Colourability of planar graphs

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Contents

What is a Graph ?

A graph (denoted as $G = (V, E)$) consists of a non-empty set of vertices or nodes V and a set of edges E

Example

Let us consider, a Graph $G = (V, E)$ where $V = a, b, c, d$ and $E = a, b, a, c, b, c, c, d$

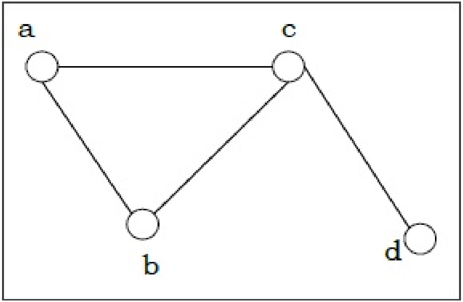


Fig 1 : Graph

Empty/Null Graph

A null graph is a graph in which there are no edges between its vertices.

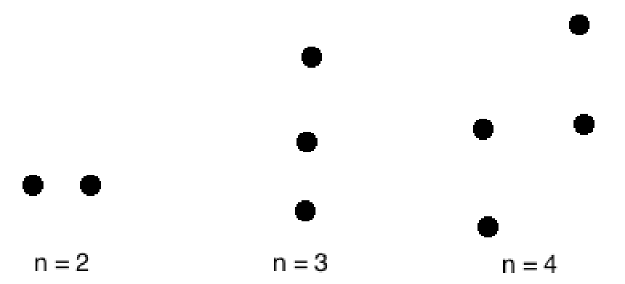


Fig 2 : Empty Graph

Simple Graph

A graph is simple if it has no loops and no two of its links join the same pair of vertices(i.e. no multiple edges). A simple graph which has n vertices, the degree of every vertex is at most n -1.

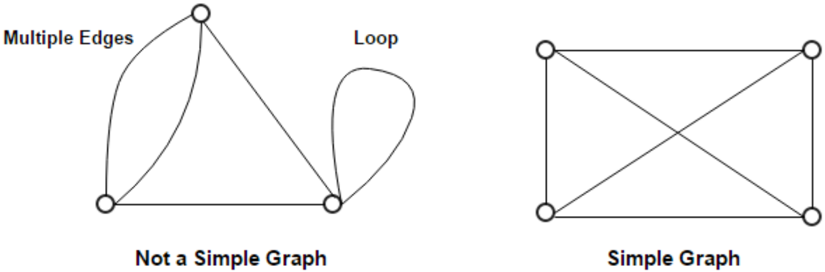


Fig 3

Connected Graphs

A connected graph is a graph in which we can visit from any one vertex to any other vertex. In a connected graph, at least one edge or path exists between every pair of vertices.

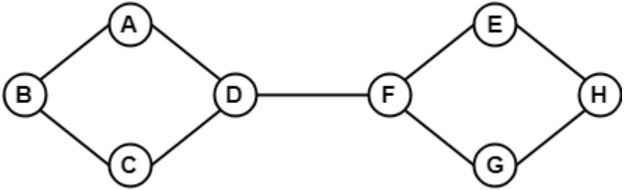


Fig 4 : Connected Graph

Bipartite Graphs

A bipartite graph is a graph in which the vertex set can be partitioned into two sets such that edges only go between sets, not within them .

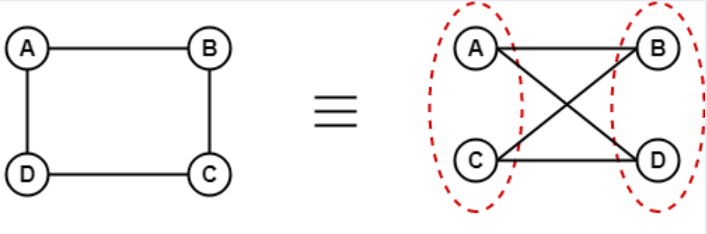


Fig 5 : Bipartite Graph

PLANAR GRAPHS

A planar graph is a graph that we can draw in a plane in such a way that no two edges of it cross each other except at a vertex to which they are incident.

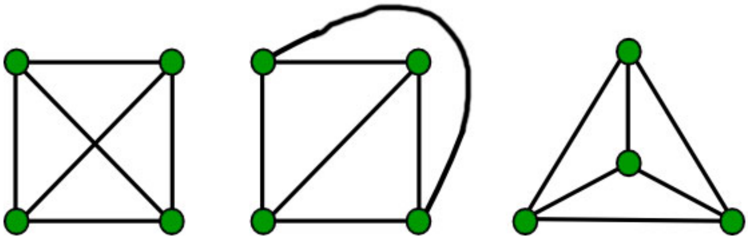


Fig 6 : Planar Graph

Euler's Formula for Planar Graphs

For any connected planar graph with v vertices, e edges and f faces, we have

$$v - e + f = 2$$

It can be proved by induction .

