1.1 Linear Equation (n variables) : $a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = b$

$\Rightarrow$ a_n, b : constants ; coefficients a are not all 0.

General Linear system of m equations in n unknowns (variables) :

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m$$

$\Rightarrow$ Special case : $b_1 = b_2 = \ldots = b_m = 0 \rightarrow$ homogeneous

$\Rightarrow$ if not, non-homogeneous

Linear system of 2 equations:

$5x + y = 3$ Soln: $x = 1, y = 2$

$2x - y = 4$

Non-linear equations:

$x + 3y^2 = 4$ } take note of the power 2

$\sin x + y = 0$ } & sin

$\Rightarrow$ at least 1 solution: linear system is CONSISTENT

$\Rightarrow$ no solution: linear system is INCONSISTENT

$\Rightarrow$ 3 possibilities : 0, 1, ∞ no. of solutions

↳ 0 : parallel lines E.g. $x - y = 4$, $3x - 3y = 6$

↳ 1 : intersection E.g. $x - y = 1$, $2x + y = 6$

↳ ∞ : coincident lines (overlap) E.g. $\underbrace{4x - 2y = 1 \, , \, 16x - 8y = 4}_{\times 4}$

1.2 Homogeneous Linear System : $\ldots a_{m1} x_1 + a_{m2} x_2 + \ldots a_{mn} x_n = 0$

$\Rightarrow$ 1 or ∞ solutions

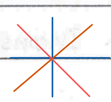
↳ 1 solution : $x_1 = 0, \ldots x_n = 0$ $\Rightarrow$ trivial solution

↳ ∞ solutions : non-trivial solution (includes trivial solution)

Homogeneous Linear System of 2 equations in 2 unknowns:

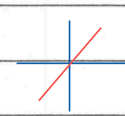
$a_1 x + b_1 y = 0$ (a_1, b_1 not both zero)

$a_2 x + b_2 y = 0$ (a_2, b_2 not both zero)



$\Rightarrow$ only trivial solution : (0,0) is the point of intersection

$\Rightarrow$ ∞ solutions + trivial : overlapping lines that passes through (0,0)



Matrix notation (Example):

$5x + y = 3$

$2x - y = 4$

coefficient matrix: $\begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}$ Augmented matrix: $\begin{bmatrix} 5 & 1 & 3 \\ 2 & -1 & 4 \end{bmatrix}$

Size of matrix: m rows × n columns

Questions to answer when doing questions:

1.3 1. system consistent (at least 1 solution)?
2. Is the solution unique?
} convert augmented matrix into a row equivalent matrix

EROs: 1. Multiply row through by a non-zero constant; 2. Interchange 2 rows;
3. Add a constant times one row to another

Row Echelon Form	Reduced Row Echelon Form
1. rows that are all 0s ⇒ grouped together at the bottom of matrix 2. two consecutive rows not all 0s ⇒ first non-zero = pivot in the lower row occurs further right than pivot of higher row.	1. matrix in row echelon form 2. In every column containing a pivot, the pivot has value 1 and all other elements in column are 0.

E.g. $\begin{bmatrix} 1 & 4 & -3 & 7 \\ 0 & 2 & 6 & 2 \\ 0 & 0 & 9 & 5 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

E.g. $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & -2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Gaussian elimination: puts matrix in row echelon form ⇒ may be not unique

Gauss-Jordan elimination: puts matrix in reduced row echelon form ⇒ unique

⇒ 2 parts: forward phase & backward phase

↳ forward phase: zeros introduced below the pivots (Gaussian)

↳ backward phase: zeros above pivots introduced and pivots transformed to 1.

* if linear system has an equation like $0x_1 + 0x_2 = 15$ ⇒ $0 = 15$

∴ Inconsistent, no solutions ↳ Exception: unknowns are used

Non-homogeneous case :

m equations < n variables ⇒ "fat" matrix

m equations > n variables ⇒ "tall" matrix

(no unique solution)

↑ for non-homogeneous

Homogeneous case :

m equations < n variables

⇒ ∞ solutions : intersecting at a line

m equations > n variables

⇒ cannot arrive to a conclusion about how many solutions

1.4 E.g. $0x_1 + 0x_2 + \dots + 0x_n = 0 \Rightarrow \infty$ solutions

E.g.
$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 & -1 \\ 0 & 0 & 5 & 10 & 0 & 15 & 5 \\ 2 & 6 & 0 & 8 & 4 & 18 & 6 \end{array}\right] \xrightarrow[\text{RREF}]{\text{EROs}} \left[\begin{array}{ccccccc} 1 & 3 & 0 & 4 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

