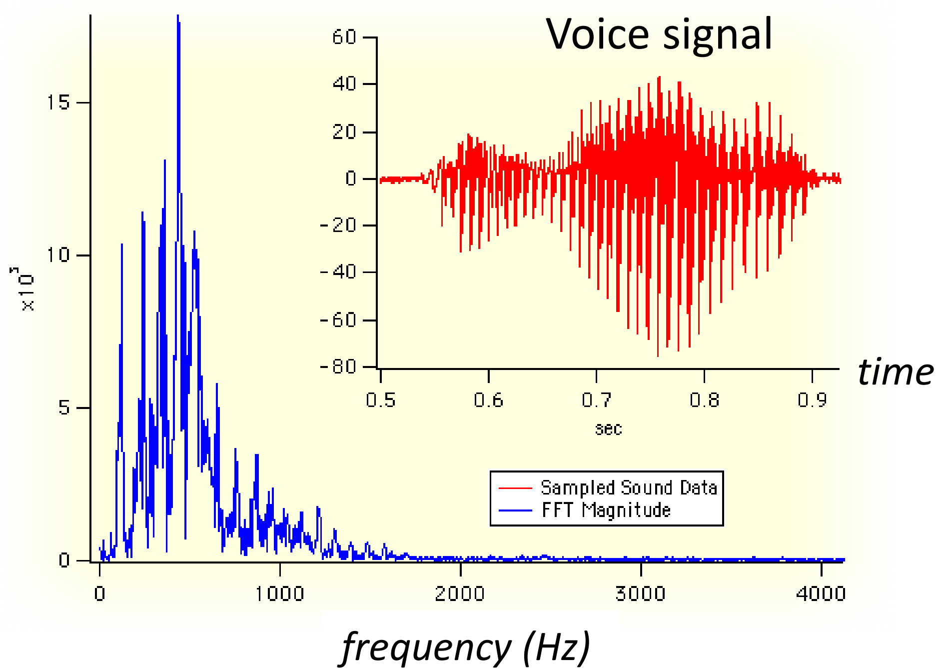
The Fourier Transform

EE 442 Analog & Digital Communication Systems

Lecture 4



Jean Joseph Baptiste Fourier

March 21, 1768 to May 16, 1830

Review: Fourier Trignometric Series (for Periodic Waveforms)

Equation (2.10) should read (time t was missing in book):

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right)$$

where $\omega_0 = \dfrac{2\pi}{T}$ and $f_0 = \dfrac{1}{T}$

and (Equations 2.12a, b, & c)

$$a_0 = \frac{1}{T}\int_0^T f(t)dt \qquad \text{(DC term)}$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)dt \quad \text{for } n = 1, 2, 3, etc.$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)dt \quad \text{for } n = 1, 2, 3, etc.$$

Agbo & Sadiku; Section 2.5 pp. 26-27

Fourier Trigonometric Series in Amplitude-Phase Format

Equations (2.13) and (2.14) should read:

$$f(t) = A_0 + \sum_{n=1}^{\infty}\left(A_n \cos(n\omega_0 t + \phi_n)\right)$$

$$a_0 = A_0$$

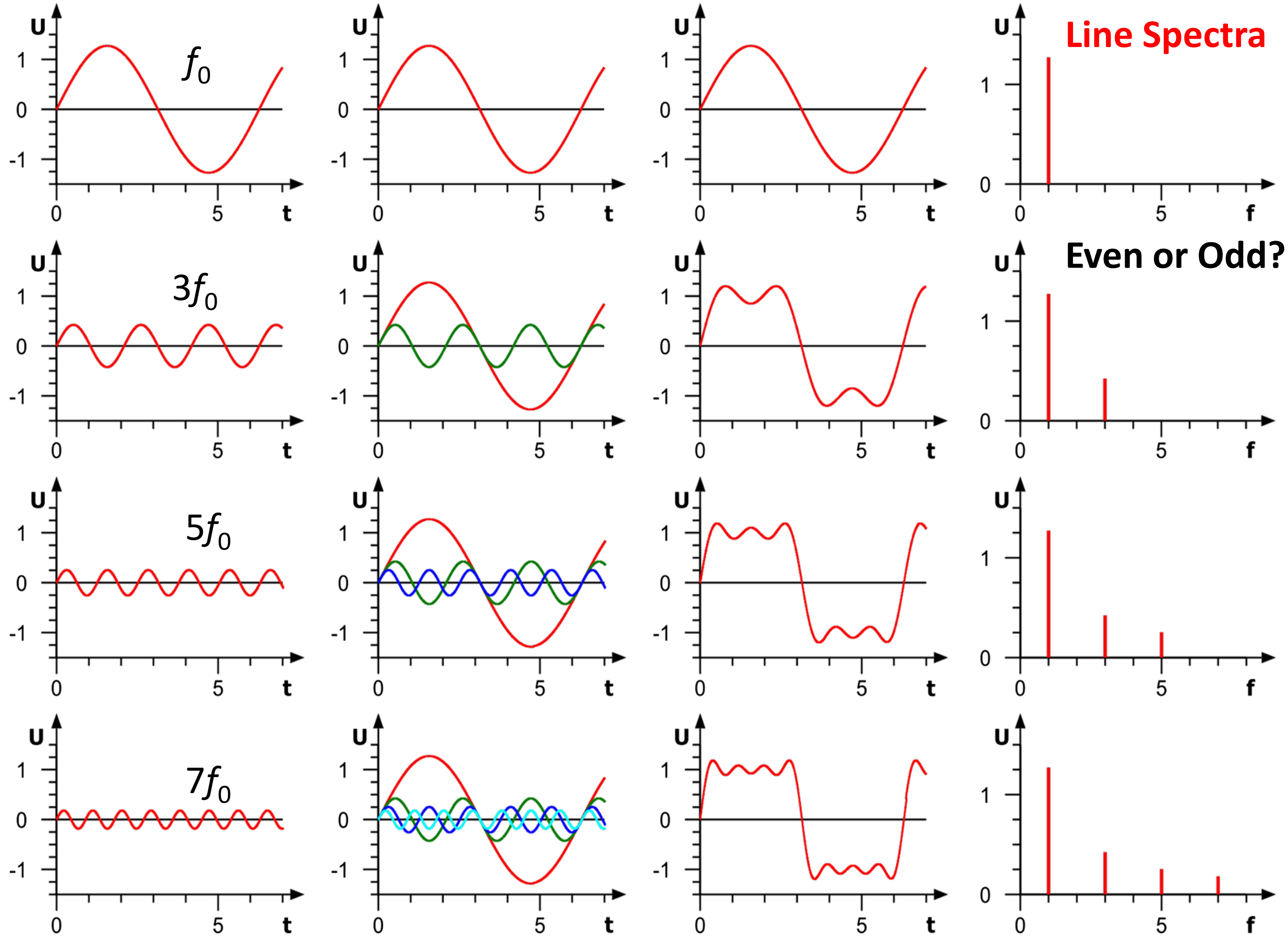
$$A_n = \sqrt{a_n^2 + b_n^2} \quad and$$

$$\phi_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Also known as polar form of Fourier series.

Agbo & Sadiku; Section 2.5 Page 27

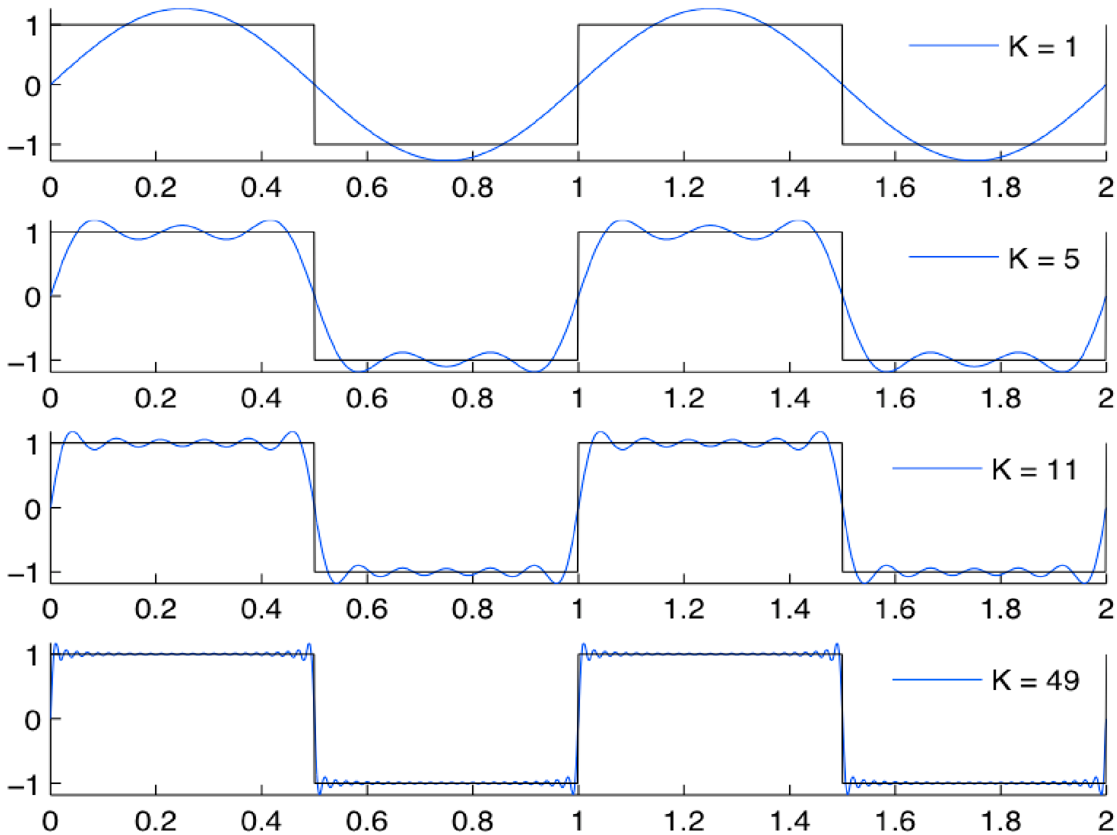
Example: Periodic Square Wave as Sum of Sinusoids



Example: Periodic Square Wave (continued)

This is an odd function

$$f(t)=\frac{4}{\pi}\left[\sin(\pi t)+\tfrac{1}{3}\sin(3\pi t)+\tfrac{1}{5}\sin(5\pi t)+\tfrac{1}{7}\sin(7\pi t)+\cdots\right]$$



Fundamental only

Five terms

Eleven terms

Forty-nine terms

$t \longrightarrow$

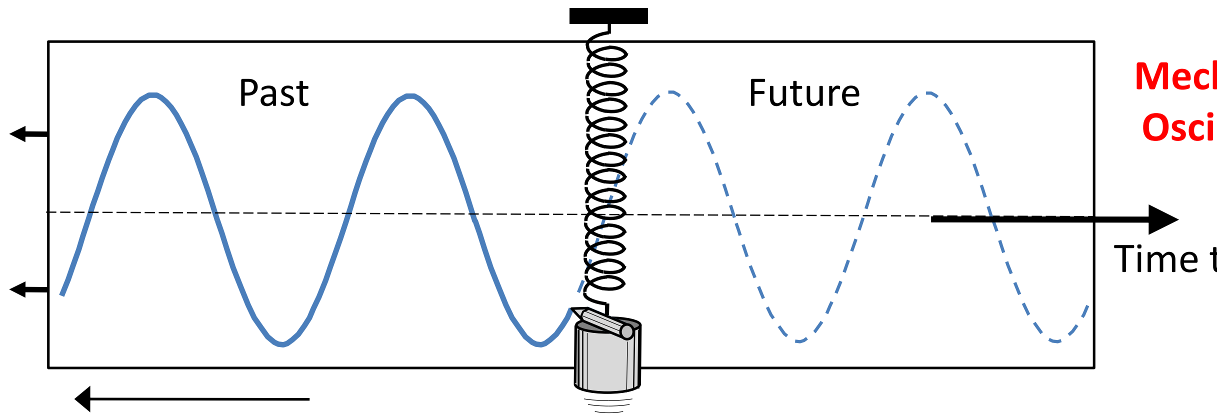
Question:

