sqrt{12} & 1/\sqrt{6} \end{bmatrix}$$

By construction, the first k columns of Q are an orthonormal basis of Span $\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$. From the proof of Theorem 12, $A = QR$ for some R. To find R, observe that $Q^TQ = I$, because the columns of Q are orthonormal. Hence

$$Q^TA = Q^T(QR) = IR = R$$

and

$$R = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -3/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} & 1/\sqrt{12} \\ 0 & -2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3/2 & 1 \\ 0 & 3/\sqrt{12} & 2/\sqrt{12} \\ 0 & 0 & 2/\sqrt{6} \end{bmatrix}$$

■

PRACTICE PROBLEMS

1. Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1/3 \\ 1/3 \\ -2/3 \end{bmatrix}$. Construct an orthonormal basis for W.

2. Suppose $A = QR$, where Q is an $m \times n$ matrix with orthogonal columns and R is an $n \times n$ matrix. Show that if the columns of A are linearly dependent, then R cannot be invertible.

---

1. Let $\mathbf{v}_1 = \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{x}_2 - 0\mathbf{v}_1 = \mathbf{x}_2$. So $\{\mathbf{x}_1, \mathbf{x}_2\}$ is already orthogonal. All that is needed is to normalize the vectors. Let

$$\mathbf{u}_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

Instead of normalizing $\mathbf{v}_2$ directly, normalize $\mathbf{v}_2' = 3\mathbf{v}_2$ instead:

$$\mathbf{u}_2 = \frac{1}{\|\mathbf{v}_2'\|}\mathbf{v}_2' = \frac{1}{\sqrt{1^2 + 1^2 + (-2)^2}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}$$

Then $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthonormal basis for W.

2. Since the columns of A are linearly dependent, there is a nontrivial vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$. But then $QR\mathbf{x} = \mathbf{0}$. Applying Theorem 7 from Section 6.2 results in $\|R\mathbf{x}\| = \|QR\mathbf{x}\| = \|\mathbf{0}\| = 0$. But $\|R\mathbf{x}\| = 0$ implies $R\mathbf{x} = \mathbf{0}$, by Theorem 1 from Section 6.1. Thus there is a nontrivial vector $\mathbf{x}$ such that $R\mathbf{x} = \mathbf{0}$ and hence, by the Invertible Matrix Theorem, R cannot be invertible.

EXAMPLE 1 Find a least-squares solution of the inconsistent system $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

SOLUTION To use normal equations (3), compute:

$$A^TA = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

Then the equation $A^TA\mathbf{x} = A^T\mathbf{b}$ becomes

$$\begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

Row operations can be used to solve this system, but since A^TA is invertible and 2×2, it is probably faster to compute

$$(A^TA)^{-1} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

and then to solve $A^TA\mathbf{x} = A^T\mathbf{b}$ as

$$\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b} = \frac{1}{84}\begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}\begin{bmatrix} 19 \\ 11 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 84 \\ 168 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad ■$$

In many calculations, A^TA is invertible, but this is not always the case. The next example involves a matrix of the sort that appears in what are called *analysis of variance* problems in statistics.

EXAMPLE 2 Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix}$$

SOLUTION Compute

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -3 \\ -1 \\ 0 \\ 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 2 \\ 6 \end{bmatrix}$$

The augmented matrix for $A^TA\mathbf{x} = A^T\mathbf{b}$ is

$$\begin{bmatrix} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 0 & 0 & -4 \\ 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 0 & 2 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 & -5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The general solution is $x_1 = 3 - x_4$, $x_2 = -5 + x_4$, $x_3 = -2 + x_4$, and x_4 is free. So the general least-squares solution of $A\mathbf{x} = \mathbf{b}$ has the form

$$\hat{\mathbf{x}} = \begin{bmatrix} 3 \\ -5 \\ -2 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad ■$$

The next theorem gives useful criteria for determining when there is only one least-squares solution of $A\mathbf{x} = \mathbf{b}$. (Of course, the orthogonal projection $\hat{\mathbf{b}}$ is always unique.)

EM 14 Let A be an $m \times n$ matrix. The following statements are logically equivalent:

a. The equation $A\mathbf{x} = \mathbf{b}$ has a unique least-squares solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

EXAMPLE 3 Given A and $\mathbf{b}$ as in Example 1, determine the least-squares error in the least-squares solution of $A\mathbf{x} = \mathbf{b}$.

SOLUTION From Example 1,

$$\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} \quad \text{and} \quad A\hat{\mathbf{x}} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$$

Hence

$$\mathbf{b} - A\hat{\mathbf{x}} = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ -4 \\ 8 \end{bmatrix}$$

and

$$\|\mathbf{b} - A\hat{\mathbf{x}}\| = \sqrt{(-2)^2 + (-4)^2 + 8^2} = \sqrt{84}$$

The least-squares error is $\sqrt{84}$. For any $\mathbf{x}$ in $\mathbb{R}^2$, the distance between $\mathbf{b}$ and the vector $A\mathbf{x}$ is at least $\sqrt{84}$. See Figure 3. Note that the least-squares solution $\hat{\mathbf{x}}$ itself does not appear in the figure. ■

Alternative Calculations of Least-Squares Solutions

The next example shows how to find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ when the columns of A are orthogonal. Such matrices often appear in linear regression problems, discussed in the next section.

