2.1 Matrix Multiplication

$A(Bx)$ is produced from x by a composition of two linear transformations.

→ by a single matrix AB

>> $A(Bx) = (AB)x$

Note : $AB \neq BA$

Special case : when A or B is I or multiples of I

2.2 Inverse of a matrix

Find inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{EROs}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & 3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

make this I — make this A^{-1} — I — A^{-1}

$$\therefore A^{-1} = \begin{bmatrix} -9/2 & 7 & 3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$$

2.3 LU Factorization

Consider solving a sequence of equations $A\mathbf{x} = \mathbf{b_1}, A\mathbf{x} = \mathbf{b_2}, \ldots, A\mathbf{x} = \mathbf{b_p}$

Inefficient solution: Compute A^{-1} and then $A^{-1}\mathbf{b_1}, \ldots, A^{-1}\mathbf{b_p}$

Efficient solution: $A_{m\times n} = L_{m\times m}U_{m\times n}$

Assumption - A can be reduced to echelon form without row interchanges

L: Unit Lower triangular

U: Upper triangular

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U \text{ (Echelon form)}}$$

$A\mathbf{x} = \mathbf{b}$

$\Rightarrow LU\mathbf{x} = \mathbf{b}$

Forward Substitution: $x_1 = 1$, $x_1 + x_2 = 2$, $x_2 + x_2 + x_3 = 2$

Backward Substitution: $y_1 + y_2 + y_3 = 2$, $y_1 = 0$, $y_2 + y_3 = 2$, $y_3 = 1$, $y_2 = 1$

Let $\mathbf{y} = U\mathbf{x}$ — 'Forward Substitution'

$L\mathbf{y} = \mathbf{b} \rightarrow$ Solve for $\mathbf{y}$

$U\mathbf{x} = \mathbf{y} \rightarrow$ Solve for $\mathbf{x}$

Easy to solve because L and U are triangular

'Backward Substitution'

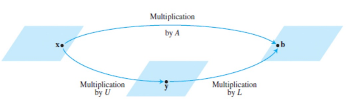


FIGURE 2 Factorization of the mapping $\mathbf{x} \mapsto A\mathbf{x}$.

Theorem 2.1

If A is an invertible $n \times n$ matrix, then for each b in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$

Theorem 2.2

Invertible Matrixes

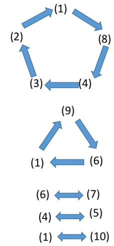
1. A is invertible, A^{-1} is invertible, $(A^{-1})^{-1} = A$
2. if A & B are invertible, AB is invertible, $(AB)^{-1} = B^{-1}A^{-1}$
3. A is invertible, A^T is invertible, $(A^T)^{-1} = (A^{-1})^T$

Theorem 2.3

An $n \times n$ matrix A is invertible ⟺ A is row equivalent to I_n, any sequence of elementary row operations that reduces A to I_n also transforms I_n to A^{-1}.

Theorem 2.4. *The Invertible Matrix Theorem*
Let A be a square $n \times n$ matrix. Then the following statements are equivalent, i.e., for a given A, the statements are either all true or all false.

1. *A is an invertible matrix.*
2. *A is row equivalent to I_n.*
3. *A has n pivot positions.*
4. $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*
5. *The columns of A form a linearly independent set.*
6. $A\mathbf{x} = \mathbf{b}$ *has ~~at least~~ only (at most) one solution for each* $\mathbf{b}$ *in* $\mathbb{R}^n$.
7. *The columns of A span* $\mathbb{R}^n$.
8. *There is an $n \times n$ matrix C such that CA=I.*
9. *There is an $n \times n$ matrix D such that AD=I.*
10. A^T *is an invertible matrix.*



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2.2 Inverse of a Matrix

Theorem 2.1. If A is an invertible $n \times n$ matrix, then for each b in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$

Proof. Let $b \in \mathbb{R}^n$

Solution exists : Substitute $A^{-1}b$ in $Ax = b$

$LHS = Ax = A(A^{-1})b = (AA^{-1})b = Ib = b = RHS$

Solution is unique : Show that if u is a solution, it must be $A^{-1}b$.

