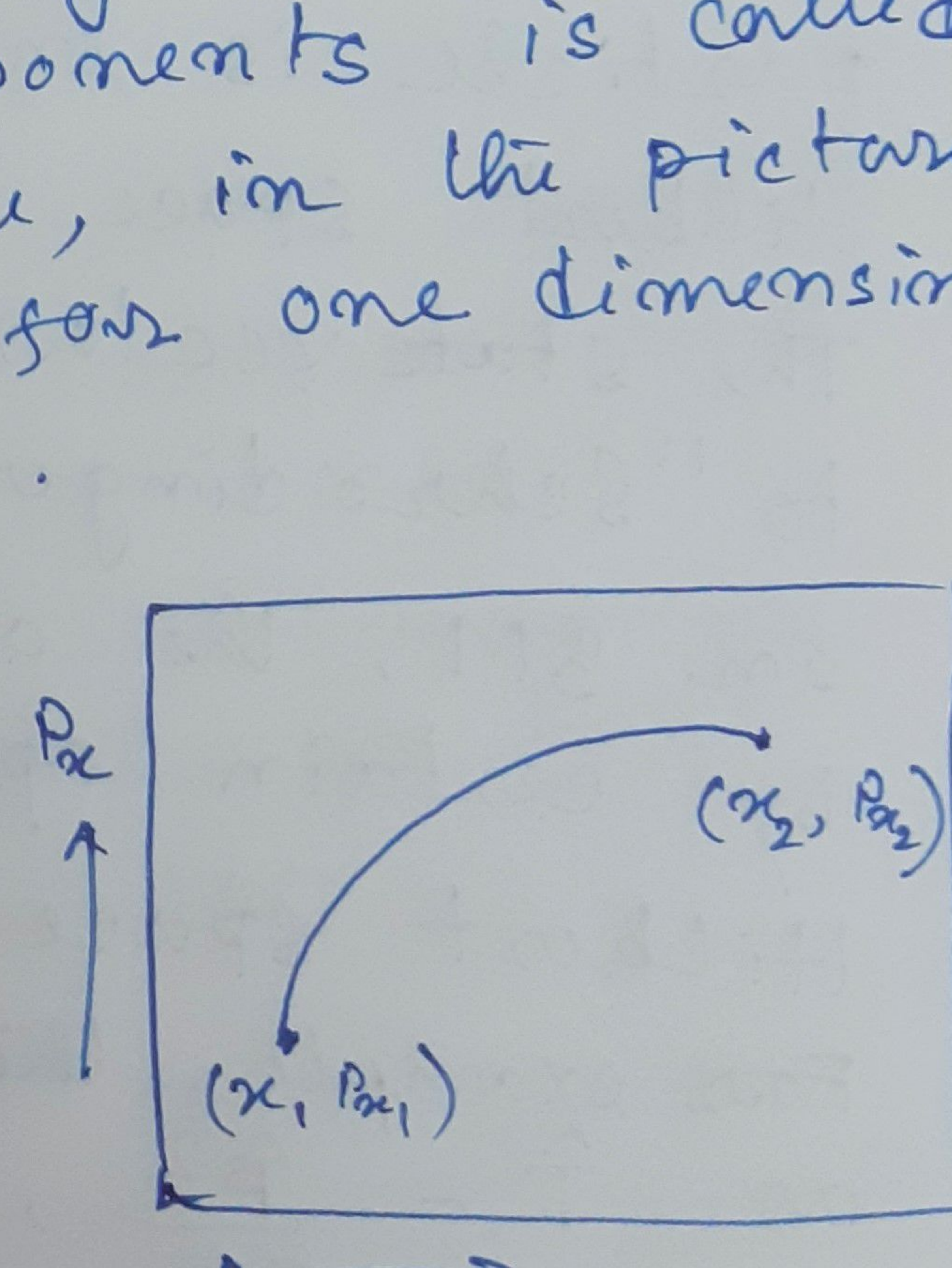
Introduction to Quantum Mechanics.

In classical mechanics, a particle's motion is governed by Newton's laws.
The equation of motions governed by Newton's Laws are Second order ordinary differential equations (ODE).
The state of motion in classical mechanics is given by the position $\vec{x}(t)$ and momentum $\vec{P}(t)$, where time 't' comes as a parameter.
Which means, that at any given time instant 't' if we know the pair $(\vec{x}(t), \vec{P}(t))$, then we know everything about the particle in consideration.
The co-ordinate space consisting all the positions and momentum components is called the phase space. For example, in the picture we draw a phase space for one dimensional motion of a single particle.



