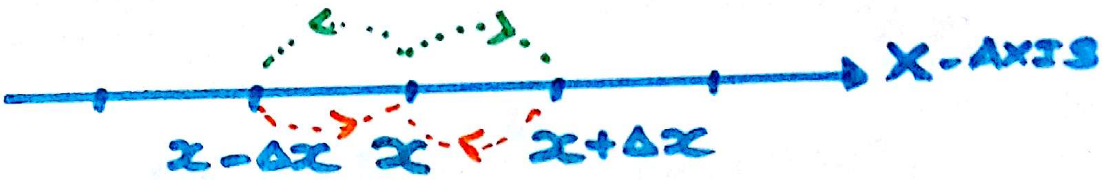
DYNAMICAL SYSTEMS

RANDOM WALK MODEL AND DIFFUSION EQUATION:



$\Rightarrow P(x,t) \Rightarrow$ PROBABILITY THAT THE RANDOM WALKER IS AT x AT TIME t

$\Delta x \Rightarrow$ STEP LENGTH IN ONE TIME STEP (Δt)

PROBABILITY FOR A RIGHT STEP $= \frac{1}{2}$
PROBABILITY FOR A LEFT STEP $= \frac{1}{2}$

$\Rightarrow$ CHANGE IN $P(x,t)$ AFTER ONE STEP:

$$\boxed{P(x, t+\Delta t) - P(x,t) = \frac{1}{2}\left[P(x+\Delta x,t) - 2P(x,t) + P(x-\Delta x,t)\right]} \quad (1)$$

$\Rightarrow$ DIVIDE BOTH SIDES OF THE EQUATION BY Δt AND $(\Delta x)^2$

$$\boxed{\frac{P(x,t+\Delta t) - P(x,t)}{\Delta t} = \frac{(\Delta x)^2}{2\Delta t}\left[\frac{P(x+\Delta x,t) - 2P(x,t) + P(x-\Delta x,t)}{(\Delta x)^2}\right]} \quad (2)$$

→ $\Delta t \to 0$; $\Delta x \to 0$

$$\frac{P(x, t+\Delta t) - P(x,t)}{\Delta t} \equiv \frac{\partial P(x,t)}{\partial t}$$

$$\frac{P(x+\Delta x, t) - 2P(x,t) + P(x-\Delta x, t)}{(\Delta x)^2} = \frac{\partial^2 P(x,t)}{\partial x^2}$$

$$D \equiv \frac{(\Delta x)^2}{2\Delta t} \Rightarrow \text{CONSTANT}$$

DIFFUSION CONSTANT

→ EQUATION 2 :

$$\boxed{\underbrace{\frac{\partial P(x,t)}{\partial t}}_{\text{TIME VARIATION}} = D \underbrace{\frac{\partial^2 P(x,t)}{\partial x^2}}_{\text{SPATIAL VARIATION}}}$$

→ FOR THREE-DIMENSIONAL SYSTEM :

$$\frac{\partial P(x,y,z,t)}{\partial t} = D_x \frac{\partial^2 P(x,y,z,t)}{\partial x^2} + D_y \frac{\partial^2 P(x,y,z,t)}{\partial y^2} + D_z \frac{\partial^2 P(x,y,z,t)}{\partial z^2}$$

$D_x = D_y = D_z = D$ (FOR ISOTROPIC DIFFUSION)

→ SIMPLE FORM :

$$\boxed{\frac{\partial P(\vec{r},t)}{\partial t} = D \nabla^2 P(\vec{r},t)}$$

HERE $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

→ BY SOLVING THIS EQUATION, WE CAN UNDERSTAND HOW $P(x,t)$ VARIES WITH x AND t

→ PROVE: $$\frac{d^2f(x)}{dx^2} = \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

$$g(x) \equiv \frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

$$\frac{dg(x)}{dx} \equiv \frac{d^2f(x)}{dx^2} = \lim_{\Delta x \to 0} \frac{g(x) - g(x-\Delta x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[\frac{df(x+\Delta x)}{dx} - \frac{df(x)}{dx}\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] - \left[\frac{f(x) - f(x-\Delta x)}{\Delta x}\right]}{\Delta x}$$

$$\frac{d^2f(x)}{dx^2} = \lim_{\Delta x \to 0} \frac{\left[f(x+\Delta x) - 2f(x) + f(x-\Delta x)\right]}{(\Delta x)^2}$$