If $Au = b$, multiply both sides by A^{-1}

$A^{-1}Au = A^{-1}b$ or $Iu = A^{-1}b \rightarrow u = A^{-1}b$

Exercise 2.3.1

Solve $A\mathbf{x} = \mathbf{b}$ if $A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}}$

and $\mathbf{b} = \begin{bmatrix} -9 \\ 5 \\ 7 \\ 11 \end{bmatrix}$.

r2 → r2 + r1
r3 → r3 − 2r1

forward substitution

$$L\mathbf{y} = \mathbf{b} : [L \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} \overbrace{1 \; 0 \; 0 \; 0}^{\text{Identity } I} & \overset{\text{RREF}}{-9} \\ 0 \; 1 \; 0 \; 0 & -4 \\ 0 \; 0 \; 1 \; 0 & 5 \\ 0 \; 0 \; 0 \; 1 & 1 \end{bmatrix} = [I \quad \mathbf{y}]$$

Number of multiplication - addition pairs to reduce L to I } 3 2 1

6 multiplications
6 additions

$$\begin{aligned} y_1 &= -9 \\ -y_1 + y_2 &= 5 \\ 2y_1 - 5y_2 + y_3 &= 7 \\ -3y_1 + 8y_2 + 3y_3 + y_4 &= 11 \end{aligned}$$

$$L\mathbf{y} = \mathbf{b} : [L \quad \mathbf{b}] = \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} = [I \quad \mathbf{y}]$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{bmatrix} \xrightarrow[r4 \to r4 + 3r1]{r2 \to r2 + r1,\; r3 \to r3 - 2r1} \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & -5 & 1 & 0 & 25 \\ 0 & 8 & 3 & 1 & -16 \end{bmatrix}$$

R3 ← R3 + 5r2
R4 ← R4 − 8r2

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xleftarrow{R4 \leftarrow R4 - 3R3} \begin{bmatrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 3 & 1 & 16 \end{bmatrix}$$

$$\begin{aligned} y_1 &= -9 && —(1) \\ -y_1 + y_2 &= 5 && —(2) \\ 2y_1 - 5y_2 + y_3 &= 7 && —(3) \\ -3y_1 + 8y_2 + 3y_3 + y_4 &= 11 && —(4) \end{aligned}$$

Sub (1) into (2):
$9 + y_2 = 5$
$y_2 = -4$ —(5)
Sub (1) & (5) into (3)
$2(-9) - 5(-4) + y3 = 7$
$-18 + 20 + y3 = 7$
$2 + y3 = 7$
$y3 = 5$

$-3(-9) + 8(-4) + 3(5) + y4 = 11$
$27 - 32 + 15 + y4 = 11$
$10 + y4 = 11$
$y4 = 1$

$$U\mathbf{x} = \mathbf{y} : [U \quad \mathbf{y}] = \begin{bmatrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = [I \quad \mathbf{x}]$$

To reduce U to I:
Number of divisions - 4
Number of additions - 6
Number of multiplications - 6

$$\begin{aligned} 3x_1 - 7x_2 - 2x_3 + 2x_4 &= -9 \\ -2x_2 - x_3 + 2x_4 &= -4 \\ -x_3 + x_4 &= 5 \\ -x_4 &= 1 \end{aligned}$$

Through LU factorization : 28 arithmetic operations or "flops" (floating point operations) - excluding cost of factorization
Through row reduction of $[A \quad \mathbf{b}]$ to $[I \quad \mathbf{x}]$: 62 flops

wh U to find x

$$\mathbf{x} = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}.$$

2.3.2 LU factorization procedure

- Row reduction of A to U produces L without extra work

RECALL: *Assumption* - A can be reduced to echelon form *without row interchanges*

There exist unit lower triangular elementary matrices $E_1, \ldots, E_p$ such that

$$E_p \ldots E_1 A = U$$

$$\Rightarrow A = (E_p \ldots E_1)^{-1} U = LU$$

[Products and inverses of unit lower triangular matrices are also unit lower triangular]

where $L = (E_p \ldots E_1)^{-1}$

Same row operations that reduce A to U also reduce L to I

$$E_p \ldots E_1 L = I$$

$L = E_2^{-1} E_1^{-1}$

Exercise 2.3.2:

Find an LU factorization of $A = \begin{bmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{bmatrix}$.

r2 ← r2 − 3r1 (6 (−3) (2))

$$E_{21}A = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{-3} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \overbrace{\begin{bmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{bmatrix}}^{A} = \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 4 & -8 & 30 \end{bmatrix}$$

r3 ← r3 − 2r1 (4 (−2) (2))

$$E_{31}(E_{21}A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boxed{-2} & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 4 & -8 & 30 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{bmatrix}$$

r3 ← r3 − 4r1 (r3 (−4) (−2))

$$E_{32}(E_{31}E_{21}A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \boxed{-4} & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{bmatrix} = \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{bmatrix} = U.$$

Since A has 3 rows, L should be 3×3

$$L = \begin{bmatrix} 1 & 0 & 0 \\ ? & 1 & 0 \\ ? & ? & 1 \end{bmatrix}$$

The row operations that create zeros in each column of A will also create zeros in each column of L.

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{bmatrix} \to \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{bmatrix} \to \begin{bmatrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{bmatrix} = U.$$

Circled entries are used to determine the sequence of transformations that transform A to U. At each pivot column, divide the encircled entries by the pivot (first element inside the circle) and place the result into L.

$$L = \begin{bmatrix} \boxed{2} & & \\ 6 & \boxed{-1} & \\ 4 & -4 & \boxed{4} \end{bmatrix} \to L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

÷2 ÷(−1) ÷4

- Row reduction of A to U produces L without extra work

$(AB)^{-1} = B^{-1}A^{-1}$

Alternately,

multiply by −1 (?)

$$L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

What kind of matrixes does this work on?

Just put the nonzero off-diagonal elements of the elementary matrices into the appropriate positions in L. Why does it become non-negative?

Exercise 2.3.3 (when below assumption is not valid)

(*Assumption* - A can be reduced to echelon form *without row interchanges*)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & -3 & -3 \end{bmatrix}$$

To switch rows 2 and 3, use **permutation matrix** $P = P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$PA = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -1 \end{bmatrix} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & -2 \end{bmatrix}}$$

In summary,
For every $n \times n$ matrix A there exists a permutation matrix P, such that PA possesses an LU-factorization, i.e., $PA = LU$, where L is a lower triangular matrix with all diagonal entries equal to 1, and U is an upper triangular matrix.

For an $n \times n$ dense matrix and for n moderately large, say $n \geq 30$,

LU factorization : about $2n^3/3$ flops
Finding A^{-1} : about $2n^3$ flops
Solving $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{y}$: $2n^2$ flops
Multiplication of $\mathbf{b}$ by A^{-1} : about $2n^2$ flops

?

***** END OF CHAPTER *****

2.1 Matrix Multiplication

Matrix as a transformation, $T: \vec{x} \to \vec{y} = A\vec{x}$

A: Represents the transformation that acts on the x vector to produce b vector.

$$\begin{tikzcd} x \arrow[r, "\cdot B"] \arrow[rr, bend right, "\cdot AB"'] & Bx \arrow[r, "\cdot A"] & A(Bx) \end{tikzcd}$$

$A(Bx)$ is produced from x by a composition of two linear compositions.

Represent the two transformation as multiplication by a single matrix AB

$A(Bx) = (AB)x$ However, do note that $AB \neq BA$

2.2 Inverse of a Matrix

Theorem 2.1. If A is an invertible $n \times n$ matrix, then for each b in R^n, the equation $Ax = b$ has the unique solution $x = A^{-1}b$

Proof. Let $b \in R^n$

Solution exists: Substitute $A^{-1}b$ in $Ax = b$

$LHS = Ax = A(A^{-1})b = (AA^{-1})b = Ib = b = RHS$

Solution is unique: Show that if u is a solution, it must be $A^{-1}b$.

