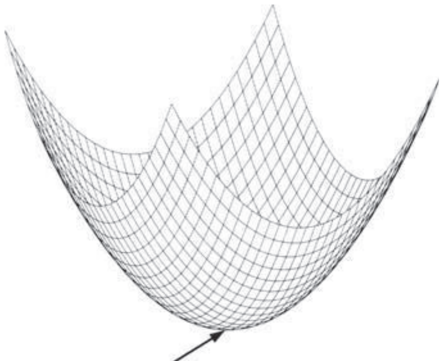

Figure 6.8 Graph of $f(\boldsymbol{x}) = x_1^2 + x_2^2$.

EXERCISES

6.1 Consider the problem

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega, \end{array}$$

where $f \in \mathcal{C}^2$. For each of the following specifications for Ω, $\boldsymbol{x}^*$, and f, determine if the given point $\boldsymbol{x}^*$ is: (i) definitely a local minimizer; (ii) definitely not a local minimizer; or (iii) possibly a local minimizer.

a. $f : \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 1\}$, $\boldsymbol{x}^* = [1, 2]^\top$, and gradient $\nabla f(\boldsymbol{x}^*) = [1, 1]^\top$.

b. $f : \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 1, x_2 \geq 2\}$, $\boldsymbol{x}^* = [1, 2]^\top$, and gradient $\nabla f(\boldsymbol{x}^*) = [1, 0]^\top$.

c. $f : \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 0, x_2 \geq 0\}$, $\boldsymbol{x}^* = [1, 2]^\top$, gradient $\nabla f(\boldsymbol{x}^*) = [0, 0]^\top$, and Hessian $\boldsymbol{F}(\boldsymbol{x}^*) = \boldsymbol{I}$ (identity matrix).

d. $f : \mathbb{R}^2 \to \mathbb{R}$, $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 \geq 1, x_2 \geq 2\}$, $\boldsymbol{x}^* = [1, 2]^\top$, gradient $\nabla f(\boldsymbol{x}^*) = [1, 0]^\top$, and Hessian

$$\boldsymbol{F}(\boldsymbol{x}^*) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

6.2 Find minimizers and maximizers of the function

$$f(x_1, x_2) = \frac{1}{3}x_1^3 - 4x_1 + \frac{1}{3}x_2^3 - 16x_2.$$

6.3 Show that if $\boldsymbol{x}^*$ is a global minimizer of f over Ω, and $\boldsymbol{x}^* \in \Omega' \subset \Omega$, then $\boldsymbol{x}^*$ is a global minimizer of f over Ω'.

6.4 Suppose that $\boldsymbol{x}^*$ is a local minimizer of f over Ω, and $\Omega \subset \Omega'$. Show that if $\boldsymbol{x}^*$ is an interior point of Ω, then $\boldsymbol{x}^*$ is a local minimizer of f over Ω'. Show that the same conclusion cannot be made if $\boldsymbol{x}^*$ is not an interior point of Ω.

6.5 Consider the problem of minimizing $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{C}^3$, over the constraint set Ω. Suppose that 0 is an *interior point* of Ω.

a. Suppose that 0 is a local minimizer. By the FONC we know that $f'(0) = 0$ (where f' is the first derivative of f). By the SONC we know that $f''(0) \geq 0$ (where f'' is the second derivative of f). State and prove a *third-order necessary condition* (*TONC*) involving the third derivative at 0, $f'''(0)$.

b. Give an example of f such that the FONC, SONC, and TONC (in part a) hold at the interior point 0, but 0 is not a local minimizer of f over Ω. (Show that your example is correct.)

c. Suppose that f is a third-order polynomial. If 0 satisfies the FONC, SONC, and TONC (in part a), then is this *sufficient* for 0 to be a local minimizer?

6.6 Consider the problem of minimizing $f : \mathbb{R} \to \mathbb{R}$, $f \in \mathcal{C}^3$, over the constraint set $\Omega = [0, 1]$. Suppose that $x^* = 0$ is a local minimizer.

a. By the FONC we know that $f'(0) \geq 0$ (where f' is the first derivative of f). By the SONC we know that if $f'(0) = 0$, then $f''(0) \geq 0$ (where f'' is the second derivative of f). State and prove a *third-order necessary condition* involving the third derivative at 0, $f'''(0)$.

b. Give an example of f such that the FONC, SONC, and TONC (in part a) hold at the point 0, but 0 is not a local minimizer of f over $\Omega = [0, 1]$.