System of ~~linear~~ equations:

$$\boxed{x_1} + 3x_2 + 4x_4 + 2x_5 = 0$$
$$\boxed{x_3} + 2x_4 = 0$$
$$\boxed{x_6} = \frac{1}{3}$$

x_1, x_3, x_6 : leading variables

x_2, x_4, x_5 : free variables ← can take any value

⇓

$$x_1 = -3x_2 - 4x_4 - 2x_5$$
$$x_3 = -2x_4$$
$$x_6 = \frac{1}{3}$$

⇒ assign free variables arbitary values r, s & t respectively

$x_1 = -3r - 4s - 2t$, $x_2 = r$, $x_3 = -2s$, $x_4 = s$,

$x_5 = t$, $x_6 = \frac{1}{3}$ ∴ Infinite value

Procedure using row reduction to solve linear system:

1. Write the augmented matrix
2. Use the row reduction algorithm (EROs) to transform matrix to row echelon form. Decide if system is consistent.
   - ⇒ consistent : go to step 3
   - ⇒ inconsistent : stop, ∴ no solution
3. Continue EROs to get reduced row echelon form
4. Write system of equations corresponding to the matrix obtained in step 3
5. Rewrite each non-zero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation

Gaussian elimination with back subsitution (Example):

Start with row echelon form

$$\left[\begin{array}{cccccc|c} 1 & 3 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{array}{l} x_1 + 3x_2 - 2x_3 + 2x_5 = 0 \\ x_3 + 2x_4 + 3x_6 = 1 \\ x_6 = \frac{1}{3} \end{array} \Rightarrow \begin{array}{ll} x_1 = -3x_2 + 2x_3 - 2x_5 & ① \\ x_3 = 1 - 2x_4 - 3x_6 & ② \\ x_6 = \frac{1}{3} & ③ \end{array}$$

work upwards

Lead variables → x_1, x_3, x_6

Free variables → x_1, x_2, x_5

Sub ③ into ②

$$x_3 = 1 - 2x_4 - 3\left(\tfrac{1}{3}\right)$$
$$= -2x_4 \quad - ④$$

Sub ④ into ①

$$x_1 = -3x_2 + 2(-2x_4) - 2x_5$$
$$= -3x_2 - 4x_4 - 2x_5$$

(right side all free variable terms)

$\therefore x_1 = -3r - 4s - 2t,\ x_2 = r,$

$x_3 = -2s,\ x_4 = s,$

$x_5 = t,\ x_6 = \frac{1}{3}$

1.5 Linear combination of vectors:

$y = c_1 v_1 + \ldots + c_p v_p$ $\Rightarrow$ $v_1 \ldots v_p$ in R^n

$\Rightarrow$ linear combination of $v_1 \ldots v_p$ $\Rightarrow$ scalars $c_1 \ldots c_p$

with weights $c_1 \ldots c_p$.

E.g. Let $a_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $a_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $b = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$.

Determine whether b can be written as a linear combination of a_1 & a_2.

$(\vec{a}_1 x_1 + \vec{a}_2 x_2 = \vec{b}) \rightarrow x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$

Transform it into a linear system

$x_1 + 2x_2 = 7$

$-2x_1 + 5x_2 = 4$

$-5x_1 + 6x_2 = -3$

Find reduced row echelon form

$\begin{matrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & 3 \end{matrix} \quad \ldots \rightarrow \ldots \quad \begin{matrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{matrix}$

$\Rightarrow x_1 = 3, \ x_2 = 2$

To check: $3 \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$

1.6 Span

If $v_1, v_2, \ldots, v_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $v_1, v_2 \ldots$ is denoted by Span $\{v_1, v_2, \ldots, v_p\}$ and is called the subset of $\mathbb{R}^n$ spanned by $v_1, v_2, \ldots, v_p$.

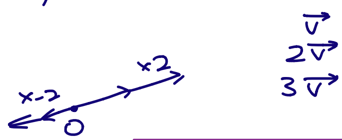
That is, Span $\{v_1, v_2, \ldots v_p\}$ is the collection of all vectors that can be written in the form $c_1v_1 + c_2v_2 + \ldots + c_pv_p$, with scalars $c_1, c_2, \ldots, c_p$.

