Operations Research → methods adopted by top management to solve Business problems

"optimizing"

↳ Maximize return

↳ minimize cost

Phases of OR

1. Problem Definition / Identification
2. Model Construction
3. Model Solution
4. Model validation
5. Solution Implementation

Scope of OR

1. Finance
2. Marketing
3. Production Planning

} Linear Programming Problem

4. Logistics & Transportation } Transportation Algorithm
5. Human Resources

Limitations of OR

1. Magnitude of computations
2. Non-quantifiable factors
3. Implementation

Linear Programming Problem (LPP)

→ it is a planning tool available to the top management
→ optimize a objective function

Procedure (how to formulate LPP)

Mathematical formulation of LPP:

1. Identify the decision variables in the problem
2. formulate the objection function to be optimized as a linear function of decision variables
3. formulate the constraints of the problem such as, resource limitation, market conditions etc. as a linear equation or in equations in terms of decision variables
4. Add the non-negative constraint such that the negative values of decision variable do not have any physical interpretation

Problem 1

Chipmont Company manufactures silicon wafer circuit for the use in microprocessors. Computer makers presently buy chipmont's entire production of following types of silicon chips; CPU, IC & memory. The following data applied. Formulate the problem as a LPP

Allocation Problem	CPU	IC	Memory	Max availability
Silicon (sheets)	0.05	0.02	0.01	10000 sheets
Sorting labour (hrs)	0.2	0.5	0.1	200000 hrs
Chemical wash (hrs)	0.1	0.4	0.15	400000 hrs
Profit /unit	\$0.25	\$0.4	\$0.15	

Formulation

Let x_1 be no. of units of CPU to be manufactured

Let x_2 be no. of units of IC " " "

Let x_3 be no. of units of Memory " " "

Objective function (z) → maximize profit

$$\text{Max; } z = 0.25x_1 + 0.4x_2 + 0.15x_3$$

Subject to : $0.05x_1 + 0.02x_2 + 0.01x_3 \le 10000$

$0.2x_1 + 0.5x_2 + 0.1x_3 \le 200000$

$0.1x_1 + 0.4x_2 + 0.15x_3 \le 400000$

$x_1, x_2, x_3 \ge 0$

Blending Problem

Problem 2

A firm products an alloy having the following specifications. Specific gravity less than or equal to 0.98, chromium percentage greater than or equal to 8. melting point of alloy ≥ 450°C. Raw materials A, B & C having the properties as shown can be used to make this alloy.

Properties of Raw materials

Property	A	B	C
Specific gravity	0.92	0.97	1.04
Chromium %	7%	13%	16%
Melting point	440°C	490°C	480°%
Cost of raw material per unit	Rs. 90	Rs. 280	Rs. 40

Find the proportions in which the raw materials A, B & C must be mixed to obtain an alloy of desired properties while keeping the cost a minimum.

$x_1 \rightarrow$ proportion of ~~no of units~~ A

$x_2 \rightarrow$ " " B

$x_3 \rightarrow$ " " C

Objective function (z) → minimize cost

$$\text{Min } z = 90x_1 + 280x_2 + 40x_3$$

Subject to : $0.92x_1 + 0.97x_2 + 1.04x_3 \le 0.98$

$0.07x_1 + 0.13x_2 + 0.16x_3 \ge 0.08$

$440x_1 + 490x_2 + 480x_3 \ge 450$

$x_1, x_2, x_3 \ge 0$

$x_1 + x_2 + x_3 = 1$

Problem 3 (Portfolio & Media Selection)

Consider the case of ABC Mutual Fund. It has developed the investment alternatives as given in the table. The ROI is expressed as the annual rate of return on the capital. The risk is the subjective assessment of the safety of the investment on a scale from 0 to 10 made by the portfolio manager. The term of the investment is the average length of time required to realize the ROI.

ABC's objective is to maximize the ROI. The guidelines for selecting a portfolio are;

1) The average risk should not exceed 2.5
2) The average term of investment should not exceed 6 years
3) Atleast 15% of the funds should be retained in the form of cash

Formulate the problem as a LPP.

Alternatives	ROI	Risk	Term (years)
Blue Chip Stock	12%	2	4
Bonds	10%	1	8
Growth stock	15%	3	2
Speculation	25%	4	10
Cash	0	0	0

Let x_1, x_2, x_3, x_4, x_5 be the percentage of investment made in blue chip stock, bonds, growth stock, speculation and cash respectively

Objective function $\longrightarrow$ maximize returns

$$\text{Max}: \; Z = 12x_1 + 10x_2 + 15x_3 + 25x_4 + 0x_5$$

Subject to,

$$2x_1 + x_2 + 3x_3 + 4x_4 + 0x_5 \leq 2.5 \quad (1)$$
$$4x_1 + 8x_2 + 2x_3 + 10x_4 + 0x_5 \leq 6 \quad (2)$$
$$x_5 \geq 15 \quad (3)$$
$$x_1 + x_2 + x_3 + x_4 + x_5 = 100$$
$$x_1, x_2, x_3, x_4, x_5 \geq 15$$

Problem 4 (Diet Problem)

A person wants to decide the constituents of a diet which will fulfill his daily requirements of protein, carbohydrates and fats at minimum cost. The choice is to be made from four different food types. The data is as given below. Formulate the problem as a LPP.

Food type	Proteins	Fats	Carbs	Cost / unit
1	3	2	6	45
2	4	2	4	40
3	8	7	7	85
4	6	5	4	65
Requirement	800	200	700	

Let x_1, x_2, x_3, x_4 be the no. of units of food type 1, 2, 3 and 4 the person needs to consume per day.

Objective function $\longrightarrow$ minimize cost

$$\text{Min}: \; Z = 45x_1 + 40x_2 + 85x_3 + 65x_4$$

Subject to,

$$3x_1 + 4x_2 + 8x_3 + 6x_4 \geq 800$$
$$2x_1 + 2x_2 + 7x_3 + 5x_4 \geq 200$$
$$6x_1 + 4x_2 + 7x_3 + 4x_4 \geq 700$$
$$x_1, x_2, x_3 \geq 0$$

Problem 5

Consider a small manufacturer making products A and B. Two resources R_1 and R_2 are required to make them. Each unit of product A requires 1 unit of R_1 and 2 units of R_2. The manu--facturer currently has 5 units and 12 units of R_2 available. The manufacturer will make a profit of Rs. 6 / unit on product A and Rs. 5/unit on Product B. Convulate the problem as a LPP.

