Int. J. Solids Structures, 1978, Vol. 14, pp. 161–172. Pergamon Press. Printed in Great Britain

BUCKMINSTER FULLER'S "TENSEGRITY" STRUCTURES AND CLERK MAXWELL'S RULES FOR THE CONSTRUCTION OF STIFF FRAMES

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(*Received* 2 *February* 1977; *revised* 18 *August* 1977)

Abstract—Maxwell has shown that b bars assembled into a frame having j joints would, in general, be simply stiff if $b = 3j-6$. Some of Buckminster Fuller's "Tensegrity" structures have fewer bars than are necessary to satisfy Maxwell's rule, and yet are not "mechanisms" as one might expect, but are actually stiff structures. Maxwell anticipates special cases of this sort, and states that their stiffness will "be of a low order". In fact, the conditions under which Maxwell's exceptional cases occur also permit at least one state of "self-stress" in the frame.

Linear algebra enables us to find the number of "incipient" modes of low-order stiffness of the frame in terms of the numbers of bars, joints and independent states of self-stress. Self-stress in the frame has the effect of imparting first-order stiffness to the frame, and it seems from experiments that a single state of self-stress can stiffen a large number of modes. It is this factor which Fuller exploits to make satisfactory structures.

INTRODUCTION

In his paper "On the Calculation of the Equilibrium and Stiffness of Frames", J. Clerk Maxwell[1] considers the mechanical performance of structures composed of straight bars connected at their ends by frictionless joints, and subjected to external forces applied at the joints. Defining a *frame* in three-dimensional space as "a system of lines connecting a number of points" and a *stiff* frame as "one in which the distance between any two points cannot be altered without altering the length of one or more of the connecting lines of the frame", he shows that a frame having j joints (points) requires in general $3j-6$ bars (lines) to render it stiff. We shall refer to this result as "Maxwell's rule". He points out that a simply stiff frame is statically determinate, i.e. that the tension in every member of the frame sustaining any arbitrary external loading may be calculated by the equations of statical equilibrium (as such, or in a graphical method), and goes on to discuss the deformation of simply-stiff frames made from elastic members. Finally he considers a "redundant" or over-stiff frame, which has more bars than are necessary to make it simply stiff, and he shows how the tensions in the members may be calculated. One of his results is the celebrated "reciprocal theorem". Maxwell's rule is well-known in structural engineering[2–5].

The subject of the present note is the behaviour of a class of frames of which an example is shown in Fig. 1. It is a minor adaptation of a so-called "Tensegrity" (≡Tension-integrity) structure of Buckminster Fuller[6, 7], illustrated as $K8$ on p. 160 of Marks[8] and as Fig. 14 of Pugh[9]. This frame, like those for which Maxwell's rule is applicable unmodified, is "free in space", i.e. not attached to a "foundation": in this note we shall not count the 6 degrees of freedom which the frame has in three-dimensional space as a "rigid body".

The frame was built from "Geo-*D*-Stix" rods and joints, and the line drawing is taken from a photograph. Geo-*D*-Stix rods are made of plastic and are about 3 mm in diameter, and the soft rubber joints are in the form of a "spider" of sockets. They are made by Geo-*D*-Stix Inc., of Spokane, Washington, U.S.A.

The frame has 12 joints and 24 bars. According to Maxwell's rule, 30 bars are required to make a frame stiff, and so we would expect the frame to be "loose", with 6 degrees of freedom. It is, however, stiff, as may readily be checked experimentally (the bars are of two distinct lengths, in the ratio 2.25:1).

The frame thus constitutes a paradoxical exception to Maxwell's rule.

But Maxwell does in fact anticipate such exceptions to his rule, for he states ([1] p. 599, Collected Papers, Vol. 1): "*In those cases where stiffness can be produced with a smaller number of lines, certain conditions must be fulfilled, rendering the case one of a maximum or*

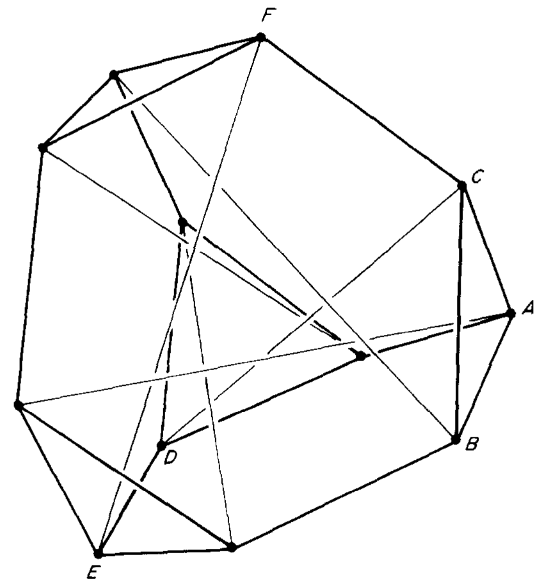

Fig. 1. A "Tensegrity" structure which is investigated in this note. The 18 "external" members all have the same length, and the 6 internal members, which are all of the same length, are as long as possible. The frame can sustain a state of self-stress in which the outer members are in tension and the inner ones are in compression.

minimum value of one or more of its lines. The stiffness of the frame is of an inferior order, as a small disturbing force may produce a displacement infinite in comparison with itself".

