Applied Multivariate analysis

Multivariate deals with collection of random Variables

Introduction:- In univariate Statistics

Central tendency — mean $\xrightarrow{\text{will be}}$ Mean vector; Mode; Median.

Dispersion — Range; Inter Quartile range; Standard deviation. (variance) $\xrightarrow{\text{will be}}$ Covariance matrix.

and also since this study is multivariable that is more than one variable so there will be correlation between variables, so we will study correlation matrix also.

Mean vector: The multivariate population is characterized (denoted) by X

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_p \end{bmatrix}_{p\times 1}$$

vector

require collect data's on p variables

$$\begin{array}{c|cccccc} i \backslash j & X_1 & X_2 & \cdots & X_j & \cdots & X_p \\ \hline 1 & x_{11} & x_{12} & \cdots & & \cdots & x_{1p} \\ 2 & x_{21} & x_{22} & \cdots & & \cdots & x_{2p} \\ X_i & x_{i1} & x_{i2} & \cdots & x_{ij} & \cdots & x_{ip} \\ \vdots & \vdots & & & & & \\ n & x_{n1} & x_{n2} & \cdots & & \cdots & x_{np} \end{array} = X_{n\times p}$$

data matrix.

$i = 1, 2, \ldots n \Rightarrow$ Number of Observations

$j = X_1, X_2 \ldots X_p \Rightarrow$ number of variables

x_{ij} means i^{th} observation on the j^{th} variable to be collected.

In probability we study if X is a random variable then $E(X) = \mu$ is the expectation. Then X_{ij} is also a random variable which means it should also have expectation, that is how we talk about mean vector.

Now X_i = i^{th} multivariate observation

$$X_i = \begin{bmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{ij} \\ \vdots \\ x_{ip} \end{bmatrix}_{p\times 1}$$

all the variables are occurring simultaneously that's why multivariate in nature.

Now x_j = observations on j^{th} variable (par...

$$x_j = \begin{bmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{ij} \\ \vdots \\ x_{nj} \end{bmatrix}_{n\times 1}$$

column wise is about v... observation on diff va...

no of observations

All information we learned is before we collect data means plan made to collect data, in this matrix are random in nature.

particular multi...

After data collection:-

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}_{p\times 1}$$

→ looks same as above but actual it makes difference that we know the values the entries. In first matrix what happens is you are expecting the values of the variables i) Mean & also different values ... mean will give variance.

Now will study what is the mean (or) expected value of each of the variable. In above there are p variables so there are p means.

Mean vector

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_i \\ \vdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} E(x_1) \\ E(x_2) \\ \vdots \\ E(x_i) \\ \vdots \\ E(x_p) \end{bmatrix}$$

Earlier w.k.T about E(x) calcula...

multivariate Expected value:

Particular variable j

$$E(x_j) = \sum_{\text{all } x_j} x_j f(x_j) \rightarrow \text{discrete.}$$

$$= \int_{-\infty}^{\infty} x_j f(x_j)\, dx_j \rightarrow \text{Continuous.}$$

$j = 1, 2, \cdots p.$

$\bar{X}$ = estimate of μ. $= \frac{1}{n}\sum_{i=1} x_i$

here $\bar{X} = \begin{bmatrix} \bar{x_1} \\ \bar{x_2} \\ \vdots \\ \bar{x_j} \\ \vdots \\ \bar{x_p} \end{bmatrix} = \begin{bmatrix} \frac{1}{n}\sum_{i=1}^{n} x_{i1} \\ \frac{1}{n}\sum_{i=1}^{n} x_{i2} \\ \vdots \\ \frac{1}{n}\sum_{i=1}^{n} x_{ij} \\ \vdots \\ \frac{1}{n}\sum_{i=1}^{n} x_{ip} \end{bmatrix}$

we will not calculate individual mean calculation Instead, we do matrix Calculation, using vector matrix concept.

how to calculate?

$X_{n\times p} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & & x_{np} \end{bmatrix}_{n\times p}$ to compute $\Rightarrow$ from this $\bar{X} = \begin{bmatrix} \bar{x_1} \\ \bar{x_2} \\ \vdots \\ \bar{x_j} \\ \vdots \\ \bar{x_p} \end{bmatrix}_{p\times 1}$

we have to calculate at one go so I take unit vector $\underline{1} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{(n\times 1)}$ we need (created) so that $(n\times p) \rightarrow (p\times 1)$

applying this unit vector with the data matrix and using computational formula to get $(p\times 1)$ we want

ie) $\underbrace{(p\times n)}$ $\cdot$ $(n\times 1) = (p\times 1)$.

take transpose of given data matrix.

Now $(n\times p)^T \cdot (n\times 1) = (p\times 1)$ we get.

$X^T \cdot \underline{1} = n\bar{X} =$ "$\sum x_{ij}$"

$\bar{x_j} = \frac{1}{n}\sum_{i=1}^{n} x_{ij}$

Eg: $p = 2$, $n = 3$.

$X_{3\times 2} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}$ $\quad \underline{1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

we do $\Rightarrow X^T \cdot \underline{1} = \begin{bmatrix} x_{11} & x_{21} & x_{31} \\ x_{12} & x_{22} & x_{32} \end{bmatrix}_{(2\times 3)} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{(3\times 1)}$ $n\bar{X}$

Mean vector is found using this formula.

$$\boxed{\bar{X} = \frac{1}{n} X^T \cdot \underline{1}}$$

$\Leftarrow = \begin{bmatrix} x_{11} + x_{21} + x_{31} \\ x_{12} + x_{22} + x_{32} \end{bmatrix}_{(2\times 1)} \rightarrow = \begin{bmatrix} \sum x_{i1} \\ \sum x_{i2} \end{bmatrix}$

This is about Mean vector from the population point of view and ~~mean~~ sample point of view - From the

(i) Sample point of view sample average is the estimate of population mean vector.

Population Covariance Matrix:-

In univariate case, W.K.T $V(x)$

$$V(x) = E(x-\mu)^2 = \sum_x (x-\mu)^2 f(x) \rightarrow \text{Discrete}$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx \rightarrow \text{continuous.}$$

case for multivariate

$$x_j \Rightarrow V(x_j) = E(x_j - \mu_j)^2$$

for $j = 1, 2, 3 \dots p$

$$= \begin{cases} \sum_{\text{for all } x_j} (x_j-\mu_j)^2 f(x_j) \rightarrow \text{discrete} \\ \int_{-\infty}^{\infty} (x_j-\mu_j)^2 f(x_j)\, dx_j \rightarrow \text{continuous.} \end{cases}$$

we get $\sigma_1^2, \sigma_2^2, \dots \sigma_p^2$.

Now we have p number of variables & assuming that they are not independent of each other which means dependent. Say for instance

if X_1 is dependent on X_2

~~(or) Say X_1 & X_2 are independent~~ eg: height versus weight of person, Non-zero correlation which means

which means if X_1 varies then X_2 & also vice versa

$\Rightarrow X_1$ & X_2 are covary.

x_j varies then x_k also varies:

$$\operatorname{cov}(x_j, x_k) = E(x_j-\mu_j)(x_k-\mu_k)$$

denoted by σ_{jk}

$$= \sum_{x_j}\sum_{x_k} (x_j-\mu_j)(x_k-\mu_k)\, P(x_j, x_k) \rightarrow \text{discrete}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x_j-\mu_j)(x_k-\mu_k)\, f(x_j, x_k)\, dx_j\, dx_k$$

When we write

$\sigma_j^2 = \sigma_{jj}$ which means variance of x_j

σ_{jk} = covariance between x_j & x_k.

using this we write Population Covariance Matrix

$$\Sigma = \begin{array}{c} \\ p \to x_1 \\ x_2 \\ \vdots \\ x_p \end{array} \begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & \cdots & & x_p \end{array} \\ \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & \vdots & & \vdots \\ \sigma_{p1} & \sigma_{p2} & \cdots & \sigma_{pp} \end{bmatrix}_{p\times p} \end{array}$$

diagonal elements

$\sigma_{11} = \sigma_1^2$ = variance

Other elements are covariance.

and also

$\sigma_{kj} = \sigma_{jk}$.

sample covariance Matrix :-

$$\underset{p\times p}{S} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1p} \\ s_{21} & s_{22} & \cdots & s_{2p} \\ \vdots & & & \vdots \\ s_{p1} & s_{p2} & \cdots & s_{pp} \end{bmatrix}$$

variance

→ others Covariance

therefore,

$$s_{jj} = \frac{1}{n-1}\sum_{i=1}^{n}(x_{ij} - \overline{x_j})^2 = \frac{1}{n-1}\sum_{i=1}^{n} x_{ij}^* \, x_{ik}^*$$

