2020101074

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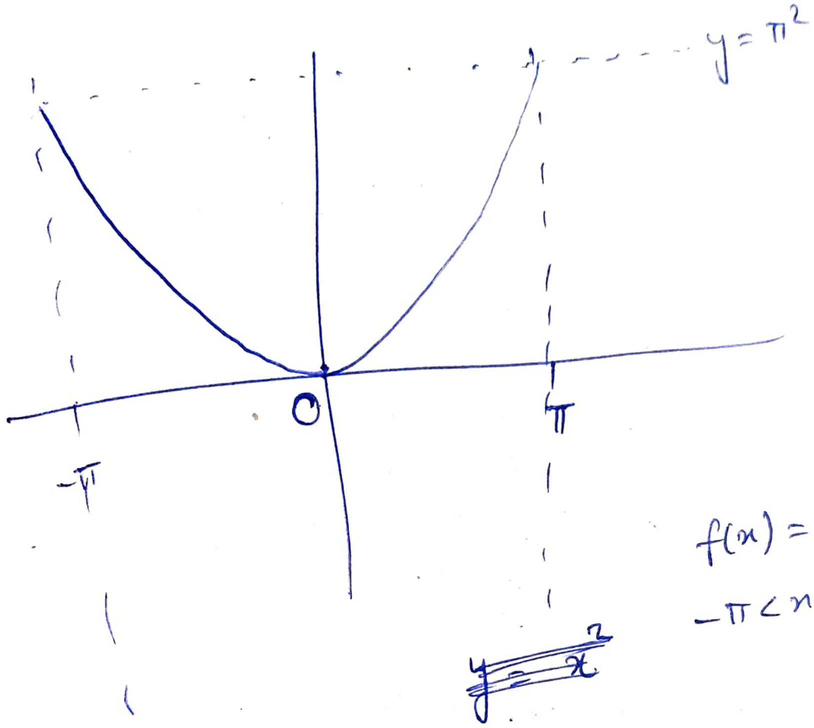
Digital Signal Analysis

Assignment-1

Fourier Series

Q1



$f(x) = x^2$

$-\pi < x < \pi$

$$T = 2\pi, \quad \therefore \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$$

$\therefore$ fourier series for $f(x) = x^2$ is :

$$f(x) = x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \int_{-\pi}^{\pi} \frac{1}{\pi} f(x)\, dx$$

$$a_n = \int_{-\pi}^{\pi} \frac{1}{\pi} f(x) \cos nx\, dx$$

$$b_n = \int_{-\pi}^{\pi} \frac{1}{\pi} f(x) \sin nx\, dx$$

$\therefore$, $$a_0 = \int_{-\pi}^{\pi} \frac{1}{\pi} x^2\, dx$$

$$= \left[\frac{x^3}{3\pi}\right]_{-\pi}^{\pi}$$

$$= \boxed{\frac{2\pi^2}{3}}$$

now, $$a_n = \int_{-\pi}^{\pi} \frac{1}{\pi} x^2 \cos nx\, dx$$

Using integration by parts,

$$\int uv\,dx = u\int v\,dx - \int u'\int v\,dx\,dx$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx = x^2\int_{-\pi}^{\pi} \cos nx\,dx - \int_{-\pi}^{\pi} 2x\,\frac{\sin nx}{n}\,dx$$

$$= \left[x^2 \frac{\sin(nx)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \frac{\sin nx}{n}\, dx$$

$$= 0 - 2x \int_{-\pi}^{\pi} \frac{\sin nx}{n}\, dx$$

$$+ \int_{-\pi}^{\pi} \frac{2}{n} \cdot \frac{(-\cos nx)}{n}\, dx$$

$$= \left(-\frac{2x}{n} \frac{(-\cos nx)}{n}\right)_{-\pi}^{\pi} + \frac{2}{n^2} \int_{-\pi}^{\pi} -\cos nx\, dx$$

$$= \left[\frac{2x}{n^2} \cos nx - \frac{2}{n^2} \frac{\sin nx}{n}\right]_{-\pi}^{\pi}$$

$$= \left(\frac{2x}{n^2} \cos nx - \frac{2}{n^3} \sin nx\right]_{-\pi}^{\pi}$$

$$= \frac{4 \cos(n\pi)}{n^2} = \boxed{\frac{4 \cdot (-1)^n}{n^2}}$$

now, $b_n = \int_{-\pi}^{\pi} x^2 \sin(nx)\,dx$

we know that,

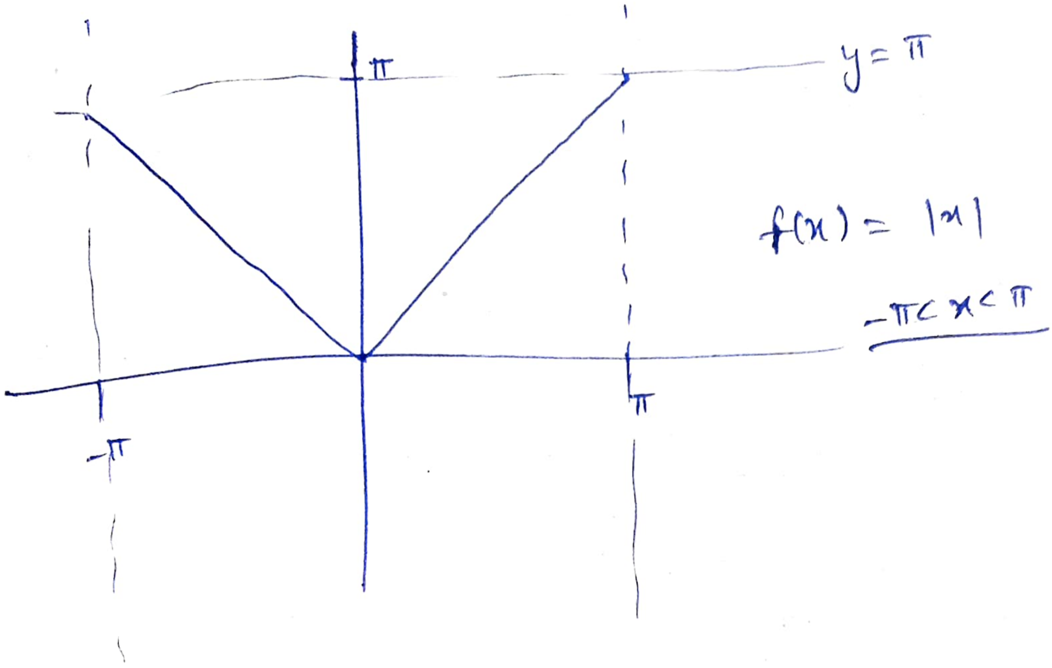
$\int_{-k}^{k} f(x)\,dx = 0$ if,

$f(2k-x) = f(x)$ i.e $f(x)$ is an odd function

$\therefore\ b_n = 0$ as $\boxed{\int_{0}^{\pi} x^2 \sin nx\,dx = -\int_{-\pi}^{0} x^2 \sin nx\,dx}$

$$\therefore\ \boxed{f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4.(-1)^n \cos nx}{n^2}}$$