6.7 Let $f : \mathbb{R}^n \to \mathbb{R}$, $\boldsymbol{x}_0 \in \mathbb{R}^n$, and $\Omega \subset \mathbb{R}^n$. Show that

$$\boldsymbol{x}_0 + \arg\min_{\boldsymbol{x} \in \Omega} f(\boldsymbol{x}) = \arg\min_{\boldsymbol{y} \in \Omega'} f(\boldsymbol{y}),$$

where $\Omega' = \{\boldsymbol{y} : \boldsymbol{y} - \boldsymbol{x}_0 \in \Omega\}$.

6.8 Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\top \begin{bmatrix} 3 \\ 5 \end{bmatrix} + 6.$$

a. Find the gradient and Hessian of f at the point $[1,1]^\top$.

b. Find the directional derivative of f at $[1,1]^\top$ with respect to a unit vector in the direction of maximal rate of increase.

c. Find a point that satisfies the FONC (interior case) for f. Does this point satisfy the SONC (for a minimizer)?

6.9 Consider the following function:

$$f(x_1, x_2) = x_1^2 x_2 + x_2^3 x_1.$$

a. In what direction does the function f *decrease* most rapidly at the point $\boldsymbol{x}^{(0)} = [2,1]^\top$?

b. What is the rate of increase of f at the point $\boldsymbol{x}^{(0)}$ in the direction of maximum decrease of f?

c. Find the rate of increase of f at the point $\boldsymbol{x}^{(0)}$ in the direction $\boldsymbol{d} = [3,4]^\top$.

6.10 Consider the following function $f : \mathbb{R}^2 \to \mathbb{R}$:

$$f(\boldsymbol{x}) = \boldsymbol{x}^\top \begin{bmatrix} 2 & 5 \\ -1 & 1 \end{bmatrix} \boldsymbol{x} + \boldsymbol{x}^\top \begin{bmatrix} 3 \\ 4 \end{bmatrix} + 7.$$

a. Find the directional derivative of f at $[0,1]^\top$ in the direction $[1,0]^\top$.

b. Find all points that satisfy the first-order necessary condition for f. Does f have a minimizer? If it does, then find all minimizer(s); otherwise, explain why it does not.

6.11 Consider the problem

$$\begin{array}{ll} \text{minimize} & -x_2^2 \\ \text{subject to} & |x_2| \leq x_1^2 \\ & x_1 \geq 0, \end{array}$$

where $x_1, x_2 \in \mathbb{R}$.

a. Does the point $[x_1, x_2]^\top = \boldsymbol{0}$ satisfy the first-order necessary condition for a minimizer? That is, if f is the objective function, is it true that $\boldsymbol{d}^\top \nabla f(\boldsymbol{0}) \geq 0$ for all feasible directions $\boldsymbol{d}$ at $\boldsymbol{0}$?

b. Is the point $[x_1, x_2]^\top = \boldsymbol{0}$ a local minimizer, a strict local minimizer, a local maximizer, a strict local maximizer, or none of the above?

6.12 Consider the problem

$$\begin{array}{ll}\text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega,\end{array}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(\boldsymbol{x}) = 5x_2$ with $\boldsymbol{x} = [x_1, x_2]^\top$, and $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1^2 + x_2 \geq 1\}$.

a. Does the point $\boldsymbol{x}^* = [0, 1]^\top$ satisfy the first-order necessary condition?

b. Does the point $\boldsymbol{x}^* = [0, 1]^\top$ satisfy the second-order necessary condition?

c. Is the point $\boldsymbol{x}^* = [0, 1]^\top$ a local minimizer?

6.13 Consider the problem

$$\begin{array}{ll}\text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega,\end{array}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(\boldsymbol{x}) = -3x_1$ with $\boldsymbol{x} = [x_1, x_2]^\top$, and $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 + x_2^2 \leq 2\}$. Answer each of the following questions, showing complete justification.

a. Does the point $\boldsymbol{x}^* = [2, 0]^\top$ satisfy the first-order necessary condition?

b. Does the point $\boldsymbol{x}^* = [2, 0]^\top$ satisfy the second-order necessary condition?

c. Is the point $\boldsymbol{x}^* = [2, 0]^\top$ a local minimizer?