(using matrix)

$$s_{jk} = \frac{1}{n-1}\sum_{i=1}^{n}(x_{ij} - \overline{x_j})(x_{ik} - \overline{x_k})$$

expressed in matrix form

→ original data matrix $p\times p$

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1j} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2j} & \cdots & x_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nj} & & x_{np} \end{bmatrix}_{n\times p} = \begin{bmatrix} x_{11} - \overline{x_1} & x_{12} - \overline{x_2} & \cdots \\ x_{21} - \overline{x_1} & \vdots & \vdots \\ \vdots & & \\ x_{n1} - \overline{x_1} & \cdots & x_{np} - \overline{x_p} \end{bmatrix}$$

x_{11}^*, x_{12}^*

every elt subtracted by its mean

→ mean computed $\overline{x_1}$ $\overline{x_2}$ $\overline{x_j}$ $\cdots$ $\overline{x_p}$

denoted by x^*

$$\Rightarrow X^* = \begin{bmatrix} x_{11}^* & x_{12}^* & \cdots & x_{1p}^* \\ \vdots & & & \\ x_{n1}^* & x_{n2}^* & \cdots & x_{np}^* \end{bmatrix}$$

where

$x_{ij}^* = x_{ij} - \overline{x_j}$

$$s_{jk} = \frac{1}{n-1}\sum_{i=1}^{n} x_{ij}^* \, x_{ik}^*$$

Using matrix concept we compute S (covariance)

(1) $X^{\text{data matrix}} = \left[\quad\right]_{n\times p}$ $\quad x_j^* = x_{ij} - \bar{x}_j$

$x^* = \left[\quad\right]_{n\times p}$

take $x^{*T} = p\times n$

$\therefore \underset{(p\times n)}{x^{*T}} \cdot \underset{(n\times p)}{x^*} = \left[\quad\right]_{p\times p} = (n-1)S$; $S_{p\times p} \Rightarrow \text{cov}(x)$

$\Rightarrow x^{*T} \cdot x^* = (n-1)S$

$\Rightarrow \boxed{S = \frac{1}{(n-1)}(x^{*T} \cdot x^*)}$

where population **Correlation Matrix**:- denoted

$\rho_{p\times p}$ population correlation matrix

$$\rho_{p\times p} = \begin{bmatrix} 1 & \rho_{12} & \rho_{13} & \cdots & \rho_{1p} \\ \rho_{12} & 1 & & \cdots & \rho_{2p} \\ \vdots & \rho_{jk} & 1 & \ddots & \\ \rho_{1p} & \rho_{2p} & \cdots & \cdots & 1 \end{bmatrix}$$

Σ Population cov matrix $\left[\quad \sigma_{jk} \quad\right]_{p\times p}$

Correlation $(x_j, x_k) = \dfrac{\text{cov between}(x_j, x_k)}{\text{Standard deviation of } x_j \cdot \text{Std deviation of } x_k} = \dfrac{\text{cov}(x, y)}{\sigma_x \sigma_y}$

$$\text{cor}(x_j, x_k) = \frac{\text{cov}(x_j, x_k)}{\sigma_{x_j} \cdot \sigma_{x_k}}$$

$$\Rightarrow \rho_{jk} = \frac{\sigma_{jk}}{\sigma_j \sigma_k} \Rightarrow \boxed{\sigma_{jk} = \rho_{jk} \cdot \sigma_j \cdot \sigma_k}$$

when $j = k \Rightarrow \sigma_{jj} = \rho_{jj}(\sigma_j)^2 = \rho_{jj}$

$\Rightarrow \boxed{\rho_{jj} = 1}$

$\rho_{jk} = +1$ +vely correlated

$\rho_{jk} = -1$ -vely correlated

$\rho_{jk} = 0$ uncorrelated

To compute Sample correlation, denoted by R. $x_{ij}^* = x_{ij} - \bar{x}_j$

$$X = \begin{bmatrix} \text{data matrix} \\ x_{ij} \end{bmatrix}_{n\times p} \quad \& \quad X^* = \begin{bmatrix} x_{ij}^* \end{bmatrix}_{n\times p}.$$

$$\tilde{x}_{ij} = \frac{x_{ij} - \bar{x}_j}{s_j} = \frac{x_{ij} - \bar{x}_j}{\sqrt{s_{jj}}}$$

$$\tilde{X} = \begin{bmatrix} \tilde{x}_{ij} \end{bmatrix}_{p\times p}.$$

Now, $$\underbrace{\tilde{X}^T_{p\times n} \cdot \tilde{X}_{n\times p}}_{p\times p} = (n-1) R$$

$$R = \frac{1}{(n-1)} \left[\tilde{X}^T \cdot \tilde{X} \right]$$ where R is correlation matrix.

also, we study formula for covar to correlation & also correlation to covariance.

given
$$R = D_s^{-1/2} \, S \, D_s^{-1/2} \quad (\text{cov to corr})$$
$$S = D_s^{1/2} \, R \, D_s^{1/2} \quad (\text{corr to cov})$$

where $$D_s = \begin{bmatrix} s_{11} & 0 & 0 & \cdots & 0 \\ 0 & s_{22} & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & \cdots & \cdots & s_{pp} \end{bmatrix}$$

Eg: To calculate Mean Vector.

$$X = \begin{bmatrix} 10 & 100 \\ 12 & 110 \\ 11 & 105 \end{bmatrix}_{n\times p} \; ; \text{ To find } \bar{X} = \begin{bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{bmatrix} = \begin{bmatrix} 11 \\ 105 \end{bmatrix}_{p\times 1}$$

33   315

$\bar{X}: \frac{33}{3} = 11 \quad \frac{315}{3} = 105$

but using $\bar{X} = \frac{1}{n} (X^T \cdot 1)$

$$\Rightarrow \; = \frac{1}{3} \left[\begin{pmatrix} 10 & 12 & 11 \\ 100 & 110 & 105 \end{pmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right]$$

$$= \frac{1}{3} \begin{bmatrix} 10+12+11 \\ 100+110+105 \end{bmatrix} = \begin{bmatrix} 11 \\ \end{bmatrix}$$

for univariate **Multivariate Normal Distribution**

w.r.t pdf $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$; $-\infty < x < \infty$

(1) $X \sim N(\mu, \sigma)$

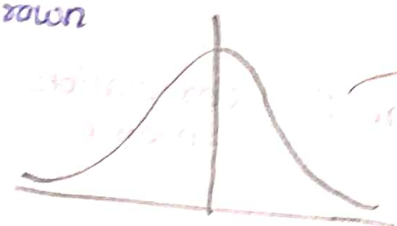
μ & σ are parameter.

in multivariate domain

$$x \to X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} \quad \& \quad \mu \to \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_p \end{bmatrix} \quad \& \quad \sigma^2 \xrightarrow{\text{cov matrix}} \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \cdots & \sigma_{2p} \\ \vdots & & & \\ \sigma_{p1} & \cdots & \cdots & \sigma_{pp} \end{bmatrix}_{p \times p}$$

we need $f(x) \sim N(\mu, \Sigma)$

univariate pdf of $f(x)$ can be drawn



(bivariate)

visualise in multivariate

in bivariate r.v. Imagine two r.v with its pdf is visualized as 3-D pictorial representation but if we do add one more variable & then visualize it is difficult.

Suppose say x_1, x_2 we have

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{p \times 1} \quad \& \quad \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}_{p \times 1} \quad \& \quad \sigma^2 = \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}_{p \times p}$$

if there is no correlation means x_1 & x_2 are independent.

$\Rightarrow \rho_{12} = 0$ (or) $\sigma_{12} = 0$

$$\Rightarrow f(x_1, x_2) = f(x_1) \cdot f(x_2)$$

$$= \underbrace{\frac{1}{\sigma_1\sqrt{2\pi}}}_{\text{component}} \underbrace{e^{-1/2\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}}_{\text{exponent part}} \times \frac{1}{\sigma_2\sqrt{2\pi}} e^{-1/2\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

$-\infty < x_j < \infty \ \forall j = 1, 2 \ldots$

$$= \frac{1}{2\pi(\sigma_1\sigma_2)} e^{-1/2\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$$

When we expand for multivariate cases you can see more number variables increased in each component

Now we derive the constant & exponent part from the population parameter.