Product	R_1	R_2	Profit /unit
A	1	2	Rs. 6
B	2	3	Rs. 5
	5	12	

Let x_1 and x_2 be the no. of unit of A and B to manufacture

Objective function $\longrightarrow$ Maximize profit

$$\text{Max}: Z = 6x_1 + 5x_2$$

Subject to,

$$1x_1 + 2x_2 \leq 5$$
$$2x_1 + 3x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

Problem 6

A firm manufactures 3 products ; A, B, C. Time to manufacture A is twice that of B and thrice that C, and they are manufactured in the ratio 3 : 4 : 5. Relevant data is given in the table below. If the whole time is engaged in the manufacturing of product A, 1600 units of product A can be manufactured. There is a demand for atleast 300, 250 and 200 for products A, B and C. The profit earned per unit is Rs. 50, Rs. 40, Rs. 70 respectively. Formulate the problem as a LPP.

	Product			
Raw Material	A	B	C	Total Available
P	6	5	9	5000
Q	4	7	8	6000

Decision Variables : Let x_1, x_2, x_3 be the number of units of products A, B and C to manufacture respectively.

Graphical Solutions to Linear Programming Problem

Problem 1

Solve the following linear programming problem graphically

Max : $z = 2x_1 + 3x_2$ (profit in '000s)

Subject to,

(1) — $6x_1 + 3x_2 \leq 36$ (Labour hours)

(2) — $10x_1 + 10x_2 \leq 70$ (Materials)

(3) — $3x_1 + 6x_2 \leq 36$ (Machine hours)

$x_1, x_2 \geq 0$

Production Planning

Eqn (1) —

$6x_1 + 3x_2 = 36$

$x_1 = 0, \; x_2 = 12$

$x_1 = 6, \; x_2 = 0$

Eqn (2) —

$10x_1 + 10x_2 = 70$

$x_1 = 0, \; x_2 = 7$

$x_1 = 7, \; x_2 = 0$

Eqn (3) —

$3x_1 + 6x_2 = 36$

$x_1 = 0, \; x_2 = 6$

$x_1 = 12, \; x_2 = 0$

Objective function,

$2x_1 + 3x_2 = 6$ → take LCM for objective function

$x_1 = 0, \; x_2 = 2$

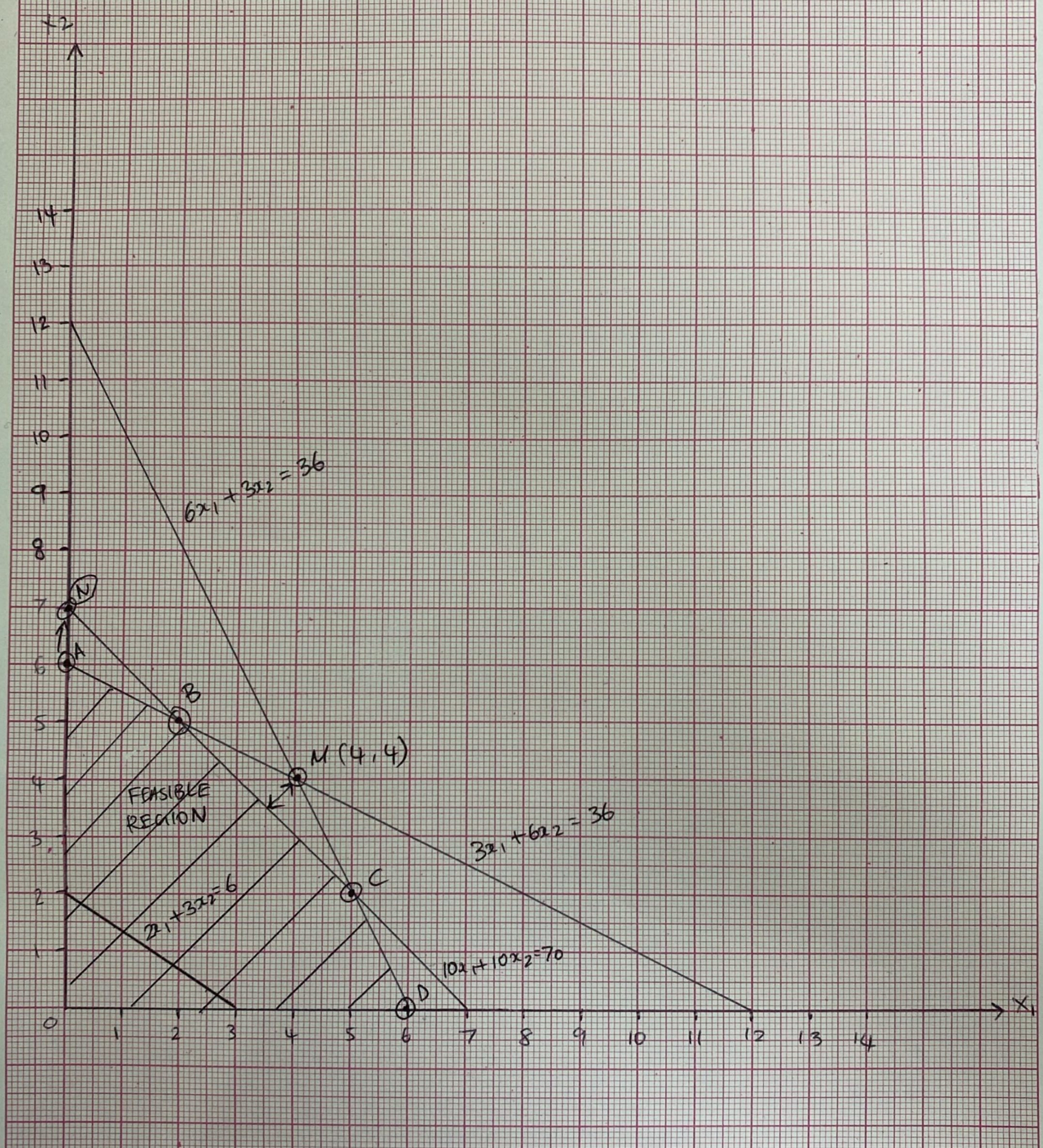
$x_1 = 3, \; x_2 = 0$

Optimal Solution can only be corner points, there are 4 corner points from the objective function line, but B is the furthest point, thus optimal solution is B.

