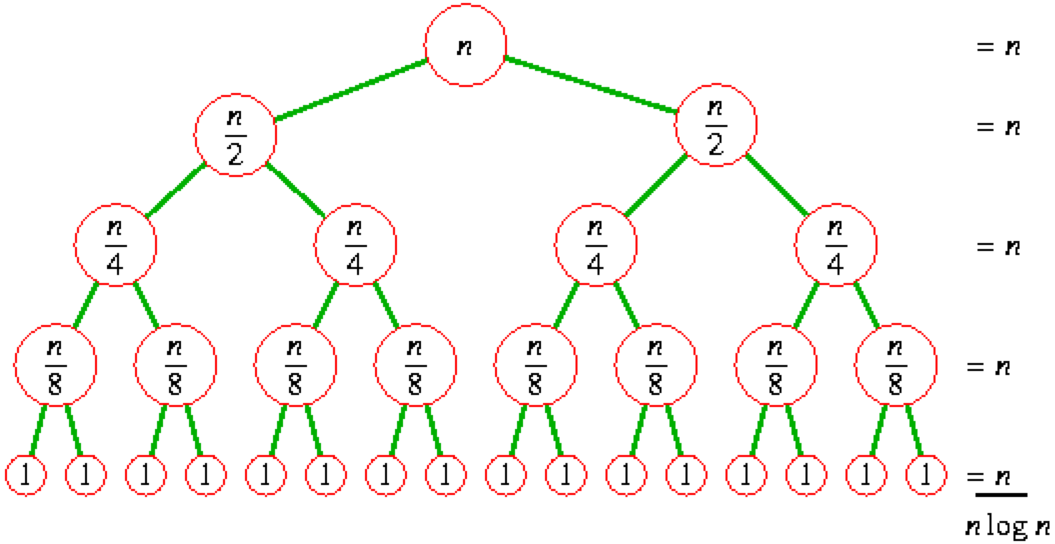
DSE 2256 DESIGN & ANALYSIS OF ALGORITHMS

Lecture 21

Divide-and-Conquer:

Multiplication of large integers
Strassen's Matrix Multiplication



Recap of L20

- Binary Search

- Binary Tree
  - Computing Height of a Binary Tree
  - Binary Tree Traversals

Multiplication of Large Integers

- Consider the problem of multiplying two (large) n-digit integers represented by arrays of their digits such as:

$$A = 12345678901357986429 \quad B = 87654321284820912836$$

The grade-school algorithm:

$$\begin{array}{r} a_1\ a_2 \ldots\ a_n \\ b_1\ b_2 \ldots\ b_n \\ \hline (d_{10})\, d_{11} d_{12} \ldots d_{1n} \quad \\ (d_{20})\, d_{21} d_{22} \ldots d_{2n} \qquad\quad \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \qquad \\ (d_{n0})\, d_{n1} d_{n2} \ldots d_{nn} \qquad\qquad\quad \\ \hline \end{array}$$

Efficiency: $\Theta(n^2)$ single-digit multiplications

Multiplication of Large Integers

$$23 * 14$$

$$23 = 2 \cdot 10^1 + 3 \cdot 10^0 \qquad 14 = 1 \cdot 10^1 + 4 \cdot 10^0$$

- Multiplying 23 and 14 :

$$23 * 14 = (2 \cdot 10^1 + 3 \cdot 10^0) * (1 \cdot 10^1 + 4 \cdot 10^0)$$

$$= \underbrace{(2 * 1)10^2 + \boxed{(2 * 4 + 3 * 1)}10^1 + (3 * 4)10^0}_{\text{4 multiplications!}} \quad \text{---------} \quad (1)$$

- Re-considering Middle part of Equation (1):

Already computed

$$2 * 4 + 3 * 1 = \underbrace{(2 + 3) * (1 + 4)}_{\text{1 multiplication instead of 2 in middle part of Eq. (1)}} - \boxed{2 * 1} - \boxed{3 * 4}$$