What would make this an even function?

http://ceng.gazi.edu.tr/dsp/fourier_series/description.aspx

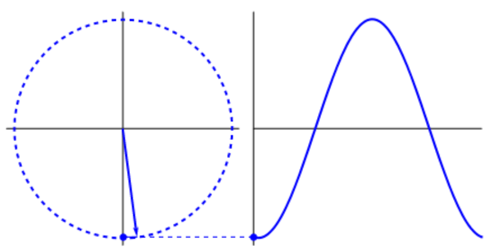
Sinusoidal Waveforms are the Building Blocks in the Fourier Series

Simple Harmonic Motion Produces Sinusoidal Waveforms

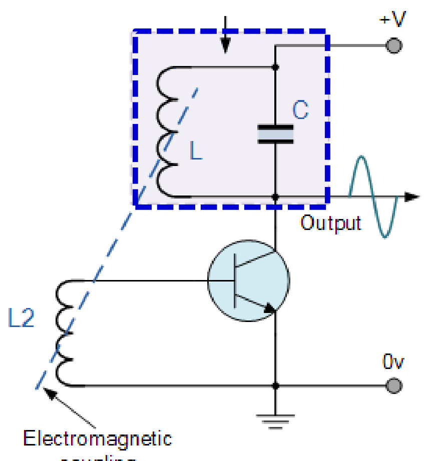


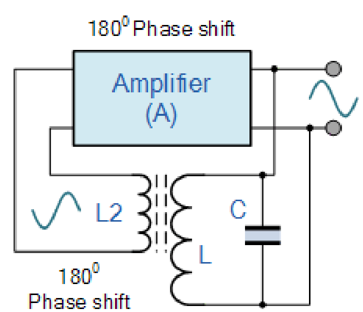
Sheet of paper unrolls in this direction

Mechanical Oscillation



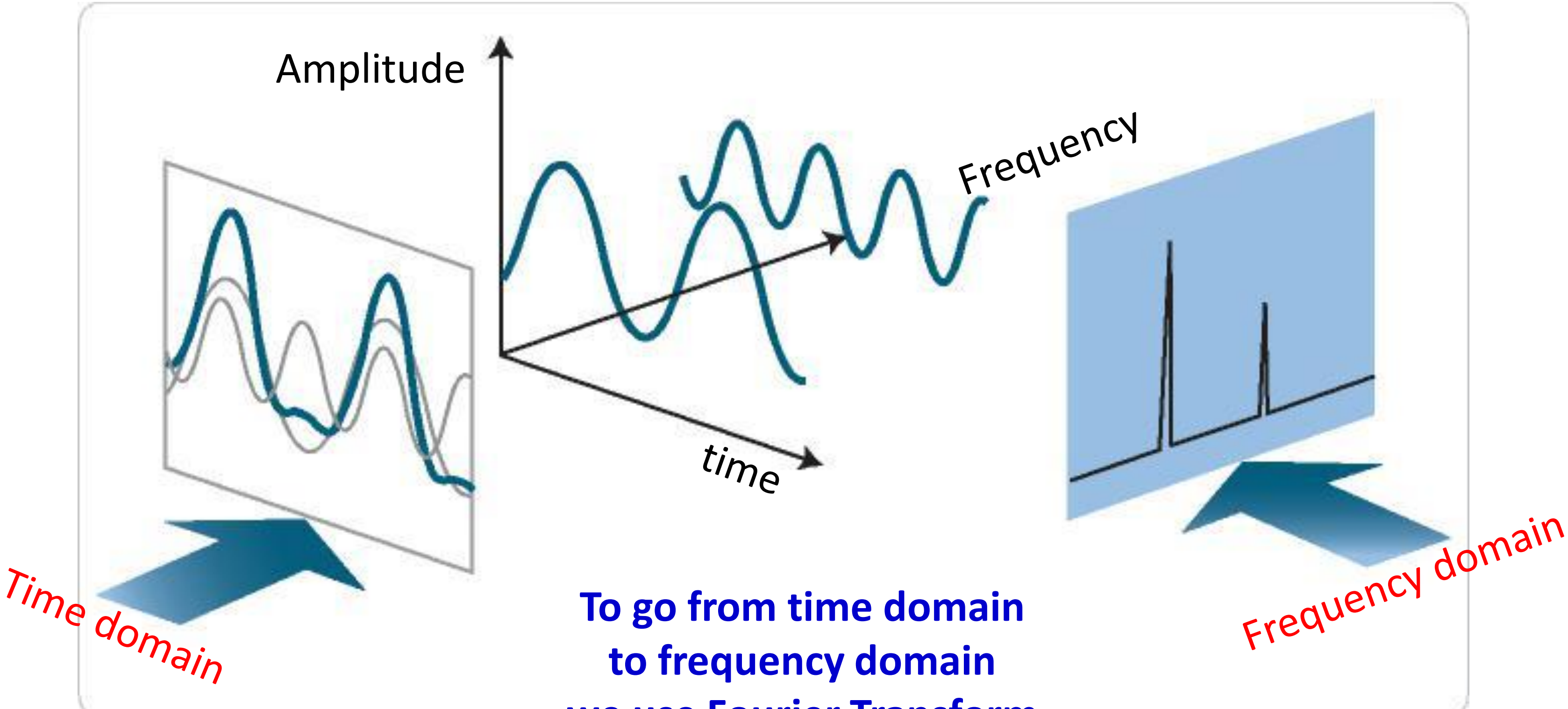
LC Tank Circuit





Electrical LC Circuit Oscillation

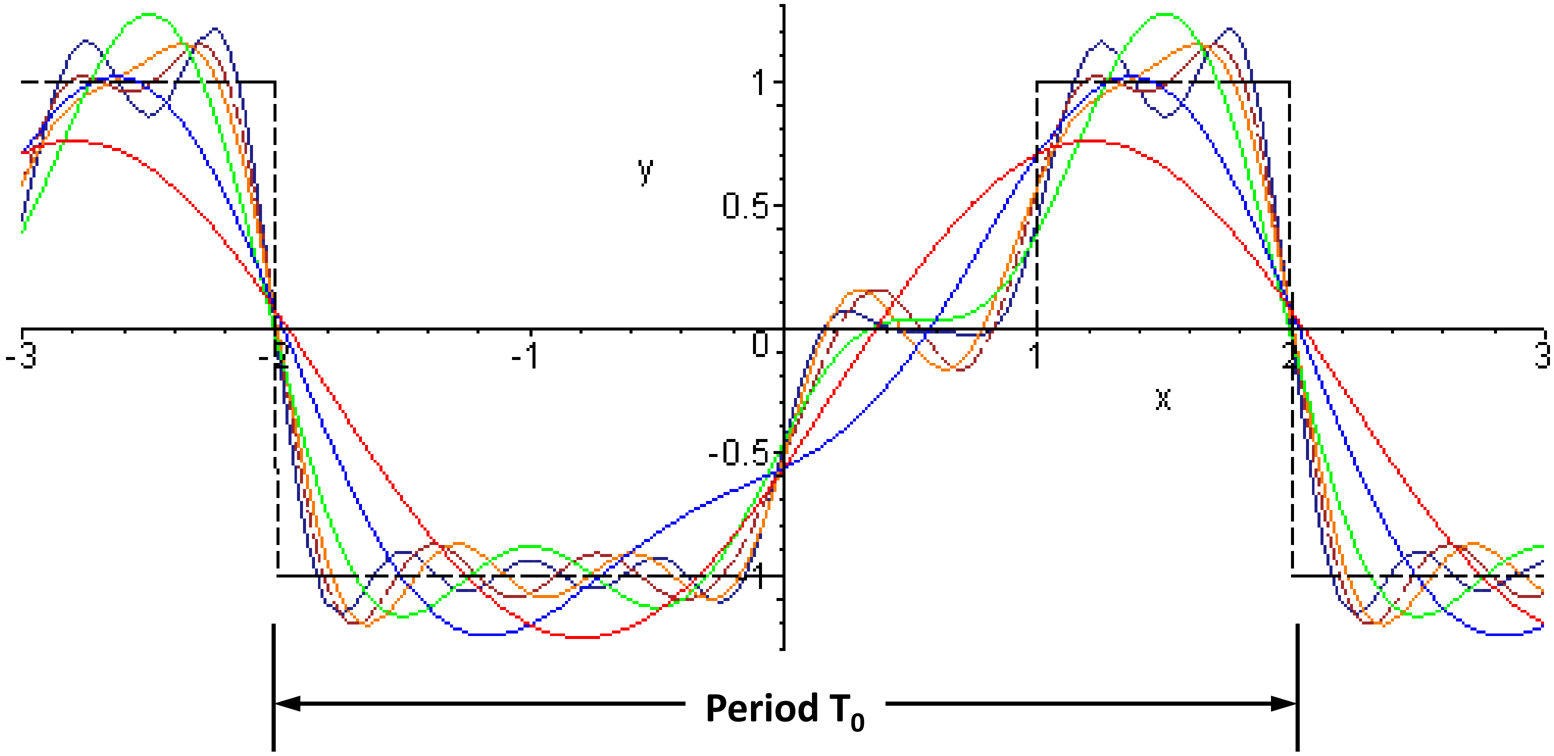
Visualizing a Signal – Time Domain & Frequency Domain



To go from time domain
to frequency domain
we use Fourier Transform

Source: ***Agilent Technologies Application Note 150***, "Spectrum Analyzer Basics"
http://cp.literature.agilent.com/litweb/pdf/5952-0292.pdf

Example: Where Both Sine & Cosine Terms are Required



Note phase shift in the fundamental frequency sine waveform.

http://www.peterstone.name/Maplepgs/fourier.html#anchor2315207

Fourier Series versus Fourier Transform

	Continuous time	Discrete time
Periodic	**Fourier Series**	**Discrete Fourier Transform**
Aperiodic	**Fourier Transform**	**Discrete Fourier Transform**

Fourier series for continuous-time periodic signals → discrete spectra
Fourier transform for continuous aperiodic signals → continuous spectra

Definition of Fourier Transform

The Fourier transform (spectrum) of *f(t)* is $F(\omega)$:

$$F(\omega) = FT\left\{ f(t) \right\} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$f(t) = FT^{-1}\left\{ F(\omega) \right\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega$$

Therefore, $f(t) \Leftrightarrow F(\omega)$ is a Fourier Transform pair

Agbo & Sadiku;
Section 2.7;
pp. 40-41

Note: Remember $\omega = 2\pi f$

Fourier Transform Produces a Continuous Spectrum

$FT\{f(t)\}$ gives a spectra consisting of a continuous sum of exponentials with frequencies ranging from - ∞ to + ∞.

$$F(\omega) = |F(\omega)| \cdot e^{j\varphi(\omega)} \ ,$$

where $|F(\omega)|$ is the continuous amplitude spectrum of *f(t)*

and

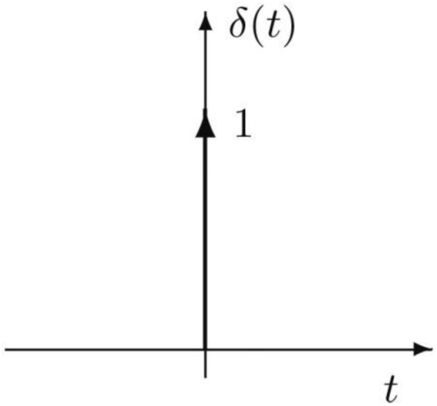
$\varphi(\omega)$ is the continuous phase spectrum of *f(t).*

Often only the magnitude of $F(\omega)$ is displayed and the phase is ignored.