NON PLANAR GRAPHS

A graph is said to be non planar if it cannot be drawn in a plane so that no edge cross.

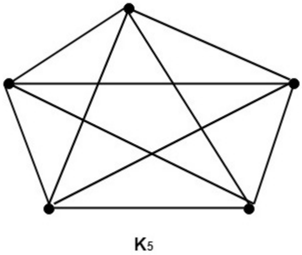


Fig 7 : Non Planar Graph

What is Vertex Colouring ?

Vertex coloring is the procedure of assignment of colors to each vertex of a graph G such that no adjacent vertices get same color.

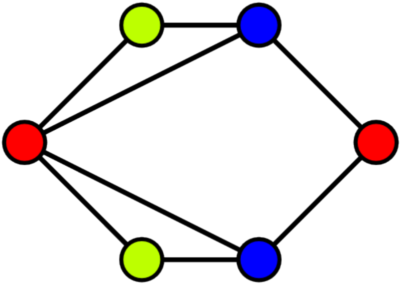


Fig 8 : Vertex Colouring

Remark

The objective is to minimize the number of colors while coloring a graph.

We will restrict our discussion to simple graphs .
Why ?

What is a subgraph ?

A graph H is a subgraph of G (written H $\subset$ G) if $V(H) \subset V(G)$ and $E(H) \subset E(G)$

What is a spanning subgraph ?

A spanning subgraph (or spanning supergraph) of G is a subgraph(or supergraph) H with $V(H) = V(G)$.

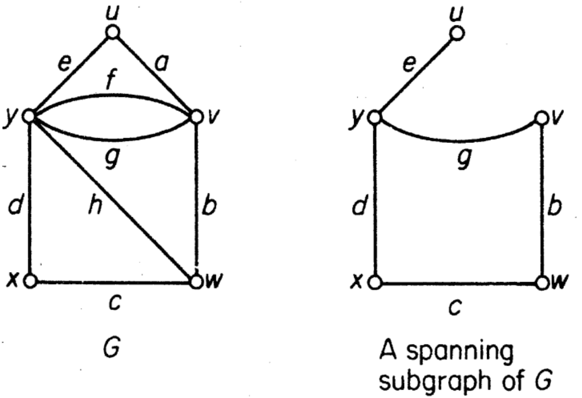


Fig 9 : Spanning subgraph

Underlying Simple Graph

By deleting from a graph G all loops and, for every pair of adjacent vertices , all but one link joining them, we obtain a simple spanning subgraph of G, called the underlying simple graph of G

A graph and its underlying simple graph

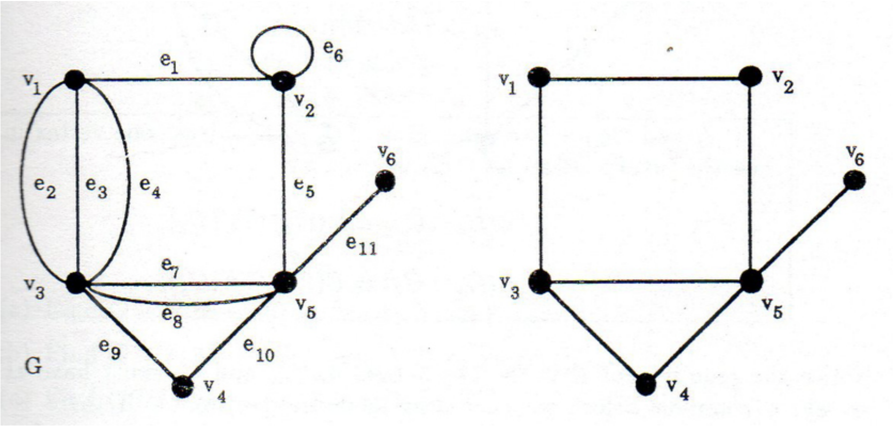


Fig 10 : Underlying Simple Graph

Remark

A graph can be vertex coloured with k colours iff its underlying simple graph can be vertex coloured with k colours .

Chromatic Number

The chromatic number $\chi(G)$ is the minimum positive integer k for which G is k colourable .
if $\chi(G) = k$, G is said to be k-chromatic.

