$Y = \mu x + \epsilon$ Ancova $Y = X\beta + \epsilon$ notation

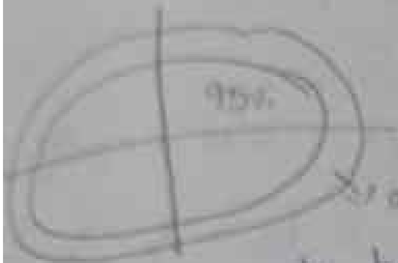
Confidence Region:-

$T^2 = n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu)$ → It is an eqn of ellipse in bivariate case

$\leq \frac{(n-1)p}{(n-p)} F_{p,n-p} = 1-\alpha$

a) $P\{ n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu)\} \leq \frac{(n-1)p}{(n-p)} F_{p,n-p}$

when we take $\alpha = 0.05$ → $1-\alpha = 95\%$ means 95% observations within on the ellipse. If I take $\alpha = 0.01$ then $1-\alpha = 99\%$



This is called confidence ellipse

we imagine to higher dimension it is called ellipsoid

all the variables have same variance then this forms circle in 2-dimension & sphere in higher dimension.

i) Sampling from multivariate normal population with known Σ $n(\bar{x}-\mu)^T \Sigma^{-1}(\bar{x}-\mu)$, χ^2_p, CR → $100(1-\alpha)\%$ $n(\bar{x}-\mu)^T \Sigma^{-1}(\bar{x}-\mu) \leq \chi^2_{p}(\alpha)$

ii) Sampling from multivariate normal population with large sample size and unknown Σ $n \geq 40$ [S is estimate of Σ]

$n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu)$, χ^2_p, CR $n(\bar{x}-\mu) S^{-1}(\bar{x}-\mu) \leq \chi^2_p(\alpha)$

iii) Sampling from m.N.P with small sample size & unknown Σ. $n-p \leq 40$ $n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu) \leq F$ distn

$n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu)$, F-distribn.

from pbm 1:-

from F-distribution table $= \frac{19}{9} \times 3.555 = 7.51$

CR ⇒ $(1.33)(10-\mu_1)^2 + (1.34)(10-\mu_1)(20-\mu_2) + (0.53)(20-\mu_2)^2$
≤ 7.51

Distribution in linear form:-

If X is multivariate normal & $a^T = [a_1, a_2 \ldots a$

linear combination is $a^T X$

Now $\bar{x} \sim N_p(\mu, \Sigma/n)$

$\Rightarrow a^T \bar{x} \sim N\left(a^T\mu, \frac{a^T \Sigma a}{n}\right)$

t-test we use can be done

for single population : $H_0 : \mu = \mu_0$

two independent polulation : $H_0 : \mu_1 = \mu_2$

two paired (dependent) " : $H_0 : \mu_1 = \mu_2$

2) let the data matrix for a random sample of size $n=3$ from a bivariate population be $X = \begin{bmatrix} 6 & 9 \\ 10 & 6 \\ 8 & 3 \end{bmatrix}$ evaluate T^2 for $\mu = [9, 5]$. What is the sampling distribution of T^2?

soln:

$$\bar{X} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{6+10+8}{3} \\ \frac{9+6+3}{3} \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$\therefore (\bar{X} - \mu) = \begin{bmatrix} 8-9 \\ 6-5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

we want $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & 9 \end{bmatrix}$

S_{ij}

$$\therefore S^{-1} = \begin{bmatrix} 1/3 & 1/9 \\ 1/9 & 4/27 \end{bmatrix}$$

$$T^2 = 3 \begin{bmatrix} 8-9 & 6-5 \end{bmatrix} \begin{bmatrix} 1/3 & 1/9 \\ 1/9 & 4/27 \end{bmatrix} \begin{bmatrix} 8-9 \\ 6-5 \end{bmatrix} = 7/9$$

Sample is selected from T^2 has the distribution

$$\frac{(3-1)2}{(3-2)} F_{2,3-2} = 4F_{2,1}$$

$Y = \beta x + \epsilon$ instead of $Y = X\beta + \epsilon$ notation

Estimation of parameters (MvLR)

$$\underset{n\times q}{Y} = \underset{n\times(p+1)}{X}\underset{(p+1)\times q}{\beta} + \epsilon \rightarrow (n\times q)$$

$$\underset{q\times n}{\epsilon^T} \cdot \underset{n\times q}{\epsilon} = q\times q \text{ order}$$

$$\Rightarrow \underset{(n\times q)}{\epsilon} = Y - X\beta$$

$\Rightarrow$ $\epsilon^T\epsilon$ will be no longer scalar, it is matrix of order $q\times q$ i.e. $\epsilon^T\epsilon$ is $SSCP_E$

and Estimation of $\hat{\beta}$ is $\hat{\beta} = [\hat{\beta}_1, \hat{\beta}_2 \cdots \hat{\beta}_q]$

$$\hat{\beta} = (X^TX)^{-1}X^T\{Y_1 : Y_2 : \cdots Y_q\}$$

$$\hat{\beta}_1 = (X^TX)^{-1}X^TY_2 \cdots\cdots \hat{\beta}_q = (X^TX)^{-1}X^TY_q$$

[in MLR we have univariate observations but in MvLR we have multivariate observations on q variable i.e. all treatments will be in multivariate domain & the test statistic are similar to ANOVA & MANOVA. we use likelihood ratio test like Wilk's lambda test]

Example - $\underset{3\times 2}{Y}$ (n obs, q → dependt variable) $= \begin{bmatrix} 10 & 100 \\ 12 & 110 \\ 11 & 105 \end{bmatrix}$ $p=2$ IV's $X = \begin{bmatrix} 9 & 62 \\ 8 & 58 \\ 7 & 64 \end{bmatrix}$

we require $\underset{3\times 3}{X_{n\times(p+1)}} = \begin{bmatrix} 1 & 9 & 62 \\ 1 & 8 & 58 \\ 1 & 7 & 64 \end{bmatrix}$
Design matrix

$\therefore$ estimate of $\hat{\beta} = (X^TX)^{-1}X^TY$

Now $X^TX = \begin{bmatrix} 1 & 1 & 1 \\ 9 & 8 & 7 \\ 62 & 58 & 64 \end{bmatrix}\begin{bmatrix} 1 & 9 & 62 \\ 1 & 8 & 58 \\ 1 & 7 & 64 \end{bmatrix} = \begin{bmatrix} 3 & 24 & 1\ldots \\ 24 & 194 & 1\ldots \\ 184 & 1470 & \end{bmatrix}$

$$\Rightarrow (X^TX)^{-1} = \frac{1}{|X^TX|}\operatorname{adj}(X^TX) = \begin{bmatrix} 320.76 & -8.16 & -4.16 \\ -8.16 & 0.56 & 0.06 \\ -4.16 & 0.06 & 0.06 \end{bmatrix}$$

$$X^TY = \begin{bmatrix} 1 & 1 & 1 \\ 9 & 8 & 7 \\ 62 & 58 & 64 \end{bmatrix}\begin{bmatrix} 10 & 100 \\ 12 & 110 \\ 11 & 105 \end{bmatrix} = \begin{bmatrix} 33 & 315 \\ 263 & 2515 \\ 2020 & 19300 \end{bmatrix}$$

Steps involved in (PCs) :-

* Each PC is a L.C of x, a $p\times 1$ variable vector (i) $a_j^T x$
* First principal component is $a_1^T x_1$, subjected to $a_1^T a_1 = 1$ that maximizes $var(a_1^T x)$
* Second principal component is $a_2^T x_2$ that maximizes $var(a_2^T x)$ & $a_2^T a_2 = 1$ & $cov(a_1^T x, a_2^T x) = 0$
* Similarly exposed for j^{th} pc vector

here what is maximize?

we have $z_j = a_j^T x$, w.r.t $a_j^T a_j = 1$

$V(z_j) = a_j^T S a_j$ $\quad \hookrightarrow a_j^T a_j - 1 = 0$

Now create a fn $L = a_j^T S a_j - \lambda(a_j^T a_j - 1)$ where λ - Lagrange multiplier

if $\Rightarrow \frac{\partial L}{\partial a_j} = 0$ results in $(S - \lambda I) a_j = 0$

to find the values of λ we have $|S - \lambda I| = 0$

$\therefore$ we can find Eigen values & Eigen vectors

Problem solving, given data → compute mean vector μ → Subtract μ from given data → calculate Σ → calculate eigen value & eigen vector → choosing component & forming a feature vector → Deriving new data set

x - $[x_i]$, Sample size n =

Principal Component Analysis

statistical procedure that takes us to summarize the information content of larger sources of data base by means of smaller

→ PCA is study of variance & co-variance of a set of variables. Main objective is to data reduction & interpretation. The data reduction is done keeping in mind (perspective) that lower dimension & orthogonal of the new dimension

Let us take 2-dimension (for instance)

$$X_{n\times 2} = \begin{array}{cc} x_1 & x_2 \end{array} \begin{bmatrix} x_{11} & x_{12} \\ \vdots & \\ x_{n1} & x_{n2} \end{bmatrix} \quad \text{we get (calculating) } cov(x) = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}_{2\times 2}$$

If we calculate correlation betn (X) (varies from -1 to 1) $= \begin{bmatrix} 1 & r_{12} \\ r_{12} & 1 \end{bmatrix}_{2\times 2}$

