Exam Date & Time: 04-Apr-2022 (04:30 PM - 05:30 PM)

MANIPAL INSTITUTE OF TECHNOLOGY
MANIPAL
(A constituent unit of MAHE, Manipal)

B-TECH IV - SEMESTER
FIRST SESSIONAL EXAMINATION APRIL- 2022
MATHEMATICAL FOUNDATION FOR DATA SCIENCE-II [MAT 2213]

Marks: 15 **Duration: 60 mins.**

Section - A(MCQ)

Answer all the questions. Section Duration: 20 mins

1) Consider the Markov chain with state space $S = \{0, 1, 2\}$ and transition probability matrix $P = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix}$, then the states are ___________ (0.5)

1)	periodic with period 2	✓	2)	periodic with period 3		3)	periodic with period 4		4)	periodic with period 1	

2) Consider a finite state discrete time Markov chain. Let the matrix Q specifies only the transition probabilities from transient states into transient states. Then which of the following statement is true. (0.5)

1)	In Q, some of its row sums are less than 1	✓	2)	In Q, all of its row sums are equal to 1		3)	In Q, all of its row sums are equal to 0		4)	In Q, some of its elements are negative	

3) For the Markov chain with state space $\{a, b, c, d\}$ and transition probability matrix

$$P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0.4 & 0.6 & 0 \\ 0.8 & 0 & 0.2 & 0 \\ 0.2 & 0.3 & 0 & 0.5 \end{bmatrix}$$

,$P[X_5 = c, X_6 = a, X_7 = c, X_8 = c | X_4 = b, X_3 = d]$ is (0.5)

1)	0.0288		2)	0.0960	✓	3)	0.1600		4)	0.6000

4) An irreducible Markov chain with finitely many states has no (0.5)

1) Null recurrent states 2) Transient states 3) null recurrent and transient states ✓ 4) Absorbing states

5) The number of students waiting for a bus at any time of day is an example for (0.5)

1) Discrete state space, discrete 2) Continuous state space, discrete 3) Discrete state space, continuous ✓ 4) continuous state space, continuous

parameter space	parameter space	parameter space	parameter space

6) If an irreducible aperiodic Markov chain has an invariant distribution, then all its states are (0.5)

1) recurrent non-null 2) Transient non-null 3) Recurrent null 4) Transient null

7) It the ultimate return to a state i having started from it, has a probability less than unity, then the state i is called (0.5)

1)	Ergodic state		2)	transient state		3)	Persistent state		4)	Absorbing state	

8) Let $\{X_n, n \geq 0\}$ be a Markov chain having state space $S = \{1, 2, 3, 4\}$ and transition probability matrix

$$P = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

Then which of the following statement is false. (0.5)

1)	State 3 is transient		2)	State 1 is transient		3)	State 4 is transient		4)	State 2 is ergodic	

9) For the Markov chain with state space $\{1,2,3,4\}$ and transition probability matrix

$$P = \begin{bmatrix} 1/2 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

, the absorbing state is (0.5)

1)	1		2)	4		3)	2		4)	3	

10) For the Markov chain with state space $\{1,2,3\}$ and transition probability matrix

$$P = \begin{bmatrix} 0 & 0.6 & 0.4 \\ 0 & 1 & 0 \\ 0.6 & 0.4 & 0 \end{bmatrix}$$

, the period of state **3** is (0.5)

1)	2		2)	1		3)	3		4)	4	

Section - B(DESCRIPTIVE)

Answer all the questions. Section Duration: 40 mins

1) Let $\{X_n, n = 0, 1, 2, \cdots\}$ be a Markov chain with state space $S = \{0, 1\}$ and one-step transition probability matrix $P = \begin{pmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{pmatrix}$. Let the initial distribution be $P(0) = (0.5, 0.5)$. Find the value of the probability $P(X_3 = 0)$. (2)

$$P = \begin{pmatrix} 0.1 & 0.9 \\ 0.3 & 0.7 \end{pmatrix} \qquad P(0) = (0.5, 0.5)$$

$$P^3 = \begin{pmatrix} 0.2440 & 0.7560 \\ 0.2520 & 0.7480 \end{pmatrix}$$

1

$$P(X_3=0) = P(X_3=0 \mid X_0=0)\, P(X_0=0) + P(X_3=0 \mid X_0=1)\, P(X_0=1)$$

$$= \frac{1}{2}(0.2440 + 0.2520)$$

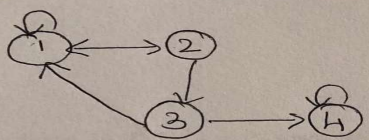
1

$$= 0.\underline{\underline{2480}}$$

2. Consider a Markov chain with state space $\{1,2,3,4\}$ and transition probability matrix $P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Find (a) the expected number of visits to state 3 beginning from state 1, and (b) the expected number of visits to state 1 beginning from state 3.