6.14 Consider the problem

$$\begin{array}{ll}\text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega,\end{array}$$

where $\Omega = \{\boldsymbol{x} \in \mathbb{R}^2 : x_1^2 + x_2^2 \geq 1\}$ and $f(\boldsymbol{x}) = x_2$.

a. Find all point(s) satisfying the FONC.

b. Which of the point(s) in part a satisfy the SONC?

c. Which of the point(s) in part a are local minimizers?

6.15 Consider the problem

$$\begin{array}{ll}\text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega\end{array}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(\boldsymbol{x}) = 3x_1$ with $\boldsymbol{x} = [x_1, x_2]^\top$, and $\Omega = \{\boldsymbol{x} = [x_1, x_2]^\top : x_1 + x_2^2 \geq 2\}$. Answer each of the following questions, showing complete justification.

a. Does the point $\boldsymbol{x}^* = [2, 0]^\top$ satisfy the first-order necessary condition?

b. Does the point $\boldsymbol{x}^* = [2, 0]^\top$ satisfy the second-order necessary condition?

c. Is the point $\boldsymbol{x}^* = [2, 0]^\top$ a local minimizer?
Hint: Draw a picture with the constraint set and level sets of f.

6.16 Consider the problem

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega, \end{array}$$

where $\boldsymbol{x} = [x_1, x_2]^\top$, $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(\boldsymbol{x}) = 4x_1^2 - x_2^2$, and $\Omega = \{\boldsymbol{x} : x_1^2 + 2x_1 - x_2 \geq 0, x_1 \geq 0, x_2 \geq 0\}$.

a. Does the point $\boldsymbol{x}^* = \boldsymbol{0} = [0, 0]^\top$ satisfy the first-order necessary condition?

b. Does the point $\boldsymbol{x}^* = \boldsymbol{0}$ satisfy the second-order necessary condition?

c. Is the point $\boldsymbol{x}^* = \boldsymbol{0}$ a local minimizer of the given problem?

6.17 Consider the problem

$$\begin{array}{ll} \text{maximize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega, \end{array}$$

where $\Omega \subset \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ and $f : \Omega \to \mathbb{R}$ is given by $f(\boldsymbol{x}) = \log(x_1) + \log(x_2)$ with $\boldsymbol{x} = [x_1, x_2]^\top$, where "log" represents natural logarithm. Suppose that $\boldsymbol{x}^*$ is an optimal solution. Answer each of the following questions, showing complete justification.

a. Is it possible that $\boldsymbol{x}^*$ is an interior point of Ω?

b. At what point(s) (if any) is the second-order necessary condition satisfied?

6.18 Suppose that we are given n real numbers, $x_1, \ldots, x_n$. Find the number $\bar{x} \in \mathbb{R}$ such that the sum of the squared difference between $\bar{x}$ and the numbers above is minimized (assuming that the solution $\bar{x}$ exists).

6.19 An art collector stands at a distance of x feet from the wall, where a piece of art (picture) of height a feet is hung, b feet above his eyes, as shown in

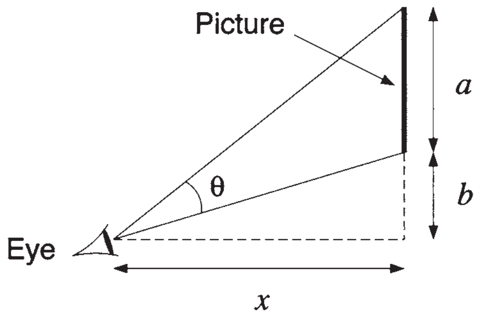

Figure 6.9 Art collector's eye position in Exercise 6.19.

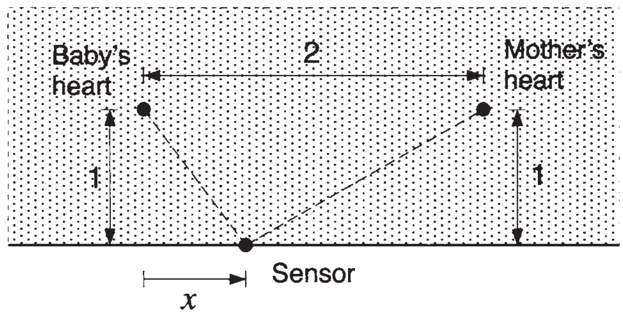

Figure 6.10 Simplified fetal heart monitoring system for Exercise 6.20.

