Hall's Theorem

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Problem Statement

A set of job applicants have applied for a set of jobs. We want to assign , to each job,an applicant who is qualified for it. Under which conditions is this possible ?

Example

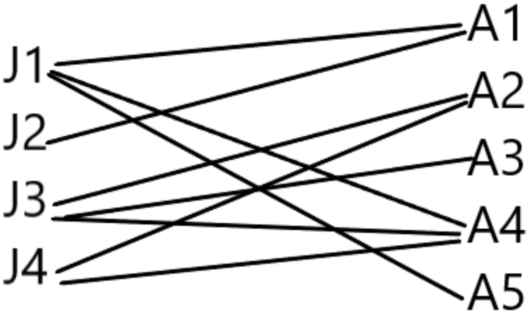
Applicants $a_1, a_2, ..., a_5$ have applied for jobs j_1, j_2, j_3 and j_4

Job	**Qualified Applicants**
j_1	a_1, a_4, a_5
j_2	a_1
j_3	a_2, a_3, a_4
j_4	a_2, a_4

A solution would be 1-1 correspondence between the set of J jobs and a subset of the set A of applicants .

Mathematical Model

Bipartite graph $G(J, A)$(i.e with J jobs , A applicants) with an edge from j $\in$ J to a $\in$ A $\iff$ a is qualified for j.


A1
A2
A3
A4
A5
J1
J2
J3
J4

A **complete matching** from J to A is a 1-1 correspondence between J and a subset of A , such that the corresponding vertices are adjacent.

Question

When does a complete matching exist ?

Necessary condition

Each set of k jobs must have at least k "jointly qualified" applicants i.e. each of them is qualified for at least one of these jobs . This must hold for every $1 \leq k \leq |J|$

The necessary condition is also sufficient

Theorem(Hall , 1935)

In the bipartite graph G(V1,V2) , there is a complete matching from V_1 to V_2 if and only if for each subset A $\subset V_1$, $|N(A)| \geq |A|$

Proof

Necessary Part

To prove : There exists a complete matching from V_1 to V_2 only if $|N(A)| \geq |A|$

$\approx$

if $|N(A)| < |A|$, then no complete matching from V_1 to V_2

$\approx$

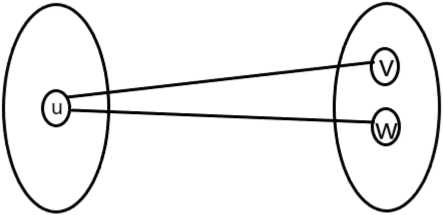
if there exists complete matching from V_1 to V_2 then $|N(A)| \geq |A|$

Let M be a complete matching from V_1 to V_2

Every node in A must be the end point of some distinct edge in M.
$\implies |N(A)| \geq |A|$

Proof(Sufficiency Condition–Induction on $|V_1|$))

We will give an existential proof . **Base Case :**$|V_1| = 1$



Given the condition , the only subset A $= \{u\}$ will have at least one neighbour . Trivially ,there exists a complete matching - An edge whose end point is u (either (u,v) or (u,w)) in this case.

Proof(Sufficiency Condition–Induction on $|V_1|$)

Inductive Hypothesis :

For every bipartite graph G = (V,E) with bipartition (V_1, V_2),where $V_1 \leq k$, such that $|N(A)| \geq |A|$ for all A $\subset V_1$, a complete matching from V_1 to V_2 exists.

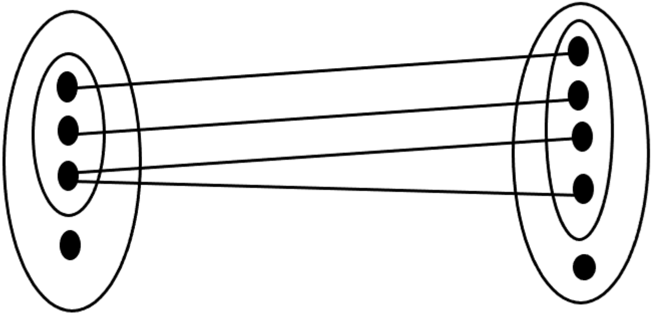
Inductive Step :

Consider a bipartite graph G with bipartition (V_1, V_2) where $|V_1| = k + 1$,such that $|N(A)| \geq |A|$ for all A $\subset V_1$

Goal :

To show the existence of a complete matching from V_1 to V_2

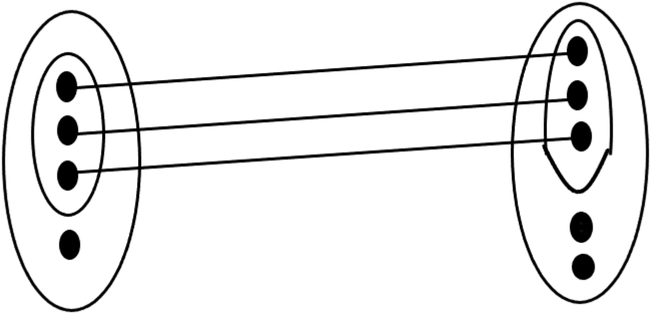
Case I : Every k-sized subset A of V_1 has at least k+1 neighbours in V_2.