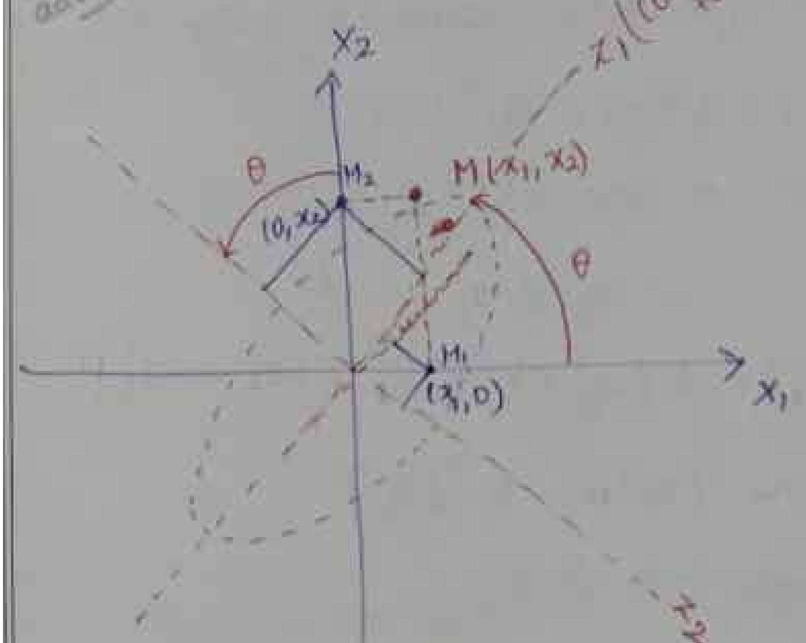
where $r_{12} \simeq 0.9$

Now consider p-dimensions $X_{n\times p}$ & $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}$

Now we convert $X_{p\times 1} \rightarrow Z_{m\times 1}$ $(m \le p)$
data reduction happens which preserves orthogonality

advantage is prediction using multiple regression, X variables (IV's) are correlated then leads to multiple li...

(i) $Y = \hat{f}(x)$ → this model is not suitable
disadvantage



based on proj...

$\therefore Z_1 = x_1 \cos\theta$ ...

$Z_2 = -x_1 \sin\theta$ ...

can be wr...

$$\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} \cos \ldots \\ -\ldots \end{bmatrix}$$

$\Rightarrow Z = A^T$ ...

Say, if we want to fit the model

$H_0: Y = X_{(0)}\beta_{(0)} + \epsilon_{(0)} \rightarrow SSCP_{\epsilon}^{1} = \hat{\Sigma}_0$

$H_1: Y = X\beta + \epsilon \rightarrow SSCP_{\epsilon} = \hat{\Sigma}$

means in one hand we have full model (x) n independent variable whereas another hand we have $X_{(0)}$ said m variables not contributing so when we say error term becomes same

H_0 is reduced model $\quad H_1$ is full model

$\therefore \Lambda = \dfrac{|\hat{\Sigma}|}{|\hat{\Sigma}_0|}$ this will have the value betn (0 to 1)

if $|\hat{\Sigma}| < |\hat{\Sigma}_0| \Rightarrow \Lambda \cong 0 \Rightarrow$ Reject H_0

if $\Lambda \simeq 1 \Rightarrow$ accept H_0.

The concept of Linear Regression:- with mean vector μ & cov matrix Σ

Partitioning

$$\mu = \begin{bmatrix} \mu_{Y\,(1\times 1)} \\ \mu_{Z\,(r\times 1)} \end{bmatrix} \quad \& \quad \Sigma = \begin{bmatrix} \sigma_{YY\,(1\times 1)} & \sigma'_{ZY\,(1\times r)} \\ \sigma_{ZY\,(r\times 1)} & \Sigma_{ZZ\,(r\times r)} \end{bmatrix}$$

with $\sigma'_{ZY} = [\sigma_{YZ_1}, \sigma_{YZ_2} \cdots \sigma_{YZ_r}]$.

$\therefore$ Single dependent Variable Y. (linear predictor)

$$Y = b_0 + b_1 Z_1 + \cdots + b_r Z_r + \epsilon = b_0 + b'Z + \epsilon$$ independent variables

& $\Rightarrow \epsilon = Y - b_0 - b'Z$

mean square error $= E(Y - b_0 - b'Z)^2 = ?$

Let the linear predictor $\beta_0 + \beta' Z$ (similar to previous notation)

with $\beta = \Sigma_{ZZ}^{-1}\sigma_{ZY}$ & $\beta_0 = \mu_Y - \beta'\mu_Z$

$\therefore E(Y - \beta_0 - \beta' Z)^2 = \sigma_{YY} - \sigma'_{ZY}\Sigma_{ZZ}^{-1}\sigma_{ZY}$

$$\text{cov}(Y, \beta_0 + \beta' Z) = \sqrt{\frac{\sigma'_{ZY}\Sigma_{ZZ}^{-1}\sigma_{ZY}}{\sigma_{YY}}}$$

[for problem Refer pg No: 405 : Eg: 7.12] //

$cov(y) = E[(x-E(x))(x-E(x))^T]$

Sampling distribution of $\hat{\beta}$

$\hat{\beta} = (XX^T)^{-1}X^T Y$

$E(\hat{\beta}) = ?$ & $cov(\hat{\beta}) = ?$

Now $E(\hat{\beta}) = E((XX^T)^{-1}X^T Y) = (XX^T)^{-1}X^T E(Y) = \underline{(XX^T)^{-1}}\,\underline{X^T X}\beta$

$= I\beta = I\beta$ { I is identity matrix }

$Y = X\beta + \epsilon$

$E(Y) = E(\underbrace{X\beta}_{const}) + \underbrace{E(\epsilon)}_{=0}$

$= X E(\beta) = X\beta$

$\Rightarrow \hat{\beta} - E(\hat{\beta})$

$= \hat{\beta} - \beta$

$= (X^T X^{-1})X^T Y - \beta$

$= (X^T X^{-1})X^T(X\beta + \epsilon) - \beta$

on simplifying,

$= (XX^T)^{-1}X^T\epsilon$

$cov(\hat{\beta}) = E\{(\hat{\beta} - E(\hat{\beta}))(\hat{\beta} - E(\hat{\beta}))^T\}$

$= E\{((XX^T)^{-1}X^T\epsilon)(\epsilon^T X(XX^T)^{-1})\}$

$\because (\hat{\beta} - E(\hat{\beta}))^T = ((XX^T)^{-1}X^T\epsilon)^T$

$= \epsilon^T X(XX^T)^{-1}$

$= (XX^T)^{-1}X^T E(\epsilon\epsilon^T)X(X^TX)^{-1}$

$= (XX^T)^{-1}X^T\, I \otimes \Sigma\, X(X^TX)^{-1}$

↳ kronecker pdt

$= (XX^T)^{-1} \otimes \Sigma$ (unknown)

Model adequacy test :-

$$\underset{n\times q}{Y} = \underset{n\times(p+1)}{X}\overset{(p+1)\times q}{\beta} + \underset{n\times q}{\epsilon}$$

$\Rightarrow \hat{Y} = X\hat{\beta}$ & also $\hat{\epsilon} = Y - \hat{Y}$.

Now

$q\times q \leftarrow \underset{q\times n}{Y^T}\underset{n\times q}{Y} = (X\beta + \epsilon)^T(X\beta + \epsilon) \longrightarrow SSCP_T$

$q\times q \leftarrow \underset{q\times n}{\hat{Y}^T}\underset{n\times q}{\hat{Y}} = (X\hat{\beta})^T(X\hat{\beta}) \longrightarrow SSCP_{B(or)\hat{Y}}$

$q\times q \leftarrow \underset{q\times n}{\hat{\epsilon}^T}\underset{n\times q}{\hat{\epsilon}} = (Y - \hat{Y})^T(Y - \hat{Y}) \longrightarrow SSCP_{\hat{E}}$

$\Rightarrow SSCP_T = SSCP_{\hat{Y}} + SSCP_{\hat{E}}$ (Similar as Manova)

[Now we take Wilk's lambda in manova (Likelyhood ratio test) $\Lambda = \frac{|SSCP_E|}{|SSCP_T|}$]

Similarly to use Likelyhood ratio test here we need the hypothesis $H_0 : \beta_{(m)} = \begin{bmatrix} \beta_{p-m} \\ \vdots \\ \beta_p \end{bmatrix} = 0$ → means the p variabl... where not...

$X = \begin{bmatrix} X_{(0)} \\ \hline X_{(m)} \end{bmatrix}$

$H_1 : \beta_{(m)} \neq 0$ for atleast $\beta_{(m)j}$

Contributing

Not Contributing

$(X^T X)^{-1} X^T Y$

$$\Rightarrow \begin{bmatrix} 330.74 & -2.75 & -4.16 \\ -2.75 & 0.35 & 0.04 \\ -4.16 & 0.04 & 0.06 \end{bmatrix}_{(3\times3)} \begin{bmatrix} 33 & 315 \\ 263 & 2515 \\ 2020 & 19300 \end{bmatrix}_{(3\times2)}$$

$$= \begin{bmatrix} 35.80 & 229 \\ -0.80 & -4.00 \\ -0.80 & -1.50 \end{bmatrix} = [\hat{\beta}_1 : \hat{\beta}_2] = \begin{bmatrix} \hat{\beta}_{10} & \hat{\beta}_{20} \\ \hat{\beta}_{11} & \hat{\beta}_{21} \\ \hat{\beta}_{12} & \hat{\beta}_{22} \end{bmatrix}$$

$$\therefore Y = X\beta + \epsilon$$

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} [\hat{\beta}_1 : \hat{\beta}_2] + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix}$$

$$\Rightarrow Y_1 = \hat{\beta}_{10} + \hat{\beta}_{11} x_1 + \hat{\beta}_{12} x_2 + \epsilon_1$$
$$Y_2 = \hat{\beta}_{20} + \hat{\beta}_{21} x_1 + \hat{\beta}_{22} x_2 + \epsilon_2$$

$$\Rightarrow Y_1 = 35.80 + (-0.80)x_1 + (-0.80)x_2 + \epsilon_1$$
$$Y_2 = 229 + (-4.00)x_1 + (-1.50)x_2 + \epsilon_2.$$