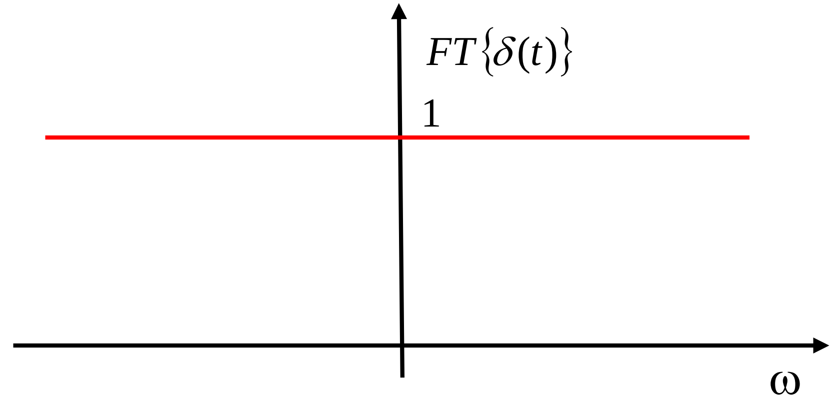
Example: Impulse Function δ(t)

$$F(\omega) = FT\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = e^{j0} = 1$$

$$\boxed{\begin{array}{c} \delta(t) \Leftrightarrow 1 \\ 1 \Leftrightarrow 2\pi\delta(\omega) \end{array}}$$



$\leftrightarrow$



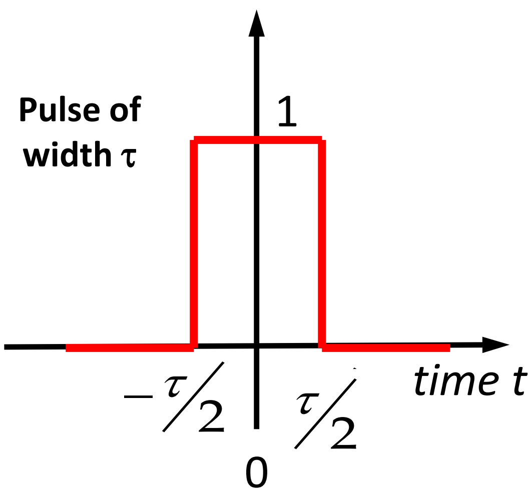
$$\delta(t) = \begin{cases} \infty \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}$$

Delta function has unity area.

Example: Fourier Transform of Single Rectangular Pulse

$$f(t) = \text{rect}(t) = \text{II}(t/\tau) = \begin{cases} 1 & \text{for } -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0 & \text{for all } |t| > \frac{\tau}{2} \end{cases}$$

$$f(t) = \text{rect}(t) = \text{II}(t/\tau)$$



Remember $\omega = 2\pi f$

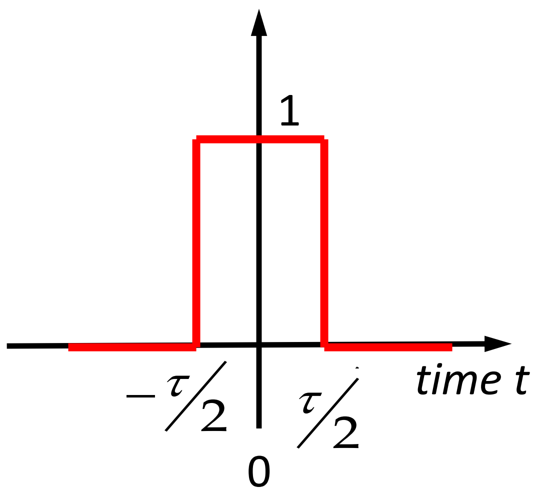
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$$

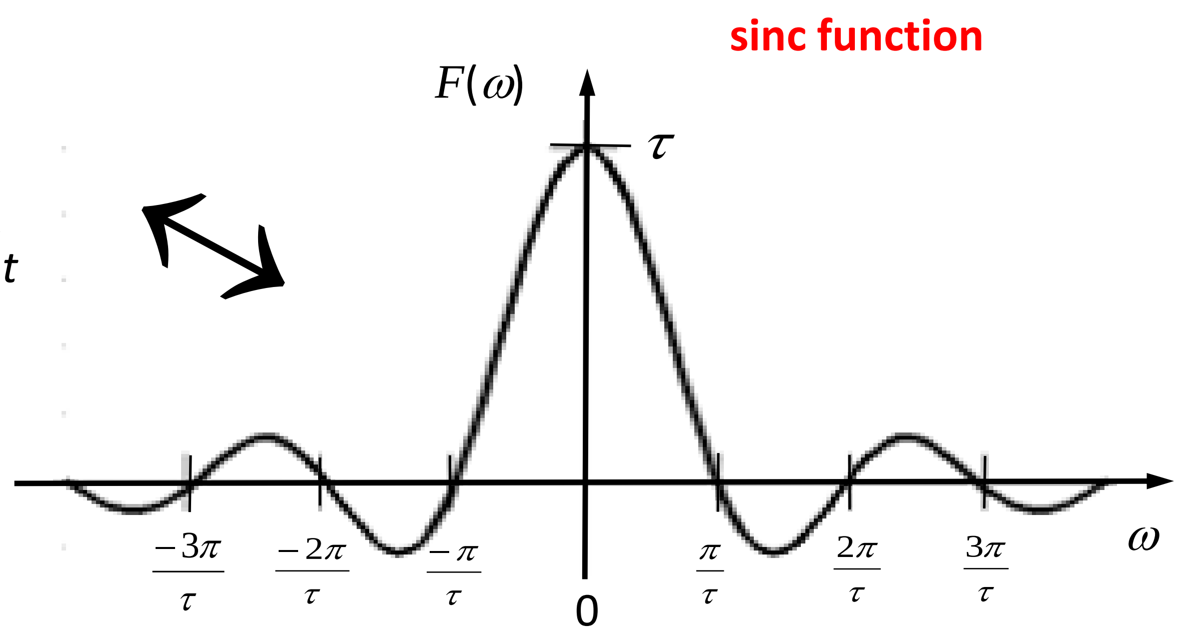
$$= \left(\frac{e^{-j\omega t}}{-j\omega} \right) \Bigg|_{-\tau/2}^{\tau/2} = \frac{e^{-j\omega\tau/2} - e^{j\omega\tau/2}}{-j\omega}$$

$$= \frac{-j2\sin(\omega\tau/2)}{-j\omega} = \tau \cdot \left[\frac{\sin(\omega\tau/2)}{(\omega\tau/2)} \right]$$

Fourier Transform of Single Rectangular Pulse (continued)

$$F(\omega) = \tau \cdot \left[\frac{\sin(\omega\tau/2)}{(\omega\tau/2)} \right] = \tau \cdot \operatorname{sinc}(\pi f \tau)$$





Note the pulse is time centered

Properties of the Sinc Function

Definition of the sinc function:

$$\text{sinc}(x) = \frac{\sin(x)}{x}$$

Sinc Properties:

1. sinc(x) is an even function of x
2. sinc(x) = 0 at points where sin(x) = 0, that is,
   sinc(x) = 0 when $x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.
3. Using L'Hôpital's rule, it can be shown that sinc(0) = 1
4. sinc(x) oscillates as sin(x) oscillates and monotonically decreases as $1/x$ decreases with increasing $|x|$
5. sinc(x) is the Fourier transform of a single rectangular pulse

Periodic Pulse Train Morphing Into a Single Pulse

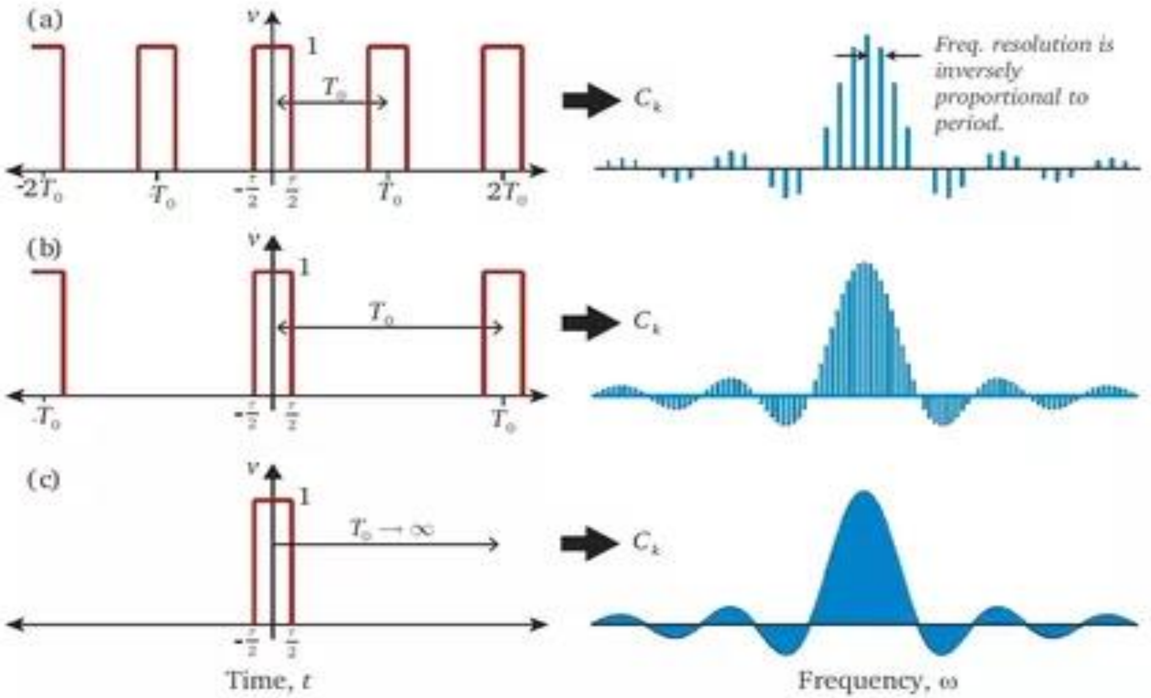
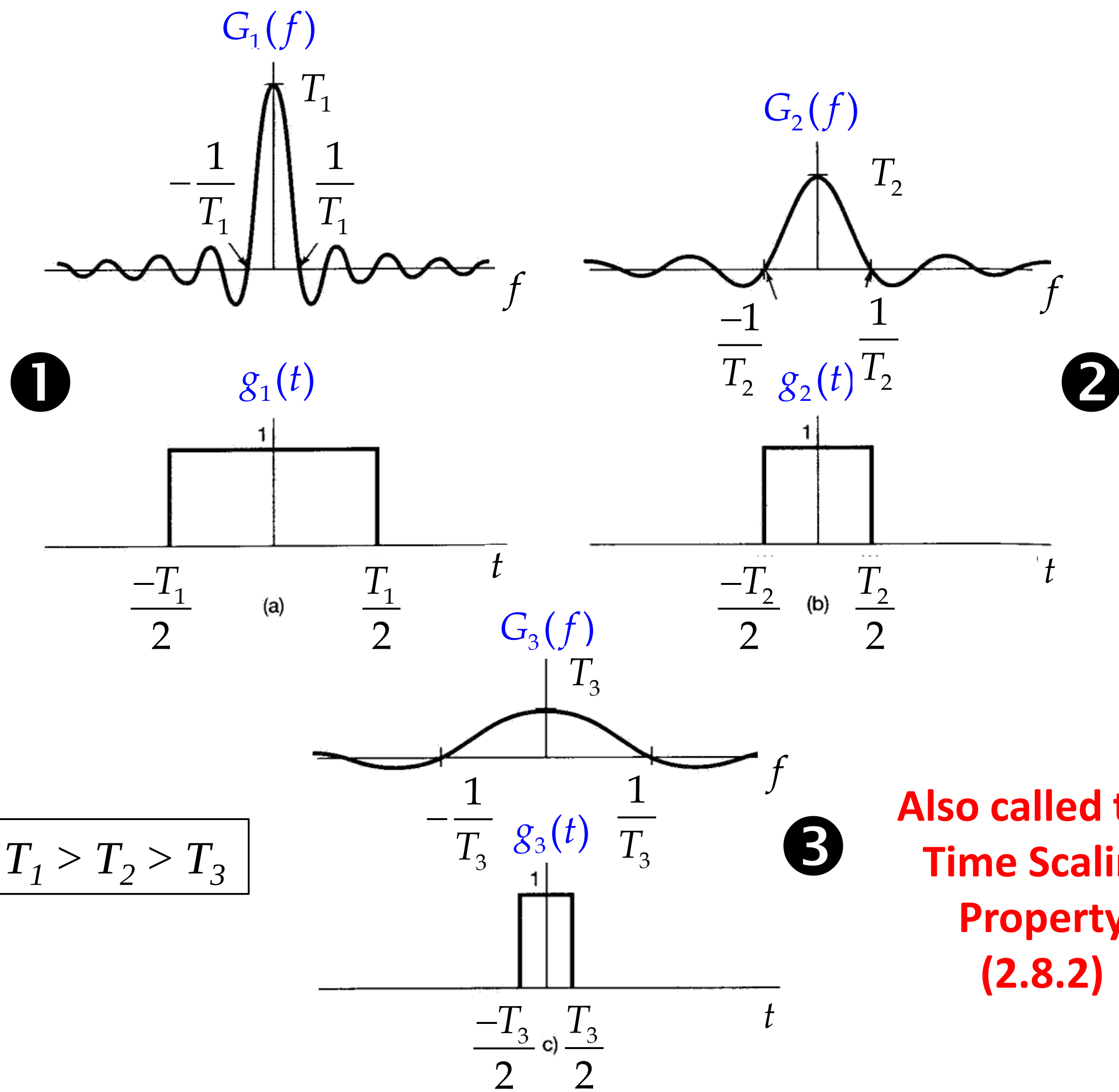