B is $(2, 5)$

$\therefore x_1 = 2$

$x_2 = 5$



x2
x1
6x1+3x2=36
3x1+6x2=36
10x1+10x2=70
2x1+3x2=6
FEASIBLE REGION
M (4,4)
N
A
B
C
D
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14

Sensitivity Analysis

↳ Does the product mix change when there is a change in the total Labours hrs, Materials, Machine hours?

$6x_1 + 3x_2$
$= 6(2) + 3(5)$
$= 27 < 36$

Labour is an Abundant Resource

$10x_1 + 10x_2$
$= 10(2) + 10(5)$
$= 70$

Materials is a Scarce Resource
Moving line to point M (refer graph)
M(4, 4)

$Z = 2x_1 + 3x_2$
$= 2(4) + 3(4)$
$= 20,000$

$10x_1 + 10x_2$
$= 40 + 40$
$= 80$

$80 - 70 = 10$ → Raw materials supply should be increased by 10 units

$3x_1 + 6x_2$
$= 3(2) + 6(5)$
$= 36$

Machine is a Scarce Resource
Moving line to point N (refer graph)
N(0, 7)

$z = 2x_1 + 3x_2$
$= 2(0) + 3(7)$
$= 21{,}000$

$3x_1 + 6x_2$
$= 3(0) + 6(7)$
$= 42$

$42 - 36 = 6 \rightarrow$ increase no. of machines by 6

Special cases of LPP

Case 1

Solve the following linear programming problem graphically

Max : $Z = 2x + 4y$

S.T, $x + 2y \leq 5$ —— (1)

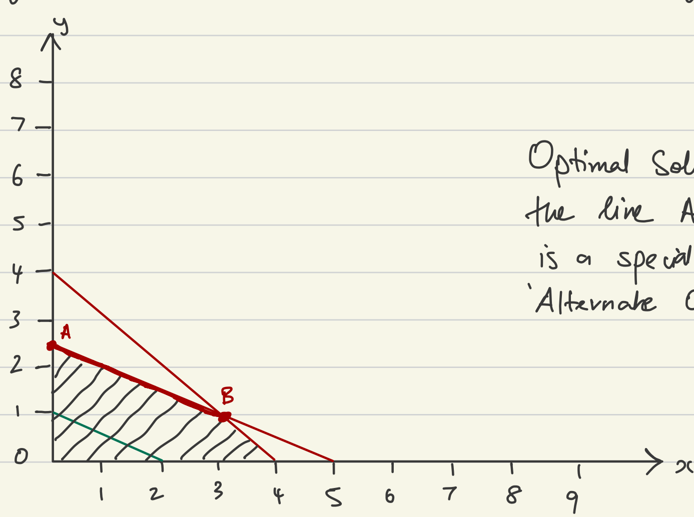
$x + y \leq 4$ —— (2)

$x, y \geq 0$

Eqn (1),
$x + 2y = 5$
$x = 0,\ y = 2.5$
$x = 5,\ y = 0$

Eqn (2),
$x + y = 4$
$x = 0,\ y = 4$
$x = 4,\ y = 0$

Objective function,
$2x + 4y = 4$
$x = 0,\ y = 1$
$x = 2,\ y = 0$



Optimal Solution lies on the line AB, thus this is a special case called 'Alternate Optimal Solution'.

Case 2

Solve the following linear programming problem graphically

Max :- $Z = 3x_1 + 2x_2$

S.T, $x_1 - x_2 \leq 1$ —— (1)

$x_1 + x_2 \geq 3$ —— (2)

$x, y \geq 0$

Eqn (1) -

$x_1 - x_2 = 1$

$x_1 = 0, x_2 = 1$

$x_1 = 1, x_2 = 0$

Eqn (2) -

$x_1 + x_2 = 3$

$x_1 = 0, x_2 = 3$

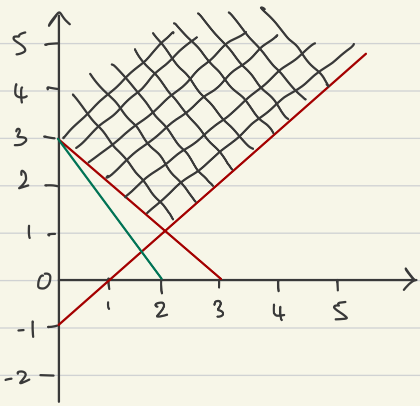
$x_1 = 3, x_2 = 0$

Objective function,

$3x_1 + 2x_2 = 6$

$x_1 = 0, x_2 = 3$

$x_1 = 2, x_2 = 0$



'Unbounded Problems'

↳ Solution to such kind of probleme is to reformulate the equations as you cannot have unbounded profit with limited resources.

Case 3

Solve the following linear programming problem graphically

Max :- $Z = 3x_1 - 2x_2$

S.T, $x_1 + x_2 \leq 1$ —— (1)

$2x_1 + 2x_2 \geq 4$ —— (2)

$x_1, y_2 \geq 0$

Eqn (1) -

$x_1 + x_2 = 1$

$x_1 = 0 \; , \; x_2 = 1$

$x_1 = 1 \; , \; x_2 = 0$

Eqn (2) -

$2x_1 + 2x_2 = 4$

$x_1 = 0, \; x_2 = 2$

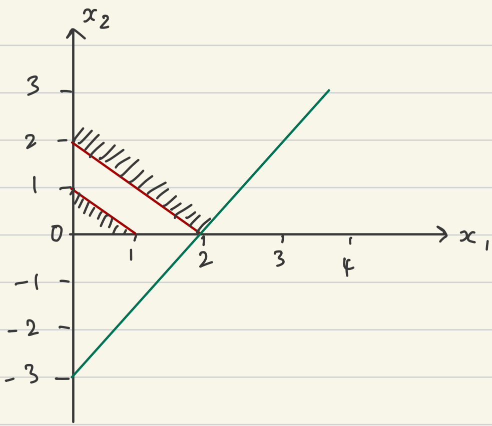
$x_1 = 2, \; x_2 = 0$

Objective function,

$3x_1 - 2x_2 = 6$

$x_1 = 0, \; x_2 = -3$

$x_1 = 2 \quad x_2 = 0$



No common area satisfying both constraints. Such problems are called 'Infeasible Solution' since no feasible region exists. Solution to this problem is to reformulate

Solution to Linear programming problems using Simplex Algorithm:

Steps in simplex algorithm:

1. Convert all the inequalities in the constraining equations to equations by adding a slack or surplus variable
2. Make all the variables appear in all the constraining equations including the objective function
3. Write the standard form of the LPP
4. Develop the first Simplex table and identify the Solution variables (the solution variables carry the identity matrix with them)
5. Check the solution for optimality by introducing c_j and $c_j - z_j$ rows
6. The solution will be optimal if all the numbers in the $c_j - z_j$ row are
   (1) Negative ; if the objective function is to maximise
   (2) Positive ; if the objective function is to minimize
7. If the solution is optimal, stop the algorithm, else revise

Steps for revision

1. Identify the incoming variables (a variable with highest positive value in the $c_j - z_j$ row becomes the incoming variable in maximization case and a variable with highest negative value in the $c_j - z_j$ row becomes the incoming variable in minimization
2. Identify the outgoing variable by calculating the critical ratio which is given by :

$$\text{critical ratio } CR = \frac{Q \text{ column no.}}{\text{key column no.}}$$

The variable against the least critical ratio becomes the outgoing variable

③ Transform the key row by dividing all the numbers in it by the key number.

④ Transform the other numbers in the key column into 0 using the key number as the pivot.

⑤ Check the second solution for optimality by introducing c_j and $c_j - z_j$ rows and repeat the above procedures until you get an optimal solution.

Example

Solve the following LPP using Simplex Algorithm

$$\text{Max } Z = 9x_1 + 6x_2 \text{ (total profit)}$$

S.T. $2x_1 + 4x_2 \leq 60$ (max hours avl on polishing machine)

$4x_1 + 2x_2 \leq 48$ (max no. of packaging hours avl)

$x_1, x_2 \geq 0$

$$\text{Max, } Z = 9x_1 + 6x_2 + 0s_1 + 0s_2$$

S.T, $2x_1 + 4x_2 + s_1 + 0s_2 = 60$

$4x_1 + 2x_2 + 0s_1 + s_2 = 48$

→ why are coefficients of s_1 & s_2 0 in the objective function?

where s_1 & s_2 is the idle/slack capacity of polishing machine and polishing hours respectively

$x_1, x_2, s_1, s_2 \geq 0$

First Simplex Table

key column / incoming variable

key value

key row / outgoing variable

critical ratio

Profit/unit	Basis/Sol^n var	c_j / Q	9 / x_1	6 / x_2	0 / s_1	0 / s_2	CR
0	s_1	60	2	4	1	0	30
0	s_2	48	4	2	0	1	12
Profit →	z_j	[0]	0	0	0	0	
Net evaluation row →	$c_j - z_j$		9	6	0	0	

Basic Feasible Solution → when the number of unknowns is more than the number of equations, we adopt this

From the first simplex table, the solution is not optimal. Hence, Revise

* Why choose the smaller critical ratio?

key column

Multiplicative factor of -2

key value

key row

make 1 as the pivot

Profit/unit	Basis/ Sol^n var	c_j Q	9 x_1	6 x_2	0 s_1	0 s_2	CR
0	s_1	36	0	3	1	$-1/2$	12
9	x_1	12	1 (-2)	$1/2$	0	$1/4$	24
	z_j	[108]	9	$4\,1/2$	0	$9/4$	
	$c_j - z_j$		0	1.5	0	$-9/4$	

The solution is not optimal, hence Revise

make this 1 as pivot

Profit/unit	Basis/ Sol^n var	c_j Q	9 x_1	6 x_2	0 s_1	0 s_2	CR
6	x_2	unit of x_2 12	0	1 $(\div \frac{1}{2})$	$1/3$	$-1/6$	
9	x_1	unit of x_1 6	1	0	$-1/6$	$1/3$	
	z_j *	[126]	9	6	$1/2$	2	
	$c_j - z_j$		0	0	$-\frac{1}{2}$	-2	

optimal profit

The solution is optimal as all the numbers in the $c_j - z_j$ are either zero or negative

Solve the following LPP using Simplex Algorithm:

Max $Z = 80x_1 + 100x_2$

S.T, $1x_1 + 1x_2 \leq 110$ (RM Avl)

$1x_1 + 2x_2 \leq 160$ (Labour Avl)

$3x_1 + 4x_2 \leq 360$ (M/C hrs Avl)

$x_1, x_2 \geq 0$

Max: $Z = 80x_1 + 100x_2 + 0s_1 + 0s_2 + 0s_3$

S.T,

$1x_1 + 1x_2 + s_1 + 0s_2 + 0s_3 = 110$

$1x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 160$

$3x_1 + 4x_2 + 0s_1 + 0s_2 + s_3 = 360$

where S_1, S_2, S_3 are the slack capacity of 3 resources respectively

$x_1, x_2, S_1, S_2, S_3 \geq 0$

First Simplex Table

Profit/unit	Basis	c_j / Q	80 / x_1	100 / x_2	0 / s_1	0 / s_2	0 / s_3	Critical Ratio
0	S_1	110	1	1	1	0	0	110
key row 0	S_2	160	1	2	0	1	0	80
0	S_3	360	3	4	0	0	1	90
	z_j	[0]	0	0	0	0	0	
	$c_j - z_j$		80	100	0	0	0	

key column

Critical ration = $\frac{\text{Q column}}{\text{Key column}}$

key number

key column → max $c_j - z_j$

key row → min CR

① Transformation of key row, and then...
② Transformation of non-key row

Second Simplex Table

Profit/unit	Basis	c_j Q	80 x_1	100 x_2	0 s_1	0 s_2	0 s_3	Critical Ratio
0	s_1	30	1/2	0	1	-1/2	0	60
100	x_2	80	1/2	1	0	1/2	0	160
0	s_3	40	1	0	0	-2	1	40
	z_j	[8000]	50	100	0	50	0	
	$c_j - z_j$		30	0	0	-50	0	

Third Simplex Table

Profit/unit	Basis	c_j Q	80 x_1	100 x_2	0 s_1	0 s_2	0 s_3	Critical Ratio
0	s_1	10	0	0	1	1/2	-1/2	20
100	x_2	60	0	1	0	3/2	-1/2	40
80	x_1	40	1	0	0	-2	1	-20
	z_j	[9200]	80	100	0	-10	30	
	$c_j - z_j$		0	0	0	10	-30	