Now estimate of $\hat{\epsilon} = Y - \hat{Y}$

fitted value $\hat{Y} = X\hat{\beta}$

$$= \begin{bmatrix} 10 & 100 \\ 12 & 110 \\ 11 & 105 \end{bmatrix} - \begin{bmatrix} & ? & \end{bmatrix}$$

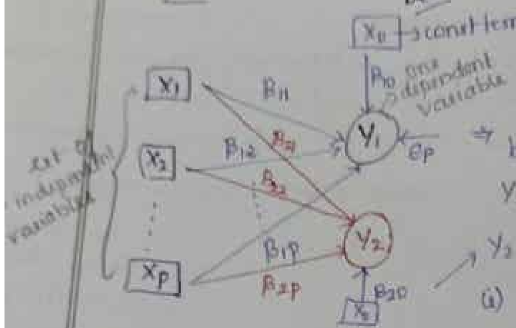
$$\Rightarrow \hat{Y} = X\hat{\beta} = \begin{bmatrix} 1 & 9 & 62 \\ 1 & 8 & 58 \\ 1 & 7 & 64 \end{bmatrix} \begin{bmatrix} \overset{35.0}{\cancel{330.74}} & +229 \\ -0.80 & -4.00 \\ -0.30 & -1.50 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 100 \\ 12 & 110 \\ 11 & 105 \end{bmatrix}$$

$$\therefore \hat{\epsilon} = Y - \hat{Y} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

this should not happen means your model is not a fitted model

MLR diff bet^n MvLR (MvLR)



Similar to last study this is MLR the eqn will be

$Y_1 = \beta_{10} + \beta_{11} x_1 + \dots + \beta_{1p}$

$Y_2 = \beta_{20} + \beta_{21} x_2^1 + \dots + \beta_{2p}$

(i) when added one dependent variable becomes MvLR that is number of dependent variable is 1 in MLR whereas in MvLR dependent variable will be >1.

In MvLR β becomes $\beta = \begin{bmatrix} \beta_{10} & \beta_{20} \\ \beta_{11} & \beta_{21} \\ \vdots & \vdots \\ \beta_{1p} & \beta_{2p} \end{bmatrix}_{(p+1)\times q}$ = for q = 2

$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_q \end{bmatrix}$ D.V's $\quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}_{p\times 1}$ I.V's $\quad \epsilon = \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \vdots \\ \epsilon_q \end{bmatrix}_{q\times 1}$ Errors $\quad \beta = \begin{bmatrix} \beta_{10} & \beta_{20} & \dots \\ \beta_{20} & \beta_{21} & \dots \\ \vdots & \vdots & \\ \beta_{1p} & \beta_{2p} & \dots \end{bmatrix}$ Regression coefficients (p

Collection Data's Representation:-

$Y_{n\times q} = \begin{bmatrix} y_{11} & y_{12} & \dots & y_{1q} \\ y_{21} & y_{22} & \dots & y_{2q} \\ \vdots & & & \\ y_{i1} & y_{i2} & \dots & y_{iq} \\ y_{n1} & y_{n2} & \dots & y_{nq} \end{bmatrix}$ & $X_{n\times(p+1)} = \begin{bmatrix} 1 & x_{11} & \dots & x_{1p} \\ 1 & x_{21} & \dots & x_{2p} \\ \vdots & & & \\ 1 & x_{i1} & \dots & x_{ip} \\ 1 & x_{n1} & \dots & x_{np} \end{bmatrix}$

1st obs of 1st variable; n observations on q variables; design matrix; $x_0 = 1$ → constant

for i^{th} observation the eqn is

$$y_{i1} = \beta_{10} + \beta_{11} x_{i1} + \dots + \beta_{1p} x_{ip} + \epsilon_{i1}$$

$$y_{i2} = \beta_{20} + \beta_{21} x_{i1} + \dots + \beta_{2p} x_{ip} + \epsilon_{i2}$$

$$\vdots$$

$$y_{iq} = \beta_{q0} + \beta_{q1} x_{i1} + \dots + \beta_{qp} x_{ip} + \epsilon_{ip}$$

$i = 1, 2 \dots n$ → Observations

$$\Rightarrow Y_{n\times q} = X_{n\times(p+1)} \cdot \beta_{(p+1)\times q} + \epsilon_{n\times q} \Rightarrow MvLR$$

We have $Z = A^T X$

then $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = [a_1 \ \ a_2]$

$a_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ then $a_1^T a_1 = [\cos\theta \ \ \sin\theta] \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = 1$

III^{ly} $a_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$ then $a_2^T a_2 = 1$

Now $A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = AA^T = A^{-1}A$ $\quad A^{-1}A = I$

orthogonal property transformation

therefore P variable, we have

$Z_1 = a_1^T X = a_{11} x_1 + a_{12} x_2 + \cdots + a_{1p} x_p$

$Z_2 = a_2^T X = a_{21} x_1 + \cdots\cdots + a_{2p} x_p$

$\vdots$

$Z_p = a_p^T X = a_{p1} x_1 + a_{p2} x_2 + \cdots + a_{pp} x_p$

$a_j^T a_j = 1 \ \forall j = 1$ to p & $Var(Z_1) \geq Var(Z_2) \geq \ldots \geq Var(Z_p)$

Using these information we will study how to Extract Principal Components.

Now W.R.T $Z_j = a_j^T X$ what is $V(Z_j) = ?$

$V(aX) = a^2 V(X)$

$V(Z_j) = V(a_j^T X) = a_j^T V(X) a_j$

Now $\left[V(X) = ? \Rightarrow cov(X) = \Sigma \right]$

$= a_j^T \Sigma a_j$

& $E(Z_j) = E(a_j^T X) = a_j^T E(X) = a_j^T \mu$

$\therefore a_j^T X \sim (a_j^T \mu, \ a_j^T \Sigma a_j)$

If population has μ, Σ then Sample has $\bar{X}, S$

Σ → pop cov

S → Sample Covariance

Multivariate linear Regression models

The classical linear regression model states that Y is composed of a mean depends on z_i's & random error ϵ. The values of the independent variable recorded from the experiment/set is treated as a fixed values. Therefore the linear regression model with a single response (dependent variable) is

MLR multiple linear regression

$$\underset{\text{dependent variable}}{Y} = \underbrace{\beta_0 + \beta_1 z_1 + \dots + \beta_r z_r}_{\text{mean (depending on } z_i\text{'s)}} + \underset{\text{error}}{\epsilon}$$

The term linear mean a linear fn of unknown parameters $\beta_0, \beta_1 \dots \beta_r$.

$\therefore$ With n independent observations on Y and the associated values of z_i becomes

$$Y_1 = \beta_0 + \beta_1 z_{11} + \dots + \beta_r z_{1r} + \epsilon_1$$
$$Y_2 = \beta_0 + \beta_1 z_{21} + \dots + \beta_r z_{2r} + \epsilon_2$$
$$\vdots$$
$$Y_n = \beta_0 + \beta_1 z_{n1} + \dots + \beta_r z_{nr} + \epsilon_n$$

Where Error terms have the following properties.

i) $E(\epsilon_n) = 0$ ii) $Var(\epsilon_n) = \sigma^2$ (constant) &

iii) $Cov(\epsilon_j, \epsilon_k) = 0 \ \forall j \neq k$.

In matrix notation.

D.V.   I.V   Regression coefficients

$$\begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_r \end{bmatrix} \begin{bmatrix} 1 & z_{11} & z_{12} & \dots & z_{1r} \\ 1 & z_{21} & z_{22} & \dots & z_{2r} \\ \vdots & & & & \vdots \\ 1 & z_{n1} & z_{n2} & \dots & z_{nr} \end{bmatrix} + \Big[$$

can be written as $\underset{(n\times1)}{Y} = \underset{(r+1)\times1}{\beta} \underset{n\times(r+1)}{Z} + \underset{(n\times1)}{\epsilon}$ of order

is called multiple linear regression.

Estimation of the parameters: (MLR)

After studying this we consider multivariate linear regression.

Treatment SSq $= \sum (\bar{x}_l - \bar{x})^2 n_l$

$= (8-4)^2 \times 3 + (1-4)^2 \times 2 + (2-4)^2 \times 3$

$= 78$ with dof $L-1 = 3-1 = 2$ (groups)

Residual SSq = Total SSq (corrected) − Treatment SSq

$= 10$ with $N-L = 8-3 = 5$

$H_0: \mu_1 = \mu_2 = \mu_3$ against H_1 = atleast 1 inequality

this is one way anova so we

simple F-statistic (i) $= \dfrac{\text{SSq(treatment)/dof}}{\text{SSq(residuals)/dof}}$

$= \dfrac{78/2}{10/5} \sim F_{L-1,\,N-L} = F_{2,5}$ 19.5 from distribⁿ table (19.5)

$= 3.78$

tabulated
$= 3.78$
< 19.5

$\Rightarrow$ Reject H_0 at 1% LOS.

tab cal
3.78 < 19.5
$T^2 > C^2 \Rightarrow$ Reject

---

consider same pbm suppose

pop 1 : $\begin{pmatrix} 9 \\ 3 \end{pmatrix} \begin{pmatrix} 6 \\ 2 \end{pmatrix} \begin{pmatrix} 9 \\ 3 \end{pmatrix}$ $n_1 = 3,\ n_2 = 2,\ n_3 = 3.$

$l = 3$ (no of groups).