Q1 (ii) $f(x) = |x| \;, \; -\pi < x < \pi$



again $\omega_0 = 1$ as $T = 2\pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx$$

as $|x| = |-x|$

$$\therefore \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_{0}^{\pi} |x|\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{\left(x^2\right)_0^{\pi}}{\pi} = \frac{\pi^2}{\pi} = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(nx)\, dx$$

again, here $f(x) = f(-x)$

as $|x| \cos nx = |-x| \cos(-nx)$

$$\therefore a_n = \frac{2}{\pi}\int_{0}^{\pi} x \cos nx\, dx$$

applying Integration by parts,

$$a_n = \frac{2}{\pi} x \int_{0}^{\pi} \cos nx\, dx - \frac{2}{\pi}\int_{0}^{\pi} 1 \cdot \frac{\sin nx}{n}$$

$$= \left[\frac{2}{\pi} x \frac{\sin nx}{n} + \frac{2}{\pi} \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}\left(\cos n\pi - 1\right)$$

now $b_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} |x| \sin(nx)\, dx$

again applying properties of definite integral,

$\displaystyle\int_{-k}^{k} f(x)\, dx = 0$ if $f(2k - x) = f(x)$

here $|x|$ is even but $\sin nx$ is odd over $-\pi$ to π

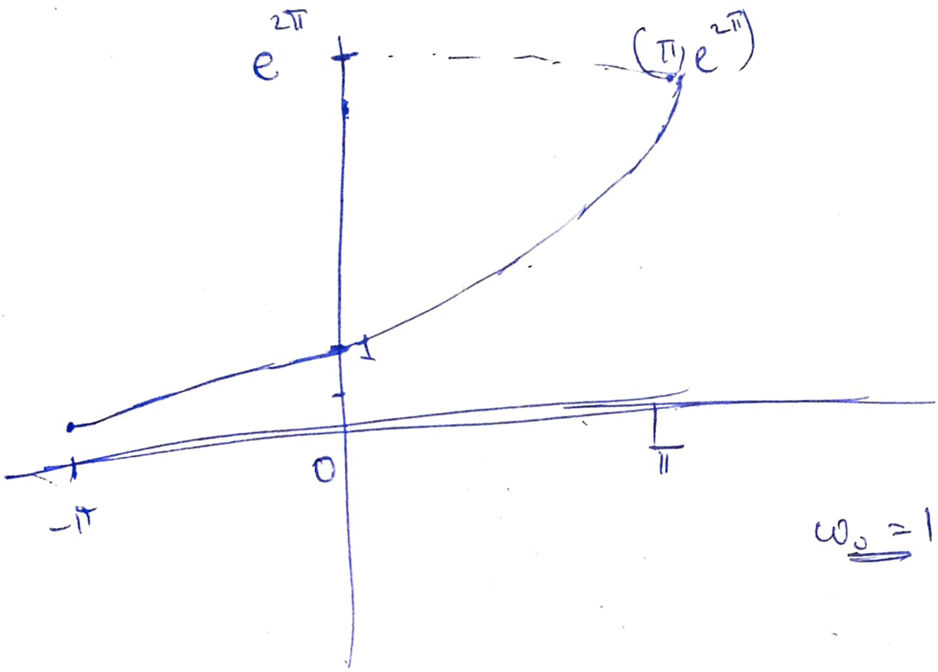
hence $b_n = 0$

$\therefore f(x) = |x| = \dfrac{\pi}{2} + \displaystyle\sum_{n=1}^{\infty} \dfrac{-2}{\pi n^2} (\cos n\pi - 1) \cos nx$

$$= \boxed{\frac{\pi}{2} - \frac{4}{\pi n^2} \sum_{n=1}^{\infty} \sin^2 n\pi/2 \; \cos nx}$$

Ans

Q1 (iii) $f(x) = e^{2x}$, $-\pi < x < \pi$



$\omega_0 = 1$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{2x}\, dx$$

$$= \left[\frac{1}{\pi} \frac{e^{2x}}{2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi} \left(e^{2\pi} - e^{-2\pi} \right)$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{2x} \cos(nx)\, dx$$

Using integration by parts,

$$\frac{1}{\pi}\left(\frac{e^{2x}\cos nx}{2} - \int -\frac{ne^{2x}\sin nx\,dx}{2}\right)$$

again using Integration by parts,

$$\frac{1}{\pi}\left(\frac{e^{2x}\cos(nx)}{2} - \left(-\frac{ne^{2x}\sin(nx)}{4} - \int -\frac{n^2e^{2x}\cos(nx)}{4}dx\right)\right)$$

now $\frac{1}{\pi}\int e^{2x}\cos nx\,dx = a_n$ itself

$\therefore$ on rearranging,

$$a_n = \frac{1}{\pi}\left(\frac{ne^{2x}\sin nx + 2e^{2x}\cos nx}{n^2+4}\right)_{-\pi}^{\pi}$$

$$= \frac{e^{2x}}{\pi(n^2+4)}\left[2\cos nx + n\sin nx\right]_{-\pi}^{\pi}$$

$$= \frac{2\cos n\pi}{(4+n^2)\pi}\left(e^{2\pi} - e^{-2\pi}\right)$$

$$b_n = \frac{1}{\pi}\int e^{2x}\sin(nx)\,dx$$

doing all steps similarly as a_n,

we get,

$$b_n = \left[\frac{2e^{2x}\sin(nx) - ne^{2x}\cos(nx)}{n^2+4}\right]_{-\pi}^{\pi}$$

$$= \frac{e^{2x}}{\pi(n^2+4)}\Big[2\sin nx - n\cos nx\Big]_{-\pi}^{\pi}$$

$$= \frac{-n\cos n\pi}{\pi(n^2+4)}\left(e^{2\pi} - e^{-2\pi}\right)$$

$$f(x) = \frac{1}{4\pi}\left(e^{2\pi} - e^{-2\pi}\right)$$

$$+ \sum_{n=0}^{\infty} \frac{2\cos n\pi}{(4+n^2)\pi}\left(e^{2\pi} - e^{-2\pi}\right) \cos nx$$

$$+ \sum_{n=0}^{\infty} \frac{-n\cos n\pi}{\pi(n^2+4)}\left(e^{2\pi} - e^{-2\pi}\right) \sin nx$$

Q2 odd / even / neither

(1) $f(x) = \begin{cases} 2 & 0 < x < 3 \\ -2 & -3 < x < 0 \end{cases}$, period = 6

as we can see that $f(x) = -f(-x)$

in ~~[illegible]~~ $x \in (-3, 0) \cup (0, 3)$

hence this is an odd function.