Multiplication of Large Integers

- To multiply two n-digit numbers, apply the divide-and-conquer strategy by dividing both numbers in the middle.

$$c = a * b = (a_1 10^{n/2} + a_0) * (b_1 10^{n/2} + b_0)$$
$$= (a_1 * b_1)10^n + (a_1 * b_0 + a_0 * b_1)10^{n/2} + (a_0 * b_0)$$
$$= c_2 10^n + c_1 10^{n/2} + c_0,$$

where

$c_2 = a_1 * b_1$ is the product of their first halves,

$c_0 = a_0 * b_0$ is the product of their second halves,

$c_1 = (a_1 + a_0) * (b_1 + b_0) - (c_2 + c_0)$ is the product of the sum of the a's halves and the sum of the b's halves minus the sum of c_2 and c_0.

Multiplication of Large Integers

Example: **2101 * 1130**

$C_2 = 21*11$, $C_0 = 01*30$

$C_1 = (21+01) * (11+30) - (C_2+C_0) = 22*41 - 21*11 + 01*30$

For 21*11

$C_2 = 2*1 = 2$

$C_0 = 1*1 = 1$

$C_1 = (2+1) * (1+1) - (2+1)$

$= 3*23 = 3$

$21*11 = 2*10^2 + 3*10^1 + 1$

$= \mathbf{231}$

For 01*30

$C2 = 0*3 = 0$

$C_0 = 1*0 = 0$

$C_1 = (0+1) * (3+0) - (0+0)$

$= 1*3 - 0 = 3$

$01*30 = 0*10^2 + 3*10^1$

$= \mathbf{30}$

For 22*41

$C2 = 2*4 = 8$

$C_0 = 2*1 = 2$

$C_1 = (2+2) * (4+1) - (8+2)$

$= 4*5 - 10 = 10$

$22*41 = 8*10^2 + 10*10^1 + 2$

$= \mathbf{902}$

Hence, $2101*1130 = 231*10^4 + (902-231-30)*10^2 + 30$

$= 2{,}374{,}130$

Multiplication of Large Integers : Analysis

- The recurrence relation for this approach would be:

$$M(n) = 3M(n/2), \quad \text{for } n>1,$$

$$M(1) = 1$$

- Solving it by backward substitution for $n = 2^k$:

$$M(2^k) = 3M(2^{k-1})$$

$$= 3M[3M(2^{k-2})] = 3^2 M(k-2) = \ldots\ldots\ldots = 3^i M(2^{k-i}) = \ldots\ldots\ldots = 3^k M(2^{k-k}) = \mathbf{3^k}$$

Since, $k = \log_2 n$:

$$M(n) = 3^{\log_2 n} = n^{\log_2 3} \approx \mathbf{n^{1.585}}$$

Matrix Multiplication

- Brute force approach for matrix multiplication

$$\begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} * \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix}$$

$$= \begin{bmatrix} a_{00} * b_{00} + a_{01} * b_{10} & a_{00} * b_{01} + a_{01} * b_{11} \\ a_{10} * b_{00} + a_{11} * b_{10} & a_{10} * b_{01} + a_{11} * b_{11} \end{bmatrix}$$

- Time complexity = $O(n^3)$

Strassen's Matrix Multiplication

- Introduced by Volker Strassen in 1969.

$$\begin{bmatrix} c_{00} & c_{01} \\ c_{10} & c_{11} \end{bmatrix} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix} * \begin{bmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{bmatrix}$$

$$= \begin{bmatrix} m_1 + m_4 - m_5 + m_7 & m_3 + m_5 \\ m_2 + m_4 & m_1 + m_3 - m_2 + m_6 \end{bmatrix}$$