pop 2 : $\begin{pmatrix} 0 \\ 4 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

pop 3 : $\begin{pmatrix} 3 \\ 8 \end{pmatrix} \begin{pmatrix} 1 \\ 9 \end{pmatrix} \begin{pmatrix} 2 \\ 7 \end{pmatrix}$

group sample mean $\bar{x}_1 = \begin{bmatrix} \frac{9+6+9}{3} \\ \frac{3+2+3}{3} \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$

$\bar{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\bar{x}_3 = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$ So that $\bar{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

Now Total SSq (uncorrected) $= 3^2 + 2^2 + \dots + 7^2 = 272$, do

Total SSq (corrected) $= (3-5)^2 + (2-5)^2 + \dots$
$+ 4^2 + 2^2 = 72.$

with dof $= 8-1 = 7.$

Eg: For same experiment of sample size $n_1 = n_2 = n_3 = \ldots$

we have $\bar{x}_1 = \begin{bmatrix} 20.20 \\ 6.60 \end{bmatrix}$, $\bar{x}_2 = \begin{bmatrix} 17.90 \\ 6.70 \end{bmatrix}$, $\bar{x}_3 = \begin{bmatrix} 20.6 \\ 7.3 \end{bmatrix}$

& $S_1 = \begin{bmatrix} 1.51 & 0.11 \\ 0.11 & 1.17 \end{bmatrix}$, $S_2 = \begin{bmatrix} 1.43 & 0.52 \\ 0.52 & 0.68 \end{bmatrix}$, $S_3 = \begin{bmatrix} 0.73 & -0. \\ -0.11 & 0. \end{bmatrix}$

Soln: To compute, $\bar{x}$ (Grand mean) $= \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1+n_2+n_3}$

$$= \frac{10(\bar{x}_1 + \bar{x}_2 + \bar{x}_3)}{30} = \frac{1}{3}[\bar{x}_1 + \bar{x}_2 + \bar{x}_3]$$

$$= \begin{bmatrix} 19.57 \\ 6.90 \end{bmatrix}$$

$$SSCP_B = \sum_{l=1}^{L} n_l (\bar{x}_l - \bar{x})(\bar{x}_l - \bar{x})^T \quad ; \quad L=3$$

$$= n_1(\bar{x}_1 - \bar{x})(\bar{x}_1 - \bar{x})^T + n_2(\bar{x}_2 - \bar{x})(\bar{x}_2 - \bar{x})^T + n_3(\bar{x}_3 - \bar{x}_3)(\bar{x}_3 - \bar{x})^T$$

$$= \begin{bmatrix} 43.47 & 7.00 \\ 7.00 & 5.60 \end{bmatrix}$$

$$SSCP_E = (n_1 - 1)S_1 + (n_2 - 1)S_2 + (n_3 - 1)S_3.$$

$$= \begin{bmatrix} 34.90 & 4.70 \\ 4.70 & 23.10 \end{bmatrix}$$

$$\therefore SSCP_T = SSCP_B + SSCP_E$$

$$= \begin{bmatrix} 77.37 & 11.70 \\ 11.70 & 28.70 \end{bmatrix}$$

Testing for two mean vectors

in multivariate :-

random variable will be a random vector.

$\bar{x}_{B1}$ & $\bar{x}_{B2}$

$\downarrow$

S_{B1} $S_{B2} \to$ cov matrix

n_{B1} $n_{B2} \to$ size

then $E(\bar{x}_{A1} - \bar{x}_2) = \vec{\mu}_1 - \mu_2$

$$= \begin{bmatrix} \mu_{11} - \mu_{21} \\ \mu_{12} - \mu_{22} \\ \vdots \\ \mu_{1p} - \mu_{2p} \end{bmatrix}$$

and $V(\bar{x}_1 - \bar{x}_2) = V(\bar{x}_1) + V(\bar{x}_2)$. $\to$

T^2 statistic for testing the equality of two means from two multivariate population can be developed using univariate concepts.

* Comparing response from one set of experimental settings (population 1) and another set of " (population
* consider a random sample size n_1 from population 1 n_2 from population 2.

population 1 $\to \bar{x}_1 = \frac{1}{n_1} \sum_{j=1}^{n1} x_{1j} \Rightarrow S_1 = \frac{1}{n_1 - 1} \sum_{j=1}^{n1} (x_{1j} - \bar{x}_1)(x_{1j}$

$x_{11}, x_{12} \ldots x_{1n_1}$

population 2 $\to \bar{x}_2 = \frac{1}{n_2} \sum_{j=1}^{n2} x_{2j} \to S_2 = \frac{1}{n_2 - 1} \sum_{j=1}^{n_2} (x_{2j} - \bar{x}_2)(x_2$

$x_{21}, x_{22} \ldots x_{2n_2}$

if we say $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$$\hat{\sigma}^2 = S_p^2 = \frac{(n_1-1)S_1 + (n_2-1)S_2}{n_1+n_2-1} \quad \text{univariate}$$

$$\rightarrow \Sigma_A = \Sigma_B = \Sigma$$

$$S = \frac{(n_A-1)S_A + (n_B-1)S_B}{n_A+n_B-1} = \hat{\Sigma} \quad \text{multivariate}$$

Pbm:- Age and risky behaviour are the two important factors that make difference between accident group (AG) & non accident group (NAG) of workers. Random samples of 20 individuals from AG & 50 individuals from NAG were collected. The sample mean vector & samp cov matrix are given. Construct 95% CR for the diff betn the two population mean vectors.

Sample - 1

$$\bar{X}_A = \begin{pmatrix} 50 \\ 6 \end{pmatrix} \quad S_A = \begin{pmatrix} 16 & -5 \\ -5 & 4 \end{pmatrix}$$

Sample - 2

$$\bar{X}_B = \begin{pmatrix} 40 \\ 8 \end{pmatrix}, \quad S_B = \begin{pmatrix} 25 & - \\ -6 & \end{pmatrix}$$

injured & non-injured → we find mean difference betn two

Here $n_1 = 20$, $n_2 = 50$, Σ is unknown.

let $\Sigma_2 = \Sigma_1 = \Sigma$ [case (ii)]

$$T^2 = \left[(\bar{X}_A - \bar{X}_B) - (\mu_A - \mu_B)\right]^T \left[\left(\frac{1}{n_A} + \frac{1}{n_B}\right) S\right]^{-1} \left[(\bar{X}_A - \bar{X}_B) - (\mu_A - \mu_B)\right]$$

now

$$S = \frac{(n_A-1)S_A + (n_B-1)S_B}{n_A+n_B-1} = \frac{(20-1)\begin{pmatrix} 16 & -5 \\ -5 & 4 \end{pmatrix} + (50-1)\begin{pmatrix} 25 & -6 \\ -6 & 9 \end{pmatrix}}{20+50-1}$$

$$= \begin{pmatrix} 23.32 & -5.72 \\ -5.72 & 7.60 \end{pmatrix} \left(\frac{1}{20} + \frac{1}{50}\right)$$

$$\left[\left(\frac{1}{n_A} + \frac{1}{n_B}\right) S\right]^{-1} = \begin{pmatrix} 0.75 & 0.57 \\ 0.57 & 2.30 \end{pmatrix}$$

Same concept if we take for inference on multivariate statistics → Two population mean vectors

In univariate case:-
→ two variables

x_1 x_2

$\downarrow$ $\downarrow$

$\bar{x}_1$ $\bar{x}_2$

s_1 s_2

n_1 n_2 → samples

Created a R.V $= \bar{x}_1 - \bar{x}_2$

$E(\bar{x}_1 - \bar{x}_2) = \mu_1 - \mu_2$

$V(\bar{x}_1 - \bar{x}_2) = V(\bar{x}_1) + V(\bar{x}_2)$

$= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

↳ from this multivariate

↳ R.V will be R.V $\bar{\underline{x}}_A - \bar{\underline{x}}_B$ in vector notation

$\bar{x}_A$ $\bar{x}_B$

S_A S_B → cov matrix for pop 1 & 2.

n_A n_B ⇒ Sample size.

$\therefore E(\bar{x}_A - \bar{x}_B) = \mu_A - \mu_B$

$$= \begin{bmatrix} \mu_{A1} - \mu_{B1} \\ \mu_{A2} - \mu_{B2} \\ \mu_{Ap} - \mu_{Bp} \end{bmatrix}$$

in multi

$S_1^2 \to \Sigma_A$

$S_2^2 \to \Sigma_B$

& $V(\bar{x}_A - \bar{x}_B) = V(\bar{x}_A) + V(\bar{x}_B)$

$$= \frac{\Sigma_A}{n_A} + \frac{\Sigma_B}{n_B}$$

Now create $z = \dfrac{R.V - E(\ )}{\sqrt{\ \ }}$ univariate $z = \frac{x-\mu}{\sqrt{V(x)}}$

$$= \frac{\bar{x}_A - \bar{x}_B - \{\mu_A - \mu_B\}}{\sqrt{\Sigma_A/n_A + \Sigma_B/n_B}}$$ in multivariate domain

Case i) Sampling from multivariate normal population with known Σ_1 & Σ_2

Case ii) Sampling from multivariate normal populations with small sample size & unknown but equal Σ

Case iii) ... with large sample size with unknown Σ_1 & Σ_2

From $Z = \frac{x-\mu}{\sigma}$ W.K.T $E(a^T \bar{x}) = a^T E(\bar{x}) = a^T \mu$