Fourth Simplex Table

Profit/unit	Basis	c_j Q	80 x_1	100 x_2	0 s_1	0 s_2	0 s_3	Critical Ratio
0	s_2	20	0	0	2	1	-1	
100	x_2	30	0	1	-3	0	1	
80	x_1	80	1	0	4	0	-1	
	z_j	[9400]	80	100	20	0	20	
	$c_j - z_j$		0	0	-20	0	-20	

Since the $c_j - z_j$ row elements contains either zero or negative values, solution is optimal

A new food called Toasties is to be marketed by the Ajax Corporation. The basic ingredients were combined and tested in a pre-production run and the analysis revealed that to each box of Toasties must be added atleast 16 units of Vitamin A and 20 units of Vitamin C. These vitamins are available in two commercial supplements whose details are as given below:

	Units of Vitamins A/C per kg of Supplement		Cost Rs/kg
	Vitamin A	Vitamin C	
Supplement 1	2	5	120
Supplement 2	4	2	80

Let x_1 be the no. of units of supplement 1 to be added
Let x_2 be the no. of units of supplement 2 to be added

substract surplus variable

$$\text{Min } Z = 120x_1 + 80x_2$$

S.T,

$$2x_1 + 4x_2 \geq 16 \Rightarrow 2x_1 + 4x_2 - S_1 = 16$$

$$5x_1 + 2x_2 \geq 20 \Rightarrow 5x_1 + 2x_2 - S_2 = 20$$

$$x_1, x_2 \geq 0$$

Standard form,

$$\text{Min } Z = 120x_1 + 80x_2 + 0S_1 + 0S_2$$

S.T,

$$2x_1 + 4x_2 - S_1 + 0S_2 = 16$$

$$5x_1 + 2x_2 + 0S_1 - S_2 = 20$$

$$x_1, x_2, S_1, S_2 \geq 0$$

Use Big M method → introduce artificial variable $A_1, A_2 \ldots$
with cost coefficients $M_1, M_2 \ldots$
where $M >$ the largest coefficient in objective function

First Simplex Table

Cost / unit	Basis	c_j Q	120 x_1	80 x_2	0 s_1	0 s_2	+1000 $\cancel{M}$ A_1	+1000 $\cancel{M}$ A_2	CR
1000	A_1	16	2	4	-1	0	1	0	8
1000	A_2	20	5	2	0	-1	0	1	4
	z_j	[3600]	7000	6000	-1000	-1000	1000	1000	
	$c_j - z_j$		-6880	-5920	1000	1000	0	0	

Introduce A_1, A_2 with coefficients M,

where M is a large positive integer

(approx 10x larger than the largest cost coefficient)

$$\text{Min } Z = 120x_1 + 80x_2 + 0s_1 + 0s_2 + M_1A_1 + M_2A_2$$

S.T,

$$2x_1 + 4x_2 - s_1 + 0s_2 + A_1 + 0A_2 = 16$$

$$5x_1 + 2x_2 + 0s_1 - s_2 + 0A_1 + A_2 = 20$$

Second Simplex Table

Cost / unit	Basis	c_j Q	120 x_1	80 x_2	0 s_1	0 s_2	$\cancel{M}$ A_1	$\cancel{M}$ A_2	CR
1000	A_1	8	0	16/5	-1	2/5	1	-2/5	5/2
120	x_1	4	1	2/5	0	-1/5	0	1/5	10
	z_j	[8480]	120	3248	-1000	376	1000	-376	
	$c_j - z_j$		0	-3168	1000	-376	0	1376	

Third Simplex Table

		c_j	120	80	0	0	$\cancel{M}$	$\cancel{M}$
Cost / unit	Basis	Q	x_1	x_2	S_1	S_2	A_1	A_2
80	x_2	5/2	0	1	-5/16	1/8	5/16	-1/8
120	x_1	3	1	0	1/8	-1/4	-1/8	1/4
	z_j	[560]	120	80	-10	-20	10	20
	$c_j - z_j$		0	0	10	20	990	980

All $c_j - z_j$ values are either zero or positive. Hence Optimal Solution reached.

$x_1 = 3$, $x_2 = 2.5$

$Z^* = 560$

Problems with Equality Constraints

Wellbuilt Products has production capacity in two areas. Area 1 has 20 hours of unused time and Area 2 has 30 hours of unused time. The firm is planning to introduce 2 new products to make use of its slack capacity. Each unit of product A requires 1 hour of Area 1 and each unit of product B requires ½ hour of Area 2. Atleast 30 units of Product B must be produced. The total combined output of the proposed products must be equal to 50 units. Product A contributes \$5 to the profit and product B \$6 to the profit. Formulate the above problem as an LPP and solve using simplex Algorithm.

Let x_1 and x_2 be the number of units of Products A and B respectively

Max : $Z = 5x_1 + 6x_2$

S.T,

$$x_1 \leq 20$$
$$\tfrac{1}{2}x_2 \leq 30$$
$$x_1 + x_2 = 50$$
$$x_2 \geq 30$$

$x_1, x_2 \geq 0$ (non-negative)

$$Z = 5x_1 + 6x_2 + 0s_1 + 0s_2 + 0s_3$$

S.T,

$$x + 0x_2 + s_1 + 0s_2 + 0s_3 = 20$$

$$0x_1 + \frac{1}{2}x_2 + 0s_1 + s_2 + 0s_3 = 30$$

$$0x_1 + x_2 + 0s_1 + 0s_2 - s_3 = 30$$

$$x_1 + x_2 + 0s_1 + 0s_2 + 0s_3 = 50$$

$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

First Simplex Table

P/unit	Basis	c_j / Q	5 / x_1	6 / x_2	0 / s_1	0 / s_2	0 / s_3	-100 ~~-M~~ / A_4	-100 ~~-M~~ / A_5	CR
0	s_1	20	1	0	1	0	0	0	0	∞
0	s_2	30	0	1/2	0	1	0	0	0	60
-100	A_4	30	0	1	0	0	-1	1	0	30
-100	A_5	50	1	1	0	0	0	0	1	50
	z_j	[-8000]	-100	-200	0	0	100	-100	-100	
	$c_j - z_j$		105	206	0	0	-100	0	0	

Second Simplex Table

P/unit	Basis	c_j / Q	5 / x_1	6 / x_2	0 / s_1	0 / s_2	0 / s_3	-100 ~~-M~~ / A_4	-100 ~~-M~~ / A_5	CR
0	s_1	20	1	0	1	0	0	0	0	∞
0	s_2	15	0	0	0	1	1/2	-1/2	0	30
6	x_2	30	0	1	0	0	-1	1	0	-30
-100	A_5	20	1	0	0	0	1	-1	1	20
	z_j	[-1820]	-100	6	0	0	-106	106	-100	
	$c_j - z_j$		105	0	0	0	106	-206	100	