The meaning of the first of these two remarks is a trifle obscure: presumably Maxwell intended to refer to a maximum or minimum value of the *length* of one or more of its lines. Certainly Fuller's "Tensegrity" structures have turnbuckles in some members and they cease to be stiff if the turnbuckles are relaxed. In the physical realisation of the frame of Fig. 1, the short outer members were of fixed length, but the length of the longer inner bars was initially adjustable. We made the adjustments so that all of the long bars were of equal length. We found that for an arbitrary length of these bars the assembly was indeed a mechanism with several (six?) degrees of freedom, but that when this length l reached a certain value, the assembly became stiff. Of course, there is some flexibility in the Geo-*D*-Stix joints, so the onset of stiffness was not expected to be startlingly abrupt: nevertheless, there was a remarkable change in the properties of the frame at this stage.

Clearly, there is a limit to the length of the interior bars within the cage or net formed by the outer members, and it was at this limit that the frame in fact became stiff.

This is presumably precisely the kind of maximum which Maxwell had in mind. Thus it seems clear that Fuller's invention corresponds to an exceptional special case anticipated by Maxwell.

In the remainder of this note we attempt to elucidate the conditions under which Maxwell's general rule may be broken, and the nature of the "inferior order" stiffness of the resulting frame.

THE ALGEBRA OF MAXWELL'S RULE

It is useful to review the matrix-algebraic basis of Maxwell's rule, and to examine the nature of the *states of self-stress* which are possible when $b > 3j - 6$ and the *mechanisms* which are possible when $b < 3j - 6$.

Consider a general frame supported in a statically determinate manner. Let it sustain $3j$ arbitrary independent components of load applied at its joints, and let these be denoted by the vector $\mathbf{p}$. The $3j$ equations of equilibrium relate the load components to the b bar tensions and the 6 reactions; let these unknown force variables be denoted by the vector $\mathbf{t}$. (Note that we are allowing external loads to be applied to all joints, even those which are connected to the support system).

The equilibrium equations for the original, undeformed frame may be written (see [10])

$$\mathbf{Ht} = \mathbf{p}. \tag{1}$$

The "equilibrium" matrix $\mathbf{H}$ has $n(\equiv 3j)$ rows and $m(\equiv b+6)$ columns.

Corresponding to $\mathbf{p}$ is the vector $\mathbf{d}$ of (small) nodal displacements; and corresponding to $\mathbf{t}$ is the vector $\mathbf{e}$ considting of b bar extensions and 6 foundation displacements, all small. These quantities are related, for small displacements, by the kinematic relations

$$\mathbf{Cd} = \mathbf{e}. \tag{2}$$

The "compatibility" matrix $\mathbf{C}$ thus has m rows and n columns.

The principle of virtual work enables us to show that $\mathbf{C} = \mathbf{H}^T$, as follows. The principle states that

$$\mathbf{p}^T\mathbf{d} = \mathbf{t}^T\mathbf{e}$$

for any $\mathbf{p}$, $\mathbf{t}$ satisfying (1) and $\mathbf{d}$, $\mathbf{e}$ satisfying (2). Transposing (1) and substituting from (2) we have

$$\mathbf{t}^T\mathbf{H}^T\mathbf{d} = \mathbf{t}^T\mathbf{Cd}$$

and since this is true for arbitrary $\mathbf{d}$, the required result follows.

Consider now the simplest case where $m = n$ (i.e. Maxwell's rule is satisfied) and the rank r of $\mathbf{H}$ is equal to n. Matrix $\mathbf{H}$ is non-singular, and may be inverted to give

$$\mathbf{t} = \mathbf{H}^{-1}\mathbf{p}. \tag{3}$$

The frame is thus *statically determinate*, since $\mathbf{t}$ is uniquely determined by the loading $\mathbf{p}$. In particular, if $\mathbf{p} = 0$, $\mathbf{t} = 0$, and consequently no states of self stress are possible. A state of self-stress (or prestress) is one in which there are some non-zero tensions in the bars even though there are no external forces acting on the frame. The matrix $\mathbf{C}(=\mathbf{H}^T)$ has the same rank as $\mathbf{H}$ and is also non-singular, so we can write

$$\mathbf{d} = (\mathbf{H}^T)^{-1}\mathbf{e}. \tag{4}$$

From this we see in particular that if $\mathbf{e} = 0$ (i.e. all bars inextensible, and supports immovable) $\mathbf{d} = 0$: the frame is *rigid*.

Next consider the same frame but with an extra bar inserted between two joints not already directly linked. The equilibrium matrix $\mathbf{H}$ (eqn 1) now nas n rows and $n+1$ columns. The rank r is unchanged, so we find by the Dimension Theorem of linear algebra (see [11]) that the dimension of the null space of the solution vector $\mathbf{t}$ is 1. In other words, there is one nontrivial solution $\mathbf{t}$ when $\mathbf{p} = 0$, and this may of course be multiplied by an arbitrary constant. This is a state of *self-stress*. We can see also that the insertion of yet another bar will give a further state of selfstress; and so on.