Figure 4.3: Take the pulse train in (a), as we increase its period, i.e., allow more time between the pulses, the fundamental frequency gets smaller, which makes the spectral lines move closer together as in (c). In the limiting case, where the period goes to ∞, the spectrum would become continuous.

https://www.quora.com/What-is-the-exact-difference-between-continuous-fourier-transform-discrete-Time-Fourier-Transform-DTFT-Discrete-Fourier-Transform-DFT-Fourier-series-and-Discrete-Fourier-Series-DFS-In-which-cases-is-which-one-used

Sinc Function Tradeoff: Pulse Duration *versus* Bandwidth



$$\boxed{T_1 > T_2 > T_3}$$

Also called the Time Scaling Property (2.8.2)

Properties of Fourier Transforms

2.8.1 Linearity (Superposition) Property
2.8.2 Time-Scaling Property
2.8.3 Time-Shifting Property
2.8.4 Frequency-Shifting Property
2.8.5 Time Differentiation Property
2.8.6 Frequency Differentiation Property
2.8.7 Time Integration Property
2.8.8 Time-Frequency Duality Property
2.8.9 Convolution Property

Agbo & Sadiku
Section 2.8;
pp. 46 to 58

2.8.1 Linearity (Superposition) Property

Given $f(t) \leftrightarrow F(\omega)$ and $g(t) \leftrightarrow G(\omega)$;

Then $f(t) + g(t) \leftrightarrow F(\omega) + G(\omega)$ **(additivity)**

also $kf(t) \leftrightarrow kF(\omega)$ and $mg(t) \leftrightarrow mG(\omega)$ **(homogeneity)**

Combining these we have,

$$\boxed{kf(t) + mg(t) \leftrightarrow kF(\omega) + mG(\omega)}$$

Hence, the Fourier Transform is a <u>linear transformation</u>.

This is the same definition for linearity as used in your circuits and systems EE400 course.

2.8.2 Time Scaling Property

$$FT\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

$$FT\{f(at)\} = \int_{-\infty}^{\infty} f(at) e^{-j\omega t} dt$$

Let $\lambda = at$ & $d\lambda = adt,$

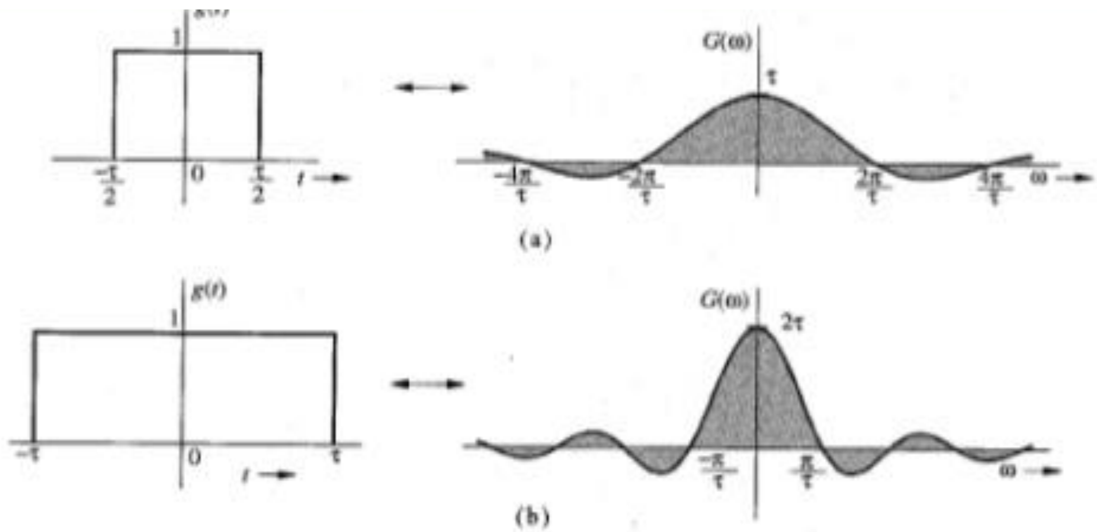
$$FT\{f(at)\} = \int_{-\infty}^{\infty} f(\lambda) e^{-j\omega t} \frac{d\lambda}{a} = \frac{1}{a} F\left(\frac{\omega}{a}\right)$$

Hence, $FT\{f(\text{-}t)\} = F(-\omega) = F^*(\omega)$

Time-Scaling Property (continued)

$$FT\{f(at)\} = \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$$

Time compression of a signal results in spectral expansion and time expansion of a signal results in spectral compression.



2.8.3 Time Shifting Property

$$FT\left\{f(t-t_0)\right\} = e^{-j\omega t_0} F(\omega)$$

$$FT\left\{f(t-t_0)\right\} = \int_{-\infty}^{\infty} f(t-t_0) e^{-j\omega t} dt$$

Let $\lambda = t - t_0, \;\; d\lambda = dt \;\; \& \;\; t = \lambda + t_0$

$$FT\left\{f(t-t_0)\right\} = \int_{-\infty}^{\infty} f(\lambda) e^{-j\omega(\lambda + t_0)} d\lambda =$$

$$= e^{-j\omega t_9} \int_{-\infty}^{\infty} f(\lambda) e^{-j\omega\lambda} d\lambda = e^{-j\omega t_0} F(\omega)$$

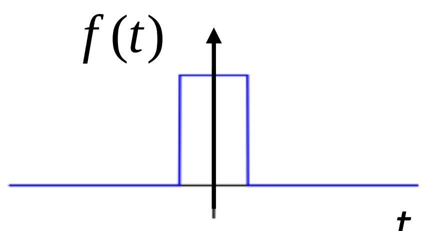
Time-Shifting Property (continued)

$$FT\{f(t-t_0)\} = e^{-j\omega t_0} F(\omega)$$

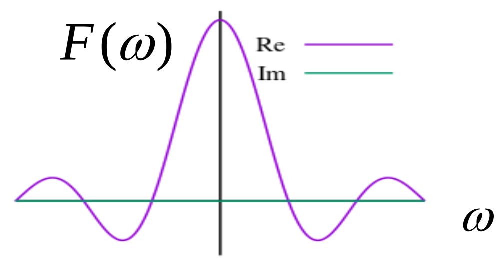
Delaying a signal by t_0 seconds does not change its amplitude spectrum, but the phase spectrum is changed by $-2\pi f t_0$.
Note that the phase spectrum shift changes linearly with frequency f.

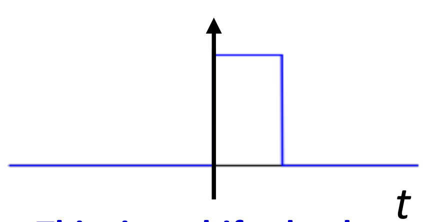
$$|F(\omega)| = \sqrt{[\mathrm{Re}(F(\omega))]^2 + [\mathrm{Im}(F(\omega))]^2}$$

A time shift produces a phase shift in its spectrum.

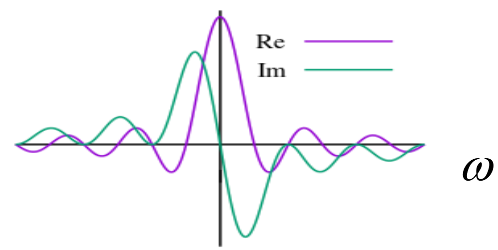


Even function.





This time shifted pulse is both even and odd.



Both must be identical.

2.8.4 Frequency Shifting Property

$$\boxed{FT\left\{f(t)e^{j\omega_0 t}\right\} = F\left(\omega - \omega_0\right)}$$

$$FT\left\{f(t)e^{j\omega_0 t}\right\} = \int_{-\infty}^{\infty} f(t)e^{j\omega_0 t}e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} f(t)e^{-j(\omega-\omega_0)t}d\lambda = F\left(\omega - \omega_0\right)$$

Special application :

Apply to $\cos\left(\omega_0 t\right) = \frac{1}{2}\left(e^{j\omega_0 t} + e^{-j\omega_0 t}\right);$

$$FT\left\{\cos\left(\omega_0 t\right)\right\} = \tfrac{1}{2}\left(F(\omega - \omega_0) + \tfrac{1}{2}\left(F(\omega + \omega_0)\right)\right)$$

Important Formula to Remember

Euler's formula

$$\exp[\pm j\theta] = \cos(\theta) \pm j\sin(\theta)$$

$$\sin(\theta) = \frac{1}{2j}\left(\exp[j\theta] - \exp[-j\theta]\right)$$

$$\cos(\theta) = \frac{1}{2}\left(\exp[j\theta] + \exp[-j\theta]\right)$$

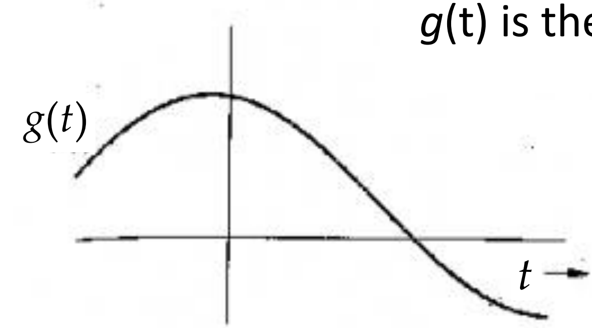
$$e^{\pm j(\pi/2)} = \pm j \quad and \quad e^{\pm jn\pi} = \begin{cases} 1 & \text{for } n\ even \\ -1 & \text{for } n\ odd \end{cases}$$

$$a + jb = re^{j\theta} \quad where \quad r = \sqrt{a^2 + b^2}, \quad \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

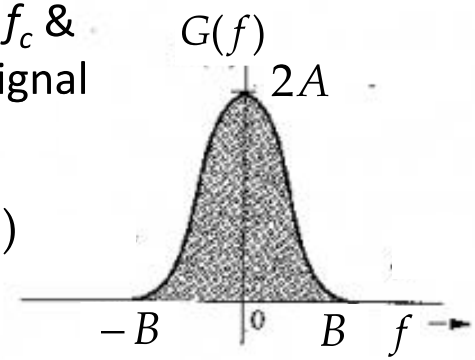
Frequency Shifting Property is Very Useful in Communications

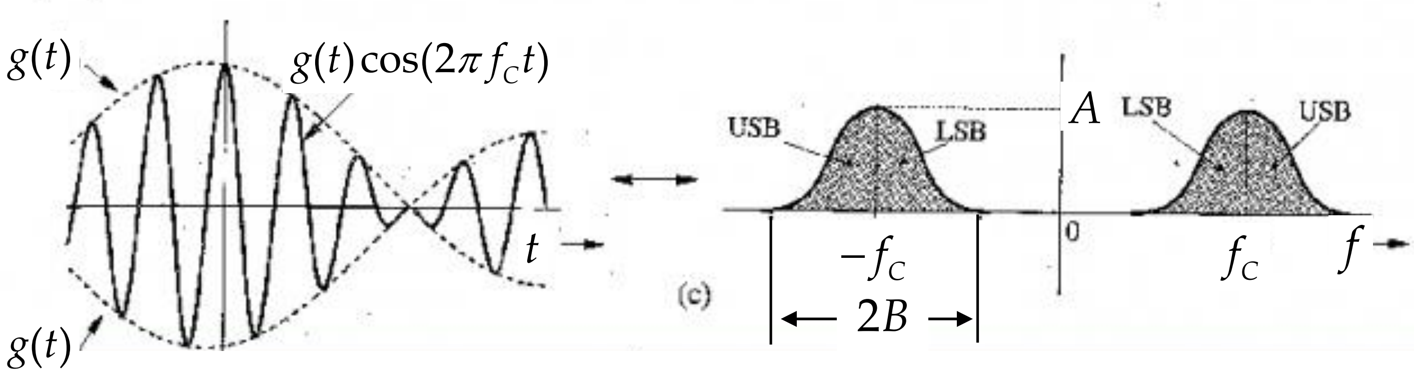
Multiplication of a signal *$g(t)$* by the factor [$\cos(2\pi f_c t)$] places *$G(f)$* centered at $f = \pm f_C$.