Geometric view of Span $\{v\}$ and Span $\{u, v\}$

Let v be a non-zero vector in R^3 (3 Dimension Vector Space). Span $\{v\}$ is the set of all scalar multiples of v, which is the set of points in R^3 through v and 0.

What is Scalar multiple?

Imagine that you have an origin O.



Any two vectors in $\mathbb{R}^2$ that are not scalar multiples of each other will span all of $\mathbb{R}^2$

a set of n vectors (columns) in $\mathbb{R}^m$ cannot span $\mathbb{R}^m$ when n variables < m equations

Exercise:

Let $a_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $a_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $b = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$.

Determine if b is in the Span $\{a_1, a_2\}$

First Step → Form the Augmented Matrix $[\vec{a_1}, \vec{a_2}, b]$

$$\begin{matrix} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{matrix} \xrightarrow[\substack{r2 = 2r1 + r2 \\ r3 = 3r1 - r3}]{\text{Convert to Echelon form}} \begin{matrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 18 & -10 \end{matrix} \longrightarrow \begin{matrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 2 \end{matrix} \Rightarrow 0x_2 = 2$$

∴ There is no solution, therefore b is not in the span of $\{\vec{a_1}, \vec{a_2}\}$

Example

Lets show that the vector $\begin{bmatrix} 20 \\ 4 \end{bmatrix}$ belongs to span $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$

$$c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 20 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 20 \\ 3 & 5 & 4 \end{bmatrix} \xrightarrow{R_2 \to -3R_1 + R_2} \begin{bmatrix} 1 & 2 & 20 \\ 0 & -1 & -56 \end{bmatrix} \xrightarrow{R_2 \to -R_2} \begin{bmatrix} 1 & 2 & 20 \\ 0 & 1 & 56 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & -92 \\ 0 & 1 & 56 \end{bmatrix}$$

∴ When $c_1 = -92$, $c_2 = 56$, $\begin{bmatrix} 20 \\ 4 \end{bmatrix}$ is in the span

$Ax = [a_1 \dots a_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 a_1 + \dots + x_n a_n$

$\Rightarrow$ E.g. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + y \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix}$

1.7 The Matrix Equation

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{matrix}\right\} Ax = b$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$, that is, $Ax = b$

A is the matrix of co-efficients

x is the vector of unknowns

b is the vector of constant terms.

Theorem 1.1

If A is an $m \times n$ matrix, with columns $a_1, a_2, \dots, a_n$ and if b is in R^m, then the matrix Equation: $(Ax = B) = (x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b) = [a_1, a_2 \dots a_n, b]$

$Ax = b$ has a solution $\Leftrightarrow$ b is a linear combination of columns of A

Exercise

$$A = \begin{matrix} 1 & 3 & 4 \\ -4 & -2 & -6 \\ -3 & -2 & -7 \end{matrix}, \quad \text{and } b = \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}$$

Form the Augmented Matrix

$$\begin{matrix} 1 & 3 & 4 & b_1 \\ -4 & -2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{matrix} \rightarrow \begin{matrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & 4b_1 - b_2 \\ 0 & 0 & 0 & -2b_1 + b_2 - 2b_3 \end{matrix}$$

$0x_1 + 0x_2 + 0x_3 = -2b_1 + b_2 - 2b_3$

In span $\{\vec{u}, \vec{v}\}$

$0 = -2b_1 + b_2 - 2b_3$

For $Ax = b$ to be consistent, $b_1 - \frac{1}{2}b_2 + b_3 = 0$

The columns of $A = [a_1\ a_2\ a_3]$ span a plane through 0 in R^3.

Theorem 1.2

Let $A = m \times n$ matrix. Then the following statements are logically equivalent, i.e for a particular A, either they are true statements or they are false.

if a == true,
c == true

a. For each b in R^m, the equation $Ax = b$ has a solution. a implies b

b. Each b in R^m is a linear combination of the columns of A. $\Rightarrow x_1\vec{a}_1 + \ldots + x_n\vec{a}_n = b$

c. The columns of A span R^m.

d. A has a pivot position in every row.