Third Simplex Table

		c_j	5	6	0	0	0	~~-M~~ -100	~~-M~~ -100
P/unit	Basis	Q	x_1	x_2	s_1	s_2	s_3	A_4	A_5
0	s_1	20	1	0	1	0	0	0	0
0	s_2	5	$-1/2$	0	0	1	0	0	$-1/2$
6	x_2	50	1	1	0	0	0	0	1
0	s_3	20	1	0	0	0	1	-1	1
	z_j	[300]	6	6	0	0	0	0	6
	$c_j - z_j$		-1	0	0	0	0	-100	-106

Solution is optimal

$x_2 = 50$, $x_1 = 0$

SENSITIVITY ANALYSIS

Sensitivity Analysis

Q) Solve the following LPP using simplex algo and perform
1) RHS sensitivity (sensitivity analysis of resources)
2) sensitivity Analysis of profit coeff.

Max ; $z = 25x_1 + 42x_2 + 30x_3$

S.T.

$3x_1 + 4x_2 + 2x_3 \le 60$ (R.M avl)

$2x_1 + x_2 + 2x_3 \le 36$ (labor)

$x_1 + 3x_2 + 2x_3 \le 62$ (M/C hrs)

$x_1, x_2, x_3 \ge 0$

Sol^n →

$z = 25x_1 + 42x_2 + 30x_3 + 0s_1 + 0s_2 + 0s_3$

$3x_1 + 4x_2 + 2x_3 + s_1 + 0s_2 + 0s_3 = 60$

$2x_1 + x_2 + 2x_3 + 0s_1 + s_2 + 0s_3 = 36$

$x_1 + 3x_2 + 2x_3 + 0s_1 + 0s_2 + s_3 = 62$

$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$

	c_j		25	42	30	0	0	0	
p/c	basis	Q	x_1	x_2	x_3	s_1	s_2	s_3	CR
0	s_1	60	3	4	2	1	0	0	15
0	s_2	36	2	1	2	0	1	0	36
0	s_3	62	1	3	2	0	0	1	23.3
	Z_j	[0]	0	0	0	0	0	0	
	$c_j - z_j$		25	42	30	0	0	0	

key value → 4

	c_j		25	42	30	0	0	0	
p/c	basis	Q	x_1	x_2	x_3	s_1	s_2	s_3	CR
42	x_2	15	3/4	1 ×(−1) ×(−3)	1/2	1/4	0	0	15
0	s_2	21	5/4	0	3/2	−1/4	1	0	0
0	s_3	17	−5/4	0	1/2	−3/4	0	1	0
	z_j	[8042]	105/2	0	63	−21/2	−420		
	$c_j - z_j$		−55/2	42	−33	21/2	−420		

to what extent is the plan flexible → the product vary remains same on fluctuation in RM, labor or M/C hrs

Sensitivity Analysis / Post-optimality Analysis

contd
Q)
Max ; $z = 25x_1 + 42x_2 + 30x_3$

S.T :

$3x_1 + 4x_2 + 2x_3 \le 60$ (RM avl.)

$2x_1 + x_2 + 2x_3 \le 36$ (labor)

$x_1 + 3x_2 + 2x_3 \le 62$ (M/C hrs)

$x_1, x_2, x_3 \ge 0$

		c_j	25	42	30	0	0	0
profit / unit	Basis	Q	x_1	x_2	x_3	s_1	s_2	s_3
42	x_2	8	1/3	1	0	1/3	−1/3	0
30	x_3	14	5/6	0	1	−1/6	2/3	0
0	s_3	10	−5/3	0	0	−2/3	−1/3	1
	z_j	[756]	39	42	30	9	6	0
	$c_j - z_j$		−14	0	0	−9	−6	0

1 → RHS Sensitivity / sensitivity analysis of the Resources

Step 1: Identify the Slack / Surplus variable that represents each resource

Step 2: Prepare a table extract it from the final simplex table showing the quantity column, the slack variable column, and the positive and negative ratio column.

Step 3: perform ~~calc~~ calculation as below:

lower bound = original resource value − the smallest no. in the positive ratio column

upper bound = original resource value − the largest no. in the negative ratio value.

Step 4: if the positive & negative ratio columns have no value. then lower bound is − infinity & the upper bound is + infinity.

→ For Previous Ques:

1 Range calculation for raw material :-

Current value = 60

Q^*	S_1	+ve Ratio	-ve ratio
8	1/3	24	-
14	-1/6	-	-84
10	-2/3	-	-15

$\Rightarrow LB = 60 - 24$
$= 36$

$\Rightarrow UB = 60 - (-15)$
$= 75$

$\therefore$ RM can fluctuate b/w 36 and 75 → product mix of x_1, x_2, x_3 will remain same. (in this ques x_2 & x_3)

③ Range calculation for labour :

Current value = 36

Q^*	S_2	+ive ratio	-ve ratio
8	-1/3	—	-24
14	2/3	21	—
10	-1/3	—	-30

$LB = 36 - 21$
$= 15$

$UB = 36 - (-24)$
$= 60$

③ Range calcⁿ for M/C hrs :

Current value = 62

Q^*	S_3	+ive ratio	-ive ratio
8	0	∞	-
14	0	∞	-
10	1	10	-

$LB = 62 - 10$
$= 52$

$UB = 62 - (-\infty)$
$= +\infty$

② → Sensitivity Analysis of profit coefficients :-

Step 1: To perform the sensitivity analysis of ~~basic~~ profit ~~val~~ coefficients of basic variables (in this case x_2, x_3, s_3) develop table extracted from the final simplex table, having the following columns.

Colⁿ 1) absolute values of the no. in the $c_j - z_j$ row
Colⁿ 2) the corresponding values against the basic variable
Colⁿ 3) positive ratio ~~& the~~ negative
Colⁿ 4) negative ratio

Step 2: calculate the LB & UB using the formulae used earlier.