If $Au = b$, multiply both sides by A^{-1}

$A^{-1}Au = A^{-1}b$ or $Iu = A^{-1}b \to u = A^{-1}b$

Theorem 2.2 Invertible Matrices

1. If A is an invertible Matrix, then A^{-1} is invertible and $(A^{-1})^{-1} = A$
2. If A and B are $n \times n$ invertible matrixes, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$ $(AB)^T = B^T A^T$
3. If A is an invertible matrix, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$

Proof

1. Find a matrix C such that $A^{-1}C = I$ and $CA^{-1} = I$. Here, C is simply A. Hence, A^{-1} is invertible and its inverse is A.

To show that a matrix x is invertible, you need to show there exist a matrix Y, such that $XY = I$ and $YX = I$

2. Find a matrix C such that $(AB)C = I$ and $C(AB) = I$.

If $C = B^{-1}A^{-1}$, then $AB(B^{-1}A^{-1})$, then $AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1}$ $= AA^{-1} = I$.

Elementary Matrix

An elementary matrix is one that is obtained by performing a single elementary row operation on an identity matrix.

Exercise 2.2.1

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \longrightarrow \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{matrix} \xrightarrow[r3-4r1]{} E_1$$

$$\text{Let } A = \begin{matrix} a & b & c \\ d & e & f \\ g & h & i \end{matrix} \qquad E_1A \quad \begin{matrix} a & b & c \\ d & e & f \\ g-4a & h-4b & i-4c \end{matrix} \rightarrow$$

This is obtained by doing the ERO applied to get E_1, which is $r3-4r1$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \longrightarrow \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix} \rightarrow E_2 \quad r2 \leftrightarrow r1 \qquad E_2A = \begin{matrix} d & e & f \\ a & b & c \\ g & h & i \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \longrightarrow \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{matrix} \rightarrow E_3 \quad 5r3 \qquad E_3A = \begin{matrix} a & b & c \\ d & e & f \\ 5h & 5i & 5j \end{matrix}$$

If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

Each elementary Matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms back into I.

Exercise 2.2.2

Find the inverse of E_1 $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{matrix}$

To tranform E_1 to I, add $+4$ times row 1 to row 3. The elementary matrix that does this is $E_1^{-1} = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{matrix}$

To get I from E_1 $r3 \leftarrow 4r1 + r3$, so apply that on your I to get E_1^{-1}

$E_2 = \begin{matrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{matrix}$ → get back to I, swapping back, so $E_2^{-1} = E_2$

$E_3 = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{matrix}$ → get back to I, $1/5$ on $R3$, Apply that on I to get $E3^{-1}$

$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$ $\quad 1/5 r3 = \quad$ $\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{matrix}$

Theorem 2.3. An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n and in this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n to A^{-1}.

Proof: $\underset{n \times n}{A} \xrightarrow{\text{EROS}} \underset{n \times n}{I_n}$ $\qquad$ $\underset{n \times n}{I_n} \xrightarrow{\text{EROS}} \underset{n \times n}{A^{-1}}$

Row operations used to transform A to I_n, are used to transform I_n to A^{-1}

Exercise 2.2.3

Find the inverse of $A = \begin{matrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{matrix}$

1st Step : Form the Augmented Matrix [A I] Concatenating A and I

$$AI = \begin{matrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ \underbrace{4 \quad -3 \quad 8}_{\substack{I \\ \text{matrix}}} \end{matrix} \quad \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \underbrace{0 \quad 0 \quad 1}_{A^{-1}} \end{matrix} \text{ (After ERO)}$$

$$A \xrightarrow{\text{ERO}} I$$

$$I \xrightarrow{\text{ERO}} A^{-1}$$

$$\begin{matrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{matrix} \xrightarrow{r3-4r1} \begin{matrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{matrix} \xrightarrow{r3+3r2} \begin{matrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{matrix} \xrightarrow{r2-r3} \begin{matrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & -4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{matrix} \xrightarrow{r3 \cdot 1.5} \begin{matrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & -4 & -1 \\ 0 & 0 & 3 & 9/2 & -6 & 3/2 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & -4 & -1 \\ 0 & 0 & 3 & 9/2 & -6 & 3/2 \end{matrix} \xrightarrow{r1-r3} \begin{matrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & -4 & -1 \\ 0 & 0 & 3 & 9/2 & -6 & 3/2 \end{matrix} \xrightarrow{\frac{1}{3} r3} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \underbrace{0 \quad 0 \quad 1}_{I} \end{matrix} \quad \begin{matrix} -9/2 & 7 & -3/2 \\ -2 & -4 & -1 \\ \underbrace{3/2 \quad -2 \quad 1/2}_{A^{-1}} \end{matrix}$$

$$\therefore A^{-1} = \begin{matrix} -9/2 & 7 & -3/2 \\ -2 & -4 & -1 \\ 3/2 & -2 & 1/2 \end{matrix}$$