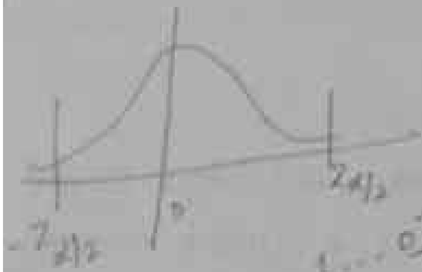
distribution $V(a^T \bar{x}) = \frac{a^T \Sigma a}{n}$

$$Z = \frac{a^T \bar{x} - E(a^T \bar{x})}{\sqrt{V(a^T \bar{x})}} \sim Z(0,1)$$

i.e will be univariate normal

$\therefore$ interval is $-Z_{\alpha/2} \le Z \le Z_{\alpha/2}$

$$\Rightarrow -Z_{\alpha/2} \le \frac{a^T \bar{x} - E(a^T \bar{x})}{\sqrt{V(a^T \bar{x})}} \le Z_{\alpha/2}$$



$$\Rightarrow a^T \bar{x} - Z_{\alpha/2} \sqrt{\frac{a^T \Sigma a}{n}} \le a^T \mu \le Z_{\alpha/2} \sqrt{\frac{a^T \Sigma a}{n}} +$$

$a^T = [0, 0, \cdots \underset{j^{th}}{1} \cdots 0] \Rightarrow \bar{x_j} - Z_{\alpha/2} \sqrt{\frac{S_{jj}}{n}} \le \mu_j \le \bar{x}_j + Z_{\alpha/2} \sqrt{\frac{S_{jj}}{n}}$

Hypothesis testing:

case i) Σ is known when null hypo is true must } means this will not wait for $\mu = \mu_0$

$H_0: \mu = \mu_0$
$H_1: \mu \ne \mu_0$ } hypo

Test statistic
$T^2 = \underbrace{n(\bar{x} - \mu_0)^T \Sigma^{-1} (\bar{x} - \mu_0)}_{A}$ } Sampling distn χ^2_p

Decision :- $A \ge \chi^2_p(\alpha) \Rightarrow$ Rejected

case ii) Σ is unknown, $n-p \ge 40$.

$H_0: \mu = \mu_0$
$H_1: \mu \ne \mu_0$ } $T^2 = n(\bar{x} - \mu_0)^T S^{-1} (\bar{x} - \mu_0)$ } χ^2_p

Decision : $T^2 \ge \chi^2_p(\alpha) \Rightarrow$ Rejected.

case iii) Σ is unknown, $n-p < 40$.

$H_0: \mu = \mu_0$
$H_1: \mu \ne \mu_0$ } $T^2 = n(\bar{x} - \mu_0)^T S^{-1} (\bar{x} - \mu_0)$ } $\frac{(n-1)p}{(n-p)} F_{p, n-p}$

Decision $T^2 \ge F$ distribution $\Rightarrow$ Reject

In pbm 1:- hypo testing if we take $\mu_0 = \begin{array}{c} \\ \begin{bmatrix} 9 \\ 18 \end{bmatrix} \end{array}\begin{array}{c} \mu_1 \\ \mu_2 \end{array}$

Σ is unknown

T^2 = Calculated by sub = 6.13 & F distn = 7.51 (using table)

$\Rightarrow 6.13 < F$-distribution

$\Rightarrow \mu_0$ is accepted

Hotelling - T-Square Statistics

Concept's on earlier topics discuss

$x \sim N_p(\mu, \Sigma)$ → population } related concepts

$\bar{x} \sim N_p(\mu, \Sigma/n)$ } samples

$S \sim W_p(n-1, \Sigma)$

In univariate case :- from point of view hypothesis concept

μ_0 is the sample mean.

null hypo $H_0 : \mu = \mu_0$ & $H_1 : \mu \neq \mu_0$ alternate hypo

$x_1, x_2, \ldots x_n$ are random sample.

small sample say < 40 → we use t-distribution ⇒ $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$ (sample mean, pop mean, sample std error, sample size)

Looking into multivariate case

$\bar{x} \to \bar{x}$, $\mu \to \mu$, $s^2 \to S$ & $n \to n$.

~~$t = \bar{x} - \mu_1$~~ Now $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}} = \sqrt{n}(\bar{x} - \mu) s^{-1}$

if we square $t^2 = n(\bar{x} - \mu)(s^2)^{-1}(\bar{x} - \mu)$

Hotelling T^2 ← $T^2 = n(\bar{x} - \mu)^T (S)^{-1} (\bar{x} - \mu)$

$\sim \dfrac{(n-1)p}{n-p} F_{p, n-p}$ $\forall$ $n - p < 40$

$\bar{x} \sim N_p(\mu, \Sigma/n)$ & $S \sim W_p(n-1, \Sigma)$

Eg (prob 1) A random sample with $n = 20$ were collected from a bivariate normal process. The sample mean vector & S cov matrix given

$\bar{x} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$ $S = \begin{pmatrix} 40 & -50 \\ -50 & 100 \end{pmatrix}$

i) Obtain Hotelling T^2 - Square

ii) what will be the distribution of it?

MANOVA Table (one - way)

Sources of Variations	dof	SSCP
Treatment (Between)	$k-1$	$\sum_{i} n_i (x_i - \bar{x})(x_i - \bar{x})^T = B$
Residual (Error) (Within)	$N-k$	$\sum_i \sum_j (x_{ij} - \bar{x}_i)(x_{ij} - \bar{x}_i)^T = W$
Total	$N-1$	$\sum \sum (x_{ij} - \bar{x})(x_{ij} - \bar{x})^T$

$H_0: \mu_1 = \mu_2 = \dots = \mu_k$ & $H_1: \mu_k \neq \mu_m$ atleast one pair not equal

Recall Λ^* Wilk's lamba $= \dfrac{|W|}{|B+W|}$ likelyhood ratio test rejects H_0 if Λ^* is small

u) Reject H_0 if $-n \ln \Lambda^* > \chi^2_p(\alpha)$

Example :- Consider the following independent samples (3 groups)

Pop 1 : 9, 6, 9 $\quad \therefore\ l = 3$

Pop 2 : 0, 2 $\quad n_1 = 3,\ n_2 = 2\ \&\ n_3 = 3$

Pop 3 : 3, 1, 2

group sample means $\bar{x}_1 = \dfrac{9+6+9}{3} = 8$

$\bar{x}_2 = \dfrac{0+2}{2} = 1,\quad \bar{x}_3 = \dfrac{3+1+2}{3} = 2$

$\therefore$ (overall mean) $\bar{x} = \dfrac{n_1\bar{x}_1 + n_2\bar{x}_2 + n_3\bar{x}_3}{n_1+n_2+n_3} = \dfrac{24+2+6}{8} = 4$

$x_{ij} = \bar{x} + (\bar{x}_i - \bar{x}) + (x_{ij} - \bar{x}_i)$

Now

Total SS_q (uncorrected) $= \sum_i \sum_j x_{ij}^2 = 9^2 + 6^2 + \dots + 1^2 + 2^2 =$

Total ssq (corrected) $= \sum_i \sum_j (x_{ij} - \bar{x})^2 =$

$= (9-4)^2 + (6-4)^2 + \dots + (1-4)^2 + (2-4)^2$

$= 88$ with dof $N-1 = 8-1 = 7$

Hypothesis $\quad H_0 : \Sigma_1 = \Sigma_2 = \dots = \Sigma_L$

$\quad H_1 : \Sigma_l \neq \Sigma_m$ for atleast one pair

(Box M test) Statistic : $D = (1-u)M$

where, $M = -2 \ln \left[\prod_{l=1} \left(\frac{|S_l|}{|S_{pooled}|} \right)^{\frac{(n_l-1)}{2}} \right]$

$= \left[\sum_l (n_l-1) \ln |S_{pooled}| \right] - \left[\sum_l \{ (n_l-1) \ln |S_l| \} \right]$

& $u = \left[\sum_l \frac{1}{(n_l-1)} - \frac{1}{\sum_l (n_l-1)} \right] \left[\frac{2p^2+3p-1}{6(p+1)(L-1)} \right]$

Decision: Reject H_0 when $D > \chi_{\alpha, \nu}$

$\nu = \frac{1}{2} p(p+1)(L-1)$

Decomposition of total sum of squares & cross pdt :-

ANOVA $\rightarrow$ $x_{il} = \mu + (\mu_l - \mu) + (x_{il} - \mu_l)$

Estimate of $\hat{\mu} = \bar{x}$ & $\hat{\mu}_l = \bar{x}_l$

Sample observation $\quad x_{il} = \bar{x} + (\bar{x}_l - \bar{x}) + (x_{il} - \bar{x}_l)$

MANOVA $\rightarrow$ $x_{il} = \mu + (\mu_l - \mu) + (x_{il} - \mu_l)$

Estimate $\hat{\mu} = \bar{x}$ & $\hat{\mu}_l = \bar{x}_l$

Sample observation $\quad x_{il} = \bar{x} + (\bar{x}_l - \bar{x}) + (x_{il} - \bar{x}_l)$

$$\left[\quad\right]_{p\times1} = \left[\quad\right]_{p\times1} + \left[\quad\right]_{p\times1} + \left[\quad\right]_{p\times1}$$

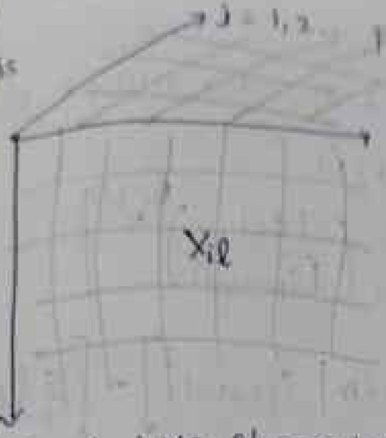
Now, $x_{il} - \bar{x} = (\bar{x}_l - \bar{x}) + (x_{il} - \bar{x}_l)$