For prev ques:

① Range calculation for profit coeff of x_2 :-

original value = 42

Abs value of no's in $c_j - z_j$ row	Corresponding value in x_2 row	+ive ratio	-ve ratio
14	1/3	42	-
9	1/3	27	-
6	-1/3	-	-18

$LB = 42 - 27$
$= 15$

$UB = 42 - (-18)$
$= 60$

② Range calcⁿ for profit coeff of x_3 :

original value = 30

Abs val of no's in $c_j - z_j$ row	corresponding values in x_3 row	+ve ratio	-ve ratio
14	5/6	84/5 = 16.8	-
9	-1/6	-	-54
6	2/3	9	-

$LB = 30 - 9$
$= 21$

$UB = 30 - (-54)$
$= 84$

TRANSPORTATION ALGORITHM

A company has 3 production facilities S_1, S_2 & S_3 with production capacity of 7, 9 and 18 units per week of a product respectively. The units are to be shipped to four warehouses D_1, D_2, D_3 and D_4 with requirements of 5, 8, 7, 14 units per week respectively. The transportation cost per unit between the factories and warehouses is as shown below. Formulate the problem as a transportation model and generate the basic feasible solution using

(i) Northwest Corner Rule

(ii) Least Cost Method

	D_1	D_2	D_3	D_4
S_1	19	30	50	10
S_2	70	30	40	60
S_3	40	8	70	20

Solution :

	D_1	D_2	D_3	D_4	Supply
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

34 → Supply = Demand (Balanced Problem)

Solution :

Developing basic feasible solution using the North West Corner rule.

	D_1	D_2	D_3	D_4	Supply
S_1	(19) 5	(30) 2	(50)	(10)	7/2/0
S_2	(70)	(30) 6	(40) 3	(60)	9
S_3	(40)	(8)	(70) 4	(20) 14	18
Demand	5/0	8/6/0	7/4/0	14	34

$Z = 19*5 + 30*2 + 30*6 + 40*3 + 70*4$
$+ 20*14$
$= 1015/-$

Check condition before proceeding:

$m + n - 1 =$ number of allocations

where $m \rightarrow$ no. of supply sources
$n \rightarrow$ no. of demand

$3 + 4 - 1 = 6$ ✓

Condition met thus non-degenerate solution

If condition is not met (degeneracy), thus solution cannot be optimized
(Degenerate Solution)

Solution using the least cost method

	D_1	D_2	D_3	D_4	Supply
S_1	(19)	(30)	(50)	(10) 7	~~7~~ 0
S_2	(70) 2	(30)	(40) 7	(60)	9/2
S_3	(40) 3	(8) 8	(70)	(20) 7	~~18~~ 10/3/0
Demand	5/2	8/0	7/0	14/7/0	34

$Z = 70 + 140 + 280 + 120 + 64 + 140$
$= 814/-$

$m + n - 1 = 6$ (non-degenerate solution)

Optimizing the Basic Feasible Solution using Stepping Stone method :

	D_1	D_2	D_3	D_4	Supply
S_1	(19) 5	(30) 2	(50)	(10)	7/2/0
S_2	(70)	(30) 6	(40) 3	(60)	9
S_3	$\boxed{(40)\ -9}$	$\boxed{(8)\ -52}$	(70) 4	(20) 14	18
Demand	5/0	8/6/0	7/4/0	14	34

$Z = 19*5 + 30*2 + 30*6 + 40*3 + 70*4$
$+ 70*14$
$= 1015/-$

Optimizing the earlier problem for which the basic feasible solution was generated using the Northwest corner rule

Stepping stone algorithm
1. Start at empty cell (+) → alternate sign
2. Change direction / no diagonal
3. Can skip cells
4. Closed path

Cost is reduced is allocation is made to these 2 cells.
Cell $S_3 D_2$ is the incoming cell.

	D_1	D_2	D_3	D_4	Supply
S_1	(19) 5	(30) 2	(50) +	(10) -32	7/2/0
S_2	(70) +	(30) 2	(40) 7	(60)	9
S_3	(40) +	(8) 4	(70) +	(20) 14	18
Demand	5/0	8/6/0	7/4/0	14	34

$Z = 807$

$S_1 D_3 \Rightarrow 50 - \cancel{30} + \cancel{30} - 40$
$= 10$

$S_1 D_4 \Rightarrow 10 - 20 + 8 - 30$
$= 18 - 50$
$= -32$

$S_2 D_1 \Rightarrow 70 - 19 + \cancel{30} - \cancel{30}$
$= +$

incoming cell

	D_1	D_2	D_3	D_4	Supply
S_1	19 5	30 (−) +	50 +	10 (+) 2	7/2/0
S_2	70 +	30 2	40 7	60 +	9
S_3	40 +	8 (+) 6	70 +	20 (−) 14	18
Demand	5/0	8/6/0	7/4/0	14	34

$Z = 743^*$

$m + n - 1 = 6$

condition satisfied

The solution is optimal as all the cell improvement indices are either 0 or positive.

Optimizing the Basic Feasible Solution using Modified Distribution Method (MODI):

Optimizing the previous basic feasible solution from the North-west corner rule.

① Calculate the dual values u_i and v_j such that $u_i + v_j = c_{ij}$ for occupied cells.

② Calculate the cell improvement indices such that cell improvement index $d_{ij} = c_{ij} - (u_i + v_j)$ for empty cells.

	$v_1 = 19$	$v_2 = 30$	$v_3 = 40$	$v_4 = -10$	
	D_1	D_2	D_3	D_4	Supply
$u_1 = 0$ S_1	19 5	30 2	50 +	10 +	7/2/0
$u_2 = 0$ S_2	70 +	30 6 (−)	40 3 (+)	60 +	9
$u_3 = 30$ S_3	40 −9	8 (+)	70 4 (−)	20 14	18
Demand	5/0	8/6/0	7/4/0	14	34

Steps

① Assume $u_1 = 0$, thus you can calculate v_1 since $c_{ij} = u_i + v_j$

	$v_1 = 19$	$v_2 = 30$	$v_3 = 40$	$v_4 = 42$	
	D_1	D_2	D_3	D_4	Supply
$u_1 = 0$ S_1	19 5	30 2 (−)	50 +	10 (+) −32	7/2/0
$u_2 = 0$ S_2	70 +	30 2	40 7	60 +	9
$u_3 = -22$ S_3	40 +	8 (+) 4	70 +	20 14 (−)	18
Demand	5/0	8/6/0	7/4/0	14	34

$Z = 807$

$m + n - 1 = 3 + 4 - 1 = 6$ allocations

Non-degenerate solution

		$v_1 = 19$	$v_2 = -2$	$v_3 = 8$	$v_4 = 10$	
		D_1	D_2	D_3	D_4	Supply
$u_1 = 0$	S_1	19 5	30 +	50 +	10 2	7/2/0
$u_2 = 32$	S_2	70 +	30 2	40 7	60 +	9
$u_3 = 10$	S_3	40 +	8 4	70 +	20 12	18
	Demand	5/0	8/6/0	7/4/0	14	34

$Z = 743$

$m+n-1 = 6$ allocations $\Rightarrow$ Non-degenerate solution

Since all the cell improvement indices are either zero or positive, optimal solution reached.