(2) $f(x) = \begin{cases} \cos x & 0 < x < \pi \\ 0 & \pi < x < 2\pi \end{cases}$

period $= 2\pi$

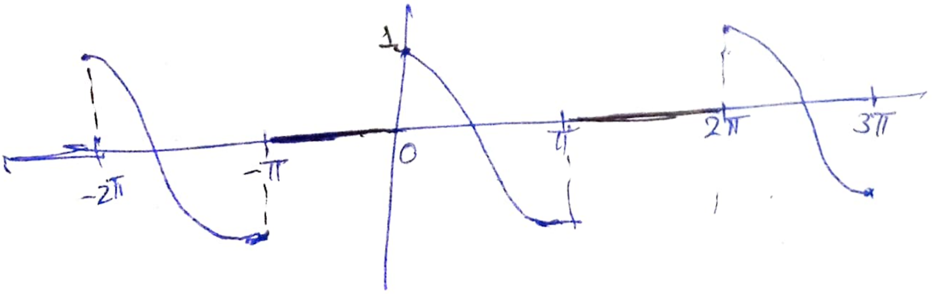
at $x = \pi/4$ $f(x) = 1/\sqrt{2}$

~~$x = -\pi/4$ $f(x)$~~

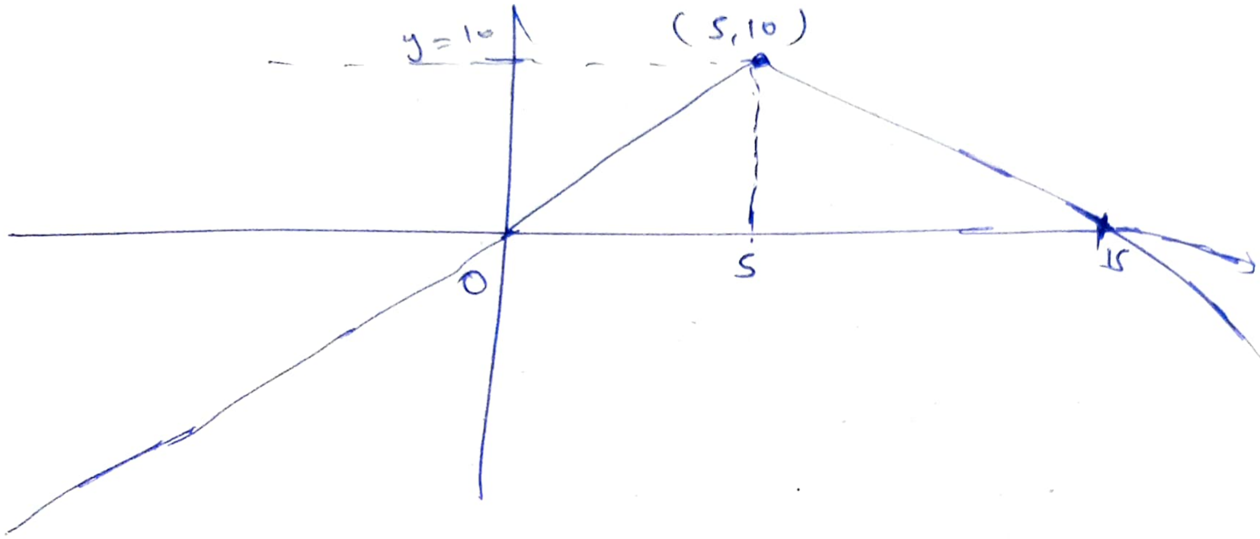
whereas at $x = -\pi/4$ $f(x) = 0$

$\therefore$ $f(x) \neq f(-x)$ & $f(x) \neq -f(-x)$

hence neither even nor odd



③ $f(x) = \begin{cases} 2x & x < 5 \\ 15 - x & x \geq 5 \end{cases}$



f(x) is neither odd nor even as,

say, $f(6) = 9$

$f(-6) = -12$

NOTE FOR Q2

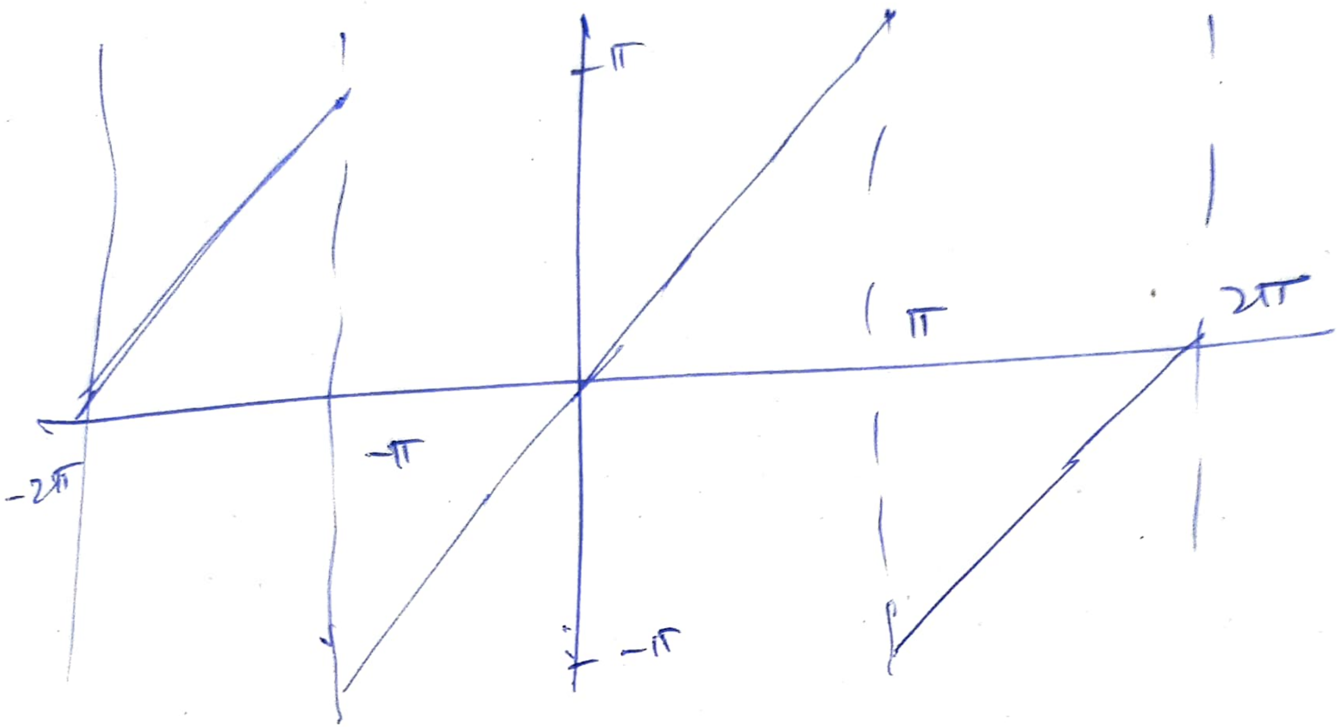
Basically used geometric visualisations or the concept that if,