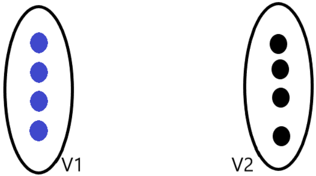
$\{k = 3$ for demonstration purpose only $\}$

Case II :

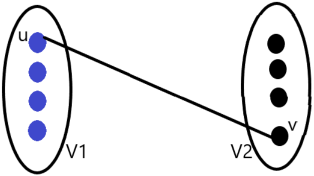
There is a k sized subset of V_1 that has exactly k neighbours.



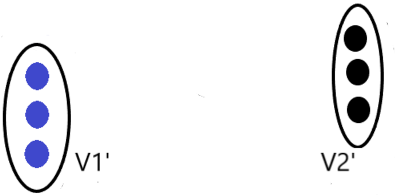
Case I



Consider any vertex u $\in V_1$ and one of its adjacent edges in V_2,say v



Let $V_1' = V_1$ - $\{u\}$ and $V_2' = V_2$ - $\{v\}$

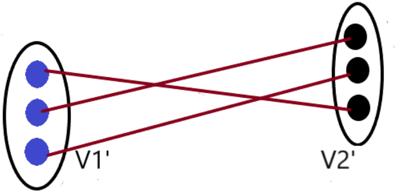


(V_1', V_2') : bipartition of the reduced graph

$|V_1'| = k$ and

Claim :$|N(A)| \geq |A|$ for all A $\subset V_1$
i.e. V_1' has at least k neighbours in V_2'

Since V_1 has at least k+1 neighbours in V_2 , so it's trivially true.

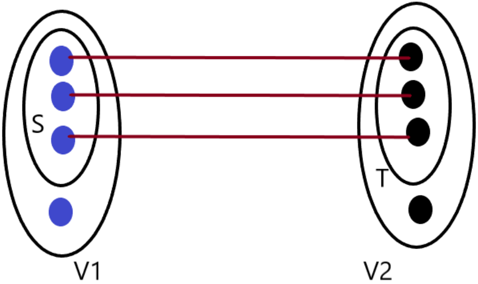


From inductive hypothesis , there is a complete matching , say M,from V_1' to V_2'

Now M $\cup\{(u, v)\}$:complete matching from V_1 to V_2

Case II

Let S $\subset V_1$ such that $|S| = k$ and T $=$ N(S) with $|T| = k$.
By Inductive Hypothesis , a complete matching , say M ,exists from S to T.

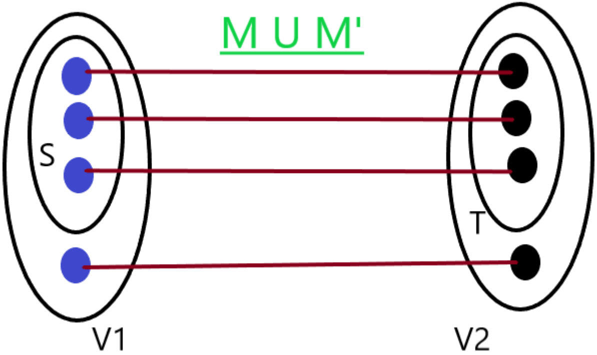


Let $V_1' = V_1$ - S and $V_2' = V_2$ - T. Now , $|V_1'| = 1$ and
<u>Claim :</u> V_1' has at least 1 neighbour in V_2'.

Otherwise It implies that V_1 has only k neighbours in V_2 , (contradiction to $|N(A)| \geq |A|$ for all A $\subset V_1$)

From Inductive Hypothesis , there is a complete matching say M' ,from V_1' to V_2' .

M $\cup$ M' is a complete matching from V_1 to V_2



Thus , Hall's Theorem is proved .

References

- Introduction to Graph Theory by Douglas B.West
- GRAPH THEORY WITH APPLICATIONS by J. A. Bondy and U. S. R. Murty

THANK YOU