Turning to the compatibility eqn (2) for the frame augmented by 1 bar we find that the $\mathbf{e}$ vector is of dimension n, even though it has $n+1$ components. Consequently we cannot specify all of the components of $\mathbf{e}$ independently, but must satisfy one equation relating them.

Now consider instead the first frame but with one bar removed. The compatibility matrix $\mathbf{C}$ now has $n-1$ rows and n columns. Its rank is $\mathbf{n}-1$, and the Dimension Theorem states that the dimension of the null space of the solution vector $\mathbf{d}$ is 1. In other words there is one nontrivial solution $\mathbf{d}$ when $\mathbf{e} = 0$. This is a *mechanism*: the frame is not rigid. There is a single mode of deformation and the elements of $\mathbf{e}$ may be multiplied by an arbitrary constant. Note, however, that our entire discussion in terms of linear algebra is a consequence of the assumption that displacements are small; consequently we have demonstrated only that there exists an *infinitesimal* mode, even if a particular frame actually has a finite mode.

Examination of the equilibrium matrix shows an analogy with the compatibility equations for the redundant structure: all the elements of **p** cannot be specified independently, and they must satisfy one relation. In other words a mechanism with one degree of freedom can only be in equilibrium under load if the components of load satisfy a certain relationship.

SPECIAL CASES ("ILL CONDITIONING") IN FRAMES SATISFYING MAXWELL'S RULE

We are now in a position to examine a simpler special case than Fuller's, and one which is well known. This is the "ill-conditioned" frame, which is apparently not stiff, even though it satisfied Maxwell's general relation. Ill-conditioned frames have been studied by Möbius[12], Bricard[13], Bennett[14], Geiringer[15, 16], Timoshenko and Young[3], Parkes[5] and Wunderlich[17].

Figure 2(a) shows a frame having 12 joints and 30 bars, which satisfies Maxwell's general rule and is also rigid. The bars form the edges of a regular icosahedron, and it is easy to show that a frame whose members form the edges of *any* polyhedron whose faces are all triangular (a "deltahedron"), *ipso facto* satisfies Maxwell's general rule, by invoking Euler's formula relating the number of faces, edges and vertices of any simply closed polyhedron. Also Cauchy[18] has shown that a frame having the form of any convex deltahedron is rigid.

Suppose now we change the lengths of all five bars meeting at a particular joint I so that they always have the same length. As the length of these bars increases joint I is pushed away from the centre of the frame, at the tip of an ever sharper "spire". Conversely, when the members are shortened, the joint is pulled in. Clearly a limiting case is reached in which all 5 members, and 5 peripheral members, lie in a common plane. It is not possible to make the 5 radial members any shorter: they simply would not connect. Now in this condition the frame exhibits a lack of stiffness, or a degree of freedom in the sense that joint I is capable of small displacements out-of-plane, at the expense of only second-order changes in length of the members. For infinitesimal displacements therefore the frame is not stiff, by Maxwell's definition.

This is shown clearly by matrix algebra. When the configuration is such that all the bars meeting at a particular joint lie in a plane, the equation of equilibrium of forces normal to the plane has all of its coefficients zero. Thus one row of **H** and one column of **C** consists of zeros; hence the rank of both **H** and **C** becomes $n - 1$. By the same argument as before, there is now an infinitesimal mode of deformation, which is in fact the mode already described. Note also that as the rank of **H** is $(n - 1)$, there is now a single state of self-stress, by the same argument as before. In the present example the state of self-stress involves tension in the 5 radial bars and compression in the 5 linking bars. A consequence of the matrix algebra is that, in general, the existence of an infinitesimal mode in a frame satisfying Maxwell's rule implies a corresponding state of self-stress. This is well known in the two-dimensional example shown in Fig. 3: part of the assembly is a mechanism and a separate part is redundant. In the example of

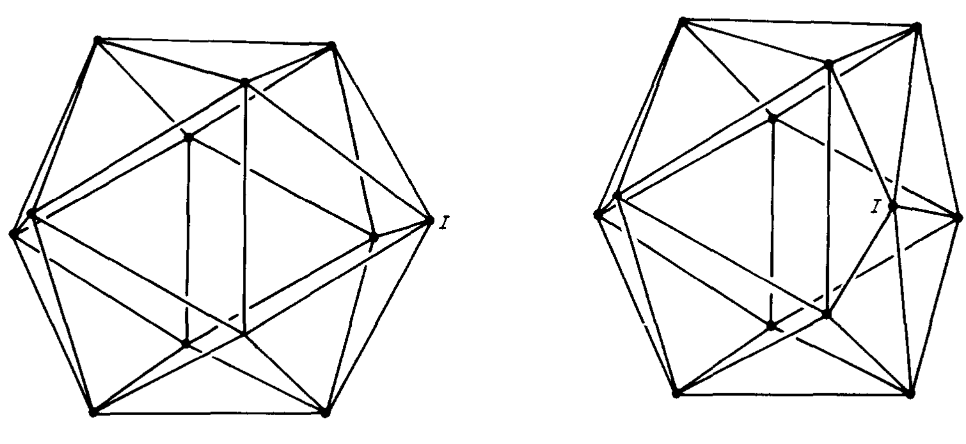

Fig. 2.(a) A frame whose members lie on the edges of a regular icosahedron. (b) as (a), except that the 5 members meeting at joint I have been shortened as much as possible, while each having the same length. The 10 bars nearest to I lie in a plane.