any function is odd → $f(x) = f(-x)$ in domain

any function is even → $f(x) = f(-x)$ in domain

$\boxed{Q3}$

(i) Plot $f(x)$



(ii) Fourier expression given by :

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \int_{-\pi}^{\pi} \frac{1}{\pi} f(x)\, dx$$

$$a_0 = \int_{-\pi}^{\pi} \frac{1}{\pi} x\, dx = \left[\frac{x^2}{2\pi} \right]_{-\pi}^{\pi}$$

$$= 0$$

$$a_n = \frac{1}{\pi} \int x \cos(nx)\, dx$$

$$\frac{1}{\pi}\left[\frac{x \sin(nx)}{n} - \int \frac{\sin(nx)}{n}\, dx \right]_{-\pi}^{\pi}$$

$$\frac{1}{\pi}\left[\frac{x \sin(nx)}{n} + \frac{\cos nx}{n^2} \right]_{-\pi}^{\pi}$$

$$\cancel{\frac{1}{\pi} \frac{2\cos nx}{n^2}} = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dn$$

$$= \frac{1}{\pi}\left[x \int_{-\pi}^{\pi} \sin(nx)\, dn + \int_{-\pi}^{\pi} \frac{\cos(nx)}{n}\, dx \right]$$

$$= \frac{1}{\pi}\left[\frac{-x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-2\pi\cos(n\pi)}{n}\right]$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n \sin nx}{n}}$$

(iii) put $x \to \pi/2$

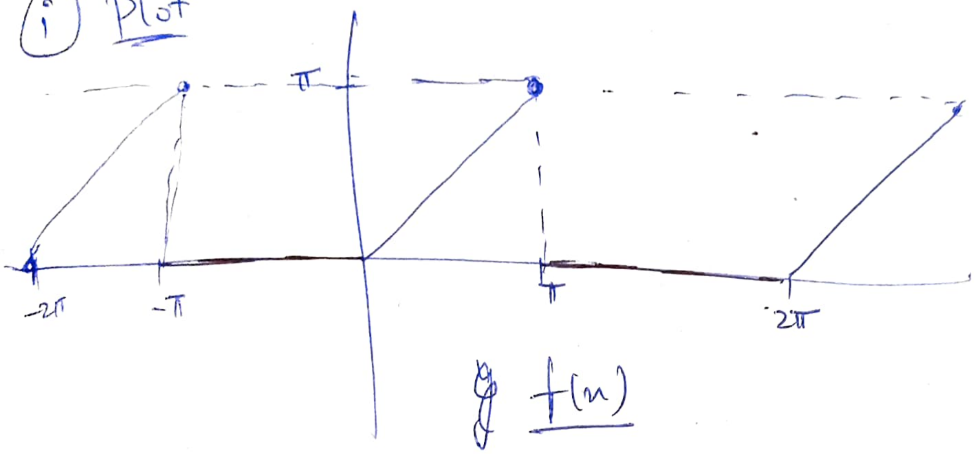
$$f(x) = x = \frac{\pi}{2} = 2 - \frac{2}{3} + \frac{2}{5} - \frac{2}{7}$$

$$\therefore \boxed{\frac{\pi}{4} = 1 - 1/3 + 1/5 - 1/7 \cdots}$$

Ans

Q4 $f(x) = \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$ 1 period $= 2\pi$

(i) Plot



of $f(x)$

(ii) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$

$$= \frac{1}{\pi}\left(\int_{-\pi}^{0} 0\,.dx + \int_{0}^{\pi} x\,.dx\right)$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2}\right] = \boxed{\pi/2}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0 \cdot \cos nx \, dx + \int_{0}^{\pi} x \cos nx \, dx \right]$$

$$= \frac{1}{\pi} \left[\int_{0}^{\pi} x \cos(nx) \cdot dx \right]$$

$$= \frac{1}{\pi} \left[\frac{x \sin(nx)}{n} - \int \frac{\sin(nx)}{n} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{x \sin(nx)}{n} + \frac{\cos(nx)}{n^2} \right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{\cos n\pi - 1}{n^2} \right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[0 + \int_{0}^{\pi} x \sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} + \int \frac{\cos(nx)}{n}\right]_{0}^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_{0}^{\pi}$$

$$= \frac{-1}{\pi}\left[\frac{\pi\cos(n\pi)}{n} - 0\right]$$

$$= \boxed{\frac{-\cos(n\pi)}{n}}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{\left((-1)^n - 1\right)}{\pi n^2}\cos(nx)$$

$$+ (-1)\sum_{n=1}^{\infty} \frac{(-1)^n \sin(nx)}{n}$$

(iii) $\frac{\pi^2}{8} = 1 + 1/3^2 + 1/5^2 + \dots$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{((-1)^n - 1)}{n^2 \pi} \cos(nx) +$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n} (-1)$$

put x as 0,

$$f(0) = \frac{\pi}{4} + \left[\frac{-2}{\pi} + 0 + \frac{-2}{9\pi} + \dots \right]$$