Theorem 1.3

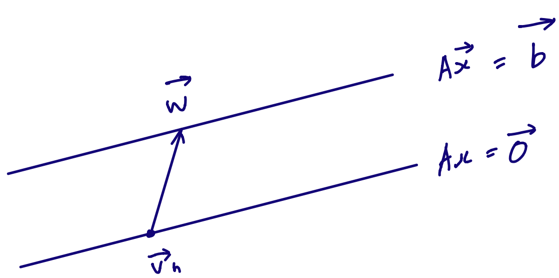
If A is an $m \times n$ matrix, u and v are vectors in R^m, and c is a scalar, then:

$A(u+v) = Au + Av$

$A(cu) = c(Au)$

Theorem 1.4 (Under 1.8)

Suppose the Equation $Ax=b$ is consistent for some given b, and let p be a solution. Then the solution set of $Ax=b$ is the set of all vectors of the form $w = p + v_n$, where v_n is any solution of the homogeneous equation $Ax=0$.



1.8 Solutions sets of Linear Systems

Solutions of Homogeneous Systems

$Ax = b$: If $b = 0$ => homogeneous Equation

$b \neq 0$ => non-homogeneous Equation

$Ax = 0$, the solution $x = 0$ is called the trivial Solution

If x = non-zero factor that satisfies $Ax = 0$, it is a non-trivial Solution.

Exercise

Determine if the following homogeneous system has a non-trivial solution

$$3x_1 + 5x_2 - 4x_3 = 0$$
$$-3x_1 - 2x_2 + 4x_3 = 0$$
$$6x_1 + x_2 - 8x_3 = 0$$

Form the Augmented Matrix

$$\begin{matrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{matrix}$$

Convert to row Echelon form

$$\begin{matrix} \boxed{3} & 5 & -4 & 0 \\ 0 & \boxed{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

No pivot here

$x_3 \rightarrow$ free variable

$A\vec{x} = \vec{0}$ has a non-trivial solution

Reduced Row Echelon form

$$\begin{matrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$x_1 - \frac{4}{3}x_3 = 0$$
$$x_2 = 0$$

$$x_1 = 4/3 x^3$$
$$x_2 = 0$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4/3 x^3 \\ 0 \\ x^3 \end{bmatrix}$$

$$\vec{x} = x3 \begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix} = x_3 \vec{v}$$

Every solution of $Ax = 0$ in this case is a scalar multiple of $v = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$

Solutions of Non-homogeneous Systems

Exercise:

Describe all solutions of $Ax = b$ where $A = \begin{matrix} 3 & 5 & -4 \\ -3 & -2 & 4 \\ 6 & 1 & -8 \end{matrix}$, and $b = \begin{matrix} 7 \\ -1 \\ -4 \end{matrix}$

Solution: Form the Augmented Matrix

$$\begin{matrix} 3 & 5 & -4 & 7 \\ -3 & -2 & 4 & -1 \\ 6 & 1 & -8 & -4 \end{matrix} \longrightarrow \begin{array}{cccc} & & \text{free} & \\ \boxed{1} & 0 & -4/3 & -1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array} \qquad \begin{aligned} x_1 + 0x_2 - 4/3x^3 &= -1 \\ 0x_1 + x_2 + 0x^3 &= 2 \\ 0 &= 0 \end{aligned}$$

Thus, $x_1 = -1 + 4/3 x_3$, $x_2 = 2$ and x_3 is free

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + 4/3 x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 4/3 x_3 \\ 0 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}}_{\uparrow p} + x_3 \underbrace{\begin{bmatrix} 4/3 \\ 0 \\ 1 \end{bmatrix}}_{\uparrow v}$$

Parametric vector form of the solution set of $Ax = b$:

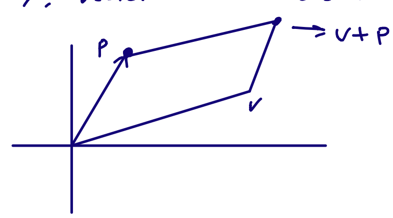
$Ax = 0$ has a non-trivial solution $\Leftrightarrow$ eqn has at least 1 free variable

$x = p + x_3 v$ or $x = p + tv$, $t \in R$

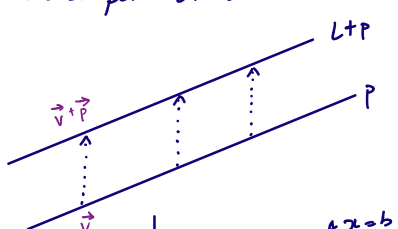
corresponding to $t = 0 \Rightarrow \vec{x} = \vec{p} + 0$

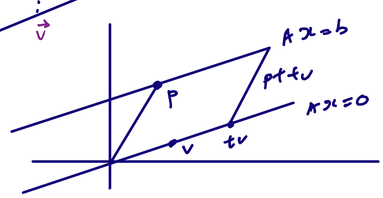
p is one particular solution of $Ax = b$, geometrically, vector addition as translation

$\Rightarrow$ v is translated by p to $v + p$.