We have $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ $\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix}$

$\therefore |\Sigma| = \begin{vmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{vmatrix} = \sigma_1^2\sigma_2^2 - \sigma_{21}\sigma_{12} = \sigma_1^2\sigma_2^2 - \sigma_{12}^2$ ($\sigma_{12}^2 = 0$ clear assump)

$= \sigma_1^2\sigma_2^2$

Constant part in
$\therefore f(x_1, x_2) = \dfrac{1}{2\pi(\sigma_1\sigma_2)} = \dfrac{1}{(2\pi)|\Sigma|^{1/2}}$

Suppose <u>three</u> variables:- $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $\Sigma = \begin{bmatrix} \sigma_1^2 & 0 & 0 \\ 0 & \sigma_2^2 & 0 \\ 0 & 0 & \sigma_3^2 \end{bmatrix}$

$f(x_1, x_2, x_3)$
$= f(x_1) \cdot f(x_2) \cdot f(x_3)$

Constant part $= \dfrac{1}{(2\pi)^{3/2}(\sigma_1^2\sigma_2^2\sigma_3^2)} = \dfrac{1}{(2\pi)^{3/2}|\Sigma|^{1/2}}$

..... continuing for p variable.

$$\boxed{f(x_1, x_2 \ldots x_p) = \dfrac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} \quad \text{for const. Part}}$$ for independent case

Now for exponent part:-

$f(x_1, x_2) = e^{-\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$

Univariate $f(x) = -\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2 = -\frac{1}{2}(\underset{\downarrow\,\mu}{\overset{X}{x}}-\mu)(\overset{\Sigma}{\sigma^2})^{-1}(x-\mu)$

in matrix form we write =

$$\boxed{\text{exponent} = -\frac{1}{2}(x-\mu)^T(\Sigma)^{-1}(x-\mu).}$$

(e) bivariate
$\to -\frac{1}{2}\begin{bmatrix} x_1-\mu_1 & x_2-\mu_2 \end{bmatrix}_{1\times2} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}_{2\times2} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}_{2\times1}$ = we get 1×1

$A^{-1}: \frac{adj A}{|A|}$

$\therefore$ for p variable,

$$f(x_1, x_2 \ldots x_p) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}\left[(x-\mu)^T \Sigma^{-1}(x-\mu)\right]}$$

$; -\infty < x_j < \infty$

$\forall j = 1, 2 \ldots p.$

Now for bivariate variables where non-independent case.

$$\therefore \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{bmatrix} ; \mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} ; x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

①

earlier defined

$$|\Sigma| = \begin{vmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{vmatrix} = \sigma_1^2\sigma_2^2 - \sigma_{12}^2 = \sigma_1^2\sigma_2^2 - [\rho_{12}\sigma_1\sigma_2]^2$$

$$= \sigma_1^2\sigma_2^2(1-\rho_{12})^2$$

$A^{-1} = \frac{adj A}{|A|}$; transpose of cofactor

$$\therefore \Sigma^{-1} = \frac{1}{\underbrace{\sigma_1^2\sigma_2^2 - \sigma_{12}^2}_{|\Sigma|}} \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix}$$

then exponent part

$$= -\frac{1}{2}\left[(x-\mu)^T \Sigma^{-1}(x-\mu)\right] = -\frac{1}{2}\left[x_1-\mu_1 \quad x_2-\mu_2\right] \underset{matrix}{\Sigma^{-1}} \begin{bmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{bmatrix}$$

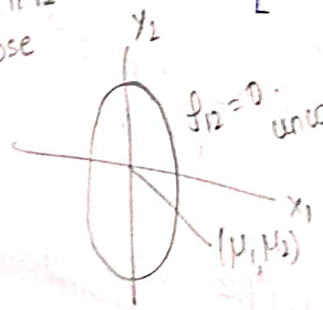
expanding & simplifying, we get.

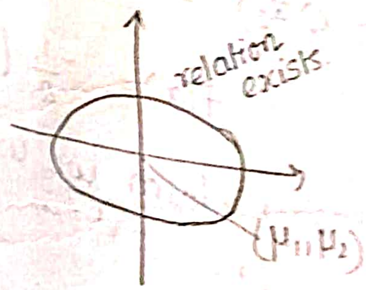
$$= \frac{-1}{2(1-\rho_{12})^2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho_{12}\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$$

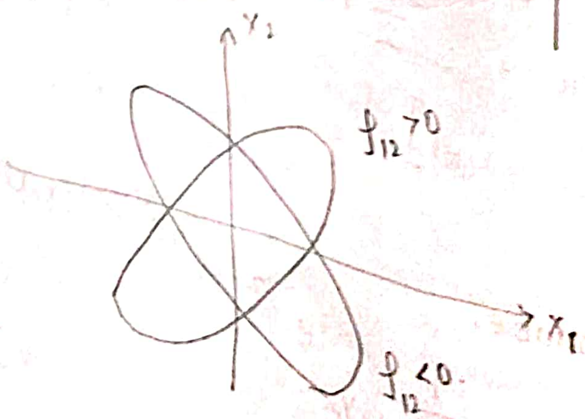
if $\rho_{12}=0$ then we get

exponent part $= -\frac{1}{2}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]$

Graphically it is a ellipse







Properties MND $\sim N_p(\mu, \Sigma)$

i) If $X_{p\times 1} \sim N_p(\mu, \Sigma)$ then x_j is $N(\mu_j, \sigma_j^2)$ $\forall x_j$; $j = 1, 2, \ldots p$.

i) multivariate then all the individually variables are univariate normal.

ii) If $X_{p\times 1} \sim N_p(\mu, \Sigma)$ then the subset of $X_{p\times 1}$,

i) $X_{q\times 1}$ is $N_q(\mu, \Sigma)$

subset is also multivariate

iii) If $X_{p\times 1} \sim N_p(\mu, \Sigma)$ then the linear combination of x_j, $j = 1, \ldots p$ is a univariate normal.

iv) If $X_{p\times 1} \sim N_p(\mu, \Sigma)$ then the q linear combination of x_j, $j = 1, 2 \ldots p$ is multivariate (q-dimension) normal.

x ——— x

i) Simultaneously say $x_1, x_2 \ldots x_p$ to be multivariate but individually $x_1, x_2 \ldots x_p$ are univariate normal.

$x_1 \Rightarrow q$ variable $\Rightarrow$ multivariate.

ii) $X = \begin{bmatrix} x_1 \\ x_2 \\ x_q \\ \hline x_p \end{bmatrix} \begin{matrix} X_1, \; q\times 1 \\ X_2 \; (p-q)\times 1 \end{matrix}$ $\Sigma = \left[\begin{array}{c|c} \Sigma_q & \\ \hline & \end{array}\right]$ $\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_q \\ \vdots \\ \mu_p \end{bmatrix}$

iii) vector $a^T = [a_1 \;\; a_2 \ldots a_p]_{1\times p}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}_{p\times 1}$

$\therefore L.C = a_1 x_1 + a_2 x_2 + \ldots + a_p x_p = a^T x \sim N(a^T\mu, a^T\Sigma a)$

$E(a^T x) = a^T E(x) = a^T \mu = a_1\mu_1 + a_2\mu_2 + \ldots a_p\mu_p$

$V(a^T x) = a^T \Sigma a = 1\times 1$

∴ Let $X_{p\times 1}$ be a random vector with mean vector $E(X) = \mu_{p\times 1}$ and the covariance matrix $Cov(X) = \Sigma_{p\times p}$

(1) then X is said to follow multivariate normal distribution

i.e) $X_{p\times 1} \sim N_p(\mu, \Sigma)$.

then the pdf of X is

$$f_X(x) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}\left[(x-\mu)^T \Sigma^{-1}(x-\mu)\right]} \quad \text{for } \Sigma > 0.$$

Computing the cov matrix:-

Find the cov matrix for two random variables X_1 & X_2 when their joint pdf is $P_{12}(x_1, x_2)$ is gn.

Eg1:

x_1	-1	0	1
$p(x_1)$	·3	·3	·4

x_2	0	1
$p(x_2)$	·8	·2

$x_1 \backslash x_2$	0	1	$P_1(x_1)$
-1	·24	·06	·3
0	·16	·14	·3
1	·40	·00	·4
$P_2(x_2)$	·8	·2	1

soln:

$$Cov = \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$\mu_1 = E(X_1) = -1(0.3) + 0 + 1(0.4)$
$= 0.1$

$\mu_2 = E(X_2) = 0(0.8) + 1(0.2) = 0.2$

$$\therefore \sigma_{11} = E(X_1-\mu_1)^2 = \sum_{\text{all } x_1} (x_1 - (0.1))^2 P_1(x_1)$$

$$= (-1-0.1)^2(0.3) + (0-0.1)^2(0.3) + (1-0.1)^2(0.4) = 0.69$$

$$\sigma_{22} = E(X_2-\mu_2)^2 = \sum_{\text{all } x_2} (x_2 - (0.2))^2 P_2(x_2)$$

$$= (0-(0.2))^2(0.8) + (1-(0.2))^2(0.2) = 0.16.$$

$$\sigma_{12} = E(x_1-\mu_1)(x_2-\mu_2)$$
$$= \sum_{\forall x_1, x_2} (x_1-(0.1))(x_2-(0.2))\, P_{12}(x_1,x_2)$$
$$= (-1-0.1)(0-0.2)(0.24)$$
$$+(-1-0.1)(1-0.2)(0.06) + (0-0.1)(0-0.2)(0.16)$$
$$+ \cdots + (1-0.1)(1-0.2)(0.00) = -0.08.$$

$$\sigma_{21} = E(x_2-\mu_2)(x_1-\mu_1) = E(x_1-\mu_1)(x_2-\mu_2) = -0.08.$$

$$\therefore \text{Cov matrix } \Sigma = \begin{bmatrix} 0.69 & -0.08 \\ -0.08 & 0.16 \end{bmatrix}.$$

x —— x

3) Computing the correlation matrix from cov matrix.