$$+ 0$$

$$0 = \frac{\pi}{4} - \frac{2}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Q5 $x(t) = \begin{cases} 1 & 1 \le |t| \le 3 \\ -1 & |t| < 1 \\ 0 & \text{otherwise} \end{cases}$

$$x(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt$$

$$= \int_{-\infty}^{-3} 0 \cdot e^{j\omega t}\, dt + \int_{-3}^{-1} 1 \cdot e^{j\omega t}\, dt +$$

$$\int_{-1}^{1} -1\, e^{j\omega t}\, dt + \int_{1}^{3} 1 \cdot e^{j\omega t}\, dt$$

$$+ \int_{3}^{\infty} 0 \cdot e^{j\omega t}\, dt$$

$$= \int_{-3}^{-1} e^{j\omega t}\, dt + \int_{-1}^{1} -e^{j\omega t}\, dt +$$

$$\int_{1}^{3} e^{j\omega t}\, dt$$

$$= \left.\frac{e^{j\omega t}}{j\omega}\right]_{-3}^{-1} + \left.\frac{-e^{j\omega t}}{j\omega}\right]_{-1}^{1} + \left.\frac{e^{j\omega t}}{j\omega}\right]_{1}^{3}$$

$$= \frac{1}{j\omega}\left[\left(e^{-j\omega} - e^{-3j\omega}\right) - \left(e^{j\omega} - e^{-j\omega}\right) + \left(e^{3j\omega} - e^{j\omega}\right)\right]$$

$$= \frac{-j}{\omega}\left[2\cdot\left(e^{-j\omega} - e^{j\omega}\right) + \left(e^{3j\omega} - e^{-3j\omega}\right)\right]$$

$$= \frac{-j}{\omega}\left(-4j\sin\omega + 2j\sin 3\omega\right)$$

$$= \frac{2\sin 3\omega - 4\sin\omega}{\omega}$$

Ans

Q6

~~(i) $x(t) = e^{-|a|t} u(t)$~~

(i) $$x(t) = \begin{cases} a & -T \le t \le T \\ 0, & \text{otherwise} \end{cases}$$

to find: magnitude & phase spectrum,

$$x(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{-T} 0 \cdot e^{-j\omega t}\, dt + \int_{-T}^{T} x(t) \cdot e^{-j\omega t}\, dt$$

$$+ \int_{T}^{\infty} 0 \cdot e^{-j\omega t}\, dt$$

$$= \int_{-T}^{T} a \cdot e^{-j\omega t}\, dt$$

$$= \frac{-a}{j\omega} \left[e^{-j\omega t} \right]_{-T}^{T}$$

$$= \frac{a}{j\omega} \left[e^{j\omega T} - e^{-j\omega T} \right]$$

~~$\frac{2a}{j\omega}\sin(j\omega T)$~~

$$= \frac{2a \sin(\omega T) \cancel{j}}{\cancel{j}\omega}$$

$$\therefore x(\omega) = 2a \sin \omega T$$

$$|x(\omega)| = \frac{2a}{\omega} \sin \omega T$$

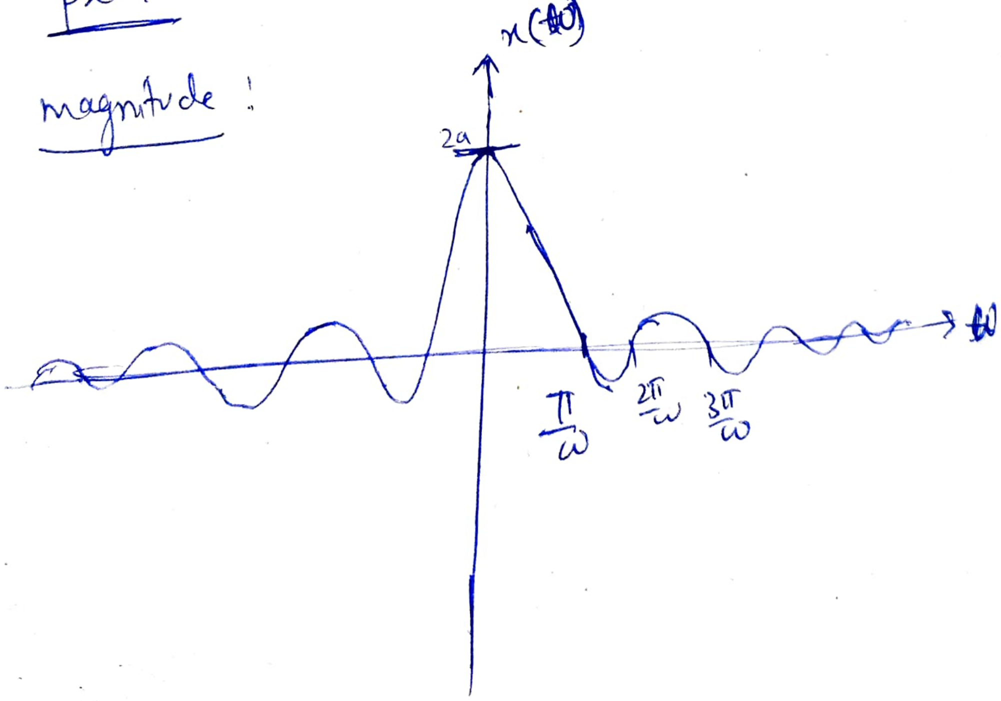
$$\text{phase spectrum} = \tan^{-1}\left(\frac{Im(x(\omega))}{Re(x(\omega))}\right)$$