Carrier frequency is f_c &
g(t) is the message signal

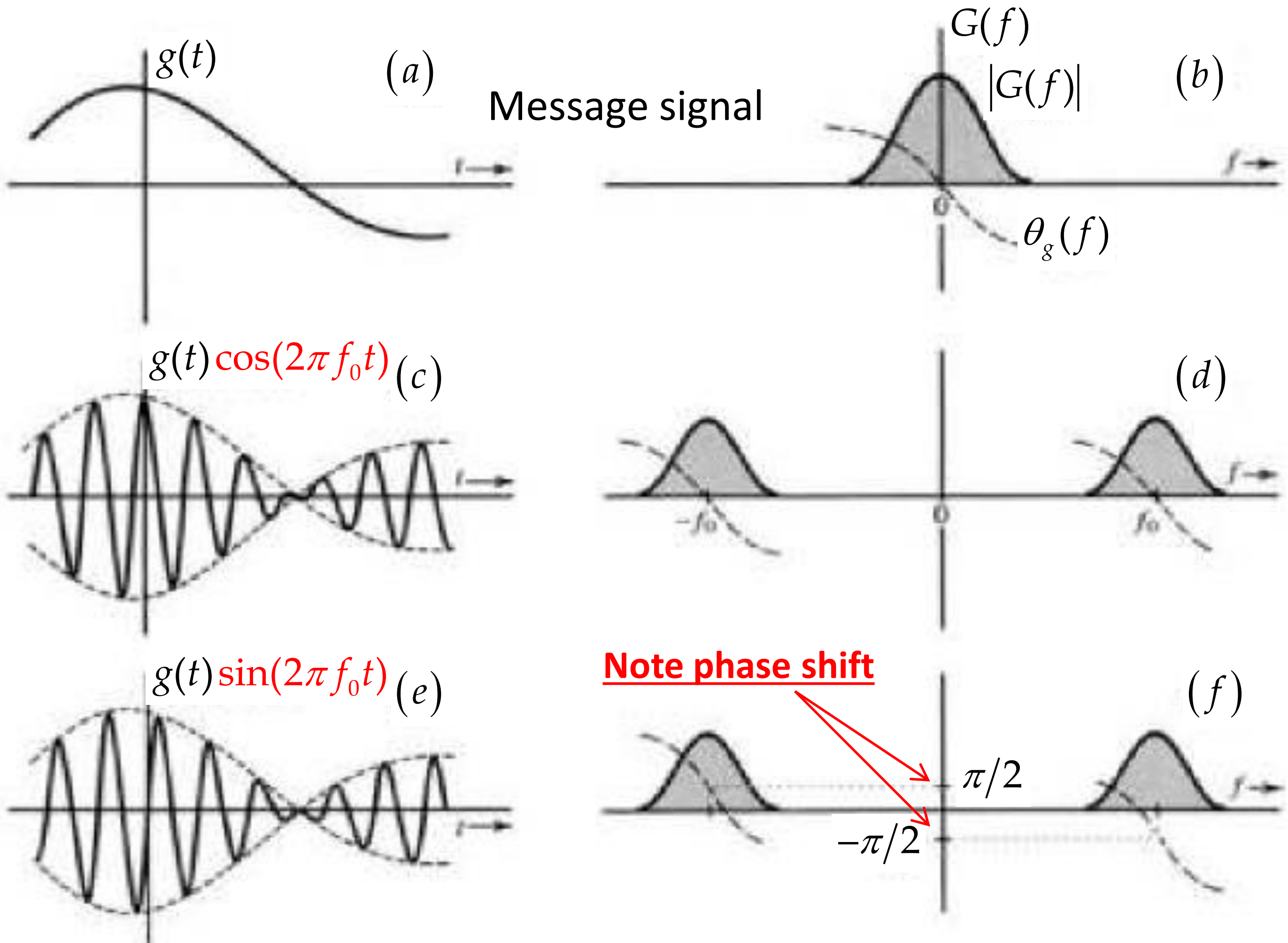


$$g(t) \Leftrightarrow G(f)$$





Frequency-Shifting Property (continued)



Multiplication of a signal *g(t)* by the factor $\cos(2\pi f_0 t)$ **places *G(f)* centered at $f = \pm f_C$.**

Modulation Comes From Frequency Shifting Property

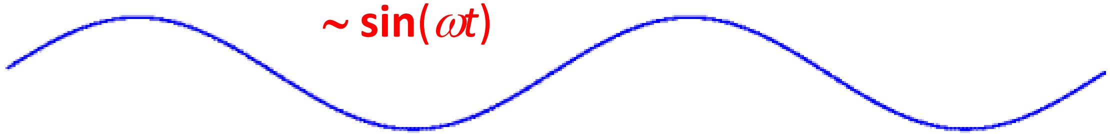
$$\text{Given FT pair:} \quad f(t) \Leftrightarrow F(\omega)$$

$$\text{then,} \quad f(t)e^{j\omega_0 t} \Leftrightarrow F(\omega - \omega_0)$$

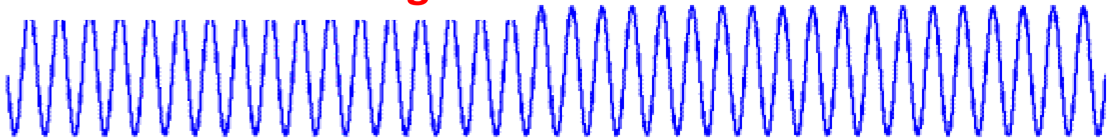
Amplitude Modulation Example:

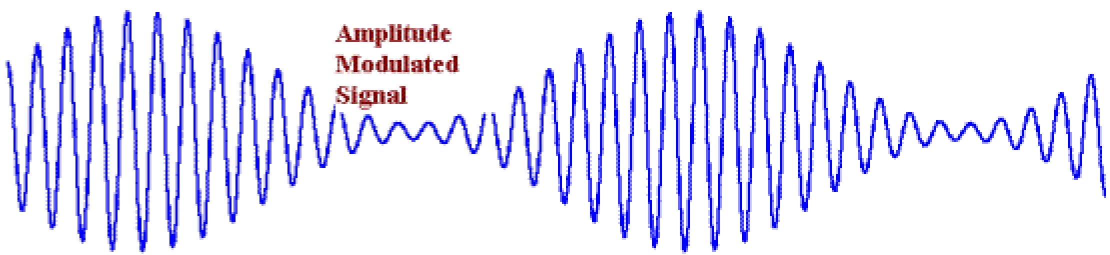
Audio tone:

~ sin(ωt)



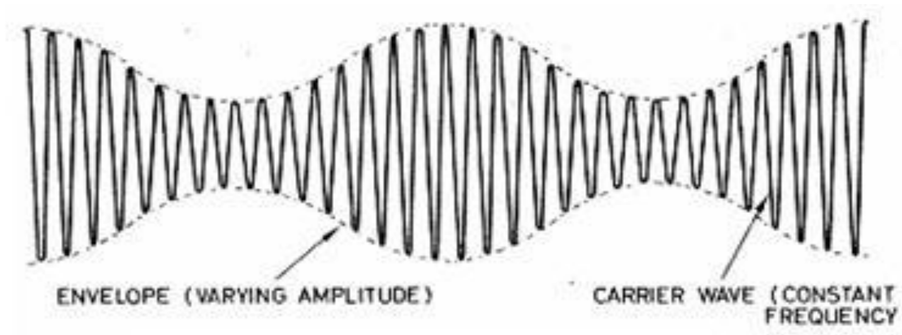
Sinusoidal carrier signal:





Fourier Transform of AM Tone Modulated Signal

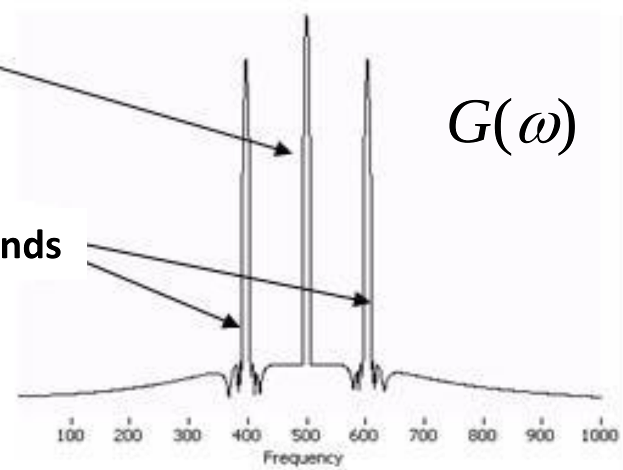
$f(t)$



Carrier signal

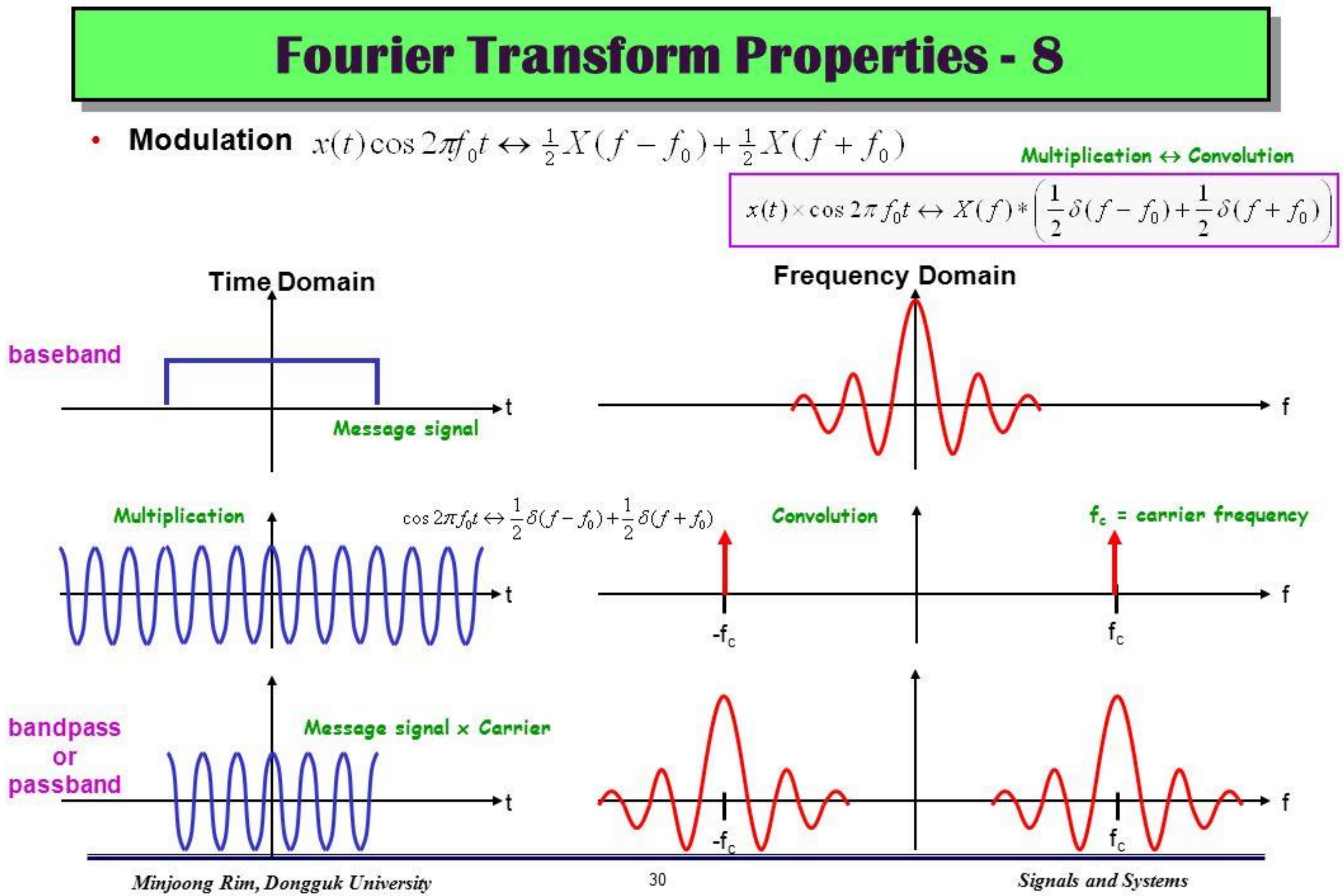
$G(\omega)$

Modulated AM sidebands



Only positive frequencies shown;
Must include negative frequencies.

Modulation of Baseband and Carrier Signals



Transform Duality Property

$$g(t) \Leftrightarrow G(f)$$

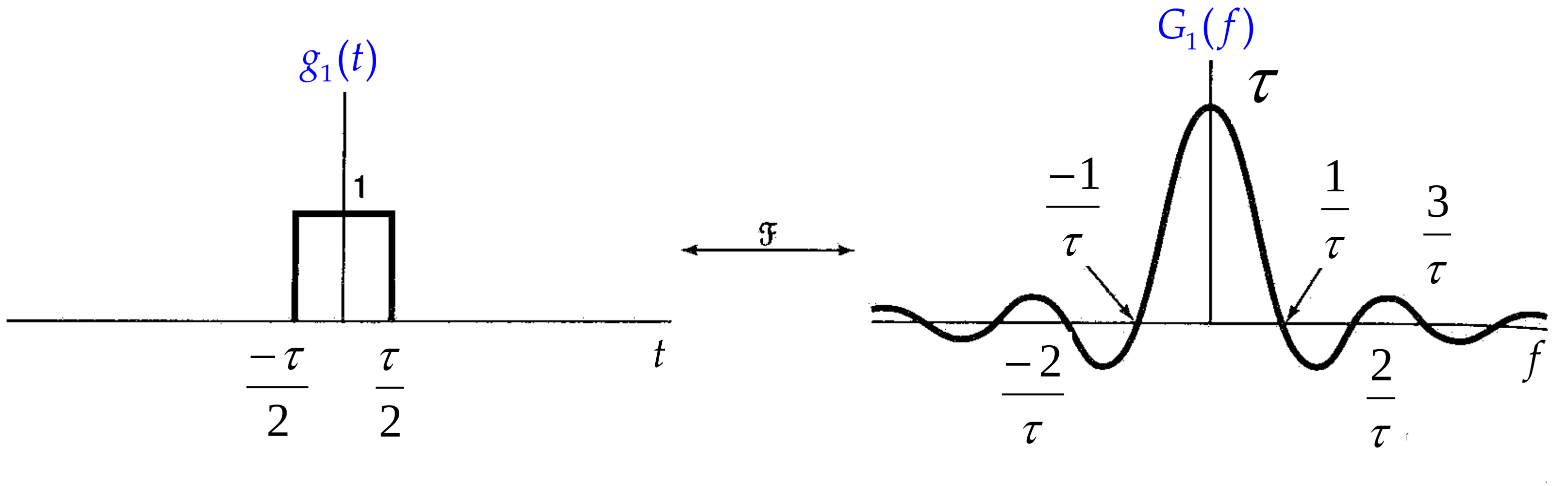
and

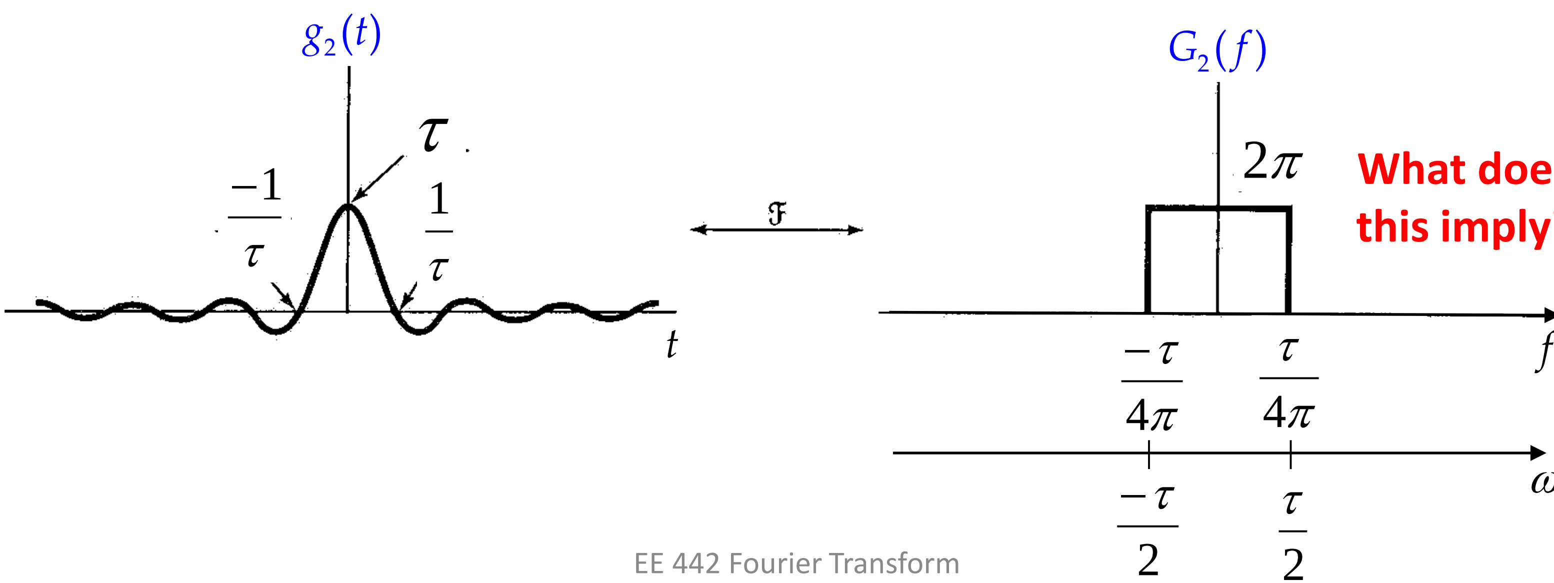
$$G(t) \Leftrightarrow g(-f)$$

Note the minus sign!

Because of the minus sign they are not perfectly symmetrical – See the illustration on next slide.

Illustration of Fourier Transform Duality





What does this imply?

Fourier Transform of Complex Exponentials

$$F^{-1}\left[\delta(f-f_c)\right]=\int_{-\infty}^{\infty}\delta(f-f_C)e^{-j2\pi f t}df$$

Evaluate for $f = f_c$

$$F^{-1}\left[\delta(f-f_c)\right]=\int_{f=f_c} e^{-j2\pi f_c t}df = e^{-j2\pi f_c t}$$

$$\therefore \boxed{\delta(f-f_c)\Leftrightarrow e^{-j2\pi f_c t}} \quad and$$

$$F^{-1}\left[\delta(f+f_c)\right]=\int_{-\infty}^{\infty}\delta(f+f_C)e^{-j2\pi f t}df$$

Evaluate for $f = -f_c$

$$F^{-1}\left[\delta(f+f_c)\right]=\int_{f=-f_c} e^{j2\pi f_c t}df = e^{j2\pi f_c t}$$

$$\therefore \boxed{\delta(f+f_c)\Leftrightarrow e^{j2\pi f_c t}}$$

Fourier Transform of Sinusoidal Functions

Taking $\delta(f - f_c) \Leftrightarrow e^{-j2\pi f_c t}$ and $\delta(f + f_c) \Leftrightarrow e^{j2\pi f_c t}$

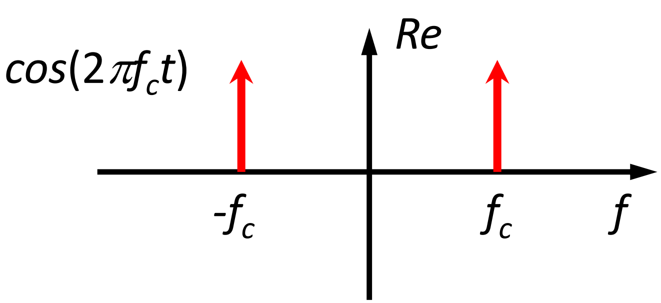
We use these results to find FT of $\cos(2\pi ft)$ and $\sin(2\pi ft)$

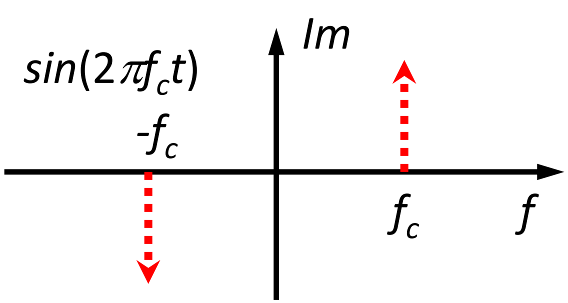
Using the identities for $\cos(2\pi ft)$ and $\sin(2\pi ft)$,

* $\cos(2\pi ft) = \frac{1}{2}\left[e^{j2\pi f_c t} + e^{-j2\pi f_c t}\right]$ & $\sin(2\pi ft) = \frac{1}{2j}\left[e^{j2\pi f_c t} - e^{-j2\pi f_c t}\right]$

Therefore,

$$\boxed{\begin{aligned} \cos(2\pi ft) &\Leftrightarrow \tfrac{1}{2}[\delta(f + f_c) + \delta(f - f_c)], \\ \sin(2\pi ft) &\Leftrightarrow \tfrac{1}{2j}[\delta(f + f_c) - \delta(f - f_c)] \end{aligned}}$$ and