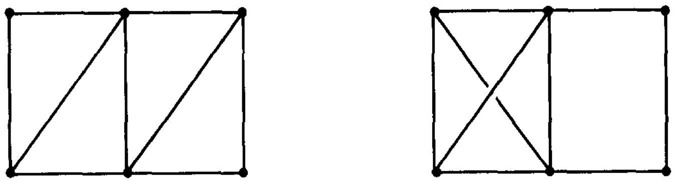

Fig. 3. Two plane frames which satisfy Maxwell's rule $b = 2j - 3$. (a) is simply stiff, but (b) is part redundant and part mechanism.

Fig. 2(b) the mechanism and the state of self-stress both occur in a localised part of the frame. In other examples, such as those cited by Timoshenko and Young ([3], p. 81), Parkes ([5], p. 44) and the infinitesimally deformable octahedron described by Wunderlich [17] the entire frame is involved in both the mechanism and the state of prestress.

The matrix algebra indicates the existence of infinitesimal modes. If we wish to investigate the response of the frame of Fig. 2(b) to a force applied at joint I normal to the plane of the junction, we are immediately in the realm of nonlinear behaviour, and the linear equations used so far are inadequate. When there is no prestress "the stiffness is of an inferior order", as Maxwell noted, and in fact the restoring force is proportional to the third power of the displacement. This is seen most easily by examining the simplest frame having essentially the same properties, which is shown in Fig. 4. There are 3 bars, and the joints are collinear in the original, undisturbed, configuration. A transverse displacement Δ of the central joint I elongates the centre-lines of the 2 short bars by an amount $\Delta^2/2a$ relative to the longer bar. The tension T in the short bars is numerically equal to the compression in the long bar, and so we find $T = AE\Delta^2/4a$. Resolving vertically at joint I we find $P = AE\Delta^3/2a^2$ for this frame. A similar expression, but with a different numerical constant, applies for the frame of Fig. 2(b).

On the other hand, the situation is different when there is a state of self-stress. For the structure of Fig. 4, a prestress involving tension T_0 in the two short bars enables joint I to support an external load of magnitude $2T_0\,\Delta/a$ when there is a small displacement Δ, and there is an analagous effect when the frame of Fig. 2(b) is prestressed. Note that in these cases the stiffness of the joint involves the level of prestress and the kinematics of the infinitesimal mode, and not the cross-sectional area of the bars or the modulus of elasticity.

It should be noted that in the case of Figs. 2(b) and 4 the self-stress must involve tension in the bars meeting at I, and not compression: although the equilibrium equations can be satisfied if the signs of the self-stress are reversed, the equilibrium will be unstable, and the frame will tend to "snap" into a stress-free, out-of-plane location for joint I. The particular sign of the self-stress is, of course, determined by the fact that we make the bars meeting at I just a little too short to meet, and connect them by straining them elastically.

Let us summarise our findings so far. A framework which satisfies Maxwell's rule and has an infinitesimal mechanism also has a corresponding state of self-stress or prestress. In the absence of prestress the mechanism has zero stiffness for infinitesimal displacements, even if the bars are rigid. In the examples considered the mode is only infinitesimal, and there is "low order" stiffness proportional to Δ^2 for small deflection Δ. The constant of proportionality depends on the cross-sectional area of the bars and the modulus of elasticity of the material. In contrast, if the assembly is prestressed, the mode (being infinitesimal) is endowed with stiffness proportional to the level of prestress.

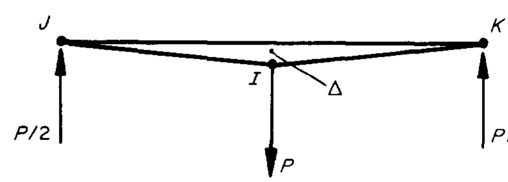

Fig. 4. A simple frame exhibiting an "infinitesimal mode". The stiffness of the frame in response to the self-equilibrating external forces depends crucially on the amount of prestress in the unloaded frame. In the undeformed configuration $IJ = IK = a$, $JK = 2a$. The cross-sectional area of all bars is A, and the Young modulus is E.