Some Important Observations

1) A simple graph is 1-colourable if and only if it is empty .

2) A simple graph is 2-colourable if and only if it is bipartite

Six Colour Theorem

The vertices of every planar graph can be colored using 6 colors in such a way that no pair of vertices connected by an edge share the same color .

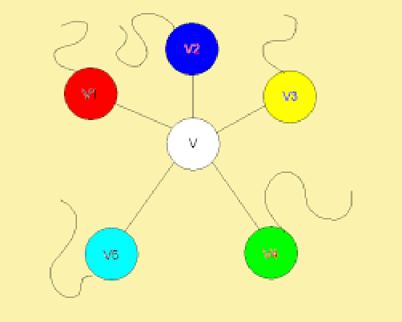


Fig 11

Lemma 1.1

Statement

In every planar graph , there exists a vertex of degree at most 5 .

Proof

Let , if possible , $\exists$ a planar graph s.t. all n vertices have degree ≥ 6 i.e. $d(v_i) \geq 6$

By First Theorem of Graph Theory , $\sum_{i=1}^{n} d(v_i) = 2e$ and

$$e \leq 3v - 6$$

$$\implies 2e \leq 6v - 12$$

Also , we assumed $d(v_i) \geq 6$, so

$$2e = \sum_{i=1}^{n} d(v_i) \geq 6v > 6v - 12$$
$$\implies 2e \geq 6v - 12$$

which is contradiction to

$$e \leq 3v - 6$$

Proof by induction

If no. of vertices (v) ≤ 6 , then it is trivially true .

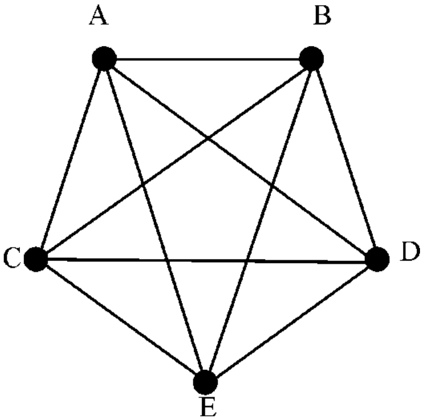


Fig 12

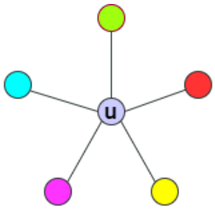
Base Case n = 1 is trivial .

Let us assume it is true for n = k vertices .

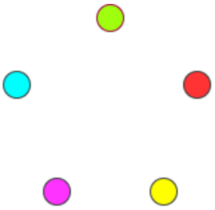
Inductive Step : For n = k+1

From Lemma 1.1 , G must contain a vertex of degree $\leq$ 5 .

Choose that vertex , denote it as u and now consider $G \setminus \{u\}$, say G'



Subgraph of G
u is the vertex with minimum degree



Subgraph of G'
Remove vertex u from G to obtain graph G'

Fig 13

Now ,clearly , a planar graph remains planar after removing a finite number of vertices , so
G' is planar with k vertices . Now , we can use our induction hypothesis to say that G' can be coloured with 6 colours .

Now we can think of this as colouring all of G except u .
But since , u has degree ≤ 5 , one of the six colours will not be used for any of the neighbours of u . Choose that colour to colour u .
Thus , G is coloured with 6 colours .
And so it is true for n = k+1 .
Since k was arbitrary , it is true for all natural numbers .

Five Colour Theorem

The vertices of every planar graph can be colored using 5 colors in such a way that no pair of vertices connected by an edge share the same color .

Proof by induction

If no. of vertices (v) $\leq$ 5 , then it is trivially true .
Base Case n = 1 is trivial .
Let us assume it is true for n = k vertices .
Inductive Step : For n = k+1
From Lemma 1.1 , G must contain a vertex of degree $\leq$ 5 .
Choose that vertex , denote it as v and now consider $G \setminus \{v\}$, say G'
Now ,clearly , a planar graph remains planar after removing a finite number of vertices , so
G' is planar with k vertices . Now , we can use our induction hypothesis to say that G' can be coloured with 5 colours .

Let's consider colouring of v

Case 1: deg(v) ≤ 4 , then it is trivial .