Squaring,

$\Rightarrow \underset{p\times1}{(x_{il} - \bar{x})} \underset{1\times p}{(x_{il} - \bar{x})^T} = \left[(\bar{x}_l - \bar{x}) + (x_{il} - \bar{x}_l) \right] \left[(\bar{x}_l - \bar{x}) + (x_{il} - \bar{x}_l) \right]^T$

$p \times p$

MANOVA is



$$X_{il} = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{iL} \end{bmatrix}_{p\times 1}$$

ANOVA

$H_0: \mu_1 = \mu_2 = \ldots = \mu_L$

$H_1: \mu_l \neq \mu_m$ for atleast one pair

$l = 1, 2 \ldots L$

$m = 1, 2 \ldots L$

$l \neq m$

MANOVA

$H_0: \mu_1 = \mu_2 = \ldots = \mu_L$

(i.e)
$$\begin{bmatrix} \mu_{11} \\ \mu_{12} \\ \vdots \\ \mu_{1p} \end{bmatrix} = \begin{bmatrix} \mu_{21} \\ \mu_{22} \\ \vdots \\ \mu_{2p} \end{bmatrix} = \ldots = \begin{bmatrix} \mu_{L1} \\ \mu_{L2} \\ \vdots \\ \mu_{Lp} \end{bmatrix}$$

$H_1: \mu_l \neq \mu_m$ for atleast one pair

$$\begin{bmatrix} \mu_{l1} \\ \mu_{l2} \\ \vdots \\ \mu_{lp} \end{bmatrix} \neq \begin{bmatrix} \mu_{m1} \\ \mu_{m2} \\ \vdots \\ \mu_{mp} \end{bmatrix}$$

initial observation vector.

X_{il} = is partioned as

$= \mu + (\mu_l - \mu) + (x_{il} - \mu_l)$

$$= \underbrace{\mu}_{\text{grand mean}} + \underbrace{\tau_l}_{\text{pop effect}} + \underbrace{\epsilon_{il}}_{\text{random errors}}$$

$$X_{il} = \mu + (\mu_l - \mu) + (x_{il} - \mu_l)$$

$$\begin{bmatrix} x_{il_1} \\ x_{il_2} \\ \vdots \\ x_{il_p} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} + \begin{bmatrix} \mu_{l1} - \mu_1 \\ \mu_{l2} - \mu_2 \\ \vdots \\ \mu_{lp} - \mu_p \end{bmatrix} + \begin{bmatrix} x_{il_1} - \mu_{l1} \\ \vdots \\ x_{il_p} - \mu_{lp} \end{bmatrix}$$

(i.e) $X_{il} = \mu + \tau_l + \epsilon_{il}$.

Assumptions:-

1) population covariances are equal
2) Errors are normally distributed
3) Errors are independent.

Y: p×1   same as Y: p×1 notation

Sampling distribution: $\frac{(n_A + n_B - 2)\,p}{(n_A + n_B) - p - 1} F_{p,\, n_A + n_B - p - 1}$ $n \times q$, $p \times q$

$$P\left\{ 0.75\,(10-\theta_A)^2 + 2 \times 0.57\,(10-\theta_A)(1-\theta_B) + 2.30\,(-2-\theta_B)^2 \in F_{2,67}(0.05) = 3.15 \right\} = 0.95$$

---

for $(1-\alpha) = 95\%$ confidence interval

testing $\alpha = 0.05 \Rightarrow \chi_2^2(0.05) = 5.99$

$\therefore T^2 = 15.66 > 5.99$ & we reject H_0

Testing for hypothesis

i) When $\Sigma_1 = \Sigma_2 = \Sigma$

$$T^2 = \{(\bar{x}_1 - \bar{x}_2) - \delta_0\}' \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{pooled}\right]^{-1} \{(\bar{x}_1 - \bar{x}_2) - \delta_0\}$$

where $\delta_0 = \mu_1 - \mu_2$ & $S_{pooled} = \dfrac{(n_1-1)S_1 + (n_2-1)S_2}{n_1+n_2-2}$

$$\sim \frac{(n_1+n_2-2)p}{(n_1+n_2-p-1)} F_{p,\, n_1+n_2-p-1} = c^2$$

$$P\{T^2 \le c^2\} = 1-\alpha \rightarrow T^2 > c^2 \Rightarrow \text{Reject}$$

When $\Sigma_1 \neq \Sigma_2$

$$\{(\bar{x}_1 - \bar{x}_2) - \delta_0\}' \left[\frac{S_1}{n_1} + \frac{S_2}{n_2}\right]^{-1} \{(\bar{x}_1 - \bar{x}_2) - \delta_0\} \le \chi_p^2(\alpha)$$

where p is dof & $\delta_0 = \mu_1 - \mu_2$.

One can test for $\mu_1 - \mu_2 = 0$ when the population covariance matrices are unequal $(\Sigma_1 \neq \Sigma_2)$

$$\{(\bar{x}_1 - \bar{x}_2)\}' \left[\frac{S_1}{n_1} + \frac{S_2}{n_2}\right]^{-1} \{(\bar{x}_1 - \bar{x}_2)\} \sim \frac{\nu p}{\nu - p + 1} F_{p,\, \nu-p+1}.$$

Where $\nu = \dfrac{p+p^2}{\sum_{i=1}^{2} \frac{1}{n_i}\left\{ tr\left[\left[\frac{1}{n_i} S_i \left(\frac{S_1}{n_1} + \frac{S_2}{n_2}\right)^{-1}\right]^2\right] + \left(tr\left[\frac{1}{n_i} S_i \left(\frac{S_1}{n_1} + \frac{S_2}{n_2}\right)^{-1}\right]\right)^2 \right\}}$

check?

$$\approx \frac{8-3-1}{3-1}\left(\frac{1-\sqrt{0.0385}}{\sqrt{0.0385}}\right) = 8.19$$

calculated $F_{2(2),\,2(4)}$ = $F_{4,8}(0.01) = 7.01$ $\Rightarrow 8.19 > 7.01$

$\Rightarrow$ Reject H_0 at 1% level of significance.

nursing home data - (Department of health & social service)

3 population's (i) private, nonprofit & government

four costs computed x_1 = cost of nursing labor x_2 = dietary, x_3 = operation & maintanence x_4 = housekeeping are given as

$l=1$ (private) $n_1 = 271$ $\bar{x}_1 = \begin{bmatrix} 2.066 \\ .048 \\ .082 \\ .360 \end{bmatrix}$ $\bar{x}_2 = \begin{bmatrix} 2.167 \\ 0.596 \\ 0.124 \\ 0.418 \end{bmatrix}$

$l=2$ (nonprofit) $n_2 = 138$

$l=3$ (govern) $n_3 = 107$ $\bar{x}_3 = \begin{bmatrix} 2.273 \\ 0.521 \\ 0.125 \\ 0.383 \end{bmatrix}$

$$S_1 = \begin{bmatrix} .291 & & & \\ -.001 & .011 & & \\ .002 & .000 & .001 & \\ .010 & .003 & .000 & .010 \end{bmatrix} \qquad S_2 = \begin{bmatrix} .561 & & & \\ .011 & .025 & & \\ .001 & .004 & .005 & \\ .037 & .007 & .002 & .019 \end{bmatrix}$$

$$S_3 = \begin{bmatrix} .261 & & & \\ .030 & .017 & & \\ 0.003 & -0.000 & .004 & \\ .018 & .006 & .001 & .013 \end{bmatrix}$$

Calculate $W = (n_1-1)S_1 + (n_2-1)S_2 + (n_3-1)S_3$.

$\bar{X} =$

$B =$ $\quad$ & $\Lambda^* = 0.7714$

$\rightarrow$ using 4th rule $= 17.67$ for $\alpha = 0.01$ LOS

tab value $= 2.51$ $\quad$ cal value $= 17.67$

$\Rightarrow$ reject H

Treatment ssq $= (-1)^2 \times 3 + (-3)^2 \times 3 + 3^2 \times 2 = 48$ with dof $3-1=2$

Residual ssq $= (-1)^2 + (-2)^2 + \dots + 1^2 = 72 - 48 = 24$
(Addition) (diff formula) with dof $= 8-3=5$

Total SSCP (uncorrected) $= 9 \times 3 + 6 \times 2 + \dots + 1 \times 9 + 2 \times 7 = 149$

Total $SSCP_T$ (corrected) $= 1(-2) + 2(-3) + \dots + (-3)(4) + (-2) \times 3 = -11$

Treatment (betn) $SSCP_B = 3 \times 4 \times (-1) + 2 \times (-3) \times (-3) + 3 \times (-2) \times (-3)$

$= -12.$

Residual (E) $SSCP_E = 1 \times (-1) + (-2) \times (-2) + \dots + (-1) \times 1 + 0 \times (-1) = 1$

one-way MANOVA

Sources of variation	dof	SSCP
Treatment	$l-1$ $\Rightarrow 3-1=2$	$\begin{pmatrix} 78 & -12 \\ -12 & 48 \end{pmatrix} = B$
Residual	$N-l$ $\Rightarrow 8-3=5$	$\begin{pmatrix} 10 & 1 \\ 1 & 24 \end{pmatrix} = W$
Total	$N-1$ $\Rightarrow 8-1=7$	$\begin{pmatrix} 88 & -11 \\ -11 & 72 \end{pmatrix} = B+W$

Now, $\Lambda^* = \dfrac{|W|}{|B+W|} = \dfrac{\begin{vmatrix} 10 & 1 \\ 1 & 24 \end{vmatrix}}{\begin{vmatrix} 88 & -11 \\ -11 & 72 \end{vmatrix}} = 0.0385$

here $p=2$, $l=3$ we have

$$\frac{(N-l-1)}{l-1} \left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}} \right) \sim F_{2(l-1),\, 2(N-l-1)}$$

Decomposition leads to

$$\sum_{l}\sum_{j}(x_{lj}-\bar{x})(x_{lj}-\bar{x})' = \sum_{l} n_l(\bar{x}_l-\bar{x})(\bar{x}_l-\bar{x})' + \sum_{l}\sum_{j}(x_{lj}-\bar{x}_l)(x_{lj}-\bar{x}_l)'$$

$$\begin{pmatrix} SSCP_T \\ \text{total corrected} \end{pmatrix} = \begin{pmatrix} SSCP_B \\ \text{(Between)} \end{pmatrix} + \begin{pmatrix} SSCP_E \\ \text{(residual within)} \end{pmatrix}$$

$N-1 \qquad L-1 \qquad N-L$

Now, the within SSCP can be expressed as

$$\sum_{l}\sum_{j}(x_{lj}-\bar{x}_l)(x_{lj}-\bar{x}_l)' = (n_1-1)S_1 + (n_2-1)S_2 + \dots + (n_L-1)S_L$$

Distribution of Wilk's lambda:-

$$\Lambda^* = \frac{|W|}{|B+W|} \quad \text{i.e.)} \quad \Lambda^* = \frac{|W|}{|B+W|} = \frac{\left|\sum_{l}\sum_{j}(x_{lj}-\bar{x}_l)(x_{lj}-\bar{x}_l)'\right|}{\left|\sum_{l}\sum_{j}(x_{lj}-\bar{x})(x_{lj}-\bar{x})'\right|}$$

this can be used to test H_0.

