3.1 Properties of determinants

1. det of an n×n identity matrix is 1.
2. det changes when two rows are exchanged.

>> if P is a permutation matrix with r row exchanges, the $|P| = 1$ for even r and $|P| = -1$ for odd r.

3. The determinant is a linear function of each row separately.

If 1 row of a matrix A is multiplied by t to get A', then $|A'| = t|A|$

$$\begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

If one row of A is added to one row of A', then the determinants add.

$$\begin{vmatrix} a+a' & b+b' \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a' & b' \\ c & d \end{vmatrix}$$

Important: *This rule applies only when the other rows do not change.*

4. If two rows of A are equal, then $|A| = 0$

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0 \Rightarrow ab - ab = 0$$

5. Subtracting a multiple of one row from another row leaves $|A|$ unchanged.

$$\begin{vmatrix} a & b \\ c - la & d - lb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$|A| = |U|$ without row exchanges and $|A| = \pm|U|$ with row exchanges.

6. A matrix with a row of zeros has $|A| = 0$

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0$$

7. If A is triangular, then $|A| = a_{11} a_{22} \cdots a_{nn}$ = product of diagonal entries

$$\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & 0 & 7 \end{vmatrix} \xrightarrow{R \leftarrow R_2 - R_1} \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 7 \end{vmatrix} = 1 \times 1 \times 7$$

$$\begin{vmatrix} 1 & 1 & 3 & 7 \\ 1 & 2 & 8 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix}$$

$(2-1)\cdot(4-3)$

$= 1$

When det = 0 → singular matrix

8. If A is singular, then $|A| = 0$

If A is invertible, then $|A| \neq 0$

Transform A to U through elimination.
If A is singular:

- U has a zero row
- From previous rules, $|A| = |U| = 0$

If A is invertible:

- U has pivots along its diagonal
- From Rule 7, product of non-zero pivots ⇒ non zero determinant
- $|A| = \pm|U| = \pm$(product of pivots)

[+ for even number of row exchanges and − for odd number of row exchanges]

Pivots of a 2 × 2 matrix ($a \neq 0$): $a \cdot d - \frac{cb}{a}$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ 0 & d - (c/a)b \end{vmatrix} = ad - bc$$

(Finally, a formula for the determinant!!)

9. $|AB| = |A||B|$

Consider the ratio $D(A) = |AB|/|B|$. If $D(A)$ satisfies rules 1, 2 and 3, then it is a determinant.

- Rule 1 (*Determinant of I*)
  - If $A = I$, then $D(A) = |B|/|B| = 1$
- Rule 2(*Sign reversal*)
  - Two rows of A are exchanged ⇒ Same two rows of $|AB|$ are exchanged ⇒ $|AB|$ changes sign ⇒ $D(A)$ changes sign
- Rule 3 (*Linearity*)
  - When 1 row of A is multiplied by t ⇒ so is 1 row of AB ⇒ $|AB|$ is multiplied by t ⇒ $D(A)$ is multiplied by t
  - When 1 row of A is added to 1 row of A' ⇒ 1 row of AB is added to 1 row of $A'B$ ⇒ determinants add ⇒ dividing by B, the ratios add

The ratio $|AB|/|B|$ has the same properties that define $|A|$.

Therefore, $|AB|/|B| = |A| \Rightarrow |AB| = |A||B|$

If $|B| = 0$, B is singular ⇒ AB is singular ⇒ $|AB| = 0$

$|A||B| = 0$

Therefore $|AB| = |A||B|$

10. $|A^T| = A$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc$$

* All properties apply to columns also

3.2 Determinants as Area or Volume

A can be transformed into a diagonal matrix by:

Interchanging two columns

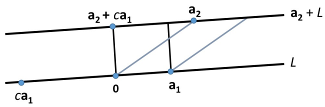
>> does not change the parallelogram

>> det (A) unchanged (Property 2)

Adding a multiple of one column to another

Prove the following geometric observation:

Let $\mathbf{a}_1$ and $\mathbf{a}_2$ be nonzero vectors. Then for any scalar c, the area of a parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2 + c\mathbf{a}_1$.

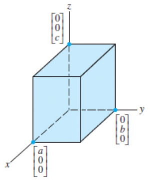


Assume $\mathbf{a}_2$ is not a multiple of $\mathbf{a}_1$.