$m_1 = (a_{00} + a_{11}) * (b_{00} + b_{11})$

$m_2 = (a_{10} + a_{11}) * b_{00}$

$m_3 = a_{00} * (b_{01} - b_{11})$

$m_4 = a_{11} * (b_{10} - b_{00})$

$m_5 = (a_{00} + a_{01}) * b_{11}$

$m_6 = (a_{10} - a_{00}) * (b_{00} + b_{01})$

$m_7 = (a_{01} - a_{11}) * (b_{10} + b_{11})$

Exercise

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 5 & 0 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 3 & 5 & 0 \end{bmatrix}$$

Recap of L21: Solution to the exercise

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 4 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 5 & 0 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 0 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 2 & 0 & 1 & 1 \\ 1 & 3 & 5 & 0 \end{bmatrix}$$

For the matrices given, Strassen's algorithm yields the following:

$$C = \left[\begin{array}{c|c} C_{00} & C_{01} \\ \hline C_{10} & C_{11} \end{array}\right] = \left[\begin{array}{c|c} A_{00} & A_{01} \\ \hline A_{10} & A_{11} \end{array}\right] \left[\begin{array}{c|c} B_{00} & B_{01} \\ \hline B_{10} & B_{11} \end{array}\right]$$

where

$$A_{00} = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}, \quad A_{01} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_{10} = \begin{bmatrix} 0 & 1 \\ 5 & 0 \end{bmatrix}, \quad A_{11} = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix},$$

$$B_{00} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \quad B_{01} = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}, \quad B_{10} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}.$$

Therefore,

$$\begin{aligned}
M_1 &= (A_{00} + A_{11})(B_{00} + B_{11}) = \begin{bmatrix} 4 & 0 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 7 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 20 & 14 \end{bmatrix}, \\
M_2 &= (A_{10} + A_{11})B_{00} = \begin{bmatrix} 3 & 1 \\ 7 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 8 \end{bmatrix}, \\
M_3 &= A_{00}(B_{01} - B_{11}) = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -5 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ -9 & 4 \end{bmatrix}, \\
M_4 &= A_{11}(B_{10} - B_{00}) = \begin{bmatrix} 3 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ 3 & 0 \end{bmatrix}, \\
M_5 &= (A_{00} + A_{01})B_{11} = \begin{bmatrix} 3 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 3 \\ 10 & 5 \end{bmatrix}, \\
M_6 &= (A_{10} - A_{00})(B_{00} + B_{01}) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix}, \\
M_7 &= (A_{01} - A_{11})(B_{10} + B_{11}) = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 6 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -9 & -4 \end{bmatrix}.
\end{aligned}$$

Recap of L21: Solution to the exercise

Accordingly,

$$
\begin{aligned}
C_{00} &= M_1 + M_4 - M_5 + M_7 \\
&= \begin{bmatrix} 4 & 8 \\ 20 & 14 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 3 & 0 \end{bmatrix} - \begin{bmatrix} 8 & 3 \\ 10 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -9 & -4 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \\
C_{01} &= M_3 + M_5 \\
&= \begin{bmatrix} -1 & 0 \\ -9 & 4 \end{bmatrix} + \begin{bmatrix} 8 & 3 \\ 10 & 5 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 1 & 9 \end{bmatrix}, \\
C_{10} &= M_2 + M_4 \\
&= \begin{bmatrix} 2 & 4 \\ 2 & 8 \end{bmatrix} + \begin{bmatrix} 6 & -3 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 5 & 8 \end{bmatrix}, \\
C_{11} &= M_1 + M_3 - M_2 + M_6 \\
&= \begin{bmatrix} 4 & 8 \\ 20 & 14 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ -9 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 2 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 7 \end{bmatrix}.
\end{aligned}
$$

That is,

$$
C = \begin{bmatrix} 5 & 4 & 7 & 3 \\ 4 & 5 & 1 & 9 \\ 8 & 1 & 3 & 7 \\ 5 & 8 & 7 & 7 \end{bmatrix}.
$$

Thank you!

Any queries?