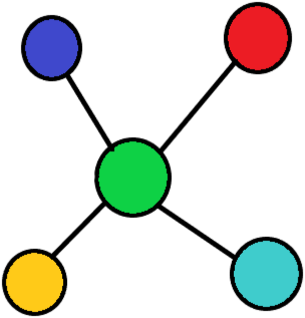
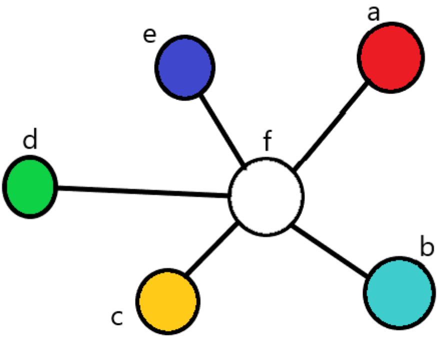


Fig 14

Case 2: $\deg(v) = 5$, then there are two subcases .



Case 1 : $\exists$ no two vertices that are directly connected by an edge.

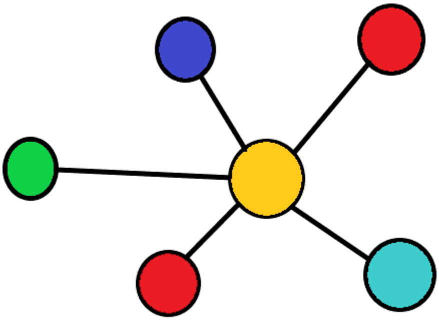
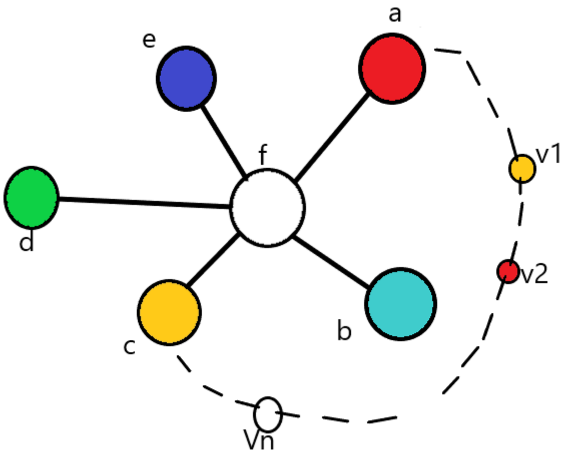
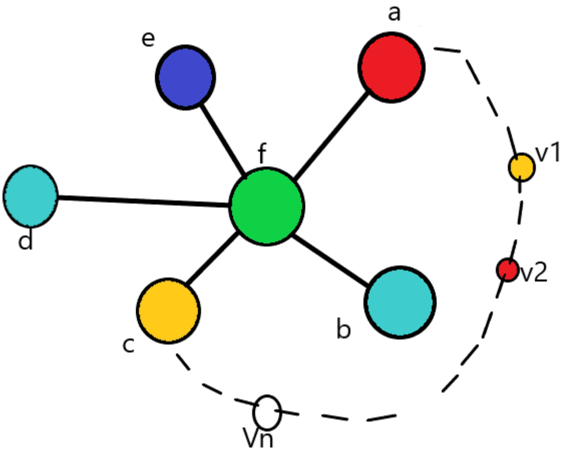


Fig 15

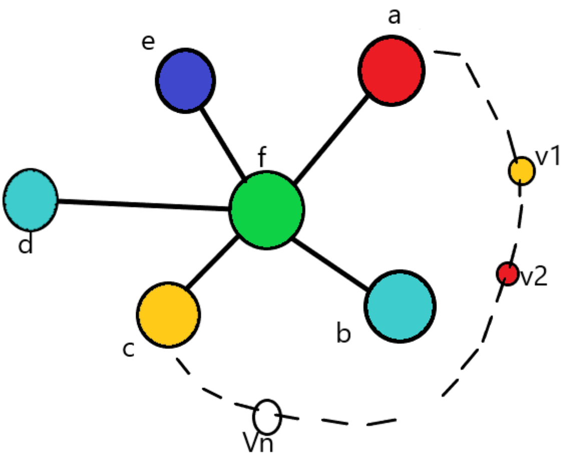
Case 2 : $\exists$ two vertices that are directly connected by an edge.



Suppose there is a path between vertices a and c, say a$\rightarrow v_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_n \rightarrow$ c.



Color alternatively a by red, v1 by yellow, v2 by red, and so on.


e
a
f
v1
d
v2
c
b
Vn

According to the property of planar graphs, there is no similar path between b and d (because it will intersect the path a and c). Hence there is no path of alternating colors p2 and p4 through vertices b and d. So, vertex d can be painted with the color p2 and vertex b is still with the color p2. Now we have color p4 left over with which we can paint vertex v.

Hence , the five colour theorem is proved .