Each point on a line translated by a vector P





Each point on a line trans by p.

Adding p to each point on L, we get the equation of the line through p parallel to v. The solution set of $Ax = b$ is a line through p parallel to the solution set of $Ax = 0$.

LI columns: $p > n$ in an $n \times p$ matrix $[\equiv]$

LD columns: $S = \{v_1, \dots v_p\}$ in R^n contains the zero vector $\Rightarrow \{0, \vec{v_1}, \vec{v_2}\}$

1.9 Linear Independence

An indexed set of vectors $\{v_1, \dots v_p\}$ in R^n is said to be linearly independent if the vector equation

$$x_1 v_1 + x_2 v_2 + \dots + x_p v_p = 0 \leftarrow x_1 = x_2 = x_3 \dots = 0$$

has only the trivial solution. The set $\{v_1, \dots, v_p\}$ is said to be linearly dependent if there exist weights $c_1, \dots, c_p$, not all zero such that

$$c_1 v_1 + c_2 v_2 + \dots + c_p v_p = 0$$

Exercise:

Let $v_1 = \begin{matrix}1\\2\\3\end{matrix}$, $v_2 = \begin{matrix}4\\5\\6\end{matrix}$ and $v_3 = \begin{matrix}2\\1\\0\end{matrix}$. Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent. If not, what is the linear dependence relation among them?

1. Check if $A\vec{x} = \vec{0}$ has only trivial solution

Form the Augmented Matrix REF

$$\begin{matrix}1 & 4 & 2 & 0\\2 & 5 & 1 & 0\\3 & 6 & 0 & 0\end{matrix} \overset{ERO}{\sim} \begin{matrix}1 & 4 & 2 & 0\\0 & -3 & -3 & 0\\0 & 0 & 0 & 0\end{matrix} \Rightarrow \begin{matrix}x_1 + 4x_2 + 2x_3 = 0\\-3x_2 - 3x_3 = 0\\x_3 \text{ is a free variable}\end{matrix}$$

$[\vec{v_1}, \vec{v_2}, \vec{v_3}]$ are linearly dependent. $\Rightarrow$ infinite solutions $\therefore$ Non-trivial

2. Convert it into Reduced Row Echelon form $\Rightarrow$ to solve for x_1, x_2, x_3

$$\begin{matrix}1 & 0 & -2 & 0\\0 & 1 & 1 & 0\\0 & 0 & 0 & 0\end{matrix} \Rightarrow \begin{matrix}x_1 - 2x_3 = 0\\x_2 + x_3 = 0\\ \end{matrix} \quad \begin{matrix}x_1 = 2x_3\\x_2 = -x_3\\x_3 \text{ is free}\end{matrix}$$

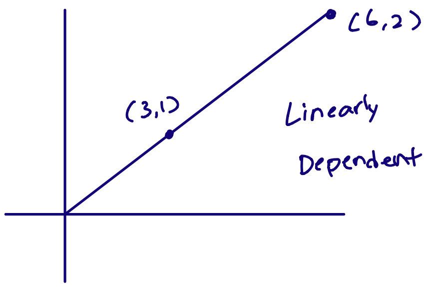
$\underbrace{x_1}_{=}\vec{v_1} + \underbrace{x_2}_{=}\vec{v_2} + \underbrace{x_3}_{=}\vec{v_3} = \vec{0}$ have to be non-zero

Choose $x_3 = 5 \Rightarrow 10\vec{v_1} - 5\vec{v_2} + 5\vec{v_3} = \vec{0}$. $\Rightarrow$ only one of the infinite relationships between v_1, v_2, v_3

The columns of matrix A are linearly independent if and only if the equation $Ax = 0$ has only the trivial solution.