Qn: $\Sigma = \begin{bmatrix} 4_{\sigma_{11}} & 1_{\sigma_{12}} & 2_{\sigma_{13}} \\ 1_{\sigma_{21}} & 9_{\sigma_{22}} & -3_{\sigma_{23}} \\ 2_{\sigma_{31}} & -3_{\sigma_{32}} & 25_{\sigma_{33}} \end{bmatrix}$ Obtain $V^{1/2}$ & ρ (→ correlation matrix) Svar.

$$V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\sigma_{22}} & \cdots & 0 \\ 0 & \cdots & & \sqrt{\sigma_{pp}} \end{bmatrix}$$

$$V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & 0 \\ 0 & \sqrt{\sigma_{22}} & 0 \\ 0 & 0 & \sqrt{\sigma_{33}} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

W.k.T $\boxed{V^{1/2}\rho V^{1/2} = \Sigma}$

$$\therefore \rho = (V^{1/2})^{-1}\Sigma(V^{1/2})^{-1}$$

$$\text{Now } (V^{1/2})^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}$$

$$\rho = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}\begin{pmatrix} 4 & 1 & 2 \\ 1 & 9 & -3 \\ 2 & -3 & 25 \end{pmatrix}\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1/2 & 1 \\ 1/3 & 3 & -1 \\ 2/5 & -3/5 & 5 \end{pmatrix}\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/5 \end{pmatrix} = \begin{pmatrix} 1 & 1/6 & 1/5 \\ 1/6 & 1 & -1/5 \\ 1/5 & -1/5 & 1 \end{pmatrix}.$$

x — x

The mean vector & cov matrix for L.C of random variables.

If x_1 is a r.v then $E(ax_1) = aE(x_1) = a\mu_1$

& $V(ax_1) = a^2 V(x_1) = a^2\sigma_{11}$.

1

If x_2 is a second random variable & a & b are constants then

$$\text{Cov}(ax_1, bx_2) = E(ax_1 - a\mu_1)(bx_2 - b\mu_2)$$
$$= ab\,E(x_1-\mu_1)(x_2-\mu_2)$$
$$= ab\,\text{cov}(x_1, x_2) = ab\,\sigma_{12}.$$

$\therefore$ for L.C of x & x_2 we have.

$$E(ax_1+bx_2) = aE(x_1) + bE(x_2) = a\mu_1 + b\mu_2$$
$$V(ax_1+bx_2) = E((ax_1+bx_2) - (a\mu_1+b\mu_2))^2$$

on Simplifying

$$= a^2\sigma_{11} + b^2\sigma_{22} + 2ab\sigma_{12}$$

Matrix representation $c' = [a \;\; b]$, ax_1+bx_2 can be written as

$$[a \;\; b]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c'x$$

$$\therefore \text{var}(ax_1+bx_2) = \text{Var}(c'x) = c'\Sigma c.$$

Partitioning the Covariance Matrix:-

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_q \\ \hline x_{q+1} \\ \vdots \\ x_p \end{bmatrix}_{p\times 1} \begin{matrix} \left.\vphantom{\begin{matrix} x \\ x \\ x \\ x \end{matrix}}\right\} q \\ \left.\vphantom{\begin{matrix} x \\ x \\ x \end{matrix}}\right\} p-q \end{matrix} = \begin{bmatrix} X^{(1)} \\ \hline X^{(2)} \end{bmatrix} \qquad \mu = E(X) = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_q \\ \hline \mu_{q+1} \\ \vdots \\ \mu_p \end{bmatrix} = \begin{bmatrix} \mu^{(1)} \\ \hline \mu^{(2)} \end{bmatrix}$$

$$\text{cov } \Sigma = E(X-\mu)(X-\mu)' = \begin{array}{c c} & \begin{array}{cc} q & p-q \end{array} \\ \begin{array}{c} q \\ p-q \end{array} & \left[\begin{array}{c|c} \Sigma_{11} & \Sigma_{12} \\ \hline \Sigma_{21} & \Sigma_{22} \end{array}\right] \end{array}$$

$$= \left[\begin{array}{ccc|ccc} \sigma_{11} & \cdots & \sigma_{1q} & \sigma_{1q+1} & \cdots & \sigma_{1p} \\ \vdots & & & \vdots & & \vdots \\ \sigma_{q1} & \cdots & \sigma_{qq} & \sigma_{qq+1} & \cdots & \sigma_{qp} \\ \hline \sigma_{q+1\,1} & \cdots & \sigma_{q+1\,q} & \sigma_{q+1\,q+1} & \cdots & \sigma_{q+1\,p} \\ \vdots & & \vdots & \vdots & & \vdots \\ \sigma_{p1} & \cdots & \sigma_{pq} & \sigma_{pq+1} & \cdots & \sigma_{pp} \end{array}\right]$$

$\therefore \text{cov}(X^{(1)}, X^{(2)}) = \Sigma_{12}$ (say)

Normal distribution - Quadratic forms:-

Quadratic forms involving normal random vectors

of $Q = X^T A X$ where X is $p\times 1 \rightarrow$ multivariate random variable

A is $p\times p$ matrix and

T is transpose.

Prior results:-

1) Orthogonal matrix $\Rightarrow A^T = A^{-1}$ (or) $A^T A = AA^T = I$

Let X be $p\times 1$ standard multivariate normal random vector & A is $p\times p$ matrix $\Rightarrow Y = AX$ i) $Y \sim N(0, I)$

2) Idempotent:-

Pbm 1: The variance covariance matrix of a 3-D r.v $X = (X_1, X_2, X_3)$ is given by $\Sigma = \begin{pmatrix} 25 & -2 & 4 \\ -2 & 4 & 1 \\ 4 & 1 & 9 \end{pmatrix}$

i) find the correlation matrix.

ii) find the correlation betn X_1 & $\frac{X_2}{2} + \frac{X_3}{2}$

Soln:

$$\text{cov}\quad \Sigma = \begin{array}{c} \\ X_1 \\ X_2 \\ X_3 \end{array}\begin{array}{c} \begin{array}{ccc} X_1 & X_2 & X_3 \\ V(X_1) & \text{cov}(X_1,X_2) & \end{array} \\ \begin{pmatrix} 25 & -2 & 4 \\ -2 & 4\,V(X_2) & 1 \\ 4 & 1 & 9\,V(X_3) \end{pmatrix} \end{array}$$

& etc.

$\to$ variance

$V = \text{dig}(\sigma_{11}, \sigma_{22}, \sigma_{33})$

$\sigma_{11} = \sigma_1^2 = 25$

$\sigma_{22} = \sigma_2^2 = 4$

$\sigma_{33} = \sigma_3^2 = 9$

$\rho_{12} = \frac{\text{cov}(X_1, X_2)}{\sqrt{\sigma_{X_1}}\sqrt{\sigma_{X_2}}} = \frac{-2}{\sqrt{\sigma_1^2} \times \sqrt{\sigma_2^2}} = \frac{-2}{5 \times 2}$

i) $\therefore$ correlation matrix of (X)

$$\overset{\rho}{\text{cov}}(X) = (V^{1/2})^{-1} \Sigma (V^{1/2})^{-1}$$

Since finding correlation w.r.t diagonal entry will be one

$\rho = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1 & \rho_{12} = \frac{-1}{5} & \frac{4}{5 \times 3} \\ \frac{-2}{5 \times 2} & 1 & \frac{1}{3 \times 2} \\ \frac{4}{5 \times 3} & \frac{1}{3 \times 2} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{pmatrix}$$

is the correlation matrix.

ii) ρ betn X_1 & $\frac{X_2}{2} + \frac{X_3}{2}$

To find

$$\Rightarrow \rho\left(X_1, \frac{X_2}{2} + \frac{X_3}{2}\right) = \frac{\text{Cov}\left(X_1, \frac{X_2}{2} + \frac{X_3}{2}\right)}{\left[\text{var}(X_1) \times \text{Var}\left(\frac{1}{2}(X_2+X_3)\right)\right]^{-1}}$$

Now

$$\text{cov}\left(X_1, \frac{X_2}{2} + \frac{X_3}{2}\right) = \text{cov}\left(X_1, \frac{X_2}{2}\right) + \text{cov}\left(X_1, \frac{X_3}{2}\right)$$

$$= \frac{1}{2}\text{cov}(X_1, X_2) + \frac{1}{2}\text{cov}(X_1, X_3)$$

$$= \frac{1}{2}(-2) + \frac{1}{2}(4) = -1 + 2 = 1.$$

$$\text{Var}\left(\frac{1}{2}(X_2+X_3)\right) = \frac{1}{4}V(X_2+X_3) + \frac{1}{4}\left[V(X_2) + V(X_3) + 2\,\text{cov}(X_2, X_3)\right]$$

$$= \frac{1}{4}\left[4 + 9 + 2(1)\right] = \frac{1}{4}(15) = \frac{15}{4}$$

$\therefore f\left(x_1, \frac{x_2}{2}+\frac{x_3}{2}\right) = \dfrac{1}{[25 \times 15/4]^{+1/2}} =$

2) Suppose $X = (x_1, x_2 \ldots x_p)'$ is a p-dimensional random vector with $E(X) = \mu$ & $cov(X) = \Sigma$. Find the cov matrix of the random vector $Z = (c_1'X, c_2'X \ldots c_k'X)$ where $c_j' \in R^p$ are vectors of constants.

new r.v of k dimension

Soln: Gn: $X_{p\times 1}$ with $E(X) = \mu$ & $cov(X) = \Sigma$

$$Z = (c_1'x,\ c_2'x,\ \ldots\ c_k'x)$$

cov betn any two i & j

Now $cov(c_i'x, c_j'x) = c_i' \Sigma c_j$

$$Cov(Z) = \begin{pmatrix} c_1'\Sigma c_1 & c_1'\Sigma c_2 & \cdots & c_1'\Sigma c_k \\ & c_2'\Sigma c_2 & \cdots & c_2'\Sigma c_k \\ \text{Same entry} & & \ddots & \\ & & & c_k'\Sigma c_k \end{pmatrix}$$

3) S.T $|S| = \underbrace{S_{11} \ldots S_{pp}}_{S_{ii}\ \forall i=1,2..p} |R|$ where S is a sample variance cov matrix & R is the same correlation matrix.