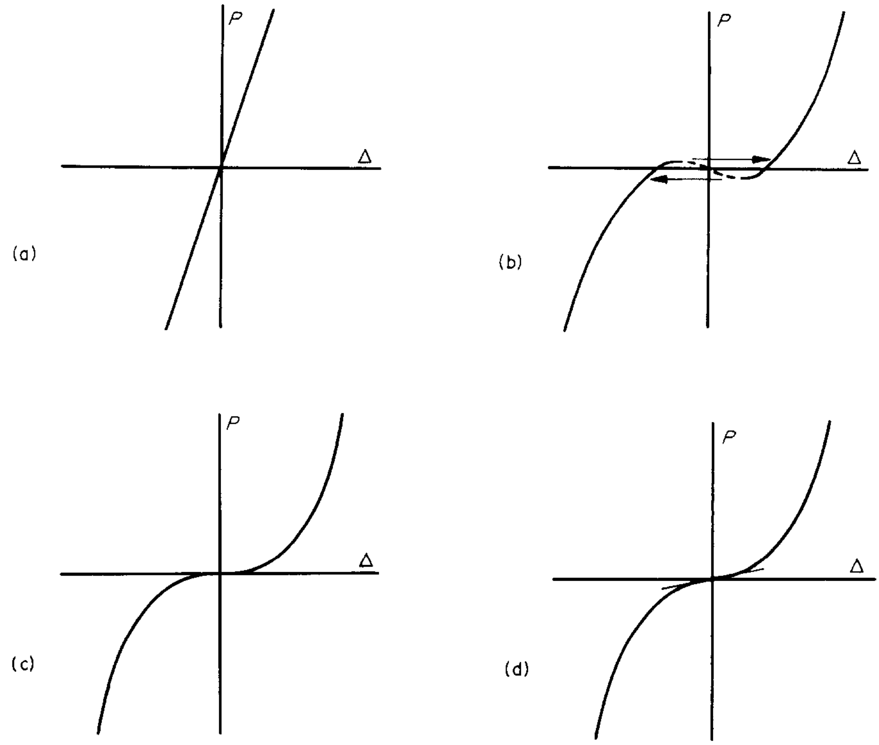

Fig. 5. Force (P), displacement (Δ) relation for joint I of the frames of Figs. 2 (and 4). (a) "Normal" case, Fig. 2(a); the P, Δ relation is linear. (b) The radial bars are a little longer than those which would meet at I in a plane. The P, Δ relationship is a cubic, with two turning-points marking the ends of a zone of unstable equilibrium (------), and snap-through into an inverted configuration can occur ($\rightarrow$). The origin is at the point corresponding to the (unstable) flat configuration. The linear curve of (a) corresponds to a short section of this curve on the left near the Δ axis. (c) The radial bars are exactly the length to intersect at I in a plane. The P, Δ relationship is $P = \text{constant}.\Delta^3$, and the slope is zero at the origin. (d) The radial bars are a little shorter than in case (c), and prestress is required to assemble the frame. The P, Δ relationship of (c) is augmented by a linear term, which has the effect of "tilting" it. Note that curve (b) has tilt in the opposite sense.

Figure 5 shows schematically the four kinds of force, displacement relation which we have encountered for joint I in the frames of Figs. 2 and 4.

BEHAVIOUR OF FRAMES WHICH HAVE TOO FEW BARS TO SATISFY MAXWELL'S RULE

We now turn to frames with fewer bars than required by Maxwell's general rule. Consider first the frame of Fig. 1, but with the inner bars removed. The members can be made to lie along the edges of a tetrahedron which has had its vertices cut off to give triangular faces. There are 18 bars and 12 joints, so the frame is short of satisfying Maxwell's rule by 12 members; it is consequently a mechanism with a large number of degrees of freedom. However, like various networks of strings, it is capable of being in a configuration of stable equilibrium under outward-directed forces applied at the nodes, which put all members in tension. In this connection Fuller[8] thinks in terms of fabric balloons being held stretched out by the constant motion of gas molecules, fish nets being expanded by fish darting back and forth, and finally fish nets being held out by interior rods in compression. Geo-*D*-Stix bars enable us to realise this idea easily. By inserting diagonals, say, 10% longer than required, we make them buckle as Euler struts, and they supply roughly constant compressive force. (For Geo-*D*-Stix bars longer than about 40 cm the Euler load is insufficient to pull the shorter bars out of their joint sockets). When the correct form has been found, the diagonals can be replaced by straight rods of exactly the right length. For the frame of Fig. 1 it is not difficult to show by geometry that the longest possible length of equal diagonals is 2.25 × the length of the (equal) shorter bars. Clearly the outer net of bars, which are in a state of tensile prestress, could be replaced by wires, as in Fuller's Tensegrity structures. In our final physical model we used Geo-*D*-Stix bars for the outer members, and specially made thicker bars for the diagonals. Each of these special bars had a projecting spigot of diameter 3 mm at each end; this made it possible to vary slightly the effective lengths of these bars, and so to adjust the prestressing force.

It is instructive to find what conclusions can be drawn from matrix algebra for a frame not necessarily satisfying Maxwell's rule. Again let $n = 3j$ and $m = b + 6$, and let r be the rank of both the equilibrium matrix $\mathbf{H}$ and the compatibility matrix $\mathbf{H}^T$. Clearly r is less than or equal to the smaller of m and n. Now let s be the number of independent states of self-stress and let q be the number of infinitesimal mechanisms of the framework. By application of the Dimension Theorem, as before, we find

$$s = m - r \geqslant 0; \; q = n - r \geqslant 0. \tag{5}$$

Eliminating r, we have $m - n = s - q$, i.e.

$$b - 3j + 6 = s - q. \tag{6}$$

For frames satisfying Maxwell's rule we recover the previous result $s = q$.