Summary of Several Fourier Transform Pairs

Table 3.1

Short Table of Fourier Transforms

	$g(t)$	$G(\omega)$	
1	$e^{-at}u(t)$	$\frac{1}{a+j\omega}$	$a>0$
2	$e^{at}u(-t)$	$\frac{1}{a-j\omega}$	$a>0$
3	$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$	$a>0$
4	$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$	$a>0$
5	$t^n e^{-at}u(t)$	$\frac{n!}{(a+j\omega)^{n+1}}$	$a>0$
6	$\delta(t)$	1	
7	1	$2\pi\delta(\omega)$	
8	$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	
9	$\cos\ \omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$	
10	$\sin\ \omega_0 t$	$j\pi[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$	
11	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$	
12	$\operatorname{sgn} t$	$\frac{2}{j\omega}$	
13	$\cos\omega_0 t\ u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$	
14	$\sin\omega_0 t\ u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$	
15	$e^{-at}\sin\omega_0 t\ u(t)$	$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$	$a>0$
16	$e^{-at}\cos\omega_0 t\ u(t)$	$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$	$a>0$
17	$\operatorname{rect}\left(\frac{t}{\tau}\right)$	$\tau\ \operatorname{sinc}\left(\frac{\omega\tau}{2}\right)$	
18	$\frac{W}{\pi}\operatorname{sinc}(Wt)$	$\operatorname{rect}\left(\frac{\omega}{2W}\right)$	
19	$\Delta\left(\frac{t}{\tau}\right)$	$\frac{\tau}{2}\operatorname{sinc}^2\left(\frac{\omega\tau}{4}\right)$	
20	$\frac{W}{2\pi}\operatorname{sinc}^2\left(\frac{Wt}{2}\right)$	$\Delta\left(\frac{\omega}{2W}\right)$	
21	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$	$\omega_0=\frac{2\pi}{T}$
22	$e^{-t^2/2\sigma^2}$	$\sigma\sqrt{2\pi}e^{-\sigma^2\omega^2/2}$	

See Agbo & Sidiku; Table 2.5, Page 54;

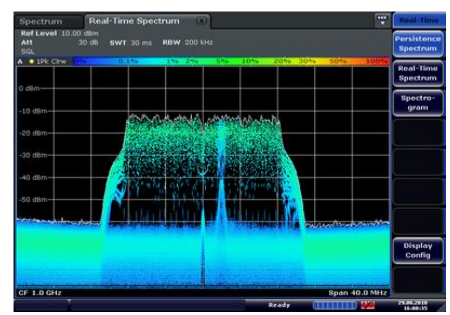
See also the Fourier Transform Pair Handout

http://media.cheggcdn.com/media/db0/db0ffe9-45f5-40a3-a05b-12a179139400/phpsu63he.png

Spectrum Analyzer Shows Frequency Domain

A **spectrum analyzer** measures the magnitude of an input signal versus frequency within the full frequency range of the instrument. It measures frequency, power, harmonics, distortion, noise, spurious signals and bandwidth.

- It is an electronic receiver
- Measure magnitude of signals
- Does not measure phase of signals
- Complements time domain

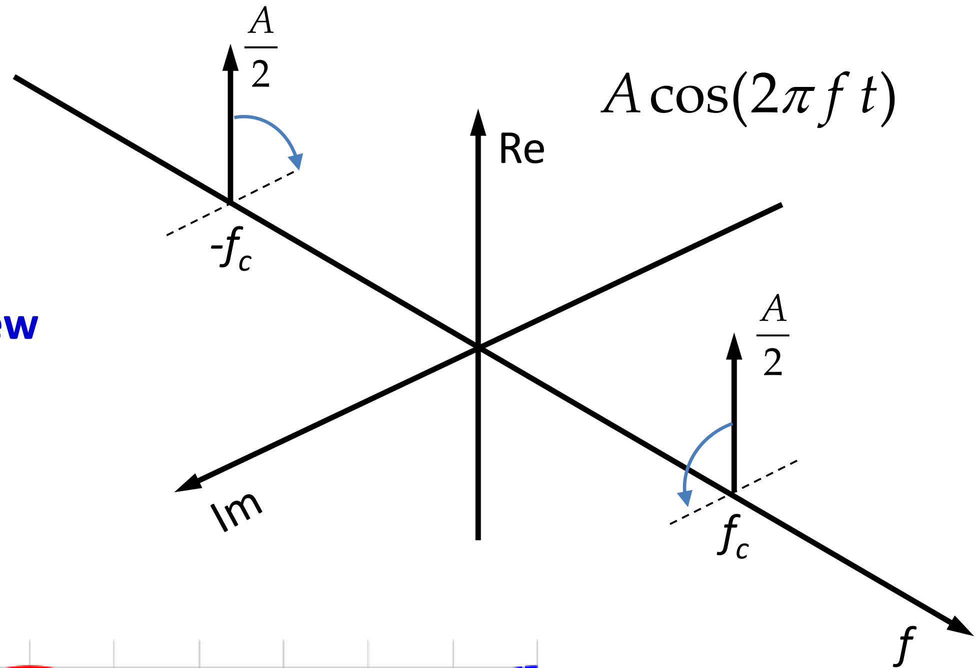


Bluetooth spectrum

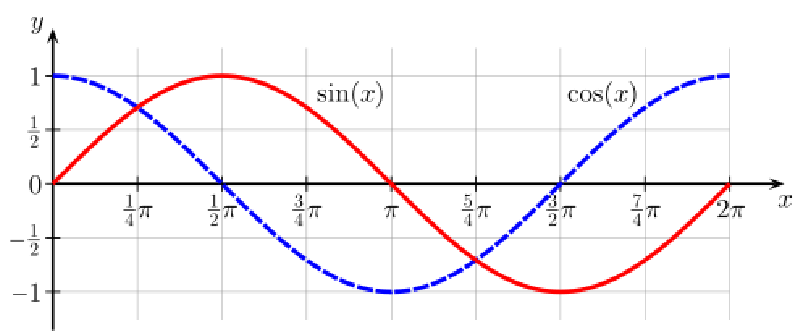
Fourier Transform of Cosine Signal

$$A\cos(2\pi f_c t) = \frac{A}{2}\left[\delta(f + f_c) + \delta(f - f_c)\right]$$

3D View

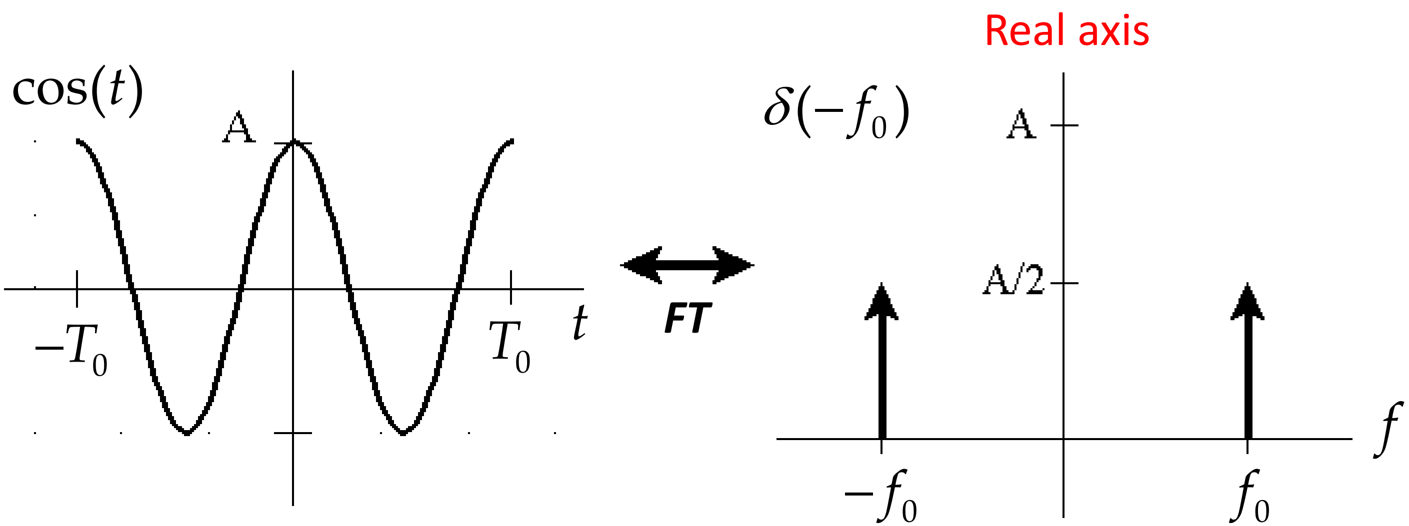


Not in
Lathi & Ding



Blue arrows indicate
positive phase directions

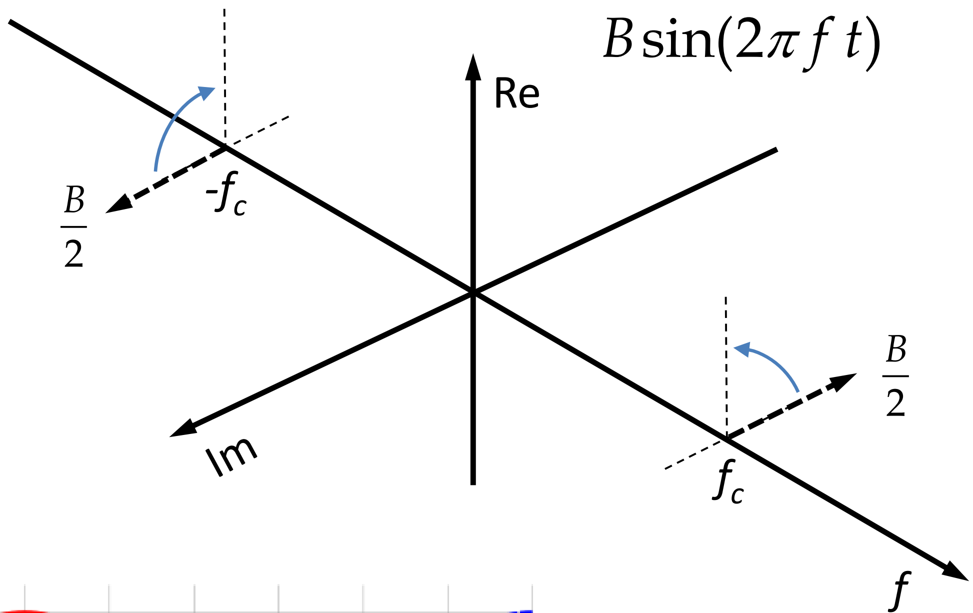
Fourier Transform of Cosine Signal (as shown in textbooks)