The trajectory of the particle in the phase space is determined by the equation of motion.
In general, for a N particle system, the phase space is
6N dimensional $\rightarrow$ 3N position co-ordinates and 3N momentum co-ordinates.
So, the bottom line in Classical mechanics is to know the instantaneous position $(\vec{x}, \vec{P})$ in the phase space, which determines the state of motion

In Quantum mechanics, the basic question remains the same → What is the state of motion?
But in this case the state of motion cannot be represented by the phase space point.
The reason behind this is the fact that, in Quantum mechanics position and momentum cannot be measured simultaneously.
Therefore, it is immpossible to determine a particular point in phase space.
Also in QM, the particle is not governed by Newton' laws.
Therefore, we need a new "space" and new "Laws of motion" for QM.

In QM, everything we could know about a particle is encoded in a vector ψ in Hilbert space $\mathcal{H}$, which is called the state vector.
The state vector evolves in time according to "Schrödinger equation."
In QM, the observables are represented by certain operators '$\hat{O}$', acting on the Hilbert space.
For example, the kinetic energy can be represented as $\hat{T} = \frac{\hat{p}^2}{2m}$, where $\hat{p}$ is the momentum operator.

The operators in quantum mechanics are

Linear maps $\hat{O}: \mathcal{H} \rightarrow \mathcal{H}$.

which means an operator $\hat{O}$ take a state vector ψ in Hilbert space $\mathcal{H}$ and maps it to ϕ, another vector in Hilbert space.

In the following lectures, we will study what Hilbert space is and what the operators do, when acted upon a vector in a given Hilbert space.

Hilbert space

The realm of QM is Hilbert space. So we will begin by exploring the properties of these.

Definition: Hilbert space is a vector space $\mathcal{H}$ over $\mathbb{C}$ (complex vector space), equipped with a complete inner product.

Saying that Hilbert space is a vector space means that it is a set on which we have an operation '+' of addition, obeying

Commutativity : $\psi + \phi = \phi + \psi$.

associativity : $\psi + (\phi + \chi) = (\psi + \phi) + \chi$

identity : $\exists!\ 0 \in \mathcal{H}$ s.t. $\psi + 0 = \psi$.

$0 \rightarrow$ null vector.

$\forall\ \psi, \phi, \chi \in \mathcal{H}$.

Linear independence : A set of vectors $\{\phi_1, \phi_2, \cdots \phi_n\}$ are linearly independent iff the only solution to $c_1\phi_1 + c_2\phi_2 + \cdots c_n\phi_n = 0$

for $c_i \in \mathbb{C}$ is $c_1 = c_2 = c_3 = \cdots = c_n = 0$.

The dimension of the vector space is the largest possible number of linearly independent vectors we can find. If there is no such largest number, we call the vector space infinite dimensional.

Orthogonality : An orthogonal set of vector $\{\phi_1, \phi_2, \cdots, \phi_n\}$ is defined by.

$$(\phi_i, \phi_j) = \begin{cases} 0 & \text{when } i \neq j \\ \text{Const.} & \text{when } i = j \end{cases}$$

If the set of vectors are additionally normalized then $(\phi_i, \phi_i) = 1$.

An orthonormal set $\{\phi_1, \phi_2 \cdots \phi_n\}$ forms a basis of n dimensional Hilbert space, if every $\psi \in \mathcal{H}$ can be uniquely expressed as a sum $\psi = \sum_{\alpha=1}^{n} c_\alpha \phi_\alpha$, with some co-efficient $c_\alpha \in \mathbb{C}$.

We can determine these co-efficients, by taking the inner product

$$(\phi_b, \psi) = (\phi_b, \sum_\alpha c_\alpha \phi_\alpha) = \sum_\alpha c_\alpha (\phi_b, \phi_\alpha)$$

$$= \sum_\alpha c_\alpha \delta_{b\alpha} = c_b.$$

here $\delta_{b\alpha} \Rightarrow$ Kronecker ~~product~~ delta function.
($\delta_{b\alpha} = 1$ for $b = \alpha$ and 0 otherwise)

We can multiply our vectors by numbers $a, b, c \in \mathbb{C}$ (complex) called scalars.

The multiplication is

distributive over $\mathcal{H}$: $c(\psi + \phi) = c\psi + c\phi$

distributive over $\mathbb{C}$: $(a+b)\psi = a\psi + b\psi$

In addition, $\mathcal{H}$ is equipped with an inner product. This is a map $(\, , \,) : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ that obeys

conjugate symmetry : $(\phi, \psi) = \overline{(\psi, \phi)}$

linearity : $(\phi, a\psi) = a(\phi, \psi)$

additivity : $(\phi, \psi + \chi) = (\phi, \psi) + (\phi, \chi)$

positive definiteness : $(\psi, \psi) \geq 0 \ \forall \ \psi \in \mathcal{H}$,

with equality iff $\psi = 0$.