Soln: T.P $|S| = S_{11} \ldots S_{pp} |R|$

S: Sample varia cov matrix

R is " correlation matrix

$V^{1/2}$ is diag matrix holding $S_{11} \ldots S_{pp}$

W.K.T Relation betn corr & cov $\Rightarrow R = (V^{1/2})^{-1} S (V^{1/2})^{-1}$ —— (1)

where $V^{1/2} = diag(S_{11} \ldots S_{pp})$.

(1) $\Rightarrow V^{1/2} R V^{1/2} = S$

$$|V^{1/2}|\,|R|\,|V^{1/2}| = |S|$$

$\hookrightarrow (S_{11}^{1/2} S_{22}^{1/2} \ldots S_{pp}^{1/2}) |R|$

(i) $\left(\prod_{i=1}^{p} S_{ii}^{1/2}\right) |R| \left(\prod_{i=1}^{p} S_{ii}^{1/2}\right) = |S|$

$$\Rightarrow |S| = S_{11} S_{22} \ldots S_{pp} |R|$$

x——x

1) Supp the mean vector & cov matrix of $X = (X_1, X_2, X_3, X_4)$ is given by

$$\mu = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix} \quad \& \quad \Sigma = \begin{array}{c|cccc} & X_1 & X_2 & X_3 & X_4 \\ \hline X_1 & 3 & 0 & 2 & 2 \\ X_2 & 0 & 1 & 1 & 0 \\ X_3 & 2 & 1 & 9 & -2 \\ X_4 & 2 & 0 & -2 & 4 \end{array}$$

let $(X)_{(1)} = (X_1, X_3)$ & $X_2 = (X_2, X_4)$ be 2 subvectors

$A = (1, 2)$ & $B = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

i) find $\operatorname{cov}(AX_{(1)})$, $\operatorname{cov}(BX_{(2)})$ & $\operatorname{cov}(AX_{(1)}, BX_{(2)})$

Ans: $X_{4\times1}$ & $E(X) = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ & $\operatorname{cov}(X) = \Sigma = \begin{pmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{pmatrix}$

Subvector $X^{(1)} = \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ & $X^{(2)} = \begin{pmatrix} X_2 \\ X_4 \end{pmatrix}$

row vector → $A = (1, 2)$ & $B = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$

$\therefore \boxed{\operatorname{cov}[AX^{(1)}] = A \operatorname{cov}(X^{(1)}) A'}$

$$\operatorname{cov}(X^{(1)}) = \begin{array}{c|cc} & X_1 & X_3 \\ \hline X_1 & 3 & 2 \\ X_3 & 2 & 9 \end{array} = \underset{1\times2}{(1, 2)} \underset{2\times2}{\begin{pmatrix} 3 & 2 \\ 2 & 9 \end{pmatrix}} \underset{2\times1}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \text{calculat} \; \underset{1\times1}{(47)}$$

& $\operatorname{cov}[BX^{(2)}] = B \operatorname{cov}(X^{(2)}) B'$ →

$$\operatorname{cov}(X^{(2)}) = \begin{array}{c|cc} & X_2 & X_4 \\ \hline X_2 & 1 & 0 \\ X_4 & 0 & 4 \end{array}$$

$$= \underset{2\times2}{\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}} \underset{2\times2}{\begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}} \underset{2\times2}{\begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}}$$

$= \text{calculate} = \underset{2\times2}{\begin{pmatrix} 17 & 10 \\ 10 & 6 \end{pmatrix}}$

i) find the Joint distribution of $AX^{(1)}$ & $BX^{(2)}$ if

Calc. $X \sim N_4(\mu, \Sigma)$. — back.

Now $\text{cov}(AX^{(1)}, BX^{(2)}) = A \,\text{cov}(X^{(1)}, X^{(2)})\, B$.

$$\therefore \text{cov}(X^{(1)}, X^{(2)}) = \begin{array}{c|cc} & X_2 & X_4 \\ \hline X_1 & \text{cov}(X_1, X_2) & \text{cov}(X_1, X_4) \\ X_3 & \text{cov}(X_2, X_3) & \text{cov}(X_3, X_4) \end{array}$$

$X^{(1)} = \begin{pmatrix} X_1 \\ X_3 \end{pmatrix}$ $\quad X^{(2)} = \begin{pmatrix} X_2 \\ X_4 \end{pmatrix}$ $\quad = \text{cov}(X_3, X_2)$

$$= \begin{pmatrix} 0 & 2 \\ 1 & -2 \end{pmatrix}$$

$$\therefore \text{cov}(AX^{(1)}, BX^{(2)}) = \underset{1\times 2}{(1 \quad 2)} \underset{2\times 2}{\begin{pmatrix} 0 & 2 \\ 1 & -2 \end{pmatrix}} \underset{2\times 2}{\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}}$$

$\rightarrow \begin{pmatrix} 4 & -2 \\ -3 & 0 \end{pmatrix}$

$= \text{calculate}.$

$= (-2 \quad -2)$

---

ii) Soln: $X \sim N_4(\mu, \Sigma)$; $\mu = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ $\quad \Sigma = \text{given}$

To find Joint distribution $AX^{(1)}$ & $BX^{(2)}$

$$Z = \begin{pmatrix} AX^{(1)} \\ BX^{(2)} \end{pmatrix}$$

$\rightarrow (1 \quad 2) \begin{pmatrix} 3 & 2 \\ 2 & 9 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (7 \quad 20) \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 7 + 40 = 47.$

$\rightarrow \begin{pmatrix} 1 & -8 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 17 & 10 \\ 10 & 6 \end{pmatrix}$

In Statistics the W.D. is a generalization to multiple dimensions of χ^2 distribution (or) in the case of non-integer degrees of freedom of the gamma distribution. It is a family of probability distribution defined over symmetric non-negative definite matrix-valued random variables. These distributions are of importance in the estimation of covariance matrices in multivariate statistics.

Defn: Supp X is $n \times p$ matrix, each row of which is independently drawn from a p-variate normal distribution with zero mean. Then the W D is the prob distribution of the $p \times p$ random matrix $S = X^T X$

①

me

matrix

$= I_p$

sity fn

where Γ

perties

i) If $M_1 \sim$

distributed

ii) If M_i

distrib

iii) If M

Wishart distribution & its properties

defn of W.D through matrix normal distribution.

Suppose $Z \sim N(0, \Sigma)$, then $\underset{m\times m}{A = Z'Z}$ is said to follow a wishart distribution (m-dimensional) with parameters n and Σ i.e. $A \sim W_m(n, \Sigma)$

Distribution of $\bar{x}$ and S_{n-1}

Distribution of $\bar{x}$:- Sample mean $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$; $x_1, x_2 \dots x_n$ are random sample follows $N_p(\mu, \Sigma)$ and also $E(\bar{x}) = \mu$.

$cov(\bar{x}) = \frac{\Sigma}{n}$.

$\Rightarrow \bar{x} \sim N_p(\mu, \frac{\Sigma}{n})$

Distribution of S_{n-1}: sample variance covariance $S_{n-1} = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})(x_i - \bar{x})^T$

this is said to have wishart distribution

i.e. $(n-1)S_{n-1}$ is said to follow p-dimensional wishart distribution with parameters $(n-1)$ and Σ

i.e. $(n-1)S_{n-1} \sim W_p(n-1, \Sigma)$. $\rightarrow$ dof $\Rightarrow (n-1)$

note: further $\bar{x}$ & S_{n-1} are independently distributed.

Wishart distribution is basically the generalization of the chi-square distribution.