Figure 6.9. Find the distance from the wall for which the angle θ subtended by the eye to the picture is maximized.
Hint: (1) Maximizing θ is equivalent to maximizing $\tan(\theta)$.
(2) If $\theta = \theta_2 - \theta_1$, then $\tan(\theta) = (\tan(\theta_2) - \tan(\theta_1))/(1 + \tan(\theta_2)\tan(\theta_1))$.

6.20 Figure 6.10 shows a simplified model of a fetal heart monitoring system (the distances shown have been scaled down to make the calculations simpler). A heartbeat sensor is located at position x (see Figure 6.10).

The energy of the heartbeat signal measured by the sensor is the reciprocal of the squared distance from the source (baby's heart or mother's heart). Find the position of the sensor that maximizes the *signal-to-interference ratio*, which is the ratio of the signal energy from the baby's heart to the signal energy from the mother's heart.

6.21 An amphibian vehicle needs to travel from point A (on land) to point B (in water), as illustrated in Figure 6.11. The speeds at which the vehicle travels on land and water are v_1 and v_2, respectively.

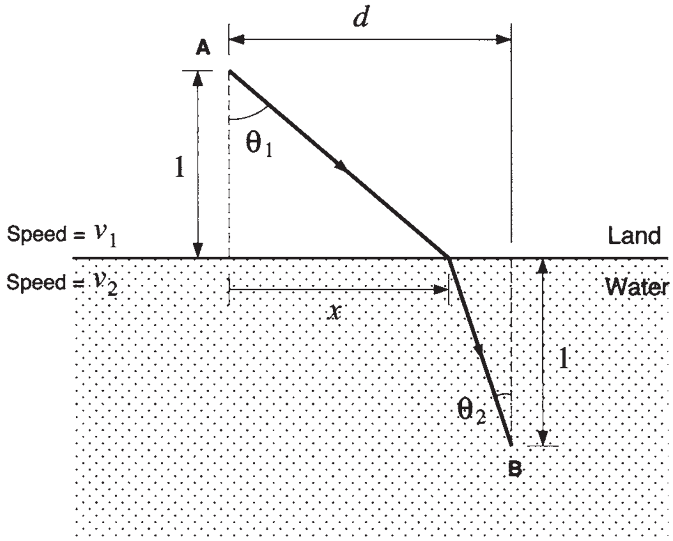

Figure 6.11 Path of amphibian vehicle in Exercise 6.21.

a. Suppose that the vehicle traverses a path that minimizes the total time taken to travel from A to B. Use the first-order necessary condition to show that for the optimal path above, the angles θ_1 and θ_2 in Figure 6.11 satisfy Snell's law:

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

b. Does the minimizer for the problem in part a satisfy the second-order sufficient condition?

6.22 Suppose that you have a piece of land to sell and you have two buyers. If the first buyer receives a fraction x_1 of the piece of land, the buyer will pay you $U_1(x_1)$ dollars. Similarly, the second buyer will pay you $U_2(x_2)$ dollars for a fraction of x_2 of the land. Your goal is to sell parts of your land to the two buyers so that you maximize the total dollars you receive. (Other than the constraint that you can only sell whatever land you own, there are no restrictions on how much land you can sell to each buyer.)

a. Formulate the problem as an optimization problem of the kind

$$\begin{array}{ll} \text{maximize} & f(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{x} \in \Omega \end{array}$$

by specifying f and Ω. Draw a picture of the constraint set.

b. Suppose that $U_i(x_i) = a_i x_i$, $i = 1, 2$, where a_1 and a_2 are given positive constants such that $a_1 > a_2$. Find all feasible points that satisfy the first-order necessary condition, giving full justification.

c. Among those points in the answer of part b, find all that also satisfy the second-order necessary condition.

6.23 Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(\boldsymbol{x}) = (x_1 - x_2)^4 + x_1^2 - x_2^2 - 2x_1 + 2x_2 + 1,$$

where $\boldsymbol{x} = [x_1, x_2]^\top$. Suppose that we wish to minimize f over $\mathbb{R}^2$. Find all points satisfying the FONC. Do these points satisfy the SONC?