Note that the first two of these ~~implies~~ imply $(a\phi, \psi) = \bar{a}(\phi, \psi)$, so that $(\, , \,)$ is antilinear in its first (leftmost) argument.

Note also that $(\psi, \psi) = \overline{(\psi, \psi)}$, so that it is ~~necessaryl~~ necessarily real.

Whenever we have inner product, we can define the norm of a state to be

$$\|\psi\| = \sqrt{(\psi, \psi)}$$

These properties ensure that the Cauchy-Schwarz inequality holds.

$$|(\phi, \psi)|^2 \leq (\phi, \phi)(\psi, \psi)$$

Quantum mechanics makes use of both finite and infinite-dimensional Hilbert space.
In the infinite dimensional case, we have to decide what we mean by an 'infinite' linear combination of vectors.
Not each such infinite sum makes sense, as infinite sum such as $\sum_{a=1}^{\infty} c_a \phi_a$ might not converge.
To prevent this, one requires that $\mathcal{H}$ is complete in the norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$.
This means that every Cauchy sequence $\{S_1, S_2 \cdots\}$ converges in $\mathcal{H}$.
We say that the sum $\sum_{a=1}^{N} c_a \psi_a$ converges to a vector ψ if

$$\lim_{N\to\infty} \|S_N - \psi\| = 0,$$

where $S_N = \sum_{a=1}^{N} c_a \psi_a$ is the sum of N terms.

Cauchy - Schwarz inequality

$$||(x,y)|| \leq (x,x)(y,y)$$

with strict inequality unless x, y are colinear.

Proof: Suppose x is not a ~~sca~~ scaler multiple of y. and neither x or y are zero. Then $x - \alpha y$ is not zero for any complex α.

Consider $$||x - \alpha y||^2 > 0$$

Expanding this, we get

$$||x||^2 - \alpha(x,y) - \bar{\alpha}(y,x) + \alpha\bar{\alpha}||y||^2$$

Let $\alpha = \mu t$ with real t and $|\mu| = 1$

So that $\mu(x,y) = ||(x,y)||$

Then $$||x||^2 - 2t||(x,y)|| + t^2||y||^2 > 0$$

The minimum of the left hand side occurs

~~when~~ $\cancel{t||y||^2 + 2||(x,y)|| = 0}$

~~which gives~~ $\cancel{t =}$

when the derivative vanishes.

i.e. $$f(x,y,t) = ||x||^2 - 2t||(x,y)|| + t^2||y||^2$$

$$\frac{\partial f}{\partial t} = -2||(x,y)|| + 2t||y||^2 = 0$$

$$\Rightarrow \quad t = \frac{||(x,y)||}{||(y)||^2}$$

putting the value of t in the ~~sam~~ inequality we get

$$||x||^2 + \left(\frac{||(x,y)||}{||y||^2}\right)^2 ||y||^2 - 2\frac{||(x,y)||}{||y||^2}||(x,y)|| > 0$$

$$\Rightarrow \quad ||(x,y)|| < ||x|| \, ||y||.$$

$$\Rightarrow \quad ||(x,y)|| < (x,x)(y,y).$$

Corollary: (Triangle inequality)

For v, w in a Hilbert space H, we have

$$\|v+w\| \leq \|v\| + \|w\|$$

with the distance function $d(v,w) = \|v-w\|$

we have the inequality

$$d(x,z) = \|x-z\| = \|x-y+y-z\|$$

$$\leq \|x-y\| + \|y-z\|$$

Proof:

$$(\|v\|+\|w\|)^2 - \|v+w\|^2$$

$$= (\|v\|^2 + \|w\|^2 + 2\|v\|\,\|w\|)$$

$$- (\|v\|^2 + \|w\|^2 + (v,w) + (w,v))$$

$$= 2\|v\|.\|w\| - (v,w) - (w+v)$$

$$= 2\|v\|.\|w\| - 2\,\mathrm{Re}\,(v,w).$$

(Now, since $\mathrm{Re}\,(v,w) \leq \|(v,w)\|$

$$\geq 2\|v\|\,\|w\| - 2\|(v,w)\|$$

$$\geq 0$$ (Using Cauchy Schwarz inequality).

Thus we prove the corollary.