Wishart distribution:- defn:-

Let $Y_1, Y_2 \dots Y_m$ be independent $N_m(0, \Sigma)$

then $A = \sum_{i=1}^{n} Y_i Y_i^T$ is said to have a wishart distribution (m-dimensional) with parameters m and Σ.

i.e. $A \sim W_m(n, \Sigma)$

Result :- i) Suppose $A_1 \sim W_m(n_1, \Sigma)$ & $A_2 \sim W_m(n_2, \Sigma)$ and are independent, then sum of two $A_1 + A_2 \sim W_m(n_1+n_2, \Sigma)$

ii) Suppose $\underset{m \times m}{A} \sim W_m(n, \Sigma)$: and let C be a $q \times m$ non-random matrix then $\underbrace{CAC'}_{q\times q} \sim W_q(n, C\Sigma C')$

⇒ X be the random vector having

$X = (X_1, X_2 \dots X_p)^T$ X_i are r.variables.

∴ distribution fn of X is given by

$$F_{X_1, X_2 \dots X_p}(x_1, x_2 \dots x_p) = \cancel{(-\infty < X_1 \le x_1, -\infty < X_2 \le x_2 \dots}$$
$$= (-\infty \le X_1 \le x_1 \dots\dots -\infty < X_p \le x_p)$$

and this classified as DRV & CRV (in multivariate cases)

multivariate distribution fn.

$\boxed{\text{Discrete}}$ ⇒ $F_{X_1, X_2 \dots X_p}(x_1, x_2, \dots x_p) = \sum\limits_{i_1 \le x_1} \dots \sum\limits_{i_p \le x_p} P(X_1 = i_1 \dots\dots X_p = i_p)$

Joint (pmf)

∴ Prob mass fn of X is defined as

$P(X_1 = x_1, X_2 = x_2 \dots X_p = x_p)$ (Joint pmf)

Marginal pmf of X_i

$$\Rightarrow P\{X_i = x_i) = \underset{\text{except } x_i}{\sum \dots \sum} P(X_1 = x_1 \dots\dots X_p = x_p)$$

Marginal pmf of X_i & X_j

$$\Rightarrow P\{X_i = x_i, X_j = x_j\} = \underset{\text{except } x_i \,\&\, x_j.}{\sum \dots \sum} P(\quad)$$

Conditional distribution of X_k given X_i & X_j

$$P(X_k = x_k \mid X_i = x_i, X_j = x_j)$$
$$= \frac{P(X_k = x_k, X_i = x_i, X_j = x_j)}{P(X_i = x_i, X_j = x_j)} \quad \begin{array}{l} \to \text{Joint pmf} \\ \to \text{Marginal pmf} \end{array}$$

Continuous

$$F_{X_1,\dots X_p}(x_1,x_2\dots x_p) = \int_{-\infty}^{x_1}\dots\int_{-\infty}^{x_p} f_{X_1,X_2\dots X_p}(x_1,x_2\dots x_p)\prod_{i=1}^{p} dx_i$$

↳ Joint pdf.

Marginal Joint pdf of X_i & X_j

$$f_{X_i,X_j}(x_i,x_j) = \int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} f_{X_1,\dots X_p}(x_1,x_2\dots x_p)\prod_{l\neq i,j}^{p} dx_l$$

except x_i, x_j

conditional density fn of X_K given by X_i & X_j

$$f_{X_K|X_i,X_j}(x_k|x_i,x_j) = \frac{f_{x_k,x_i,x_j} \to \text{Joint}}{f_{x_i,x_j} \Rightarrow \text{Marginal}}$$

Now $X=(X_1,X_2\dots X_p)^T$ is set of independent random variables then Joint distribution is

$$P(X_1=x_1\dots X_p=x_p) = \prod_{i=1}^{p} P(X_i=x_i) \Rightarrow \text{discrete.}$$

$$f_{X_1,X_2\dots X_p}(x_1,x_2\dots x_p) = \prod_{i=1}^{p} f_{X_i}(x_i) \Rightarrow \text{continuous.}$$

Expectation vector of X

$$\mu = E(X) = \begin{pmatrix} E(X_1)\\ \vdots \\ E(X_p)\end{pmatrix}$$

we make transformation

$$X_{p\times 1} \to Y = \underset{q\times p}{A}X + \underset{q\times 1}{b}.$$

$$\therefore \mu_y = E(Y) = A\,E(X)+b$$
$$= A\mu + b$$

Cov of X:-

$$\Sigma = \operatorname{cov}(X) = E(X-\mu)(X-\mu)^T$$

for $(i,j)^{th}$ element $\sigma_{ij} = E(X_i-\mu_i)(X_j-\mu_j)$

$$\sigma_{ii} = E(X_i-\mu_i)^2$$

Correlation matrix of X:-

$$\rho = \operatorname{corr}(X) = (V^{1/2})^{-1}\Sigma(V^{1/2})$$

where $V = \operatorname{diag}(\sigma_{11},\sigma_{22}\dots\sigma_{pp})$

ii) $X \sim N_4(\mu, \Sigma)$ $\mu = \begin{pmatrix} 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}$; $\Sigma = \begin{pmatrix} 3 & 0 & 2 & 2 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 9 & -2 \\ 2 & 0 & -2 & 4 \end{pmatrix}$

To find Joint distn of $Ax^{(1)}$ & $Bx^{(2)}$

$x^{(1)} = \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}$ $x^{(2)} = \begin{pmatrix} x_2 \\ x_4 \end{pmatrix}$

i) $\boxed{\underset{3\times 1}{Z} = \begin{pmatrix} \underset{1\times 1}{Ax^{(1)}} \\ \underset{2\times 2}{Bx^{(2)}} \end{pmatrix}}$ → scalar r. variate; $A = 1\times 2$, $B = 2\times 2$

W.K.T from previous $cov(x^{(1)}) = \begin{array}{c|cc} & x_1 & x_3 \\ x_1 & 3 & 2 \\ x_3 & 2 & 9 \end{array}$

& $cov(x^{(2)}) = \begin{array}{c|cc} & x_2 & x_4 \\ x_2 & 1 & 0 \\ x_4 & 0 & 4 \end{array}$

Now $\alpha^T Z$ is linear combination of element of X.
using property L.C ~ univariate $\sim N_1 \; \forall \alpha \in \mathbb{R}^3$

$A = 1\times 2$
$B = 2\times 2$

$\Rightarrow \boxed{Z \sim N_3(\underset{\mu_z}{E(z)}, \underset{\Sigma_z}{Cov(z)})}$

$\mu_z = E(z) = \begin{pmatrix} A\,E(x^{(1)}) \\ B\,E(x^{(2)}) \end{pmatrix} = \begin{pmatrix} \underset{1\times 2}{(1 \;\; 2)} \underset{2\times 1}{\begin{pmatrix} 4 \\ 2 \end{pmatrix}} \\ \underset{2\times 2}{\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}} \underset{2\times 1}{\begin{pmatrix} 3 \\ 1 \end{pmatrix}} \end{pmatrix} = \begin{pmatrix} 8 \\ \begin{pmatrix} 1 \\ 5 \end{pmatrix} \end{pmatrix}$ → compute

$E(x^{(1)}) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$
$E(x^{(2)}) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

& $Cov(z) = Cov\begin{pmatrix} Ax^{(1)} \\ Bx^{(2)} \end{pmatrix} = \begin{pmatrix} A\,cov(x^{(1)})A' & A\,cov(x^{(1)}, x^{(2)})B' \\ \text{same} & B\,cov\,x^{(2)} B' \end{pmatrix}$

$= \begin{pmatrix} \text{already computed} \end{pmatrix} = \begin{pmatrix} 47 & (\;\;) \\ & \begin{pmatrix} 17 & 10 \\ 10 & 6 \end{pmatrix} \end{pmatrix}$

2×2

iv) with $X \sim N_4(\mu, \Sigma)$ find the marginal distributions of $X^{(1)}$ & $X^{(2)}$ & conditional distributions of $X^{(1)}$ given $X^{(2)} = (2,1)^T$

$$X^{(1)} \sim N_2(\mu^{(1)}, \Sigma^{(1)}) \;;\; \mu^{(1)} = E\begin{pmatrix} X_1 \\ X_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\Sigma^{(1)} = cov(X^{(1)}) = \begin{pmatrix} 3 & 2 \\ 2 & 9 \end{pmatrix}$$

Marginal already computed

$$X^{(2)} \sim N_2(\mu^{(2)}, \Sigma^{(2)}) \text{ where}$$

$$\mu^{(2)} = E(X^{(2)}) = E\begin{pmatrix} X_2 \\ X_4 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\Sigma^{(2)} = Cov(X^{(2)}) = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

Marginal ~~condit~~ already computed.