6.24 Show that if $\boldsymbol{d}$ is a feasible direction at a point $\boldsymbol{x} \in \Omega$, then for all $\beta > 0$, the vector $\beta\boldsymbol{d}$ is also a feasible direction at $\boldsymbol{x}$.

6.25 Let $\Omega = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$. Show that $\boldsymbol{d} \in \mathbb{R}^n$ is a feasible direction at $\boldsymbol{x} \in \Omega$ if and only if $\boldsymbol{A}\boldsymbol{d} = \boldsymbol{0}$.

6.26 Let $f : \mathbb{R}^2 \to \mathbb{R}$. Consider the problem

$$\begin{array}{ll} \text{minimize} & f(\boldsymbol{x}) \\ \text{subject to} & x_1, x_2 \geq 0, \end{array}$$

where $\boldsymbol{x} = [x_1, x_2]^\top$. Suppose that $\nabla f(\boldsymbol{0}) \neq \boldsymbol{0}$, and

$$\frac{\partial f}{\partial x_1}(\boldsymbol{0}) \leq 0, \qquad \frac{\partial f}{\partial x_2}(\boldsymbol{0}) \leq 0.$$

Show that $\boldsymbol{0}$ cannot be a minimizer for this problem.

6.27 Let $\boldsymbol{c} \in \mathbb{R}^n$, $\boldsymbol{c} \neq \boldsymbol{0}$, and consider the problem of minimizing the function $f(\boldsymbol{x}) = \boldsymbol{c}^\top\boldsymbol{x}$ over a constraint set $\Omega \subset \mathbb{R}^n$. Show that we cannot have a solution lying in the interior of Ω.

6.28 Consider the problem

$$\begin{array}{ll} \text{maximize} & c_1 x_1 + c_2 x_2 \\ \text{subject to} & x_1 + x_2 \leq 1 \\ & x_1, x_2 \geq 0, \end{array}$$

where c_1 and c_2 are constants such that $c_1 > c_2 \geq 0$. This is a *linear programming* problem (see Part III). Assuming that the problem has an optimal feasible solution, use the first-order necessary condition to show that the *unique* optimal feasible solution $\boldsymbol{x}^*$ is $[1, 0]^\top$.

Hint: First show that $\boldsymbol{x}^*$ cannot lie in the interior of the constraint set. Then, show that $\boldsymbol{x}^*$ cannot lie on the line segments $L_1 = \{\boldsymbol{x} : x_1 = 0, 0 \le x_2 < 1\}$, $L_2 = \{\boldsymbol{x} : 0 \le x_1 < 1, x_2 = 0\}$, $L_3 = \{\boldsymbol{x} : 0 \le x_1 < 1, x_2 = 1 - x_1\}$.

6.29 *Line Fitting.* Let $[x_1, y_1]^\top, \ldots, [x_n, y_n]^\top$, $n \ge 2$, be points on the $\mathbb{R}^2$ plane (each $x_i, y_i \in \mathbb{R}$). We wish to find the straight line of "best fit" through these points ("best" in the sense that the average squared error is minimized); that is, we wish to find $a, b \in \mathbb{R}$ to minimize

$$f(a,b) = \frac{1}{n}\sum_{i=1}^{n}(ax_i + b - y_i)^2 .$$

a. Let

$$\begin{aligned}
\overline{X} &= \frac{1}{n}\sum_{i=1}^{n} x_i,\\
\overline{Y} &= \frac{1}{n}\sum_{i=1}^{n} y_i,\\
\overline{X^2} &= \frac{1}{n}\sum_{i=1}^{n} x_i^2,\\
\overline{Y^2} &= \frac{1}{n}\sum_{i=1}^{n} y_i^2,\\
\overline{XY} &= \frac{1}{n}\sum_{i=1}^{n} x_i y_i.
\end{aligned}$$