No of variables	No of groups	Sampling distribution
$p=1$	$l \geq 2$	$\left(\frac{N-L}{L-1}\right)\left(\frac{1-\Lambda^*}{\Lambda^*}\right) \sim F_{L-1,\,N-L}$
$p=2$	$l \geq 2$	$\left(\frac{N-L-1}{L-1}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\sqrt{\Lambda^*}}\right) \sim F_{2(L-1),\,2(N-L-1)}$
$p \geq 1$	$l=2$	$\left(\frac{L-p-1}{p}\right)\left(\frac{1-\Lambda^*}{\Lambda^*}\right) \sim F_{p,\,L-p-}$
$p \geq 1$	$l=3$	$\left(\frac{L-p-2}{p}\right)\left(\frac{1-\sqrt{\Lambda^*}}{\Lambda^*}\right) \sim F_{2p,\,2(L-p-}$

Wilk's lamba can also be expressed as a functio[n] of eigenvalues of $\hat{\lambda}_1, \hat{\lambda}_2, \dots \hat{\lambda}_s$ of $W^{-1}B$ as $\Lambda^* = \prod_{j=1}^{s}\left(\frac{1}{1+\dots}\right.$

$$\sum_{i=1}^{n_l}(x_{il}-\bar{x})(x_{il}-\bar{x})^T =$$

$$\sum_{i=1}^{n_l}(\bar{x}_l-\bar{x})(\bar{x}_l-\bar{x})^T + \sum_{i=1}^{n_l}(\bar{x}_l-\bar{x})(x_{il}-\bar{x}_l)^T$$

$$+ \sum_{i=1}^{n_l}(x_{il}-\bar{x}_l)(\bar{x}_l-\bar{x})^T$$

$$+ \sum_{i=1}^{n_l}(x_{il}-\bar{x}_l)(x_{il}-\bar{x}_l)^T$$

$$\sum_{l=1}^{L}\sum_{i=1}^{n_l}(x_{il}-\bar{x})(x_{il}-\bar{x})^T = \sum_{l=1}^{L}\sum_{i=1}^{n_l}(\bar{x}_l-\bar{x})(\bar{x}_l-\bar{x})^T + \sum_{l=1}^{L}\sum_{i=1}^{n_l}(x_{il}-\bar{x}_l)(x_{il}-\bar{x}_l)^T$$

$$\sum_{l=1}^{L} n_l(\bar{x}_l-\bar{x})(\bar{x}_l-\bar{x})^T + \sum_{l=1}^{L}\sum_{i=1}^{n_l}(x_{il}-\bar{x}_l)(x_{il}-\bar{x}_l)^T$$

Summation notation not required since in matrix form

Total = betⁿ + Error

$$\Rightarrow \begin{bmatrix} SSCP_T \end{bmatrix}_{p\times p} = \begin{bmatrix} SSCP_B \end{bmatrix}_{p\times p} + \begin{bmatrix} SSCP_E \end{bmatrix}_{p\times p}$$

(Sum square cross product)

df $\Rightarrow$ $N-1 = L-1 + N-L$

$$N = \sum_{l=1}^{L} n_l$$

$SSCP_B$ is computed using $\sum_{l=1}^{L} n_l(\bar{x}_l-\bar{x})(\bar{x}_l-\bar{x})^T$

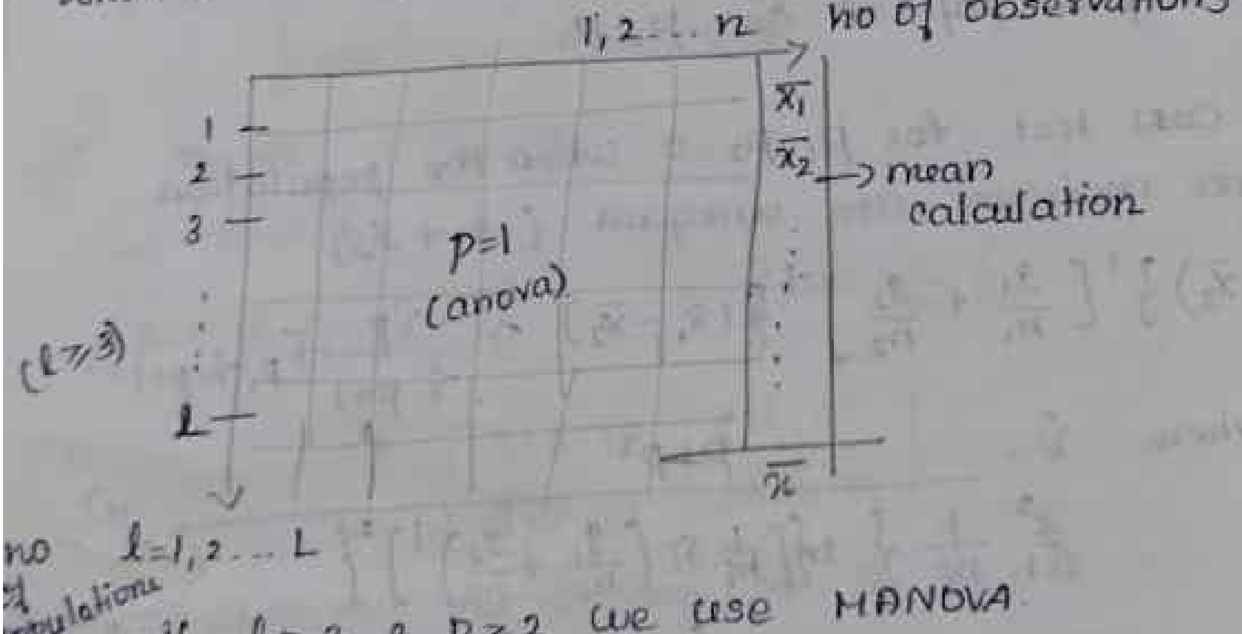
$SSCP_E$ " $= (n_1-1)S_1 + (n_2-1)S_2 + \cdots + (n_l-1)S_l$

$\therefore\ SSCP_T = SSCP_B + SSCP_E.$

Multivariate population means (MANOVA)

Populations (l)	Variables (p)	hypothesis	Statistic's test
$l = 2$	$p = 1$ (univariate)	$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$	t-test
$l = 2$	$p \geqslant 2$ (multivariate)	"	Hotelling T^2
$l \geqslant 2$	$p = 1$ (uni)	$H_0: \mu_1 = \mu_2 \dots = \mu_l$ H_1: atleast one pair $(\mu_l = \mu_m)$ is not equal	Anova.
$l \geqslant 2$	$p \geqslant 2$ (multi)	"	Manova.

What is Anova? (ANOVA)



if $l \geqslant 3$ & $p \geqslant 2$ we use MANOVA

for instance say $\left.\begin{matrix} l = 3 \\ p = 2 \end{matrix}\right\}$ MANOVA

The two sample situation when $\Sigma_1 \neq \Sigma_2$

When $\Sigma_1 \neq \Sigma_2$ we cannot use 'distance' measures like T^2 whose distributions does not depend on the unknowns Σ_1 & Σ_2. Bartlett's test is used to test the equality of Σ_1 & Σ_2.

$n_1 - p$ & $n_2 - p$ are large, then an approximation $100(1-\alpha)\%$ confidence ellipsoid for $\mu_1 - \mu_2$ is given by all $\mu_1 - \mu_2$ satisfying

$$\{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)\}^T \left[\frac{S_1}{n_1} + \frac{S_2}{n_2}\right]^{-1} \{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)\} \leq \chi^2_p$$

p to be dof.