Problem

A Dairy farm has three plants located in a state. The daily milk production at each plant is as follows:

Plant	1	2	3
Milk supply /million litres	6	1	10

Each day the firm must fulfill the needs of its four distribution centres. The requirement at each centre is as follows:

Centre	1	2	3	4
Demand /million litres	7	5	3	2

The cost of shipping 1 million litre from each plant to distribution to distribution centre is as given in table :

Distribution Centre

	D_1	D_2	D_3	D_4
P_1	2	3	11	7
P_2	1	0	6	1
P_3	5	8	15	9

Formulate the case as a transportation problem and solve to minimize the total transportation cost.

	D_1	D_2	D_3	D_4	Supply
S_1	2	3	11	7	6
S_2	1	0	6	1	1
S_3	5	8	15	9	10
Demand	7	5	3	2	17

Overcoming the problem of a degenerate solution:

Problem

	D_1 ($v_1 = 0$)	D_2 ($v_2 =$)	D_3 ($v_3 = 4$)	Supply
$u_1 = 0$ S_1	[0] 10	[5]	[4] 20	~~30~~ ~~20~~ 0
$u_2 =$ S_2	[2]	[9] 20	[8]	~~20~~ 0
u_3 S_3	[5]	[11] 10	[10]	10
Demand	~~10~~ 0	~~30~~ ~~10~~ 0	20	

$Z = 10(0) + 20(4)$
$+ 20(9) + 10(11)$
$= 80 + 180 + 110$
$= 370$

$m + n - 1 = 3 + 3 - 1 = 5 \neq 4$

Solution is degenerate

Unable to proceed with MODI Method since there is one allocation less
∴ the solution is degenerate

Why does degeneracy exist in the above problem?
- When allocating $S_1 D_3$ 20 units, both demand and supply was exhausted simultaneously therefore there is one allocation less

To overcome this problem:

	D_1 ($v_1 = 0$)	D_2 ($v_2 = 5$)	D_3 ($v_3 = 4$)	Supply
$u_1 = 0$ S_1	[0] (−) 10	[5] ϵ (+)	[4] 20	~~30~~ ~~20~~ 0
$u_2 = 4$ S_2	[2] (+) −2	[9] (−) 20	[8] 0	~~20~~ 0
$u_3 = 6$ S_3	[5] −1	[11] 10	[10] 0	10
Demand	~~10~~ 0	~~30~~ ~~10~~ 0	20	

Add a new variable epsilon (ϵ)
(1) At any empty cell
(2) At an empty cell with least cost
(3) At empty cell along the row/column where the degenerate cell lies

	D_1	D_2	D_3	Supply
S_1	[0]	[5]	[4]	30
S_2	[2]	[9]	[8]	20
S_3	[5]	[11]	[10]	10
Demand	10	30	20	

	D_1	D_2	D_3	Supply
S_1	0	5	4	30
S_2	2	9	8	20
S_3	5	11	10	10
Demand	10	30	20	

Problem

Solve the following the transportation algorithm :

$V_1 = 3$ $V_2 = 6$ $V_3 = 2$

		D_1	D_2	D_3	Supply
$U_1 = 0$	S_1	3 (−) 50	6 (+) 30	8 (+)	~~80~~ ~~30~~ 0
$U_2 = -5$	S_2	6 (+)	1 40	2 (+)	~~40~~ 0
$U_3 = 1$	S_3	4 (+) 0	7 (−) 20	3 10	~~30~~ 20
	Demand	~~50~~ 0	~~90~~ ~~50~~ 20	~~10~~ 0	150

$Z = 50(3) + 30(6) + 40(1) + 20(7) + 10(3)$
$= 150 + 180 + 40 + 140 + 30$
$= 210 + 150 + 180$
$= 540$

$m + n - 1 = 3 + 3 - 1 = 5$ allocations

Solution is non-degenerate

$V_1 = 3$ $V_2 = 6$ $V_3 = 2$

		D_1	D_2	D_3	Supply
$U_1 = 0$	S_1	3 30	6 50	8	80
$U_2 = -5$	S_2	6	1 40	2	40
$U_3 = 1$	S_3	4 20	7 0	3 10	30
	Demand	50	90	10	150

$Z = 30(3) + 50(6) + 40(1) + 20(4) + 0(7)$
$= 540$

$m + n - 1 = 3 + 3 - 1 = 5$ allocations

Solution is non-degenerate

Whenever there is a cell with cell improvement index of 0, you will get multiple optimal solutions.

Real life implication → in case flexibility of solution is needed i.e road is blocked / warehouse is under some repair etc..

Problem

Solve the following transportation problem with an objective to maximize the total sales revenue.

MARKET

PRODUCT		$V_1 = 20$: 1	$V_2 = 15$: 2	$V_3 = 20$: 3	Supply
$U_1 = 0$	I	[20] 16	[15] 8	[13] −	~~24~~ ~~8~~ 0
$U_2 = 1$	II	[19] −	[12] −	[21] 11	~~11~~ 0
$U_3 = -2$	III	[17] −	[13] 12	[18] 3	~~15~~ 12
	Demand	~~16~~ 0	~~20~~ 12	~~14~~ 3	50

Using Maximum profit cell method

$Z = 881$

$m + n - 1 = 3 + 3 - 1 = 5$ allocations

Solution is non-degenerate

The solution is optimal as all the cell-improvement indices are all zero or negative

Problem

Solve the following transportation problem with the objective of minimizing the total transportation cost

DESTINATION

SOURCE	A	B	C	Supply
1	[2]	[2]	[3] 10	10/0
2	[4]	[1] ϵ 15	[2]	15/0
3	[1] 20	[3]	[100] 20	40/20/0
Demand	20/0	15/0	~~30~~ 20/0	65

- Source 3 to Destination C is blocked
- Thus use the Big M method, add a positive coefficient
  ↳ 10x larger than the biggest cost.

In this case assume cost is ~~M~~ 100