$$= \tan^{-1}\left(\frac{0}{\frac{2a}{\omega}\sin(\omega t)}\right)$$

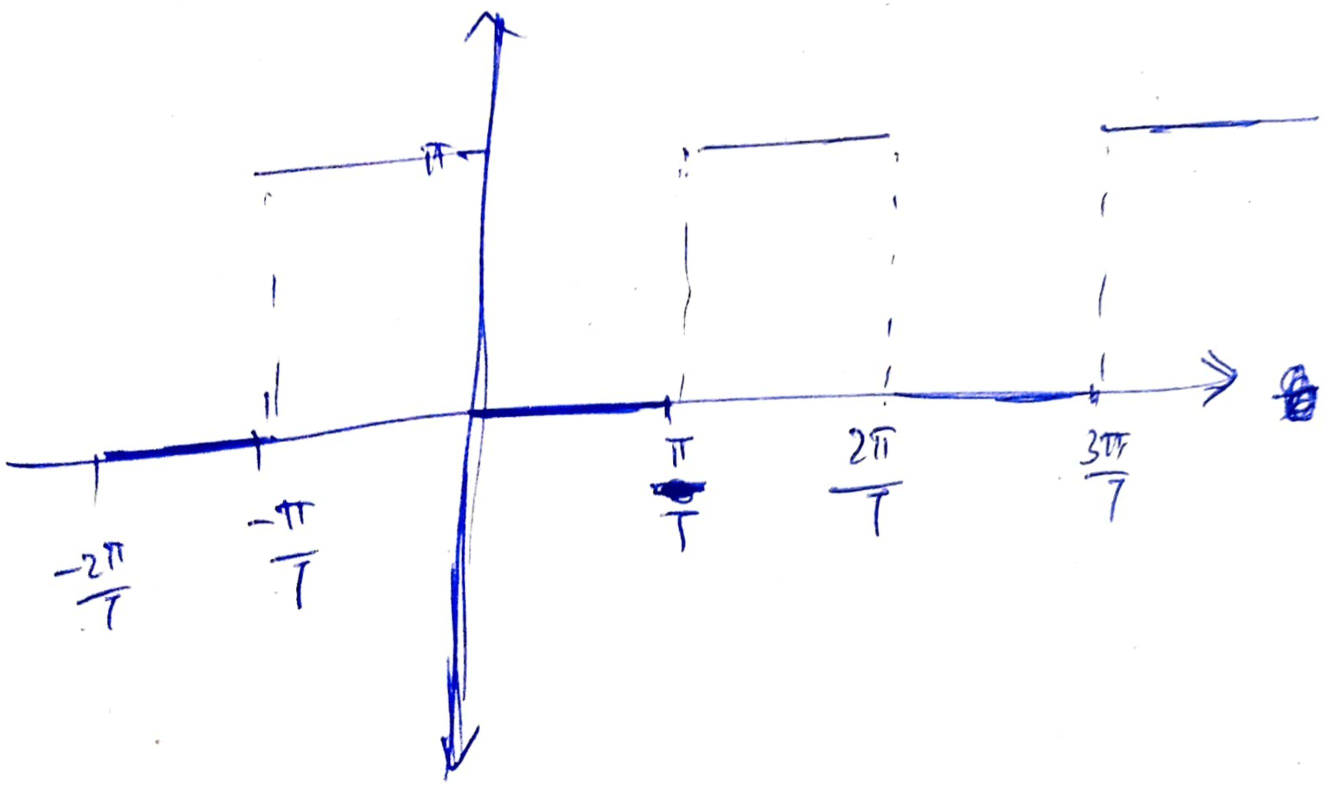
$$= 0$$

plot

magnitude :



plot of phase spectrum:



$$\phi(\omega) = 0, \quad \text{when } \frac{2n\pi}{T} < \omega < \frac{(2n+1)\pi}{T}$$

$$\phi(\omega) = \pi, \quad \text{when } \frac{(2n-1)\pi}{T} < \omega < \frac{2n\pi}{T}$$

Q6 (b) $x(t) = \delta(t-a)$

by use of the property:

$$x(t) \cdot \delta(t-t_0) = x(t_0) \cdot \delta(t-t_0)$$

thus, $$X(j\omega) = \int_{-\infty}^{\infty} \delta(t-a)\, e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} \delta(t-a)\, e^{-j\omega a}\, dt$$

$$= e^{-j\omega a} \int_{-\infty}^{\infty} \delta(t-a)\, dt$$

let $x = t-a$

$dx = dt$

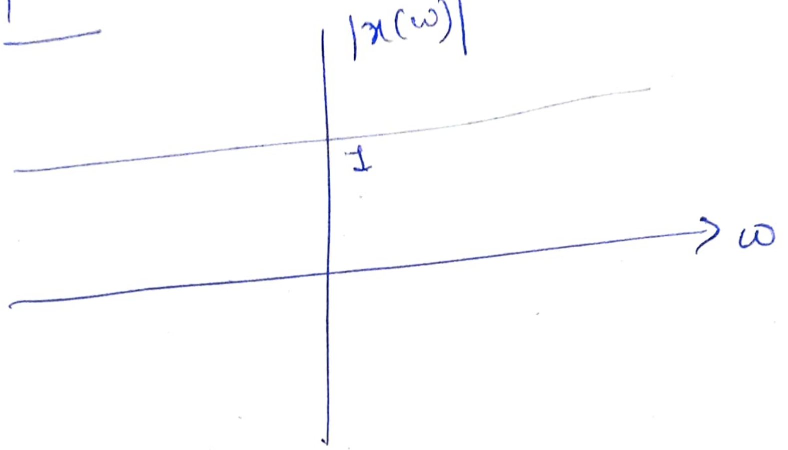
$$= e^{-j\omega a} \int_{-\infty}^{\infty} \delta(x)\, dx$$

$$= \boxed{e^{-j\omega a}}$$

$\therefore \; |x(\omega)| = |e^{-j\omega a}|$

$= 1 \; (\text{as } |e^{-j\theta}| = 1)$

plot :

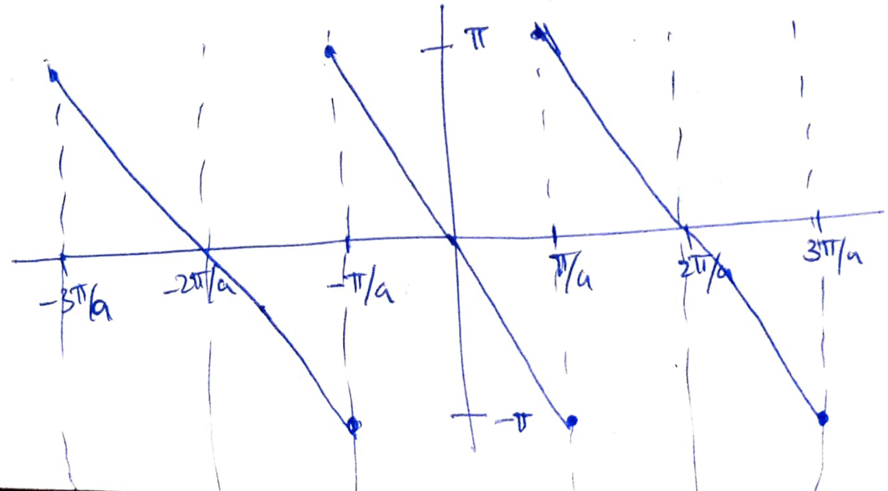


phase spectrum

$\phi(x(\omega)) = \tan^{-1}(\tan(\omega a))$

$= -\omega a$

phase spectrum :