By Inspection for simple cases

$v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$



Linearly Dependent

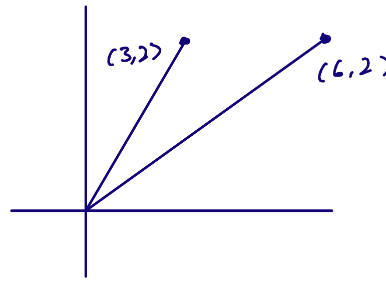
$2\vec{v_1} - \vec{v_2} = \vec{0}$

$x_1 \vec{v_1} + x_2 \vec{v_2} = \vec{0}$

$x_1 = 2$
$x_2 = -1$

They are dep on each other to make it become a 0

$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$



Linearly Independent

$x_1 \vec{v_1} + x_2 \vec{v_2} = \vec{0}$

so either $x_1 = 0$ / $x_2 = 0$, hence they are linearly independent.

① e.g. $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

$\Rightarrow \vec{v_3} = x_1\vec{v_1} + x_2\vec{v_2}$

↓ can be put into this form meaning linearly dependent

② e.g. $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

$v_1 \neq 0$

$j = 3$,

$v_{j=3} = x_1\vec{v_1} + x_2\vec{v_2}$

v_1, v_2 is the preceding vectors

Characterization of linearly dependent sets

① An indexed set $S = \{v_1, \dots, v_p\}$ of two or more vectors is linearly dependent if and only if at least one of the vectors in S is a linear combination of the others. ② In fact, if S is linearly dependent and $v_1 \neq 0$, then some v_j (with $j > 1$) is a linear combination of the preceding vectors $v_1, \dots, v_{j-1}$

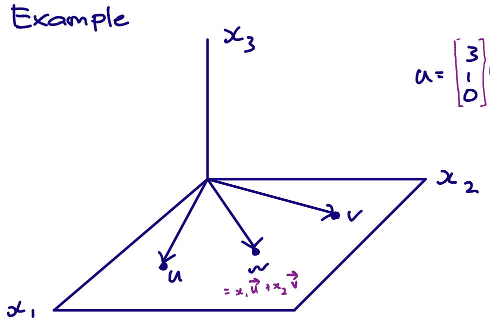
What does it mean?

Given a Span $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$, if they are linearly dependent, it can be written as

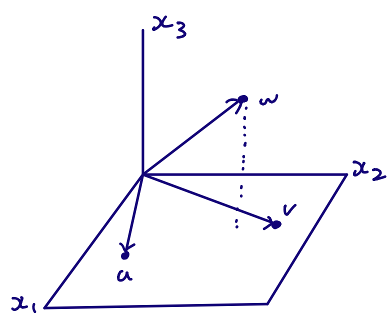
$\vec{v_3} = x_1\vec{v_1} + x_2\vec{v_2}$

Example

$u = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 6 \\ 0 \end{bmatrix}$



Linearly dependent w in span $\{u, v\}$



Linearly independent w not in span $\{u, v\}$

w is in Span $\{u, v\}$, if and only if $\{u, v, w\}$ is linearly dependent.

Theorem 1.5. If a set contains more vectors than there are entries in each vector, then the set is linearly dependent, i.e., any set $\{v_1, \dots, v_p\}$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

e.g. $\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}, \begin{bmatrix} a_3 \\ b_3 \end{bmatrix} \Rightarrow \mathbb{R}^2$

3 vectors, 2 entries ⇒ linear dependent

<u>Proof</u>

Let $A = [v_1 \dots v_p]$. A is $n \times p$.

$Ax = 0$ corresponds to n equations in p unknowns

"fat" matrix

$p > n \Rightarrow$ more variables than equations $\Rightarrow$ there must be a free variable $\Rightarrow$

$Ax = 0$ has non-trivial solutions $\Rightarrow$ columns of A are linearly dependent.

Theorem 1.6.

If a set $S = \{v_1, \dots, v_p\}$ in R^n contains the zero vector, then the set is linearly dependent.

↳ $\{\vec{0}, \vec{v_1}, \vec{v_2}\}$

Proof

By renumbering the vectors, we may suppose $v_1 = 0$. Then the equation $1v_1 + 0v_2 + \dots + 0v_p = 0$ shows that S is linearly dependent.

↓

non-zero
while the rest is
∴ linearly dependent

$p = 3$, $n = 2$

$p > n$

Example

The vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ are linearly dependent. Note: none of the vectors is a multiple of one of the other vectors.

↳ but is still linearly dependent cause $p > n$

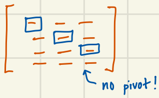
(3) Could a set of 3 vectors in $\mathbb{R}^4$ span all of $\mathbb{R}^4$?