Now if we find a frame, such as that of Fig. 1, which has $b - 3j + 6 < 0$ and is apparently stiff, we must conclude (a) that the geometrical relationship of the member lengths is such that the modes of distortion are only infinitesimal and (b) that there is one state of prestress (at least) which endows the modes with an appreciable first-order stiffness.

To illustrate this conclusion let us consider first the two-dimensional four-bar assemblies shown in Fig. 6: these constitute the simplest example of such a frame. For two-dimensional frames the equation corresponding to (6) is

$$b - 2j + 3 = s - q. \tag{7}$$

For the frames of Fig. 6, $b = j = 4$; so $q = s + 1$. In the frame of Fig. 6(a) no two consecutive bars are co-linear: therefore it is only possible to satisfy the joint equilibrium equations for zero external forces if all of the bar tensions are zero. Thus $s = 0$, and consequently $q = 1$, which is a well-known result for a two-dimensional "four-bar chain". On the other hand, the frame of Fig. 6(b) has $s = 1$. Here all of the members are connected to their neighbours with included angles 0 or 180°, and the state of selfstress under zero external forces involves compression in the long member AD and tension in the three short members. It follows from (7) that $q = 2$. The two infinitesimal modes involve independent small displacements of joints B and C normal to AD, and they are both given a first-order stiffness, in the manner of Fig. 5(d), by prestress in the frame.

The obvious special feature of the frame shown in Fig. 6(b) is that all of the members are co-linear. However, it is more instructive, in relation to the understanding of the behaviour of more complex frames, to think in terms of a transformation of a frame like that shown in Fig. 6(a) to configuration (b) by means of a progressive elongation of (any) one member while all others retain their original dimensions. This process must stop when (in the notation of Fig. 6(b)) $AD = AB + BC + CD$. It is obvious that any attempt to make member AD longer than this

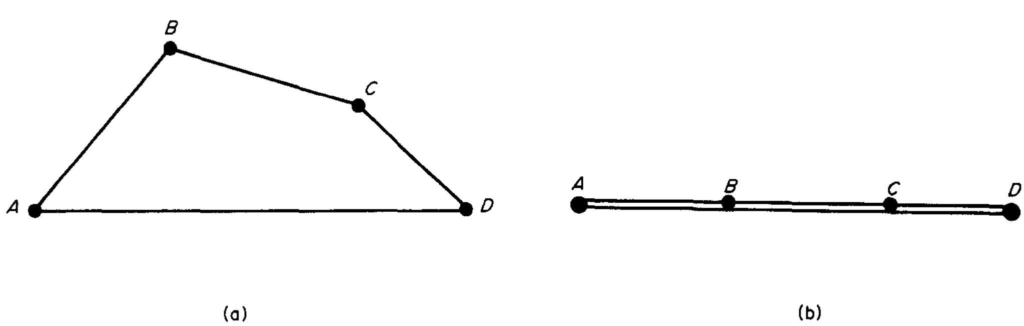

Fig. 6. A simple two-dimensional frame which has a single mode in configuration (a) but two (infinitesimal) modes—and a corresponding state of prestress—in configuration (b).

will be resisted by the setting up of a state of prestress of the kind which we have described. Note particularly that a condition of co-linearity for the frame in which $AB + BC = AD + CD$ (not illustrated) would also admit of a state of prestress; but the prestress could readily be "escaped from" by a lateral displacement of the two compressive members. Prestress in such a frame would *not* stabilise the infinitesimal modes. Neither, indeed, would a state of prestress in the frame of Fig. 6(b) in which the long member AD is in tension and the other members are in compression. The key to the situation (Maxwell's[1] remarks) is the idea that the geometry of the layout is such that the length of one member is *maximum*; and the corresponding state of prestress is achieved by attempting to make the length of this bar even greater.

Returning to the frame of Fig. 1, we find by inspection of the physical realisation that there is only one state of prestress possible, with each of the long diagonals carrying an equal compressive force. Consequently eqn (6) indicates that there are seven distinct infinitesimal modes of the framework. These will be most clearly seen in a physical model when the frame is adjusted so that it has zero prestress.

It is not difficult to find these seven modes, with a little help from the obvious symmetry of the frame. The simplest mode involves small rotation of the triangle ABC (Fig. 1) about the axis of 3-fold symmetry passing through its centre, with relatively small rotations of the other triangles. As there are four 3-fold axes, we have found four of the seven mechanisms. The remaining three mechanisms are associated with the three axes of 2-fold symmetry of the framework. One of these axes passes through the mid-points of members FC and ED in Fig. 1, and the mechanism is characterised by a distortion of the warped quadrilateral $CDEF$ with bar CD moving a small distance in the general direction CD while bar EF moves correspondingly in the direction EF.