- L is the line through $\mathbf{0}$ and $\mathbf{a}_1 \Rightarrow \mathbf{a}_2 + L$ is the line through $\mathbf{a}_2$ and parallel to L
- Points $\mathbf{a}_2$ and $\mathbf{a}_2 + c\mathbf{a}_1$ have the same perpendicular distance to L
- Hence, two parallelograms have the same area (base X height)

<u>Proof for $\mathbb{R}^3$ (i.e., 3×3 matrix):</u>

True for a 3×3 diagonal matrix $\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}$



Can we transform any 3×3 matrix $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ into a diagonal matrix without change in volume of the associated parallelepiped or in $|A|$?

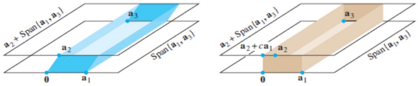
A can be transformed into a diagonal matrix by:

- Interchanging two columns (same as row operations on A^T)
  - Does not change the parallelepiped

- Adding a multiple of one column to another

  In the figure below:

  - Volume of parallelepiped = area of base $\times$ height
  - Base is in Span$\{\mathbf{a}_1, \mathbf{a}_3\}$ Height = $\mathbf{a}_2 +$ Span$\{\mathbf{a}_1, \mathbf{a}_3\}$
  - $\mathbf{a}_2 + c\mathbf{a}_1$ lies in the plane $\mathbf{a}_2+$Span$\{\mathbf{a}_1, \mathbf{a}_3\}$, which is parallel to Span$\{\mathbf{a}_1, \mathbf{a}_3\}$
  - Hence, any vector $\mathbf{a}_2 + c\mathbf{a}_1$ has the same height as $\mathbf{a}_2$
  - Therefore, the volume of the parallelepiped is unchanged when $[\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ is changed to $[\mathbf{a}_1 \quad \mathbf{a}_2 + c\mathbf{a}_1 \quad \mathbf{a}_3]$

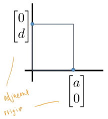


Theorem 3.1

if A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is |A|. If |A| is a 3×3 matrix, the volume of the parallelopiped determined by the columns of A is |A|.

Proof. True for a 2×2 diagonal matrix:

$\text{abs}(\begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}) = \text{abs}(ad) =$area of rectangle



must be adjacent to origin

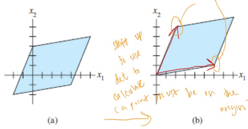
<u>Exercise 3.2.1</u>

Calculate the area of the parallelogram determined by the points $(-2, -2), (0, 3), (4, -1), (6, 4)$.

<u>Solution</u>

Translate the parallelogram to one having the origin as a vertex, e.g., subtract (-2, -2) from each of the four vertices.

adjacent vertices



shift up to use det to calculate (a point must be on the origin)

(a) (b)

Translating a parallelogram does not change its area

New vertices are at $(0, 0), (2, 5), (6, 1), (8, 6)$.

This parallelogram is determined by the columns of $A = \begin{bmatrix} 2 & 6 \\ 5 & 1 \end{bmatrix}$

adjacent matrixs

$\text{abs}(|A|) = |-28|$

Therefore, area of the parallelogram is 28.

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3.3 Linear Transformations

- How does the area (or volume) of a transformed set compare with the area (or volume) of the original

Theorem 3.2. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then area of $T(S) = abs(|A|) \times$ area of S.*
If T is determined by a 3×3 matrix A, and if S is a parallelepiped in $\mathbb{R}^3$, then volume of $T(S) = abs(|A|) \times$ volume of S.

Proof.
Consider the 2×2 case, $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{bmatrix}$
A parallelogram at the origin in $\mathbb{R}^2$ determined by the vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ has the form
$S = \{s_1\mathbf{b}_1 + s_2\mathbf{b}_2 : 0 \le s_1 \le 1, 0 \le s_2 \le 1\}$

The image of S under T consists of the points of the form

$$T(s_1\mathbf{b}_1 + s_2\mathbf{b}_2) = s_1T(\mathbf{b}_1) + s_2T(\mathbf{b}_2) = s_1A\mathbf{b}_1 + s_2A\mathbf{b}_2,$$

where $0 \le s_1 \le 1, 0 \le s_2 \le 1$.

$T(S)$ is the parallelogram determined by columns of $\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix} = AB$ where $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix}$.

$$\text{area of } T(S) = abs(|AB|) = (abs|A|)(abs|B|) = (abs|A|)(\text{area of } S)$$

Now for the general case:

An arbitrary parallelogram has the form $\mathbf{p} + S$
where $\mathbf{p}$ is a vector and S is a parallelogram at the origin.