(2)

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} .5 & .5 & 0 & 0 \\ .5 & 0 & .5 & 0 \\ .5 & 0 & 0 & .5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



States 1, 2 & 3 are transient & 4 is recurrent.

(a) $\mu_{13} = \sum_{n=1}^{\infty} n f_{13}^{(n)} = 2$ ______ (1 mark)

(b) $\mu_{31} = \sum_{n=1}^{\infty} n f_{31}^{(n)} = 4.$ ______ (1 mark)

Aliter:

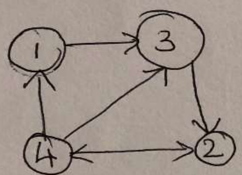
$$\left(I - \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} .5 & .5 & 0 \\ .5 & 0 & .5 \\ .5 & 0 & 0 \end{bmatrix} \end{array} \right)^{-1} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 8 & 4 & 2 \\ 6 & 4 & 2 \\ 4 & 2 & 2 \end{bmatrix} \end{array}$$

μ_{13}

μ_{31}

3. Consider a Markov chain with state space $\{1,2,3,4\}$ and transition probability matrix $P = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/4 & 1/8 & 1/8 & 1/2 \end{bmatrix}$. Examine the nature of the states, clearly identifying the recurrent, null/non-null, transient states and their periods.

(13)



Chain is irreducible.
∴ All states have same nature.
$p_{44} = 1/2 > 0$; aperiodic. ______ (1 mark)

$f_{44}^{(1)} = 1/2$; $f_{44}^{(2)} = p_{42} p_{24} = \frac{1}{8} \times 1 = 1/8$;

$f_{44}^{(3)} = p_{43} p_{32} p_{24} = \frac{1}{8} \times 1 \times 1 = 1/8$; $f_{44}^{(n)} = 0 \; \forall n \geq 5$

$f_{44}^{(4)} = p_{41} p_{13} p_{32} p_{24} = \frac{1}{4} \times 1 \times 1 \times 1 = 1/4$;

$F_{44} = \sum_{n=1}^{\infty} f_{44} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + 0 + 0 + \cdots = 1$ ∴ Recurrent — (1 mark)

$\mu_{44} = \sum_{n=1}^{\infty} n f_{44}^{(n)} = \frac{17}{8} < \infty$ ∴ +vely recurrent

State 4 is ergodic. Hence all states are ergodic ______ (1 mark).

4. For his yearly vacation, a business executive selects one of three places – the Bahamas, Europe, or Hawaii – using the following rule: If he has been to the Bahamas the past year, he will choose Europe with probability $2/3$ and Hawaii with probability $1/3$. If he has been to Europe the past year, he will choose the Bahamas, Europe again, and Hawaii with probabilities $3/8$, $1/8$, and $1/2$, respectively. If he has spent his vacation in Hawaii, the Bahamas and Europe are equally likely to be chosen this year. How would you rate his preferences after a sufficiently long time? Interpret the results.

1. Let the three states be B (the Bahamas), E (Eu and H (Hawaii)

TPM is

$$P = \begin{array}{c} \\ B \\ E \\ H \end{array} \begin{array}{c} \begin{array}{ccc} B & E & H \end{array} \\ \begin{pmatrix} 0 & 2/3 & 1/3 \\ 3/8 & 1/8 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \end{array} \quad (1/2)$$

calculating the long-run probabilities by solving

$\pi P = \pi \qquad \pi = (\pi_B \ \pi_E \ \pi_H)$

& $\Sigma \pi_i = 1$

ie $(\pi_B \ \pi_E \ \pi_H) = (\pi_B \ \pi_E \ \pi_H) \begin{pmatrix} 0 & 2/3 & 1/3 \\ 3/8 & 1/8 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$ (2)

$\Rightarrow$
$$\left.\begin{aligned} \pi_B &= \tfrac{3}{8}\pi_E + \tfrac{1}{2}\pi_H \\ \pi_E &= \tfrac{2}{3}\pi_B + \tfrac{1}{8}\pi_E + \tfrac{1}{2}\pi_H \\ \pi_H &= \tfrac{1}{3}\pi_B + \tfrac{1}{2}\pi_E \end{aligned}\right\} \circledast$$

solving $\circledast$ simultaneously with the condition $\pi_B + \pi_E + \pi_H =$

$\Rightarrow \quad \pi_B = 0.3, \ \pi_E = 0.4, \ \pi_H = 0.3$ (1/2)

$\therefore$ after a sufficiently long time he would prefer Europe with probability 0.4 & the Bahamas & Hawaii with probability 0.3 each.