Conditional distribution of $X_{(1)}$ given $X_{(2)} = (2,1)^T$

$$\left(X^{(1)} / X^{(2)} = x^{(2)}\right) \sim N_2\left(\mu^{(1)} + \Sigma_{12}\Sigma_{22}^{-1}(x_2 - \mu_2),\ \Sigma_{11.2}\right)$$

$$\text{where } \Sigma_{11.2} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$$\Sigma_{11} = cov(X^{(1)}),\ \Sigma_{22} = cov(X^{(2)})$$

$$\Sigma_{12} = cov(X^{(1)}, X^{(2)})$$

Calculate

Conditional distrib in bivariate

$$\text{mean} = \mu_1 + \frac{\sigma_{12}}{\sigma_{22}}(x_2 - \mu_2)$$

$$\text{var} = \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}$$

---

3) Suppose $X \sim N_2(\mu, \Sigma)$ with $\mu = (2,2)'$ & $\Sigma = I_2$ consider $A = \{1,1\}$ & $B = \{1,-1\}$ verify whether AX & BX are independent? to say zero as well distri follows the multiv

Sol:

$$\mu = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad A = (1,1) \qquad B = (1,-1)$$

$$\text{distribution of } Z = \begin{pmatrix} AX \\ BX \end{pmatrix} = X\begin{pmatrix} A_{1\times 2} \\ B_{1\times 2} \end{pmatrix} = C_{2\times 2}X$$

multivariate normal $\sim N$ using property

$$= CX \sim N_2(E(CX),\ cov(CX))$$

$$\cong N_2(C\mu,\ C\overset{cov(X)}{I_2}C')$$

Z_1 & Z_2 $cov(Z_1, Z_2) = 0$ $\Rightarrow$ independent

$$\therefore Cov(AX, BX) = AI_2B'$$

$$= (1,1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(1\ -1) = (1,1)\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0.$$

$\Rightarrow$ independent

Hotelling - T-Square Statistics

Concept's on earlier topics discuss.

$X \sim N_p(\mu, \Sigma) \rightarrow$ population

$\bar{X} \sim N_p(\mu, \Sigma/n)$ } sample's

$S \sim W_p(n-1, \Sigma)$

} related concepts.

In univariate case :- from point of view hypothesis concept

μ_0 is the sample mean.

$H_0 : \mu = \mu_0$ & $H_1 : \mu \neq \mu_0$.

null hypo

alternate hypo

$x_1, x_2 \ldots x_n$ are random sample.

small sample say < 40 $\rightarrow$ we use t-distribution $\Rightarrow t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}}$

sample mean, pop mean, sample std devia, sample size.

looking into multivariate case

$\bar{x} \rightarrow \bar{X}$, $\mu \rightarrow \mu$, $s^2 \rightarrow S$ & $n \rightarrow n$.

~~$t = \bar{X} - \mu$~~ Now $t = \dfrac{\bar{x} - \mu}{s/\sqrt{n}} = \sqrt{n}(\bar{x} - \mu)s^{-1}$

if we square $t^2 = n(\bar{x} - \mu)(s^2)^{-1}(\bar{x} - \mu)$

Hotelling T^2 $\leftarrow T^2 = n(\bar{X} - \mu)^T (S)^{-1}(\bar{X} - \mu)$

$\bar{X} \sim N_p(\mu, \Sigma/n)$ & $S \sim W_p(n-1, \Sigma)$

$\sim \dfrac{(n-1)p}{n-p} F_{p, n-p} \quad \forall \; n-p < 4$

Eg: (pbm1) A random sample with $n = 20$ were collected from a bivariate normal process. The sample mean vector & S cov matrix given.

$\bar{X} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$ $\quad S = \begin{pmatrix} 40 & -50 \\ -50 & 100 \end{pmatrix}$

i) obtain Hotelling T^2 - Square

ii) What will be the distribution of it?

W.K.T $T^2 = n(\bar{x}-\mu)^T S^{-1} (\bar{x}-\mu)$.

$\bar{x} = \begin{pmatrix} 10 \\ 20 \end{pmatrix}$ $S = \begin{pmatrix} 40 & -50 \\ -50 & 100 \end{pmatrix}$

$\bar{x}-\mu = \begin{bmatrix} 10 \\ 20 \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = \begin{bmatrix} 10-\mu_1 \\ 20-\mu_2 \end{bmatrix}$

$S^{-1} = \frac{1}{|S|} \text{adj}(S) = \frac{1}{\underset{\parallel 1500}{|S|}} \begin{bmatrix} \text{transpose} \\ \text{of cofactor} \end{bmatrix} = \frac{1}{1500} \begin{bmatrix} 100 & 50 \\ 50 & 40 \end{bmatrix}$

$\therefore T^2 = 20 \begin{bmatrix} 10-\mu_1 & 20-\mu_2 \end{bmatrix} \cdot \frac{1}{1500} \begin{bmatrix} 100 & 50 \\ 50 & 40 \end{bmatrix} \begin{bmatrix} 10-\mu_1 \\ 20-\mu_2 \end{bmatrix}$

Ans: $(1.33)(10-\mu_1)^2 + (0.34)(10-\mu_1)(20-\mu_2) + (0.53)(20-\mu_2)^2$.

ii) $T^2 \sim \frac{(n-1)p}{n-p} F_{p,n-p} = \frac{(20-1)2}{20-2} F_{2,\,20-2}$.

$= \frac{19 \times 2}{18} F_{2,18} = \frac{19}{9} F_{2,18}$

$S = \frac{1}{n-1} (x_i-\bar{x})^T (x_i-\bar{x})$

$\rightarrow$ $T^2 = \sqrt{n}(\bar{x}-\mu)^T \left[\frac{1}{n-1} (x_i-\bar{x})(x_i-\bar{x})^T\right]^{-1} \sqrt{n}(\bar{x}-\mu)$

$= \begin{pmatrix} \text{multivariate} \\ \text{normal} \\ \text{vector} \end{pmatrix}^T \begin{pmatrix} \frac{\text{Wishart random matrix}}{\text{dof}} \end{pmatrix}^{-1} \begin{pmatrix} \text{multivariate} \\ \text{normal} \\ \text{vector} \end{pmatrix}$

$= N_p(0,\Sigma)^T \left[\frac{W_{p,n-1}(\Sigma)}{n-1}\right]^{-1} N_p(0,\Sigma)$.

Eg2: Let the data matrix for a random sample of size $n=3$ from bivariate normal distribution be.

$X = \begin{bmatrix} 6 & 9 \\ 10 & 6 \\ 8 & 3 \end{bmatrix}$. Evaluate the observed T^2 for $\mu_0' = [9, 5]$ $= 7/9$

What is the sampling distribution of T^2 in this case?

$4F_{2,1}$.

t-test we use can be done through Hotelling T^2

$\downarrow$

for Single population : $H_0 : \mu = \mu_0$

two independent polulation : $H_0 : \mu_1 = \mu_2$

two paired (dependent) " : $H_0 : \mu_1 = \mu_2$

2) let the data matrix for a random Sample of size $n=3$ from a bivariate population be $X = \begin{bmatrix} 6 & 9 \\ 10 & 6 \\ 8 & 3 \end{bmatrix}$

Evaluate T^2 for $\mu = [9, 5]$. What is the sampling distribution of T^2?

Soln:

$$\bar{X} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{6+10+8}{3} \\ \frac{9+6+3}{3} \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

$$\therefore (\bar{X} - \mu) = \begin{bmatrix} 8-9 \\ 6-5 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

we want $S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -3 & 9 \end{bmatrix}$

$$\therefore S^{-1} = \begin{bmatrix} 1/3 & 1/9 \\ 1/9 & 4/27 \end{bmatrix}$$

$$T^2 = 3 \begin{bmatrix} 8-9 & 6-5 \end{bmatrix} \begin{bmatrix} 1/3 & 1/9 \\ 1/9 & 4/27 \end{bmatrix} \begin{bmatrix} 8-9 \\ 6-5 \end{bmatrix} = 7/9.$$

Sample is selected from T^2 has the distribution

$$\frac{(3-1)2}{(3-2)} F_{2, 3-2} = 4 F_{2,1}$$

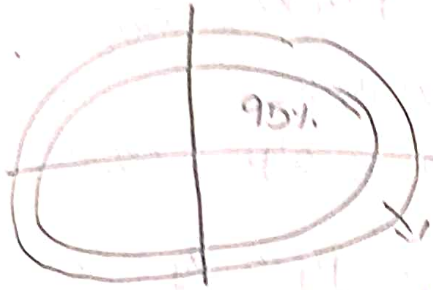
Confidence Region:-

$$T^2 = n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu) \Rightarrow$$ It is an eqn of ellipse in bivariate case.

$$\leq \frac{(n-1)p}{(n-p)} F_{p,n-p} = 1-\alpha$$

(i) $P\{ n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu)\} \leq \frac{(n-1)p}{(n-p)} F_{p,n-p}$

When we take $\alpha = 0.05 \Rightarrow 1-\alpha = 95\%$ means 95% observations within (or) on the ellipse. If I take $\alpha = 0.01$ then $1-\alpha = 99\%$



This is called confidence ellipse

If we imagine to higher dimension it is called ellipsoid

If all the variables have same variance then this forms a circle in 2-dimension & sphere in higher dimension.