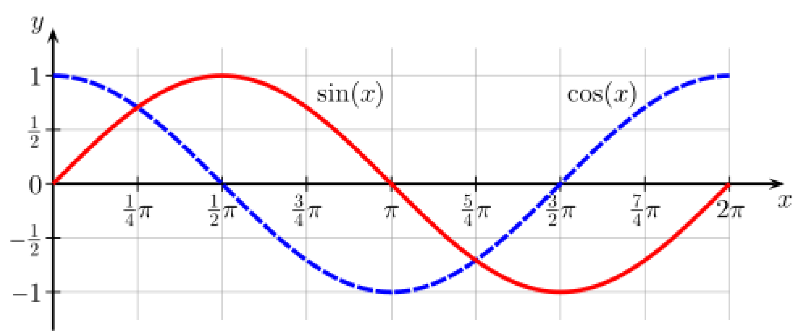


$$f_0 = \frac{1}{T_0}$$

Fourier Transform of Sine Signal

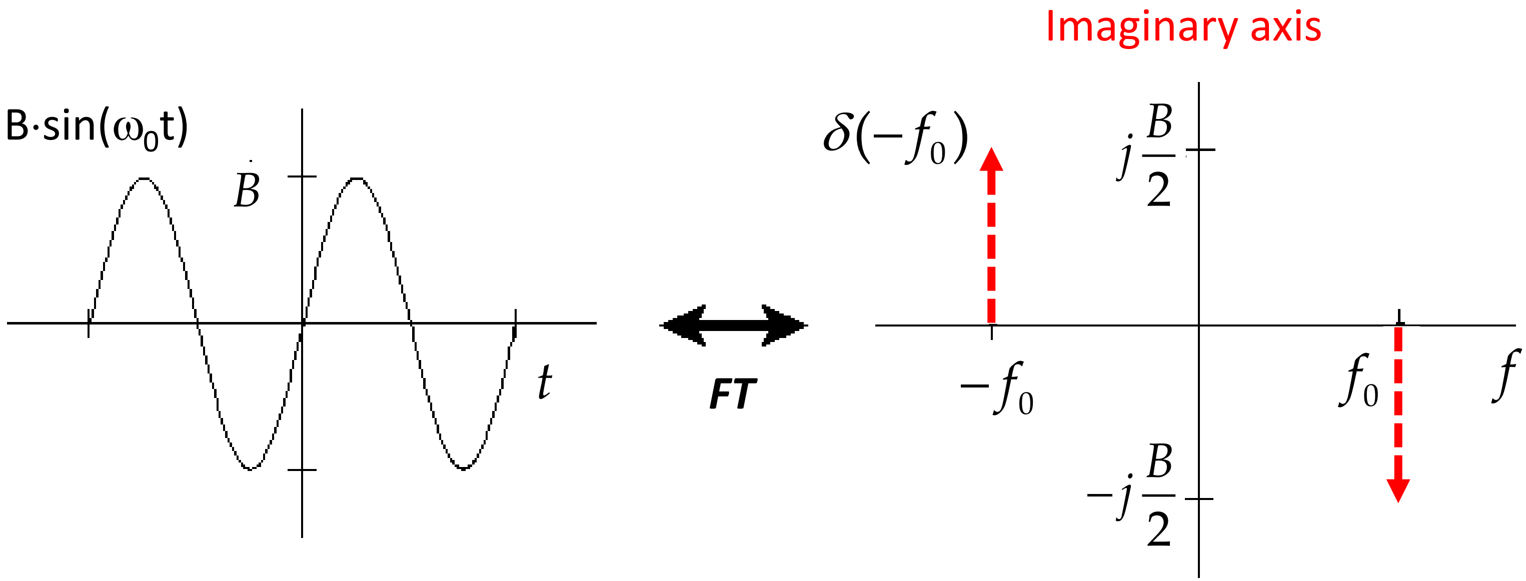
$$B\sin(2\pi f_c t) = j\frac{B}{2}\left[\delta(f+f_c) - \delta(f-f_c)\right]$$



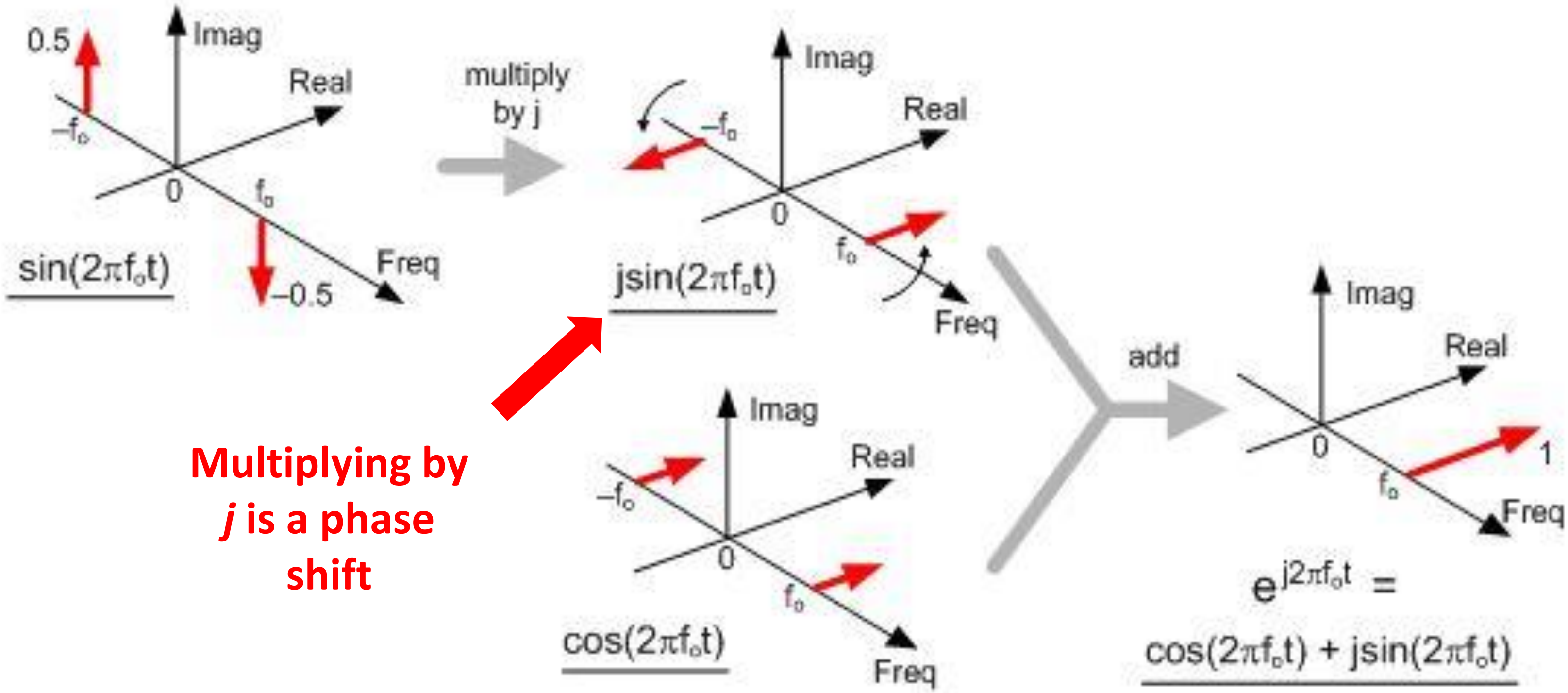


We must **subtract** 90°
from cos(x) to get sin(x)

Fourier Transform of Sine Signal (as usually shown in textbooks)



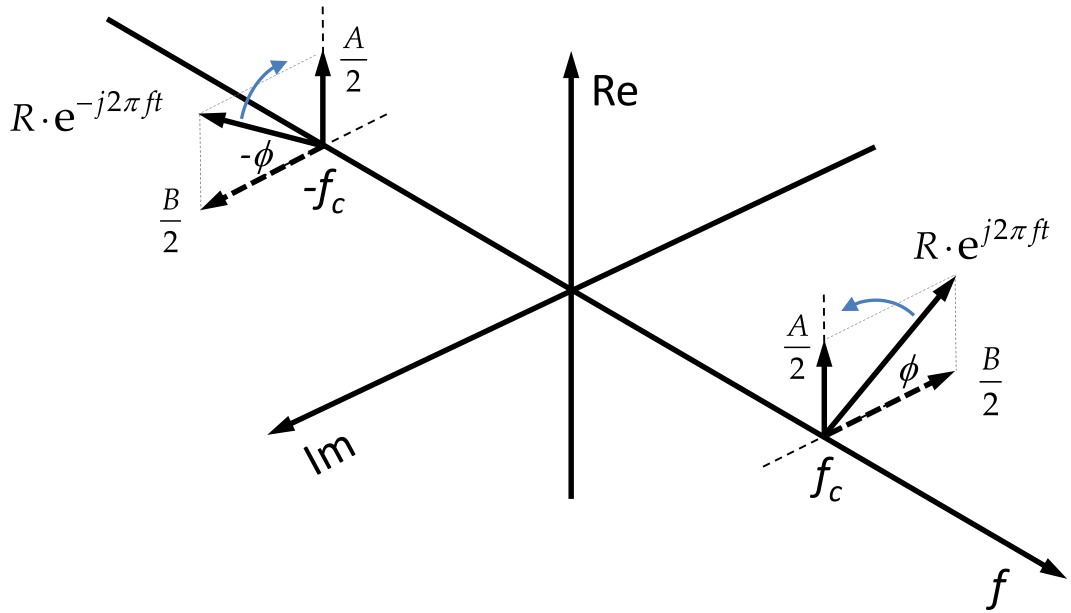
Visualizing Fourier Spectrum of Sinusoidal Signals



Multiplying by *j* is a phase shift

Fourier Transform of a Phase Shifted Sinusoidal Signal
(with phase information shown)

$$R \cdot e^{j\phi}\, e^{j2\pi ft} + R \cdot e^{-j\phi}\, e^{-j2\pi ft}$$



$$R = \sqrt{\left(\frac{A}{2}\right)^2 + \left(\frac{B}{2}\right)^2} \quad \textit{and} \quad \phi = \tan^{-1}\left(-\frac{A}{B}\right)$$

Selected References

1. Paul J. Nahin, **The Science of Radio**, 2nd edition, Springer, New York, 2001. A novel presentation of radio and the engineering behind it; it has some selected historical discussions that are very insightful.

2. Keysight Technologies, Application Note 243, **The Fundamentals of Signal Analysis**; http://literature.cdn.keysight.com/litweb/pdf/5952-8898E.pdf?id=1000000205:epsg:apn

3. Agilent Technologies, Application Note 150, **Spectrum Analyzer Basics**; http://cp.literature.agilent.com/litweb/pdf/5952-0292.pdf

4. Ronald Bracewell, **The Fourier Transform and Its Applications**, 3rd ed., McGraw-Hill Book Company, New York, 1999. I think this is the best book covering the Fourier Transform (Bracewell gives many insightful views and discussions on the ***FT*** and it is considered a classic textbook).

Auxiliary Slides For Introducing Sampling

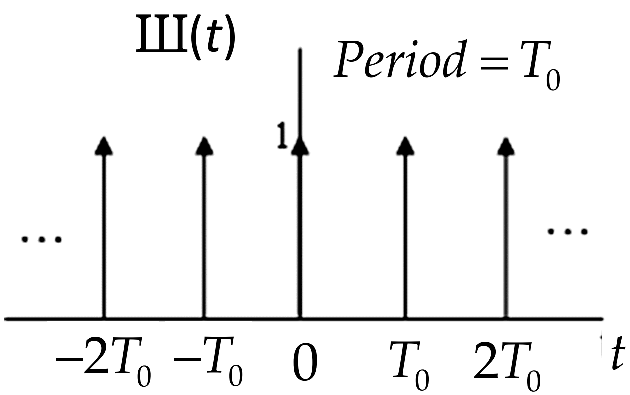
Fourier Transform of Impulse Train $\delta(t)$ (Shah Function)

aka "Dirac Comb Function," Shah Function & "Sampling Function"

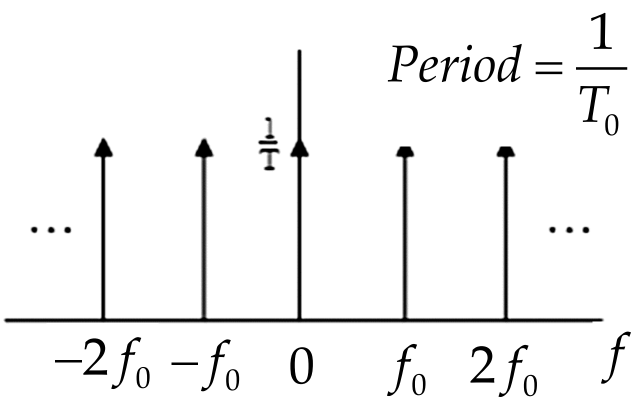
Shah function ($\text{Ш}(t)$)**:**

$$\text{Ш}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0) = \sum_{n=-\infty}^{\infty} \delta(t + nT_0)$$

$$\int_{n-\frac{1}{2}}^{n+\frac{1}{2}} \text{Ш}(t)\, dt = 1$$



(a)



(b)

Shah Function (Impulse Train) Applications

The sampling property is given by

$$\text{III}(t)\, f(t) = \sum_{n=-\infty}^{\infty} f(n)\delta(t - nT_0)$$

The "replicating property" is given by the convolution operation:

$$\text{III}(t) \star f(t) = \sum_{n=-\infty}^{\infty} f(t - nT_0)$$

Convolution

Convolution theorem:

$$g_1(t) \star g_2(t) \quad \Leftrightarrow \quad G_1(f)G_2(f)$$ and

$$g_1(t)g_2(t) \quad \Leftrightarrow \quad G_1(f) \star G_2(f)$$

Sampling Function in Operation

$$Ш(t)\,f(t)=\sum_{n=-\infty}^{\infty} f(n)\delta(t-nT_0)$$