2.3 Matrix Factorizations

Matrix multiplication => Synthesis of Data

A expressed as a product of two or more matrices => analysis of data

2.3.1 The LU factorization

Consider solving $Ax = b_1, \ldots\ldots Ax = b_p$

Inefficient Solution: Compute $A^{-1}b_1, \ldots A^{-1}b_p$

Efficient Solution: $A_{m\times n} = L_{m\times n} U_{m\times n}$

Assumption - A can be reduced to echelon form without row interchanges

L: Unit Lower Triangular

U: Upper Triangular

$$A = \underbrace{\begin{matrix} 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 \\ * & * & 1 & 0 \\ * & * & * & 1 \end{matrix}}_{L} \quad \underbrace{\begin{matrix} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{matrix}}_{U}$$

$Ax = B \Rightarrow LUx = B$

Forward Substitution

Let $y = Ux$, $\quad Ly = b \rightarrow$ Solve for y

$\qquad\qquad\quad Ux = y \rightarrow$ Solve for x

} Easy to solve cuz ω are triangular

Backward Substitution

$y_1 + y_2 + y_3 = 2 \qquad\qquad x_1 = 1$

$\quad y_2 + y_3 = 2 \rightarrow$ Backward Substitution $\qquad x_1 + x_2 = 2 \longrightarrow$ Forward sub from top to bottom

$\quad y_3 = 1 \qquad\qquad x_1 + x_2 + x_3 = 2$

Exercise 2.3.1

Solve $Ax=b$ if $A = \begin{matrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & 5 & 12 \end{matrix} = \begin{matrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{matrix} \cdot \begin{matrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{matrix}$

$b = \begin{matrix} -9 \\ 5 \\ 7 \\ 11 \end{matrix}$

ERO METHOD

$Ly = b : [L \ b] = \begin{matrix} 1 & 0 & 0 & 0 & -9 \\ -1 & 1 & 0 & 0 & 5 \\ 2 & -5 & 1 & 0 & 7 \\ -3 & 8 & 3 & 1 & 11 \end{matrix} \quad \begin{matrix} r2+r1 \\ r3-2r1 \\ r4+3r1 \end{matrix} \quad \begin{matrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & -5 & 1 & 0 & 25 \\ 0 & 8 & 3 & 1 & -16 \end{matrix}$

$\begin{matrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & -5 & 1 & 0 & 25 \\ 0 & 8 & 3 & 1 & -16 \end{matrix} \quad \begin{matrix} r3+5r2 \\ r4-8r2 \end{matrix} \quad \begin{matrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 3 & 1 & 16 \end{matrix} \quad r4-3r1 \quad \begin{matrix} 1 & 0 & 0 & 0 & -9 \\ 0 & 1 & 0 & 0 & -4 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 1 \end{matrix}$

Forward Sub Method

$y1 = -9$

$y =$

$-y_1 + y_2 = 5$

$2y_1 - 5y_2 + y_3 = 7$

$-3y_1 + 8y_2 + 3y_3 + y_4 = 11$

$Ux = y : [U \; y] = \begin{matrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{matrix}$