Size of A : 4×3 matrix

But there are only 3 columns

$\Rightarrow$ there cannot be a pivot in every row.

$\Rightarrow$ columns do not span $\mathbb{R}^4$



Theorem 1.2 :

→ col of A span all of $\mathbb{R}^n$

→ Every row has a pivot

∴ In general, a set of n vectors in $\mathbb{R}^m$ cannot span $\mathbb{R}^m$ when $n < m$

(4) Describe all solutions of $Ax = 0$ in parametric form where row equivalent form of A is $\begin{bmatrix} 1 & 3 & 0 & -4 \\ 2 & 6 & 0 & -8 \end{bmatrix}$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \begin{bmatrix} 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_2 - 4x_4 = 0$

$x_1 = -3x_2 + 4x_4$

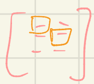
general solution : $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 4 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_4$

$\Rightarrow x_2, x_3, x_4$: free variables

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3x_2 + 4x_4 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} 4 \\ 0 \\ 0 \\ 1 \end{pmatrix} x_4$$

Extract all the x_2 values

(5) A is a 3×2 matrix has 2 pivot positions



(a) does $Ax = 0$ have a non trivial solution?

There are no free variables [no columns without pivots]

$\Rightarrow$ no infinite solutions

$\Rightarrow$ no non-trivial solutions

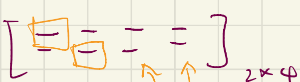
(b) Does $Ax = b$ have at least one solution for every b?

1.2 Theorem

Since every row does not have a pivot,

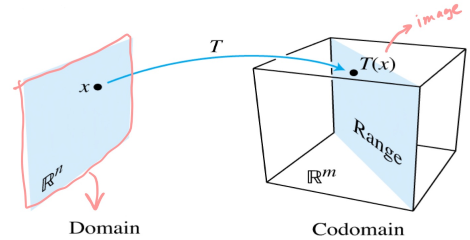
Answer is <u>NO</u>.

(6) A is a 2×4 with 2 pivots



(a) 2 free variables $\Rightarrow$ non trivial solutions

(b) Pivot in every row $\Rightarrow$ Yes



Domain, codomain, and range of $T: \mathbb{R}^n \to \mathbb{R}^m$.

1.10 Introduction to Linear Transformation

Think of $Ax = b$ as:

matrix A as an object that "acts" on a vector x by multiplication to produce a new vector called Ax.

A transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector x in $\mathbb{R}^n$ a vector $T(x)$ in $\mathbb{R}^m$.

Set of all images $T(x)$ - range of T

$\mathbb{R}^n$ - domain of T, $\mathbb{R}^m$ - co-domain of T, $T(x)$ - image of x, all images, range of x.

Exercise

Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$.

a. Find $T(u)$, b. Find an x in $\mathbb{R}^2$ whose image under T is b, c. Is there more than one x whose image under T is b? d. Determine if C is in the range of T

(a) $T(\vec{x}) = A\vec{x}$

$T(\vec{u}) = A\vec{u} = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \xrightarrow{ERO} \begin{bmatrix} 1 & 0 & -1.5 \\ 0 & 1 & -0.5 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -1.5 / x_2 = -0.5 \quad \therefore \vec{x} = \begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix}$

no free variables

(c) $A\vec{x} = b$ has a unique solution, $\therefore$ therefore there is exactly one $\vec{x}$ whose image is $\vec{b}$

(d) Range: set of all images

$c = T(x)$ for some $\vec{x}$?

Is $A\vec{x} = \vec{c}$ consistent?

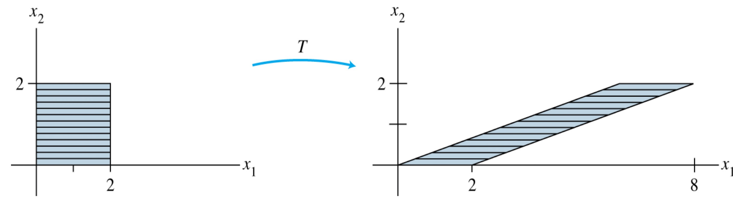
$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ 1 & 7 & 5 \end{bmatrix}$$

EROS

$$\begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

Last Eqn is $0 = -35 \Rightarrow$ inconsistent

C is not in the range of T.



Examples:

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. What is this transformation?