It is important to realise that the number of infinitesimal mechanisms is not a consequence of the symmetry of the framework but is, as we have shown, a function of the numbers of bars and joints. This is readily demonstrated with a Geo-*D*-Stix model. By replacing a few outer members of the frame of Fig. 1 by bars of different length and making sure that the inner bars are as long as possible we can make a non-symmetric frame which has seven modes of infinitesimal distortion which are of precisely the same general character as those described above. The role of symmetry was merely to provide clues in the search for seven distinct modes.

It is of course remarkable that a single state of prestress should stabilise all seven infinitesimal modes of the frame shown in Fig. 1. We saw in the example of Fig. 6(b) how a single state of prestress could stabilise two infinitesimal modes. Indeed, that example may readily be generalised by the insertion of additional bars in the chain of short members which cannect the ends A, D of the single long member. Each additional joint introduces an extra degree of freedom, but whatever the number of infinitesimal modes, each is given some first-order stiffness by the single state of prestress.

It is difficult to appreciate the modes of distortion of the frame of Fig. 1 without a physical model. For the benefit of those who are not in a position to construct such a model we give in Fig. 7 a simpler example, of a frame whose "outer" members form the edges of a cube and whose inner members are the four space diagonals. A frame of this sort can be assembled with shorter diagonals, but it is a "floppy" mechanism with two degrees of freedom: $b = 16$ and $j = 8$, so $b - 3j + 6 = 2$, and as no prestress is possible, $q = 2$. However, when the diagonals—which we make of equal length for convenience—are lengthened, they reach a maximum length, consistent with the integrity of the outer members, when the outer members form a cube, as is easily shown by elementary calculus. At this point we have a "Tensegrity" structure, and a single state of prestress is possible, with the diagonals in equal compression and the edge bars in equal tension. Consequently $s = 1$ and $q = 3$.

(There is of course a problem over the assembly of a frame of this sort, as the diagonals tend to interfere at the centre. The difficulty can be overcome in principle by putting short bends in the diagonals, as indicated in Fig. 7. This circumstance makes the example of Fig. 1 better for classroom demonstrations.)

One of the three independent mechanisms may be visualised by supposing that joints A, A', C and C' (Fig. 7) are held fixed, while the other four joints move equal amounts in the direction BB'. It is easy to check that the lengths of all bars are unchanged, to a first order of

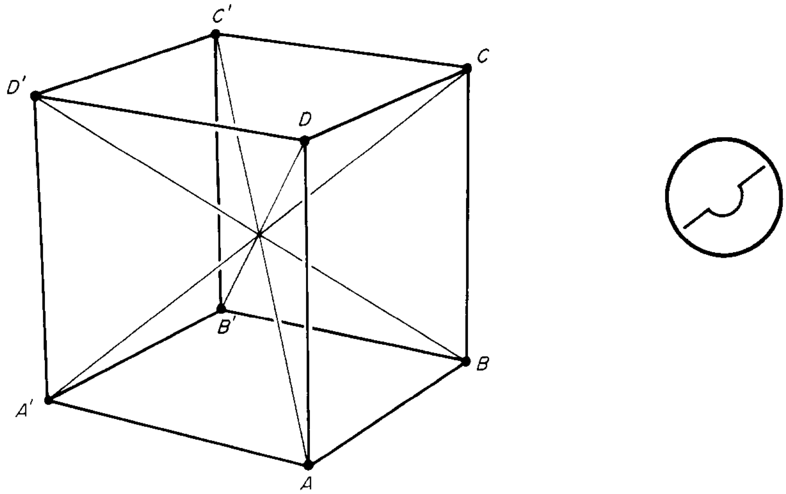

Fig. 7. A frame in the form of a cube with four space diagonals. Inset is a schematic sketch of the detailed modification of the diagonals necessary to avoid interference at the centre.

approximation, by a mechanism of this sort. Clearly there are three independent mechanisms of this sort, with motion parallel to the three perpendicular edge directions, respectively.

It is easy to see by inspection of one of these mechanisms that prestress has a stiffening effect: this comes essentially from the inclination of the bars originally in the planes normal to the common displacement direction.

DISCUSSION

So far we have discussed two particular examples of "Tensegrity" structures. In Table 1 we record particulars of the frame of Fig. 1 (at (a)) and four other examples shown on pp. 160–161 of [8]. We have listed the numbers of tensile wires (w) and compressive rods (r) separately, although they both count as "lines" in Maxwell's terminology.

In examples (a), (c) and (d) we established $s = 1$ by direct experimentation on physical models. If $s = 1$ a state of prestress is not possible if a single wire is cut, and the frame is rendered floppy. This is precisely what happens. In example (c) (which is not exactly icosahedral in spite of its name) the single mode is very clear and involves equal motion of the three sets of parallel bars towards each other. The single mode for (d) is equally striking, with the surface triangles rotating clockwise and anti-clockwise like a set of meshed gear wheels. In fact examples (c) and (d) are closely related. They have identical outer nets and differ only in the arrangement of pairs of joints spanned by the diagonals. This relationship is not brought out by Fuller's picturesque nomenclature. In example (e) the outer members would, if made of rods, constitute an example of well-behaved framework satisfying Maxwell's rule. The three diagonals are redundant bars in the usual sense.