Q7 (i) $x(t) = e^{-|a|t} \cdot u(t)$

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$x(\omega) = \int_{-\infty}^{0} e^{-|a|t} \cdot e^{-j\omega t} \cdot 0 \cdot dt$$

$$+ \int_{0}^{\infty} e^{-|a|t} \cdot e^{-j\omega t} \cdot 1 \cdot dt$$

$$= \int_{0}^{\infty} e^{-|a|t} \cdot e^{-j\omega t}$$

$$x(\omega) = \int_{0}^{\infty} e^{-(|a|+j\omega)t}$$

$$= \left. \frac{e^{-(|a|+j\omega)t}}{-(|a|+j\omega)} \right]_{0}^{\infty}$$

$$X(\omega) = \frac{1}{(|a|+j\omega)}$$

$\therefore$ (a) $|x(\omega)| = \left| \dfrac{1}{|a| + j\omega} \right|$

$= \dfrac{1}{||a| + j\omega|}$

$$\boxed{|x(\omega)| = \frac{1}{\sqrt{|a|^2 + \omega^2}}}$$

(b) $\angle x(\omega) = \tan^{-1}\left(\dfrac{(Im(x(\omega)))}{(Re(x(\omega)))}\right)$

now $x(\omega) = \dfrac{1}{|a| + j\omega}$

$= \dfrac{|a| - j\omega}{|a|^2 + \omega^2}$

$$\therefore \quad Im(x(\omega)) = \left(\frac{-\omega}{|a|^2 + \omega^2}\right)$$

$$Re(x(\omega)) = \frac{|a|}{|a|^2 + \omega^2}$$

$$\therefore \quad \angle x(\omega) = \tan^{-1}\left(\frac{-\omega}{|a|}\right)$$

(c) $$Re(x(\omega)) = \frac{|a|}{|a|^2 + \omega^2}$$

(d) $$Im(x(\omega)) = \frac{-\omega}{|a|^2 + \omega^2}$$

Q7 (ii) $x(t) = e^{(-1+2j)t} \cdot u(t)$

$$X(\omega) = \int_0^\infty e^{(2j-1)t} \cdot e^{-j\omega t}\, dt$$

$$= \int_0^\infty e^{(2j-1-j\omega)t}\, dt$$

$$= \left. \frac{e^{(2j-1-j\omega)t}}{2j-1-j\omega} \right|_0^\infty$$

$$= 0 - \frac{1}{2j-1-j\omega}$$

$$= \frac{1}{1+j\omega-2j}$$

$$= \frac{1}{1+j(\omega-2)}$$

$$|X(\omega)| = \frac{1}{\sqrt{1+(\omega-2)^2}}$$

$$x(\omega) = \frac{1}{1+j(\omega-2)}$$

$$= \frac{1-j(\omega-2)}{1+(\omega-2)^2}$$

$$\angle x(\omega) = \tan^{-1}\left(\frac{\frac{2-\omega}{1+(\omega-2)^2}}{\frac{1}{1+(\omega-2)^2}}\right)$$

$$= \tan^{-1}(2-\omega)$$

(c) $$Re\{x(\omega)\} = \frac{1}{1+(\omega-2)^2}$$

(d) $$Im(x(\omega)) = \frac{j(2-\omega)}{1+(\omega-2)^2}$$

Q8) (i) $x[n] = \left(\frac{1}{5}\right)^n u(n+1)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

as at $n < -1$, $u(n+1) = 0$

$$\therefore X(\omega) = \sum_{n=-1}^{\infty} \left(\frac{1}{5} e^{-j\omega}\right)^n$$

$$= 5e^{j\omega} + 1 + \frac{1}{5e^{j\omega}} + \frac{1}{25e^{2j\omega}} + \cdots$$

$$X(\omega) = \frac{25\, e^{2j\omega}}{5e^{j\omega} - 1}$$ Ans

(ii) $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$$= x[0] \cdot e^{-j\omega \cdot 0} + x[1] e^{-j\omega(1)} + x[2] e^{-j\omega(2)}$$

$$= \boxed{1 + e^{-j\omega} + e^{-2j\omega}}$$ Ans

=

(iii) $x[n] = \left(\frac{1}{2}\right)^{n+2} u(n)$

for $n<0 \quad x[n] = 0$

$n>0 \quad x[n] = 1$

$$x[n] = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+2} \cdot 1 \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n+2} e^{-j\omega n}$$

$$= \boxed{\frac{1/4}{1 - \frac{1}{2} \cdot e^{-j\omega}}}$$ Ans

(iv) $x[n] = (1/2)^n u(n-4)$

$$= \sum_{n=4}^{\infty} \left(\frac{1}{2}\right)^n \cdot 1 \cdot e^{-j\omega n}$$

$$= \frac{\left(1/2 \, e^{-j\omega}\right)^4}{1 - \frac{1}{2e^{j\omega}}}$$

Q9 8 point DFT

$$X_N[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi}{N}kn}$$

(i) $$X_8[k] = \sum_{n=0}^{7} \frac{\delta(n)}{4}\, e^{-j\frac{2\pi}{8}kn}$$

$$= \frac{\delta(0)}{4}\, e^{j\frac{\pi}{4}k(0)}$$

$$= \frac{1}{4}$$

$$X_8[k] = \left\{\tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{4}\right\}$$

(ii) $x[n] = \{1, -1, 1, -1, 1\}$

~~taking~~ for 8-point, take

$x[n] = \{1, -1, 1, -1, 1, 0, 0, 0\}$

$$X_8[k] = \sum_{n=0}^{7} x(n)\, e^{-j\frac{2\pi nk}{8}}$$

$$= 1 \cdot e^{0} + (-1)\, e^{-jk\pi/4} + 1 \cdot e^{-jk\pi/2}$$

$$+ (-1)\, e^{-\frac{3jk\pi}{4}} + 1 \cdot e^{-jk\pi}$$

$$+ 0 + 0 + 0$$

$$\boxed{1 + e^{-jk\pi/2} + e^{-jk\pi} - e^{-jk\pi/4} - e^{-3jk\pi/4}}$$

Ans

(iii) $x[n] = \cos\frac{\pi n}{4}$

$$X_8(k) = \sum_{n=0}^{7} x[n]\, e^{\frac{-j2\pi nk}{8}}$$

$$= \sum_{n=0}^{7} \cos\frac{\pi n}{4}\, e^{-j\pi nk/4}$$

$$= \sum_{n=0}^{7} \left(\frac{e^{j\pi n/4} + e^{-j\pi n/4}}{2}\right) e^{-j\pi nk/4}$$

$$= \frac{1}{2}\sum_{n=0}^{7} e^{\frac{j\pi n}{4}(1-k)} + e^{-j\pi n(1+k)}$$

at $n=0$, term is $\frac{1+1}{2}$

$$\therefore \frac{1}{2}\left[\left(\frac{1\cdot\left(1-e^{2j\pi(1-k)}\right)}{1-e^{j\pi(1-k)/4}}\right) + 1\cdot\left(\frac{1-e^{-2j\pi(1+k)}}{1-e^{-j\pi(1+k)/4}}\right)\right]$$

