Appendix B Properties of Expectations and Variances

Let Y denote a random variable that takes on values according to some probability density function if Y is continuous or some probability mass function if Y is discrete. The *expected value*, or *expectation*, of Y is simply its *mean* or average value and is usually denoted by

$$\mu = E(Y).$$

It is often referred to as the first *moment* of Y, since it describes the location of the center of the distribution. The precise definition of the expectation of Y is that it is a *weighted* average of all the possible values of Y, with weights determined by the probabilities associated with each possible value.

The *variance* of Y, often denoted by $\sigma^2 = \text{Var}(Y)$, is a measure of the dispersion or variability around the mean or expected value of Y. The variance is often referred to as the second *central moment* of Y and is defined as

$$\sigma^2 = \text{Var}(Y) = E\{Y - E(Y)\}^2.$$

The variance is a weighted average of the squared deviations of Y around its mean. Because the variance is expressed in squared units of Y, a measure of variability in the original units of Y is given by the *standard deviation*

$$\sigma = \sqrt{\text{Var}(Y)}.$$

Finally, the covariance between two random variables, X and Y, is defined as

$$\text{Cov}(X, Y) = E\left[\{X - E(X)\}\{Y - E(Y)\}\right],$$

and is a measure of the *linear dependence* between X and Y. If X and Y are *independent*, then $\text{Cov}(X, Y) = 0$. Note that the covariance of a variable with itself is the variance, $\text{Cov}(Y, Y) = \text{Var}(Y)$.

Properties of Expectations and Variances

Next we consider some properties of expectations and variances. Let X and Y be two (possibly dependent) random variables and let a and b denote non-random constants. Then the expectation operator, $E(\cdot)$, has the following five important properties:

1. $E(a) = a$
2. $E(b\,X) = b\,E(X)$
3. $E(a + b\,X) = a + b\,E(X)$
4. $E(a\,X + b\,Y) = a\,E(X) + b\,E(Y)$
5. $E(XY) \neq E(X)\,E(Y)$ (unless X and Y are *independent*)

Thus expectation is a linear operator in the sense that it respects or preserves the arithmetic operations of addition and multiplication by a constant. As a result the expected value of a linear function of Y (e.g., $a + b\,Y$) is simply the same linear function of the expected value of Y (e.g., $a + b\,E(Y)$).

The variance operator, $\text{Var}(\cdot)$, has the following five important properties:

1. $\text{Var}(a) = 0$
2. $\text{Var}(b\,Y) = b^2\,\text{Var}(Y)$
3. $\text{Var}(a + b\,Y) = b^2\,\text{Var}(Y)$
4. $\text{Var}(a\,X + b\,Y) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) + 2\,a\,b\,\text{Cov}(X, Y)$
5. $\text{Var}(a\,X - b\,Y) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y) - 2\,a\,b\,\text{Cov}(X, Y)$

In particular, if X and Y are dependent, then

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) + 2\,\text{Cov}(X, Y)$$

and

$$\text{Var}(X - Y) = \text{Var}(X) + \text{Var}(Y) - 2\,\text{Cov}(X, Y).$$

Finally, we note that the expectation and variance operators can also be applied to vectors of random variable. For example, let Y be a $n \times 1$ (column) response vector (e.g., repeated measurements at n different occasions),

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix},$$

then

$$E(Y) = \begin{pmatrix} E(Y_1) \\ E(Y_2) \\ \vdots \\ E(Y_n) \end{pmatrix},$$

and

$$\mathrm{Cov}(Y) = \begin{pmatrix} \mathrm{Var}(Y_1) & \mathrm{Cov}(Y_1, Y_2) & \dots & \mathrm{Cov}(Y_1, Y_n) \\ \mathrm{Cov}(Y_2, Y_1) & \mathrm{Var}(Y_2) & \dots & \mathrm{Cov}(Y_2, Y_n) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{Cov}(Y_n, Y_1) & \mathrm{Cov}(Y_n, Y_2) & \dots & \mathrm{Var}(Y_n) \end{pmatrix}.$$