Eg: Experimental observations for sample sizes $n_1 = 45$ & $n_2 = 55$; $\bar{x}_1 = \begin{bmatrix} 204.4 \\ 556.6 \end{bmatrix}$ $\bar{x}_2 = \begin{bmatrix} 130.0 \\ 355.0 \end{bmatrix}$

& $S_1 = \begin{bmatrix} 13825.3 & 23823.4 \\ 23823.4 & 73107.4 \end{bmatrix}$ & $S_2 = \begin{bmatrix} 8632.0 & 19616.7 \\ 19616.7 & 55964.5 \end{bmatrix}$

Soln: $n_1 = 45$ & $n_2 = 55$

first calculate

$$\frac{1}{n_1} S_1 + \frac{1}{n_2} S_2 = \frac{1}{45}\left[\quad\right] + \frac{1}{55}\left[\quad\right]$$

$$= \begin{bmatrix} 464.17 & 886.08 \\ 886.08 & 2642.15 \end{bmatrix}$$

T^2 statistic for testing $H_0: \mu_1 - \mu_2 = 0$ is

$$T^2 = [(\bar{x}_1 - \bar{x}_2)]' \left[\frac{1}{n_1} S_1 + \frac{1}{n_2} S_2\right]^{-1} [\bar{x}_1 - \bar{x}_2]$$

$$= \begin{bmatrix} 204.4 - 130.0 \\ 556.6 - 355.0 \end{bmatrix}^T \begin{bmatrix} 464.17 & 886.08 \\ 886.08 & 2642.15 \end{bmatrix}^{-1} \begin{bmatrix} 204.4 - 130.0 \\ 556.6 - 355.0 \end{bmatrix}$$

$$= 15.66$$

$\rightarrow$ This formula for $\mu_1 - \mu_2 \neq 0$, if $\mu_1 - \mu_2 = 0$

then $T^2 = [(\bar{x}_1 - \bar{x}_2)]' \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{pooled}\right]^{-1} (\bar{x}_1 - \bar{x}_2)^T$

$\rightarrow$ Someone interested in finding the confidence interval without knowing the μ_1 & μ_2 then use the concept of finding c^2 using the eigen values & vectors of S_{pooled} (or) S_{pooled}^{-1}. then evaluate [Refer book]

Result:- if $\Sigma_1 \neq \Sigma_2$ we cannot apply T^2, instead we use Bartlett's test.

$\rightarrow$ Let $(n_1 - p)$ & $(n_2 - p)$ are large then approxi $100(1-\alpha)\%$ confidence ellipsoid for $\mu_1 - \mu_2$ is given by

$$[(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)]' \left[\frac{1}{n_1} S_1 + \frac{1}{n_2} S_2\right]^{-1} [(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)] \leq \chi^2_p(\alpha)$$

i) When n_1 & n_2 are large then S_1 close to Σ_1 & S_2 close to Σ_2. Replace S_1 & S_2 by Σ_1 & Σ_2 in the above.

Conclusion

No of population	no of variables	hypothesis	Techniq
Small sample: $l = 1$	$p = 1$	$H_0: \mu = \mu_0$, $H_1: \mu \neq \mu_0$	t-test
Small sample: $l = 1$ (single population)	$p \geq 2$	$H_0: \mu = \mu_0$, $H_1: \mu \neq \mu_0$	Hotellir
$l = 2$	$p = 1$	$H_0: \mu_l = \mu_m$, $H_1: \mu_l \neq \mu_m$	t-tes
	$p \geq 2$	$H_0: \mu_l = \mu_m$, $H_1: \mu_l \neq \mu_m$	T^2 Ho
from	hypothesis testing point of view.		

Result: $\bar{x}_1$ from $N_p(\mu_1, \Sigma)$ of size n_1 & $\bar{x}_2$ from $N_p(\mu_2, \Sigma)$ of size n_2

then sampling distribution is

$$T^2 = [(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)]' \left[\left(\frac{1}{n_1} + \frac{1}{n_2}\right) S_{pooled}\right]^{-1} [(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)]$$

follows $\frac{(n_1+n_2-2)p}{(n_1+n_2-p-1)} F_{p,\, n_1+n_2-p-1}$

$\therefore P\{T^2 \le c^2\} = 1-\alpha$ where $c^2 = \frac{(n_1+n_2-2)p}{(n_1+n_2-p-1)} F \ldots$

Eg: fifty bars of soap are manufactured in each of two ways. Two characteristics x_1 = lather, x_2 = mildness, are measured. The statistics for bars produced by method 1 & 2 are

$\bar{x}_1 = \begin{bmatrix} 8.3 \\ 4.1 \end{bmatrix}$ $S_1 = \begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$ obtain a 95% confidence region for μ_1 & μ_2

$\bar{x}_2 = \begin{bmatrix} 10.2 \\ 3.9 \end{bmatrix}$ $S_2 = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$

Soln: We notice S_1 & S_2 are approx. equal so that it is reasonable to pool them. $n_1 = 50$, $n_2 = 50$

$$S_{pooled} = \frac{49 S_1 + 49 S_2}{98} = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}$$

$$= \frac{(50-1)\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix} + (50-1)\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}}{98 \ (n_1+n_2-2)}$$

$$\bar{x}_1 - \bar{x}_2 = \begin{bmatrix} 8.3 \\ 4.1 \end{bmatrix} - \begin{bmatrix} 10.2 \\ 3.9 \end{bmatrix} = \begin{bmatrix} -1.9 \\ 0.2 \end{bmatrix}$$

Obtain the expression for T^2 & compare with c^2 value to decide the accept (or) reject H_0

Calculating,

$x_{11}, x_{12} \ldots x_{1n_1}$ → size n_1 → with mean vector μ_1 & covariance matrix Σ_1

$x_{21}, x_{22} \ldots x_{2n}$ ⇒ size n_2 → with mean vector μ_2 & covariance matrix Σ_2.

$$E(\bar{x}_1 - \bar{x}_2) = \mu_1 - \mu_2$$

$$V(\bar{x}_1 - \bar{x}_2) = V(\bar{x}_1) + V(\bar{x}_2) = \frac{\Sigma_1}{n_1} + \frac{\Sigma_2}{n_2}$$

Create $Z = \frac{RV - E(\ )}{\sqrt{V(\ )}}$

$$\Rightarrow Z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\Sigma_1}{n_1} + \frac{\Sigma_2}{n_2}}}$$ → in multivariate domain

cases:) Sampling from multivariate normal populations

i) with known Σ_1 & Σ_2

ii) with small sample size & unknown but equal Σ
$\Sigma_1 = \Sigma_2 = \Sigma$

iii) with large sample size & unknowns $\Sigma_1 = \Sigma_2$.

In case of $\Sigma_1 = \Sigma_2 = \Sigma$ is an estimation of $(n_1-1)\Sigma$ & $(n_2-1)\Sigma$ we can pool the information in both sample in order to estimate the common covariance Σ.

$$S_{pooled} = \frac{\sum_{j=1}^{n_1}(x_{1j} - \bar{x}_1)(x_{1j} - \bar{x}_1)' + \sum_{j=1}^{n_2}(x_{2j} - \bar{x}_2)(x_{2j} - \bar{x}_2)'}{n_1 + n_2 - 1.}$$ → has (n_1-1) dof
(n_2-1) dof

combin

$$= \frac{n_1 - 1}{n_1 + n_2 - 2} S_1 + \frac{n_2 - 1}{n_1 + n_2 - 2} S_2$$

$$= \frac{(n_1 - 1) S_1 + (n_2 - 1) S_2}{n_1 + n_2 - 2}$$ ⇒ Estimation o

W.K.T. $T^2 = n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu)$

$$\bar{x} = \begin{pmatrix} 10 \\ 20 \end{pmatrix} \qquad S = \begin{pmatrix} 40 & -50 \\ -50 & 100 \end{pmatrix}$$

$$\bar{x}-\mu = \begin{bmatrix} 10 \\ 20 \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 10-\mu_1 \\ 20-\mu_2 \end{bmatrix}$$

$$S^{-1} = \frac{1}{|S|}\,\text{adj}(S) = \frac{1}{\underset{1500}{|S|}}\begin{bmatrix} \text{transpose} \\ \text{of cofactor} \end{bmatrix} \quad \frac{1}{1500}\begin{bmatrix} 100 & 50 \\ 50 & 40 \end{bmatrix}$$

$$T^2 = 20\begin{bmatrix} 10-\mu_1 & 20-\mu_2 \end{bmatrix} \frac{1}{1500}\begin{bmatrix} 100 & 50 \\ 50 & 40 \end{bmatrix}\begin{bmatrix} 10-\mu_1 \\ 20-\mu_2 \end{bmatrix}$$

<u>Ans:</u> $(1.33)(10-\mu_1)^2 + (1.34)(10-\mu_1)(20-\mu_2) + (0.53)(20-\mu_2)^2$

ii) $T^2 \sim \dfrac{(n-1)p}{n-p} F_{p,\,n-p} = \dfrac{(20-1)2}{20-2} F_{2,\,20-2}$

$$= \frac{19\times 2}{18} F_{2,18} = \frac{19}{9} F_{2,18}$$

$S = \frac{1}{n-1}(x_i-\bar{x})^T(x_i \ldots$

$$\rightarrow \quad T^2 = \sqrt{n}(\bar{x}-\mu)^T \left[\frac{1}{n-1}(x_i-\bar{x})(x_i-\bar{x})^T\right]^{-1} \sqrt{n}(\bar{x}-\mu)$$

$$= \begin{pmatrix} \text{multivariate} \\ \text{normal} \\ \text{vector} \end{pmatrix}^T \begin{pmatrix} \dfrac{\text{Wishart random matrix}}{\text{dof}} \end{pmatrix}^{-1} \begin{pmatrix} \text{multivariate} \\ \text{normal} \\ \text{vector} \end{pmatrix}$$

$$= N_p(0,\Sigma)^T \left[\frac{W_{p,\,n-1}(\Sigma)}{n-1}\right]^{-1} N_p(0,\Sigma)$$

<u>Eg2:</u> Let the data matrix for a random sample of size $n=3$ from bivariate normal distribution be

$X = \begin{bmatrix} 6 & 9 \\ 10 & 6 \\ 8 & 3 \end{bmatrix}$ Evaluate the observed T^2 for $\mu_0' = [9$ $=7/9$

What is the sampling distribution of T^2 in this case?

$4 F_{2,1}$.