Thus we see that example (e) is an orthodox redundant framework in which prestress has been used to enable 4/5 of the members to be made of wire rather than rods. Examples (c) and (d) are classical "ill-conditioned" frames of this kind studied by Möbius and others[12]. Fuller has exploited the possibility of prestress again to make 4/5 of the members of wire, but at the expense of introducing an infinitesimal mechanism which is stiffened only by the prestress.

The number of states of prestress in example (b) is at least 1 and possibly more. It would be pleasant if we could use symmetry properties to exclude certain possibilities. For example if $s = 3$, $q = 57$. The number of mechanisms for a frame having icosahedral symmetry may be expressed by $q = 6i + 10j + 15k$, where i, j and k are positive integers. $q = 57$ is certainly a possibility (i, j, k = 2, 0, 3), so we cannot exclude $s = 3$ on these grounds. In fact we cannot exclude *any* value of s on these grounds alone, as all integers above 29 may be generated in this way.

We could, in principle, find whether $s = 1$ by building a physical model and cutting a string. We have not done this. In this connection we observe that the easiest way to build a model is to use elastic strings for the tension members. With a model of this sort the test of cutting a string to find the value of s can be misleading, as the lengths of the other strings can easily adjust and

Table 1.

	MARKS' (1960) designation	Originator	Description	w	r	b = w+r	j	b − 3j + 6 = s−q	s	q
a	K8	della Sala	tensegrity tetrahedron	18	6	24	12	-6	1	7
b	K11	Hogden	tensegrity tricontahedron	90	30	120	60	-54	1(?)	55(?)
c	K9	Fuller	tensegrity icosahedron	24	6	30	12	0	1	1
d	K10	Moelman	tensegrity vector equilibrium	24	6	30	12	0	1	1
e	K8A	Pope	tensegrity octahedron	12	3	15	6	3	3	0

adapt to the loss of a member by changing the configuration (i.e. by changing the design of the entire structure). Our theory, of course, presupposes that the strings, like the bars, are strictly inextensible, and this presents a severe difficulty in the study of large frameworks. The most promising approach to finding the value of for a large assembly is to investigate the rank of the relevant matrices. This is a topic for further research.

In the course of preparation of a revised version of this paper, J. Bunce brought to our attention the work of Buchholdt, Davies and Hussey[19] on the analysis of cable nets which includes a derivation of an expression which is the counterpart of eqn (6) for a cable net anchored to rigid abutments at its periphery. In this kind of problem the number of independent prestressing states is readily determined. These workers do not seem to have taken advantage of the fact that the stiffness of each mode is determined mainly by the level of prestress, with the elastic properties of the material and cross-sectional area of the wires having only second-order effects.

In the present paper we have used linear algebra to determine the basic relation (6) between the number of bars, joints, mechanisms and states of prestress, and have then used intuitive methods and illustrative examples to study the essentially nonlinear mechanics of deformation in a typical mode. All of our examples illustrating modes of deformation of frameworks admitting prestress have involved only *infinitesimal* mechanisms in contrast to the more usual kind of *finite* mechanism which is commonly found when a bar is removed from a just-stiff framework. Such a mechanism involves absolutely no change in length of any member, and in consequence no state of prestress can impart stiffness. We have not found any but the most trivial examples of this circumstance in any frameworks "free in space", but there are some simple examples in the work of Föppl[20] of frameworks connected to a rigid foundation which satisfy the appropriate form of Maxwell's rule and yet constitute a finite mechanism. The simplest example consists of "tripod" of members connecting a joint to three points on a rigid foundation. When the three foundation points are collinear the entire plane frame is free to pivot about this line. Prestress is possible, but it gives no stiffness to the mode. The "trivial" examples referred to above are ones in which the three foundation points are joints of an orthodox rigid frame "free in space". Geo-*D*-Stix physical models provide the most direct way of investigating these exceptions to the "tensegrity principle".

Clearly the "Tensegrity" structures display Fuller's ingenuity in designing large but easily packed and portable structures: an obvious saving in weight accrues from the fact that about three-quarters of the members are wires rather than rods.

On the other hand, if the aim is to design economical but stiff engineering structures it is not clear that there is much point in making the outer network so sparse that the resulting frame has a number of infinitessimal modes whose stiffness is necessarily low. It would be interesting to study experimentally a frame such as (c) with and without the additional wires which could make the "net" itself satisfy Maxwell's rule. One area for investigation would be how difficult it would be to control the prestress in a genuinely redundant structure.

Acknowledgements—I thank Dr. A. Klug, who aroused my interest in Buckminster Fuller's structures: he has used these to illustrate some points in relation to the construction of spherical virus coats[21]. The present work may conceivably have a bearing on this sort of problem if it can be extended beyond the realm of bars and wires to cover assemblages of three-dimensional blocks. I also thank Dr. R. P. Burn, Dr. R. A. Crowther, Dr. H. T. Croft, Prof. J. Heyman, Prof. M. S. Longuet-Higgins, Dr. A. C. Palmer, Dr. J. M. T. Thompson and Dr. J. M. Prentis for helpful discussions. Finally, I thank Dr. C. W. F. Everitt for stimulating my interest in Clerk Maxwell.

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