$T(\vec{x}) = A\vec{x}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \longrightarrow$ Projection onto a Plane $x_1 - x_2$

random input of numbers

Let $A = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$. What is this transformation?

$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$ (Shear Transformation) $\Rightarrow$ based is fixed but skewed to the side

1.11 Linear Transformations

Contraction: $0.5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$

Dilation: $2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

A transformation T is linear if:

i. $T(u+v) = T(u) + T(v)$ for all u, v in the domain of T;

ii $T(cu) = cT(u)$

Generalization:

$T(c_1v_1 + \dots + c_pv_p) = c_1T(v_1) + \dots + c_pT(v_p)$

Examples:

$T(x) = rx$, $T: R^2 \rightarrow: R^2$ is called a contraction when $0 \le r \le 1$ and a dilation when $r > 1$.

Show that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a linear transformation by finding the images of $u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$T(\vec{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$; $T(\vec{v}) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$

$T(\vec{u}) + T(\vec{v}) = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$

$\vec{u} + \vec{v} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

$T(\vec{u} + \vec{v}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$

$\therefore T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ (first condition)

$c\vec{u} = \begin{bmatrix} 4c \\ c \end{bmatrix}$;

$T(c\vec{u}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4c \\ c \end{bmatrix} = \begin{bmatrix} -c \\ 4c \end{bmatrix}$

$cT(\vec{u}) = c \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -c \\ 4c \end{bmatrix}$ (second condition)

$\therefore$ Satifies both conditions of the transformation rule.

1.12 The Matrix of a Linear Transformation

Given the description of a linear transformation 'in words', how to obtain the corresponding matrix?

Key observation: T is completely determined by what it does to the columns of the $n\times n$ identity matrix I_n.

Exercise

Let $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Suppose T is a linear transformation from R^2 to R^3 such that $T(e_1) = \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix}$ and $T(e_2) = \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix}$. Find the formula for the image of an arbitrary x in R^2.

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} T(\vec{u}) &= T(x_1 \vec{e_1} + x_2 \vec{e_2}) \\ &= x_1 T(\vec{e_1}) + x_2 T(\vec{e_2}) \\ &= x_1 \begin{bmatrix} 5 \\ -7 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 8 \\ 0 \end{bmatrix} \\ &= \begin{matrix} 5x_1 - 3x_2 \\ -7x_1 + 8x_2 \\ 2x_1 \end{matrix} \quad // \\ &= \begin{bmatrix} 5 & -3 \\ -7 & 8 \\ 2 & 0 \end{bmatrix} \end{aligned}$$

Theorem 1.7

Let $T: R^n \to R^m$ be a linear transformation. Then there exists a unique matrix A such that $T(x) = Ax$ for all x in R^n.

In fact, A is the $m\times n$ matrix whose j^{th} column is the vector $T(e_j)$, where e_j is the jth column of the identity matrix in R^n:

$$A = [\,T(e_1) \dots T(e_n)\,]$$

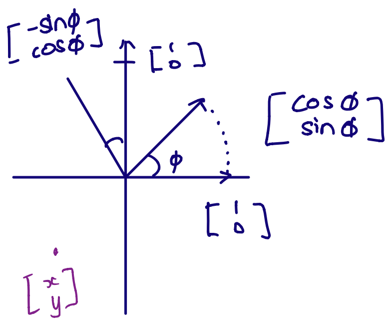
A is called the standard matrix for the linear transformation of T.

Exercise:

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about the origin through an angle ϕ, with counterclockwise rotation for a positive angle

Find the standard Matrix A for this transformation

$A = [\, T(e_1) ,\ T(e_2) \,]$ $\qquad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$



$T(e_1) = \begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}$

$T(e_2) = \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$

$A = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$ //

$A = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$

↘ To find out where x, y go in transformation T

Question:

Determine values of h such that $\begin{bmatrix} 1 & 3 & -2 \\ -4 & h & 8 \end{bmatrix}$ is the augmented matrix of a consistent linear system?

$\begin{bmatrix} 1 & 3 & -2 \\ -4 & h & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & -2 \\ 0 & h+12 & 0 \end{bmatrix}$

if $h = -12$, equation are row equivalent & have infinitely many solutions.

if h is any other number, lines are not parallel and will meet at one point so there will still be 1 solution

∴ All h is possible → consistent.

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow$ unique solution

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ 1 solution

$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow$ no solution