Edge Coloring

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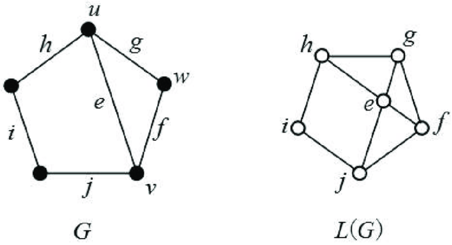
What is edge coloring ?

An edge coloring of a graph G is a mapping f :E(G) $\rightarrow$ S.The elements of S are colors . The edges colored by same color foem a color class . If $|S| = k$, then f is a k-edge coloring of G.

Line Graph

A line graph L(G) of a simple graph G is obtained by associating a vertex with each edge of the graph and connecting two vertices with an edge iff the corresponding edges of G have a vertex in common



Important Points

- Every edge coloring problem can be converted into a vertex coloring problem.
- Coloring the edges of a graph G is same as coloring the vertices of the line graph L(G)

In general , $\chi(G) \geq \Delta$ where $\Delta =$ max. degree of a vertex in G for any graph G .

Bipartite Graphs

A bipartite graph is a graph in which the vertex set can be partitioned into two sets such that edges only go between sets, not within them.

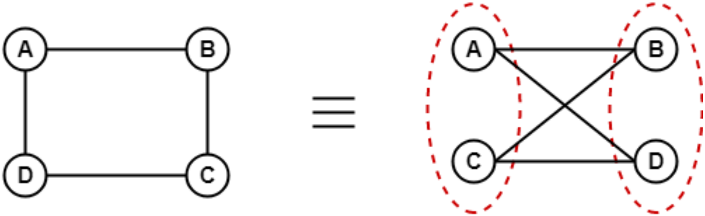
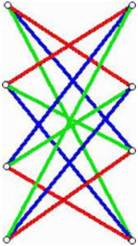


Figure: Fig 5 : Bipartite Graph

Bipartite Graphs

Konig's Theorem

If G is bipartite , then $\chi(G) = \Delta$ where $\Delta =$ max. degree of a vertex in G



Let m = no. of edges in G
Base Case: m = 1
It is trivially true

So , let's assume the statement is true for any graph G with m edges

Inductive Step: Let there be m+1 edges in a graph G with max. degree = Δ

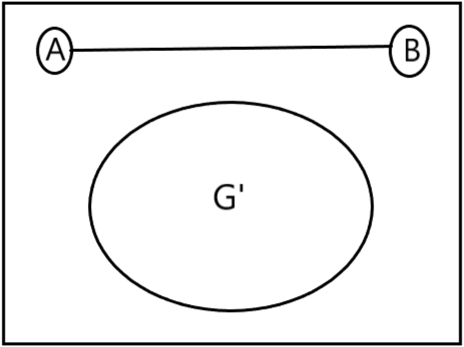
Claim: $\chi(G) = \Delta$

Proof

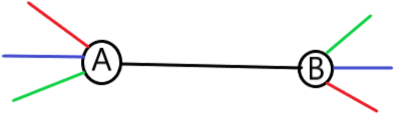
Remove an arbitrary edge say (A,B) from G , call this subgraph G'

By Induction Hypothesis , G' can be colored with Δ colours .

G

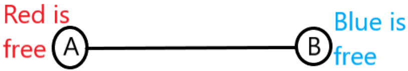


Since max. degree $= \Delta$, deg(A) $\leq \Delta - 1$
and deg(B) $\leq \Delta - 1$ in G'
It means A and B can also use Δ colors each and so , both A and B each have one unused color , α and β respectively.

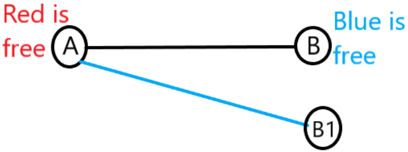


Case 1 : $\alpha = \beta$,
then we can use this colour to colour (A,B)

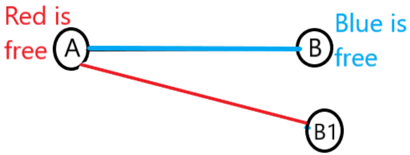
Case 2 : $\alpha \neq \beta$



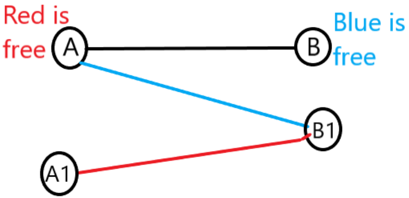
Clearly , there is a blue edge incident on A (otherwise it would be case 1) , let that edge be incident on some vertex B1.



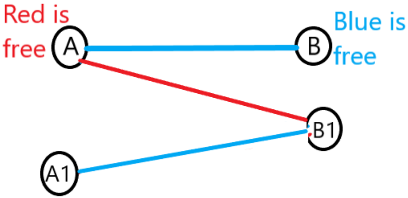
Subcase 2.1 : There is no red edge incident on B1 . It means Red is available at B1 .
So , color (A,B1) with red .



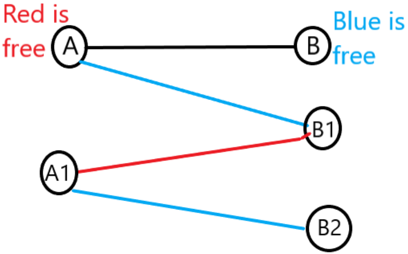
Subcase 2.2 : There is a red edge incident on B1 .

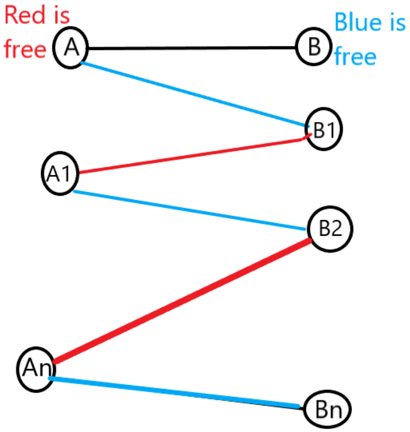


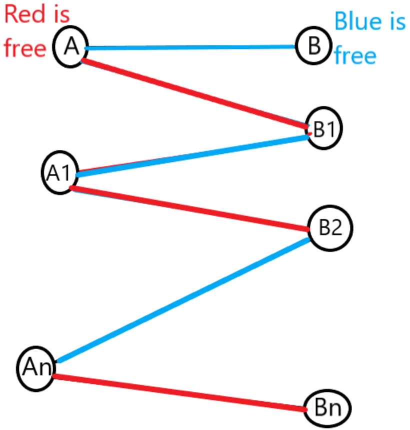
Subcase 2.2.1 : There is no blue edge incident on A1 .



Subcase 2.2.2 : There is a blue edge incident on A1 .




Red is
free
A
B
Blue is
free
B1
A1
B2
An
Bn



So , (A,B) gets coloured without using any extra color .
Therefore $\chi(G) = \Delta$

Thus , the proof of Konig's Theorem is complete .

References

- Graph Theory with Applications to Engineering and Computer Science by Narsingha Deo
- GRAPH THEORY WITH APPLICATIONS by J. A. Bondy and U. S. R. Murty

THANK YOU