EXAMPLE 4 Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 1 & -6 \\ 1 & -2 \\ 1 & 1 \\ 1 & 7 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 1 \\ 6 \end{bmatrix}$$

SOLUTION Because the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ of A are orthogonal, the orthogonal projection of $\mathbf{b}$ onto Col A is given by

$$\hat{\mathbf{b}} = \frac{\mathbf{b}\cdot\mathbf{a}_1}{\mathbf{a}_1\cdot\mathbf{a}_1}\mathbf{a}_1 + \frac{\mathbf{b}\cdot\mathbf{a}_2}{\mathbf{a}_2\cdot\mathbf{a}_2}\mathbf{a}_2 = \frac{8}{4}\mathbf{a}_1 + \frac{45}{90}\mathbf{a}_2 \tag{5}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \\ 1/2 \\ 7/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 5/2 \\ 11/2 \end{bmatrix}$$

Now that $\hat{\mathbf{b}}$ is known, we can solve $A\hat{\mathbf{x}} = \hat{\mathbf{b}}$. But this is trivial, since we already know what weights to place on the columns of A to produce $\hat{\mathbf{b}}$. It is clear from (5) that

$$\hat{\mathbf{x}} = \begin{bmatrix} 8/4 \\ 45/90 \end{bmatrix} = \begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$$ ■

In some cases, the normal equations for a least-squares problem can be *ill-conditioned*; that is, small errors in the calculations of the entries of A^TA can sometimes

PRACTICE PROBLEMS

1. Let $A = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$. Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$, and compute the associated least-squares error.
2. What can you say about the least-squares solution of $A\mathbf{x} = \mathbf{b}$ when $\mathbf{b}$ is orthogonal to the columns of A?

EXAMPLE 1 Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the data points $(2, 1)$, $(5, 2)$, $(7, 3)$, and $(8, 3)$.

SOLUTION Use the x-coordinates of the data to build the design matrix X in (1) and the y-coordinates to build the observation vector $\mathbf{y}$:

$$X = \begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix}, \quad \mathbf{y} = \begin{array}{c} y \\ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} \end{array}$$

For the least-squares solution of $X\beta = \mathbf{y}$, obtain the normal equations (with the new notation):

$$X^TX\beta = X^T\mathbf{y}$$

That is, compute

$$X^TX = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 5 \\ 1 & 7 \\ 1 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}$$

$$X^T\mathbf{y} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 5 & 7 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

The normal equations are

$$\begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 9 \\ 57 \end{bmatrix}$$

Hence

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 4 & 22 \\ 22 & 142 \end{bmatrix}^{-1}\begin{bmatrix} 9 \\ 57 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 142 & -22 \\ -22 & 4 \end{bmatrix}\begin{bmatrix} 9 \\ 57 \end{bmatrix} = \frac{1}{84}\begin{bmatrix} 24 \\ 30 \end{bmatrix} = \begin{bmatrix} 2/7 \\ 5/14 \end{bmatrix}$$

Thus the least-squares line has the equation

$$y = \frac{2}{7} + \frac{5}{14}x$$

See Figure 2. ■

SOLUTIONS TO PRACTICE PROBLEMS

1. First, compute

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & 7 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 9 & 0 \\ 9 & 83 & 28 \\ 0 & 28 & 14 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 1 \\ -3 & 5 & 7 \\ -3 & 1 & 2 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix} = \begin{bmatrix} -3 \\ -65 \\ -28 \end{bmatrix}$$

Next, row reduce the augmented matrix for the normal equations, $A^TA\mathbf{x} = A^T\mathbf{b}$:

$$\begin{bmatrix} 3 & 9 & 0 & -3 \\ 9 & 83 & 28 & -65 \\ 0 & 28 & 14 & -28 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & -1 \\ 0 & 56 & 28 & -56 \\ 0 & 28 & 14 & -28 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & -3/2 & 2 \\ 0 & 1 & 1/2 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The general least-squares solution is $x_1 = 2 + \frac{3}{2}x_3$, $x_2 = -1 - \frac{1}{2}x_3$, with x_3 free. For one specific solution, take $x_3 = 0$ (for example), and get

$$\hat{\mathbf{x}} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

To find the least-squares error, compute

$$\hat{\mathbf{b}} = A\hat{\mathbf{x}} = \begin{bmatrix} 1 & -3 & -3 \\ 1 & 5 & 1 \\ 1 & 7 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ -5 \end{bmatrix}$$

It turns out that $\hat{\mathbf{b}} = \mathbf{b}$, so $\|\mathbf{b} - \hat{\mathbf{b}}\| = 0$. The least-squares error is zero because $\mathbf{b}$ happens to be in Col A.

2. If $\mathbf{b}$ is orthogonal to the columns of A, then the projection of $\mathbf{b}$ onto the column space of A is $\mathbf{0}$. In this case, a least-squares solution $\hat{\mathbf{x}}$ of $A\mathbf{x} = \mathbf{b}$ satisfies $A\hat{\mathbf{x}} = \mathbf{0}$.