ERO Method: From the bottom to the top

$$\begin{matrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & -1 & 1 \end{matrix} \xrightarrow{-r4} \begin{matrix} 3 & -7 & -2 & 2 & -9 \\ 0 & -2 & -1 & 2 & -4 \\ 0 & 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & 1 & -1 \end{matrix} \xrightarrow[r3-r4]{r1-2r4 \;\; r2-2r4} \begin{matrix} 3 & -7 & -2 & 0 & -7 \\ 0 & -2 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{matrix}$$

$$\begin{matrix} 3 & -7 & -2 & 0 & -7 \\ 0 & -2 & -1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{matrix} \xrightarrow{-r3} \begin{matrix} 3 & -7 & -2 & 0 & -7 \\ 0 & -2 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{matrix} \xrightarrow{r1+2r3 \;\; r2+r3} \begin{matrix} 3 & -7 & 0 & 0 & -19 \\ 0 & -2 & 0 & 0 & -8 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{matrix}$$

$$\begin{matrix} 3 & -7 & 0 & 0 & -19 \\ 0 & -2 & 0 & 0 & -8 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{matrix} \xrightarrow{r2/-2} \begin{matrix} 3 & -7 & 0 & 0 & -19 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{matrix} \xrightarrow{r1+7R2} \begin{matrix} 3 & 0 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{matrix} \xrightarrow{1/3 r1} \begin{matrix} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & -6 \\ 0 & 0 & 0 & 1 & -1 \end{matrix}$$

$3x_1 - 7x_2 - 2x_3 + 2x_4 = -9$ Use Backwards substitution to solve.

$-2x_2 - x_3 + 2x_4 = -4$

$-x_3 + x_4 = 5$

$-x_4 = 1$

$$X = \begin{bmatrix} 3 \\ 4 \\ -6 \\ -1 \end{bmatrix}$$

2.3.2 LU Factorization Procedure

Row Reduction of A to U produces L without extra work.

There exist unit lower triangular elementary matrices $E_1, \dots, E_p$ such that

$E_p \dots E_1 A = U$

$\Rightarrow A = (E_p \dots E_1)^{-1} U = LU$

where $L = (E_p \dots E_1)^{-1}$

Same row operations that reduce A to U also reduce L to I

$E_p \dots E_1 L = I$

Find the LU Factorization of $A = \begin{matrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{matrix}$ Make the Lower Triangle 0

$$E_{21}A = \begin{matrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \cdot \begin{matrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{matrix} = \begin{matrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 4 & -8 & 30 \end{matrix}$$

$$E_{31}(E_{21}A) = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{matrix} \cdot \begin{matrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 4 & -8 & 30 \end{matrix} = \begin{matrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{matrix}$$

$$E_{32}(E_{31}E_{21}A) = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{matrix} \cdot \begin{matrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{matrix} = \begin{matrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{matrix} = U$$

Since A has 3 rows, L should be 3x3

$$L = \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

The row operations that creates zeros in each column of A will also create zeros in each column of L

$$A = \begin{matrix} 2 & -2 & 3 \\ 6 & -7 & 14 \\ 4 & -8 & 30 \end{matrix} \xrightarrow[r3-2r1]{r2-3r1} \begin{matrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & -4 & 24 \end{matrix} \xrightarrow{r3-4r2} \begin{matrix} 2 & -2 & 3 \\ 0 & -1 & 5 \\ 0 & 0 & 4 \end{matrix} = U$$

Circled entries are used to determine the sequence of transforms from A to U

At each pivot column, divide the encircled entries by the pivot (first element inside the circle) and place the result in L.

$$L = \begin{matrix} 2 & 0 & 0 \\ 6 & -1 & 0 \\ 4 & -4 & 4 \end{matrix} \longrightarrow L = \begin{matrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{matrix}$$

Alternatively,

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{matrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \cdot \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{matrix} \cdot \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 4 & 1 \end{matrix} = \begin{matrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 4 & 1 \end{matrix}$$

$$A = \begin{matrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & -1 \end{matrix} \longrightarrow \begin{matrix} 1 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & -3 & -3 \end{matrix}$$

In summary, for every nxn matrix A, there exists a permutation matrix P, such that PA = LU, where L is a lower triangular matrix with all diagonal entries equal to 1, and U is an upper triangular matrix.

$Ax = b$ other solution.

$Ax_1 = b$ some solution

Ax

Ax_1

Ax_0

$b - b = 0$

$Ax - Ax_1$

$Ax_0 = 0$

$$A \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{x}$$

$$\begin{matrix} 1 \\ 2 \\ 3 \end{matrix} + \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} = \begin{matrix} 1 \\ 2 \\ 3 \end{matrix}$$

$$A \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{x_1}$$

$$A \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{x_0} = 0$$