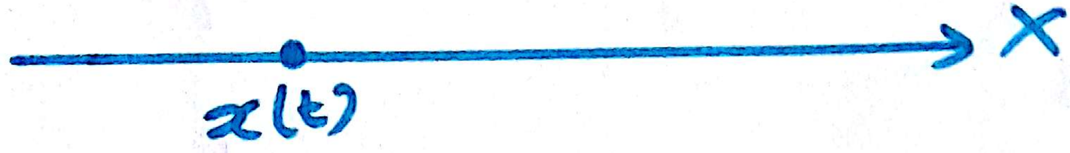
BROWNIAN MOTION



- CONSIDER A PARTICLE OF MASS m MOVING IN ONE-DIMENSIONAL AXIS/SPACE.
- $x(t) \Rightarrow$ POSITION OF THE PARTICLE AT TIME t

  $V(t) \Rightarrow$ VELOCITY AT TIME t
- EQUATION OF MOTION :

DAMPING FORCE (MEDIUM)

$$m\frac{dv}{dt} = -\alpha v(t) + F(t)$$

FRICTION CONSTANT (POSITIVE)

RAPIDLY FLUCTUATING FORCE (NOISE/STOCHASTIC FORCE)

$$V \equiv \dot{x}$$

$$\boxed{\dot{x} \equiv \frac{dx}{dt}}$$

$$\frac{dv}{dt} = \frac{d\dot{x}}{dt}$$

$$m\frac{d\dot{x}}{dt} = -\alpha\dot{x} + F(t)$$

→ MULTIPLY BY x ON BOTH SIDES

$$m x \frac{d\dot{x}}{dt} = -\alpha x \dot{x} + x F(t)$$

$$m\left[\frac{d}{dt}(x\dot{x}) - \dot{x}^2\right] = -\alpha x\dot{x} + x F(t)$$

→ TAKE AVERAGE ON BOTH SIDES:

$$m\left[\left\langle \frac{d}{dt}(x\dot{x})\right\rangle - \langle \dot{x}^2\rangle\right] = -\alpha \langle x\dot{x}\rangle + \cancel{\langle xF\rangle}$$

ZERO (UNCORRELATED NOISE)

$\langle xF\rangle = \langle x\rangle\langle F\rangle = 0$

FROM EQUIPARTITION THEOREM:

$$\frac{1}{2} m \langle \dot{x}^2\rangle = \frac{1}{2} k_B T$$

$$m \frac{d\langle x\dot{x}\rangle}{dt} = k_B T - \alpha \langle x\dot{x}\rangle$$

$$\boxed{\gamma = \frac{\alpha}{m}}$$

SOLUTION:

$$\langle x\dot{x}\rangle = C e^{-\gamma t} + \frac{k_B T}{\alpha}$$

C → CONSTANT OF INTEGRATION

NOTE: $\langle x\dot{x}\rangle = \frac{1}{2}\frac{d}{dt}\langle x^2\rangle$

INITIAL CONDITION:

$$t=0\ ;\ x=0$$

$$C=-\frac{k_BT}{\alpha}$$

$$\Rightarrow \langle x\dot{x}\rangle=\frac{1}{2}\frac{d}{dt}\langle x^2\rangle=\frac{k_BT}{\alpha}\left(1-e^{-\Gamma t}\right)$$

INTEGRATE THIS:

$$\langle x^2\rangle=\frac{2k_BT}{\alpha}\left[t-\frac{\left(1-e^{-\Gamma t}\right)}{\Gamma}\right]$$

CASE 1: $\quad t\ll\frac{1}{\Gamma}\Rightarrow\Gamma t\ll 1$

$$\langle x^2\rangle=\frac{\cancel{2}k_BT}{\alpha}\left[\cancel{t}-\frac{1}{\cancel{\Gamma}}\left(\cancel{1}-\cancel{1}+\cancel{\Gamma t}-\frac{1}{\cancel{2}}\Gamma^2t^2+\ldots\right)\right]$$

$$\simeq\frac{k_BT}{\cancel{\alpha}}\left(\frac{\cancel{\alpha}}{m}\right)t^2$$

$$\boxed{\langle x^2\rangle\simeq\frac{k_BT}{m}t^2}$$

SHORT TIME LIMIT

→ MOVES LIKE A FREE PARTICLE WITH A CONSTANT THERMAL VELOCITY $\left(\frac{k_BT}{m}\right)^{1/2}$

CASE 2 : $t \gg \frac{1}{\Gamma}$; $\Gamma t \gg 1$

$$e^{-\Gamma t} \to 0$$ LONG TIME LIMIT

$$\langle x^2 \rangle = \frac{2 k_B T}{\alpha} t$$

$\Rightarrow$ MOVES LIKE A RANDOM WALKER

$\langle x^2 \rangle \propto t$