(iv) $x[n] = \{j, 0, -j, 1\}$

$$x[n] = \{j, 0, -j, 1, 0, 0, 0, 0\}$$

$$X_8[k] = \sum_{n=0}^{7} x[n]\, e^{-2\pi j n k/8}$$

$$= j \cdot e^{0} + 0 + -j\, e^{-j\pi k/2} + 1 \cdot e^{-3\pi j k/4} + 0 + \ldots$$

$$= \boxed{j + -j\, e^{-j\pi k/2} - e^{-3\pi j k/4}}$$

Q10 ② ①

$$x(e^{j\omega}) = \frac{e^{-j\omega} - 1/4}{1 - \frac{1}{4}e^{-j\omega}}$$

$$= \frac{e^{-j\omega}}{1 - \frac{1}{4}e^{-j\omega}} - \frac{1/4}{1 - \frac{1}{4}e^{-j\omega}}$$

$$= e^{-j\omega}\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n\left(e^{-j\omega}\right)^n - \frac{1}{4}\left(\sum_{n=0}^{\infty}\left(\frac{1}{4}e^{-j\omega}\right)^n\right)$$

$$= 4\cdot\sum_{n=0}^{\infty}\left(\frac{1}{4}e^{-j\omega}\right)^{n+1} - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{4}e^{-j\omega}\right)^n$$

$$= 4\cdot\sum_{n=1}^{\infty}\left(\frac{1}{4}e^{-j\omega}\right)^n - \frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{4}e^{-j\omega}\right)^n$$

$$\cancel{= 4\left(\frac{1}{4}\right)^n u(n-1) - \frac{1}{4}\left(\frac{1}{4}\right)^n u(n)}$$

$$\therefore \boxed{x[n] = 4\cdot\left(\frac{1}{4}\right)^n u(n-1) - \frac{1}{4}\left(\frac{1}{4}\right)^n u(n)}$$

$$\boxed{x[n] = \left(\frac{1}{4}\right)^{n-1} u(n-1) - \left(\frac{1}{4}\right)^{n+1} u(n)}$$

~~Ans~~

(ii) $$x(e^{j\omega}) = \frac{1 - \left(\frac{1}{2}\right)^4 e^{-4j\omega}}{1 - \left(\frac{1}{2}\right) e^{-j\omega}}$$

$$= \frac{\cancel{\left(1 - \frac{1}{2} e^{-j\omega}\right)}\left(1 + \frac{1}{2} e^{-j\omega}\right)\left(1 + \left(\frac{1}{2}\right)^2 e^{-2j\omega}\right)}{\cancel{\left(1 - \frac{1}{2} e^{-j\omega}\right)}}$$

$$= 1 + \frac{1}{2} e^{-j\omega} + \left(\frac{1}{2}\right)^2 e^{-2j\omega} + \left(\frac{1}{2}\right)^3 e^{-3j\omega} + \cdots$$

$$x(t - t_0) \rightleftharpoons e^{-j\omega t_0}$$

$$\therefore x[n] = \delta[n] + \frac{1}{2} \delta[n-1] + \frac{1}{2^2} \delta[n-2] + \cdots$$

y

(iii) $x(e^{j\omega}) = \cos^3 3\omega + \sin^2\omega$

$$= \left(\frac{1+\cos 6\omega}{2}\right) + \left(\frac{1-\cos 2\omega}{2}\right)$$

$$= 1 + \tfrac{1}{2}\left(\cos 6\omega - \cos 2\omega\right)$$

$$= 1 + \frac{1}{2}\left(\frac{e^{j6\omega}+e^{-j6\omega}}{2} - \left(\frac{e^{j2\omega}+e^{-j2\omega}}{2}\right)\right)$$

$$= 1 + \frac{1}{4}\left[e^{j6\omega} + e^{-j6\omega} - e^{j2\omega} - e^{-j2\omega}\right]$$

$$\delta(t-t_0) \rightleftharpoons e^{-j\omega t_0}$$

$$\therefore x[n] = \delta(n) + \frac{1}{4}\left(\delta[n+6]+\delta[n-6] - \delta[n+2] - \delta[n-2]\right)$$

Q12 (a) $\displaystyle\int_{-\pi}^{\pi} |x(e^{j\omega})|^2 \, d\omega$

by Parseval's theorem,

if $x[n]$ & $X(e^{j\omega})$ are Inverse pairs,

then,

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\omega})|^2 \, d\omega$$

$$\therefore \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega = 2\pi \left(\sum_{n=-\infty}^{\infty} |x[n]|^2 \right)$$

$$= 2\pi \left(1^2 + 2^2 + 7^2 + 8^2 + 10^2 \right)$$

$$= \boxed{436\pi}$$

(b) $\int_{-\pi}^{\pi} \left| \frac{d X(e^{j\omega})}{d\omega} \right|^2 d\omega$

if $x[n] \rightleftharpoons X(e^{j\omega})$

$\Rightarrow \; n\,x[n] \rightleftharpoons j \frac{d}{d\omega} X(e^{j\omega})$

by Parseval's theorem,

$$\int_{-\pi}^{\pi} \left| j \frac{d}{d\omega} X(e^{j\omega}) \right|^2 d\omega$$

$$= 2\pi \left(\sum_{n=-\infty}^{\infty} |n|^2 \, |x[n]|^2 \right)$$

$$= 2\pi \left((-2)^2 (1)^2 + (-1)^2 (-2)^2 + (0)^2 (7^2) + 2^2 (8)^2 + 3^2 (-10)^2 \right)$$

$$= \boxed{2328\pi}$$