$T(\mathbf{p} + S) = T(\mathbf{p}) + T(S)$

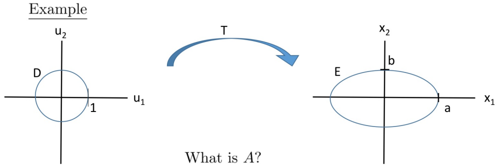
Translation does not affect the area of a set

$$\begin{aligned} \text{area of } T(\mathbf{p} + S) &= \text{area of } (T(\mathbf{p}) + T(S)) \\ &= \text{area of } T(S) \\ &= abs(|A|) \times \text{area of } S \\ &= abs(|A|) \times \text{area of } \mathbf{p} + S \end{aligned}$$

Proof for 3×3 is analogous.

Thereom 3.2 is applicable for arbitrary shapes also.

Example

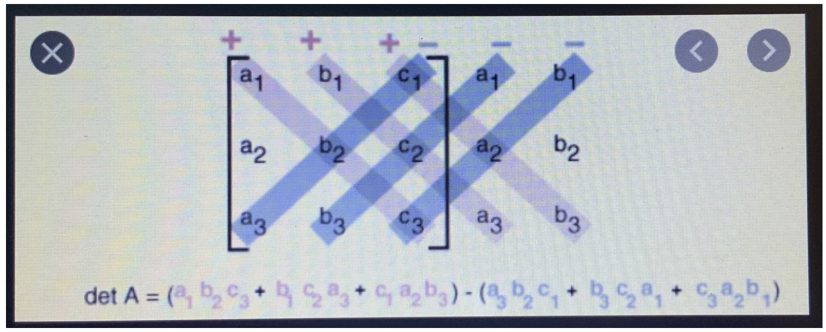


What is A?

$$\begin{aligned} \text{area of ellipse} &= \text{area of } T(D) \\ &= abs(|A|) \times \text{area of } D \\ &= ab \times \pi 1^2 = \pi ab \end{aligned}$$

If $\mathbf{x} = A\mathbf{u}$ with $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, equation of ellipse given by $\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$, what is $\mathbf{u}$?

"Shoelace" method



1 Determinants

2×2 matrices

Given a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, its **determinant** is $\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3×3 matrices

- Direct calculation

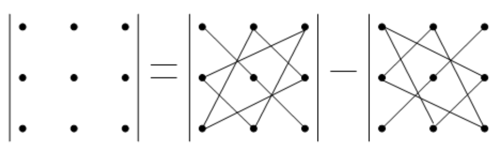
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

- Expansion by first row (recursive definition)

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \overset{+}{a_1}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} \overset{-}{-} a_2\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} \overset{+}{+} a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

- Rule of Sarrus

  This is mostly a mnemonic device. Each line corresponds to a multiplication term of three (connected) matrix elements. Compare this to the explicit formula to find all corresponding terms.



Larger matrices

For larger matrices, a recursive definition is used via a row (or column) expansion. Since you have a choice which row/column to use, it is important to know what sign is attached to the determinant of a minor matrix. Below is the pattern: compare this to the recursive definition (expansion by the first row) of the determinant of a 3×3 matrix.

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix}$$

Properties

- A determinant of a **triangular** matrix, T, has as its value the multiplication of the values along the diagonal, i.e., $det(T) = \prod_{i=1}^{n} T_{ii}$
- A determinant of a matrix with two identical rows has value zero.
- If A and B are $n \times n$ matrices then $\det(AB) = \det(A)\det(B)$.
- Interchanging two rows in a matrix changes the sign of the determinant.
- Multiplying one row by k has the effect of multiplying the determinant by k.
- Adding a multiple of one row to another row does not change the determinant.

2 Inverse matrix

Definition 1 (Inverse)**.** *A $k \times k$ matrix A is said to be* ***invertible*** *if there exists a $k \times k$ matrix B such that*

$$AB = I_k \quad \textit{and} \quad BA = I_k$$

If such a matrix B exists, it is often written A^{-1} and called the ***inverse*** *of A*

Explicit formula for inverse (case of 3×3 matrices)

Recall the notion of cofactors from the formula for the determinant:

$$\begin{aligned} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} &= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} \end{aligned}$$

- The A_{ij} numbers are called the **cofactors**
- They are used to calculate the **adjoint** matrix:

$$\text{adj}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^T = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$$

Theorem 1. $A^{-1} = \frac{1}{\det A} adj\,(A)$

Theorem 2. *Let A be a $n \times n$ matrix. The following are equivalent:*

- *A is invertible*
- *The set of* ***homogenous*** *simultaneous equations (those which can be written $A\mathbf{x} = 0$) has no solution other than $\mathbf{x} = 0$*
- *The set of simultaneous equations which can be written $A\mathbf{x} = c$ has a unique solution*
- *The determinant of A is non-zero.*