Show that $f(a,b)$ can be written in the form $\boldsymbol{z}^\top \boldsymbol{Q}\boldsymbol{z} - 2\boldsymbol{c}^\top \boldsymbol{z} + d$, where $\boldsymbol{z} = [a,b]^\top$, $\boldsymbol{Q} = \boldsymbol{Q}^\top \in \mathbb{R}^{2\times 2}$, $\boldsymbol{c} \in \mathbb{R}^2$ and $d \in \mathbb{R}$, and find expressions for $\boldsymbol{Q}$, $\boldsymbol{c}$, and d in terms of $\overline{X}$, $\overline{Y}$, $\overline{X^2}$, $\overline{Y^2}$, and $\overline{XY}$.

b. Assume that the x_i, $i = 1, \ldots, n$, are not all equal. Find the parameters a^* and b^* for the line of best fit in terms of $\overline{X}$, $\overline{Y}$, $\overline{X^2}$, $\overline{Y^2}$, and $\overline{XY}$. Show that the point $[a^*, b^*]^\top$ is the only local minimizer of f.
Hint: $\overline{X^2} - (\overline{X})^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{X})^2$.

c. Show that if a^* and b^* are the parameters of the line of best fit, then $\overline{Y} = a^*\overline{X} + b^*$ (and hence once we have computed a^*, we can compute b^* using the formula $b^* = \overline{Y} - a^*\overline{X}$).

6.30 Suppose that we are given a set of vectors $\{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(p)}\}$, $\boldsymbol{x}^{(i)} \in \mathbb{R}^n$, $i = 1, \ldots, p$. Find the vector $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ such that the average squared distance (norm) between $\bar{\boldsymbol{x}}$ and $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(p)}$,

$$\frac{1}{p}\sum_{i=1}^{p}\|\bar{\boldsymbol{x}} - \boldsymbol{x}^{(i)}\|^2,$$

is minimized. Use the SOSC to prove that the vector $\bar{\boldsymbol{x}}$ found above is a strict local minimizer. How is $\bar{\boldsymbol{x}}$ related to the centroid (or center of gravity) of the given set of points $\{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(p)}\}$?

6.31 Consider a function $f : \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is a convex set and $f \in \mathcal{C}^1$. Given $\boldsymbol{x}^* \in \Omega$, suppose that there exists $c > 0$ such that $\boldsymbol{d}^\top \nabla f(\boldsymbol{x}^*) \geq c\|\boldsymbol{d}\|$ for all feasible directions $\boldsymbol{d}$ at $\boldsymbol{x}^*$. Show that $\boldsymbol{x}^*$ is a strict local minimizer of f over Ω.

6.32 Prove the following generalization of the second-order sufficient condition:

Theorem: Let Ω be a convex subset of $\mathbb{R}^n$, $f \in \mathcal{C}^2$ a real-valued function on Ω, and $\boldsymbol{x}^*$ a point in Ω. Suppose that there exists $c \in \mathbb{R}$, $c > 0$, such that for all feasible directions $\boldsymbol{d}$ at $\boldsymbol{x}^*$ ($\boldsymbol{d} \neq \boldsymbol{0}$), the following hold:

1. $\boldsymbol{d}^\top \nabla f(\boldsymbol{x}^*) \geq 0$.
2. $\boldsymbol{d}^\top \boldsymbol{F}(\boldsymbol{x}^*)\boldsymbol{d} \geq c\|\boldsymbol{d}\|^2$.

Then, $\boldsymbol{x}^*$ is a strict local minimizer of f.

6.33 Consider the quadratic function $f : \mathbb{R}^n \to \mathbb{R}$ given by

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top \boldsymbol{Q}\boldsymbol{x} - \boldsymbol{x}^\top \boldsymbol{b},$$

where $\boldsymbol{Q} = \boldsymbol{Q}^\top > 0$. Show that $\boldsymbol{x}^*$ minimizes f if and only if $\boldsymbol{x}^*$ satisfies the FONC.

6.34 Consider the linear system $x_{k+1} = ax_k + bu_{k+1}$, $k \geq 0$, where $x_i \in \mathbb{R}$, $u_i \in \mathbb{R}$, and the initial condition is $x_0 = 0$. Find the values of the control inputs $u_1, \ldots, u_n$ to minimize

$$-qx_n + r\sum_{i=1}^{n} u_i^2,$$

where $q, r > 0$ are given constants. This can be interpreted as desiring to make x_n as large as possible but at the same time desiring to make the total input energy $\sum_{i=1}^n u_i^2$ as small as possible. The constants q and r reflect the relative weights of these two objectives.