Final Value Theorem

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Introduction

Final Value Theorem is a mathematical property which can helps to determine $t = \infty$ values which can be useful for engineering purposes like steady state calculations.

Proof

From derivative property of Laplace transform;

$$sX(s) = \int_{0^-}^{\infty} x'(t)e^{-st}dt \ + x(0^-)$$

Taking limit s goes to zero both side;

$$\lim_{s\to 0} sX(s) = \lim_{s\to 0} \int_{0^-}^{\infty} x'(t)e^{-st}dt \ + x(0^-)$$

$$\lim_{s\to 0} sX(s) = \int_{0^-}^{\infty} x'(t)e^{-0t}dt \ + x(0^-)$$

$$\lim_{s\to 0} sX(s) = x(\infty) - x(0^-) + x(0^-)$$

$$\lim_{s\to 0} sX(s) = x(\infty)$$

Conditions for proof

There is a pole or poles at RHP

We can't use final value theorem for that has unstable pole. For example $T(s) = \frac{1}{s-5}$ is a plant with unity feedback. We know that since we have positive pole time response has positive exponential and become infinite when t goes to infinite However;

$$\lim_{s \to 0} \frac{s}{s-5} = x(\infty) = 0$$

Final value theorem gives us wrong answer. Then we can not use in this region.

Poles at the Imaginary axis

We know that when the system has poles at the imaginary axis time response is oscillatory. Then final value of oscillatory function is meaningless. For example $T(s) = \frac{1}{s^2+4}$;

$$\lim_{s \to 0} \frac{s}{s^2+4} = x(\infty) = 0$$

Zero is the average final value of the system however, still final value theorem meaningless at that region.

Poles at LHP

$$T(s) = \frac{1}{s+5}$$

$$\lim_{s \to 0} \frac{s}{s+5} = x(\infty) = 0$$

Which is true for system has negative poles.

Poles at the origin

$$T(s) = \frac{1}{s}$$

$$\lim_{s \to 0} \frac{s}{s} = x(\infty) = 1$$

Which is make sense since time response for transfer function is 1.

System Types and Error Coefficients

System types is the number of poles of the origin.

Type 0 System

There is no integrator term in the system. All poles at the left half plane. For example;

$$G(s) = \frac{1}{s^2 + 6s + 8}$$

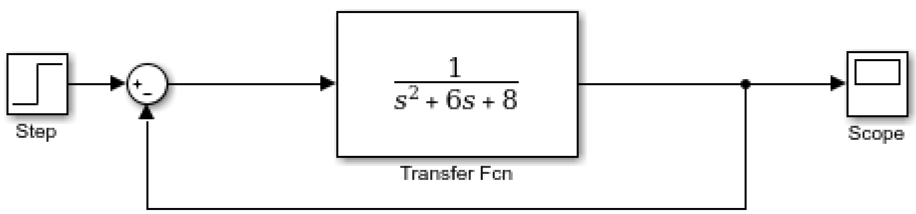

Figure 1: Type 0 system

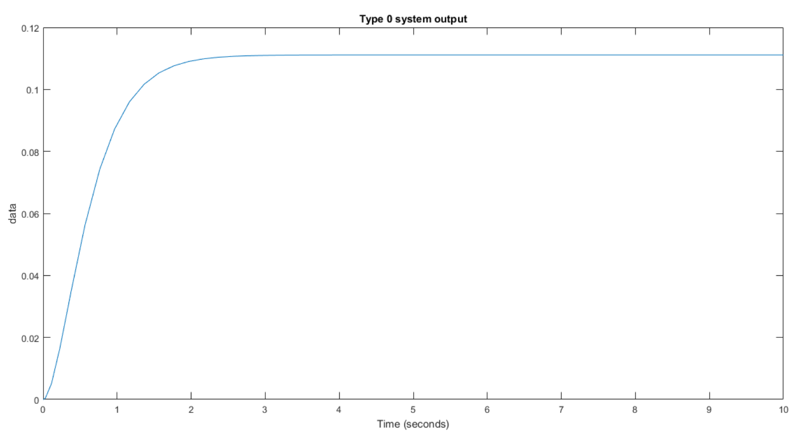

Figure 2: Type 0 system response to step input

Position Error Coefficient K_d

Position error occurs when we give step input to the system. Type 0 systems has finite position error.

$$K_p = lim_{x \to 0} G(s)H(s) = \frac{1}{s^2 + 6s + 8} = \frac{1}{8}$$

$$e_{ss} = \frac{1}{1 + K_p} = \frac{8}{9}$$

Type 1 system

There is one integrator term occur in the transfer function. Type one system has no position steady state error.(Or we can say $K_p = \infty$). However it has finite velocity error. Velocity error occurs when we give ramp input to the system.

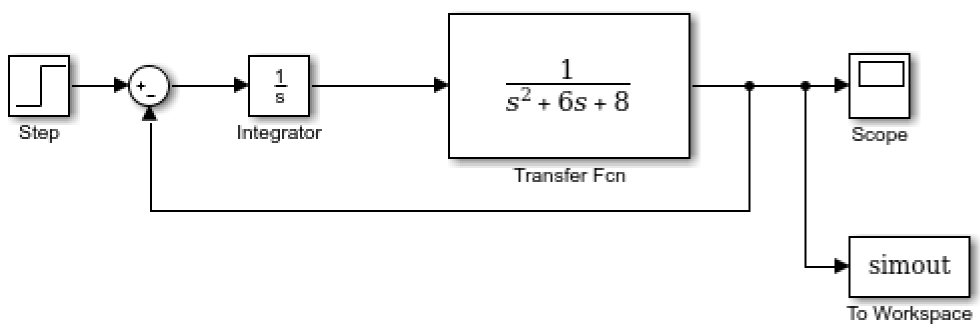

Figure 3: Type 1 system

Now G(s)H(s) has one integrator term then it is Type 1 system. Adding integrator to type 0 system can be used as improving position error of the system. Following figure we can see effect of the adding integrator to the type 0 system.

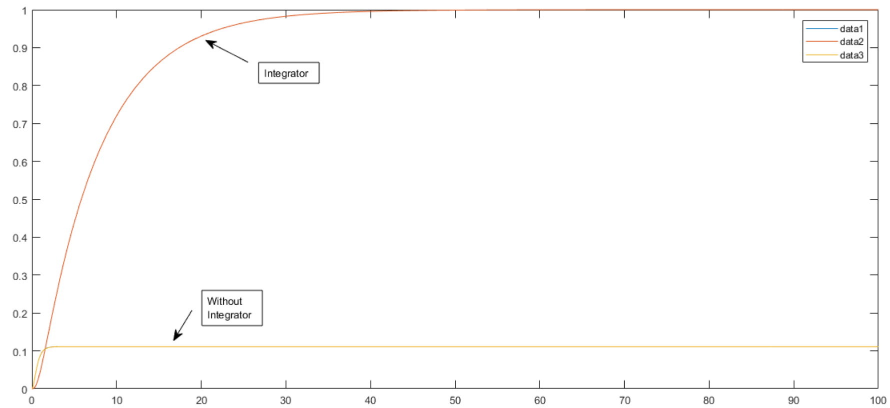

Figure 4: Comparing type 1 and type 0 time response