Case i) Sampling from multivariate normal population with known Σ $n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu)$, χ^2_p, CR $\rightarrow$ $100(1-\alpha)\%$ $n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu) \leq \chi^2_p(\alpha)$

Case ii) Sampling from multivariate normal population with large sample size and unknown Σ. $n \geq 40$ | S is estimate of Σ |

$n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu)$, χ^2_p, CR $n(\bar{x}-\mu) S^{-1}(\bar{x}-\mu) \leq \chi^2_p(\alpha)$

Case iii) Sampling from m.N.P with small sample size & unknown Σ. $n-p \leq 40$

$n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu)$, F-distribn, $n(\bar{x}-\mu)^T S^{-1}(\bar{x}-\mu) \leq$ F distri.

Eg: from pbm 1:-

from F-distribution table $= \frac{19}{9} \times 3.555 = 7.51$

$$CR \Rightarrow (1.33)(10-\mu_1)^2 + (1.34)(10-\mu_1)(20-\mu_2) + (0.53)(20-\mu_2^2) \leq 7.51.$$

Distribution in Linear form:-

If X is multivariate normal & $a^T = [a_1, a_2 \dots a_p]$

Linear combination is $a^T X$

now $\bar{X} \sim N_p(\mu, \Sigma/n)$

$$\Rightarrow a^T \bar{X} \sim N\left(a^T\mu, \frac{a^T \Sigma a}{n}\right)$$ $\rightarrow$ variance

u can write l.c of $a^T\bar{x}$ & using property i.e will be univariate normal from $Z = \frac{X-\mu}{\sigma}$ W.K.T

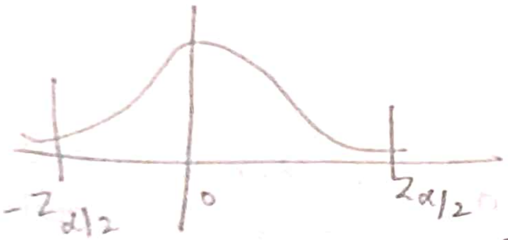
$E(a^T\bar{x}) = a^T E(\bar{x}) = a^T$

$\Rightarrow$ distribution $V(a^T\bar{x}) = \frac{a^T \Sigma a}{n}$.

$$Z = \frac{a^T\bar{x} - E(a^T\bar{x})}{\sqrt{V(a^T\bar{x})}} \sim Z(0,1)$$

i.e will be univariate normal.

$\therefore$ interval is $-Z_{\alpha/2} \le Z \le Z_{\alpha/2}$



$$\Rightarrow -Z_{\alpha/2} \le \frac{a^T\bar{x} - E(a^T\bar{x})}{\sqrt{V(a^T\bar{x})}} \le Z_{\alpha/2}$$

$$\Rightarrow a^T\bar{x} - Z_{\alpha/2}\sqrt{\frac{a^T\Sigma a}{n}} \le a^T\mu \le Z_{\alpha/2}\sqrt{\frac{a^T\Sigma a}{n}}$$

$a^T = [0, 0 \ldots 1 \ldots 0]$ (j^{th})

$$\Rightarrow \bar{x}_j - Z_{\alpha/2}\sqrt{\frac{S_{jj}}{n}} \le \mu_j \le \bar{x}_j + Z_{\alpha/2}\sqrt{}$$

Hypothesis testing:

Case i) Σ is known when null hypo is ~~true~~ must true } means this will not work for $\mu < \mu$

$H_0: \mu_0 = \mu_0$ (hypo)
$H_1: \mu \neq \mu_0$

} Test statistic $T^2 = n(\bar{x}-\mu_0)^T \Sigma^{-1}(\bar{x}-\mu_0)$ } Sampling distn of χ^2_p.

Decision :- $A \ge \chi^2_p(\alpha) \Rightarrow$ Rejected

Case ii) Σ is unknown, $n-p \ge 40$.

$H_0: \mu = \mu_0$
$H_1: \mu \neq \mu_0$

} $T^2 = n(\bar{x}-\mu_0)^T S^{-1}(\bar{x}-\mu_0)$ } χ^2_p.

Decision : $T^2 \ge \chi^2_p(\alpha) \Rightarrow$ Rejected.

Case iii) Σ is unknown, $n-p < 40$.

$H_0: \mu = \mu_0$
$H_1: \mu \neq \mu_0$

} $T^2 = n(\bar{x}-\mu_0)^T S^{-1}(\bar{x}-\mu_0)$ } $\frac{(n-1)p}{(n-p)} F_{p,n-p}$.

Decision $T^2 \ge F$ distribution $\Rightarrow$ Reject

Eg: In pbm 1:- hypo testing if we take $\mu_0 = \begin{bmatrix} 9 \\ 18 \end{bmatrix}$ (μ_1, μ_2)
Σ is unknown

T^2 = Calculated by sub $= 6.13$ & F distr $= 7.51$ (using table)

$\Rightarrow 6.13 <$ F-distribution

$\Rightarrow H_0$ is accepted

Same concept if we take for inference on multivariate statistics ⇒ Two population mean vectors.

in univariate case :-

→ two variables

x_1 x_2 Created a R.V = $\bar{x}_1 - \bar{x}_2$

↓ ↓

$\bar{x}_1$ $\bar{x}_2$ $E(\bar{x}_1 - \bar{x}_2) = \mu_1 - \mu_2$

s_1 s_2 $V(\bar{x}_1 - \bar{x}_2) = V(\bar{x}_1) + V(\bar{x}_2)$

σ_1 σ_2 $= \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

n_1 n_2 → samples

mean and deviation

↳ from this multivariate.

↳ R.V will be R.V $\bar{X}_A - \bar{X}_B$ in vector notation

↓

p variables

from 1st population $\bar{X}_A$ $\bar{X}_B$

S_A S_B → cov matrix for pop 1 & 2.

n_A n_B → Sample Size.

$\therefore E(\bar{X}_A - \bar{X}_B) = \mu_A - \mu_B$

$$= \begin{bmatrix} \mu_{A1} - \mu_{B1} \\ \mu_{A2} - \mu_{B2} \\ \vdots \\ \mu_{Ap} - \mu_{Bp} \end{bmatrix}$$

univ	multi
σ_A^2	→ Σ_A
σ_B^2	→ Σ_B

& $V(\bar{X}_A - \bar{X}_B) = V(\bar{X}_A) + V(\bar{X}_B)$

$$= \frac{\Sigma_A}{n_A} + \frac{\Sigma_B}{n_B}.$$

Now create $\overset{r.v.}{Z} = \frac{R.V - E(\;)}{\sqrt{\;\;}}$ univariate $Z = \frac{X - \mu}{\sqrt{V(x)}}$.

$$= \frac{\bar{X}_A - \bar{X}_B - \{\mu_A - \mu_B\}}{\sqrt{\Sigma_A/n_A + \Sigma_B/n_B}}$$ in multivariate domain

Case i) Sampling from multivariate normal population with known Σ_1 & Σ_2

Case ii) Sampling from multivariate normal populations with Small sample size & unknown (a) $\Sigma_1 = \Sigma_2 = \Sigma$ but equal Σ

Case iii) .... with large sample size with unknown Σ_1 & Σ_2

if We say $\sigma_1^2 = \sigma_2^2 = \sigma^2$

$$\hat{\sigma}^2 = S_p^2 = \frac{(n_1-1)S_1 + (n_2-1)S_2}{n_1+n_2-1} \quad \text{univariate}$$

$$\rightarrow \Sigma_A = \Sigma_B = \Sigma$$

$$\therefore S = \frac{(n_A-1)S_A + (n_B-1)S_B}{n_A+n_B-1} = \hat{\Sigma} \Big\} \text{ multivariate}$$

Pbm 1:- Age and risky behaviour are the two important factors that make difference between accident group (AG) & non accident group (NAG) of workers. Random samples of 20 individuals from AG & 50 individuals from NAG were collected. The sample mean vector & Samp cov matrix are given. construct 95% CR for the diff betn the two population mean vectors.

Sample - 1

$$\bar{X}_A = \begin{pmatrix} 50 \\ 6 \end{pmatrix} \quad S_A = \begin{pmatrix} 16 & -5 \\ -5 & 4 \end{pmatrix}$$

Sample - 2

$$\bar{X}_B = \begin{pmatrix} 40 \\ 8 \end{pmatrix}, \quad S_B = \begin{pmatrix} 25 & -6 \\ -6 & 9 \end{pmatrix}$$

injuried & non-injured → we find mean differences betn two.

Here $n_1 = 20$, $n_2 = 50$, Σ is unknown.

let $\Sigma_2 = \Sigma_1 = \Sigma$ [case(ii)]

$$\therefore T^2 = \left[(\bar{X}_A - \bar{X}_B) - (\mu_A - \mu_B)\right]^T \left[\left(\frac{1}{n_A} + \frac{1}{n_B}\right) S\right]^{-1} \left[(\bar{X}_A - \bar{X}_B) - (\mu_A - \mu_B)\right]$$

NOW

$$S = \frac{(n_A-1)S_A + (n_B-1)S_B}{n_A+n_B-1} = \frac{(20-1)\begin{pmatrix} 16 & -5 \\ -5 & 4 \end{pmatrix} + (50-1)\begin{pmatrix} 25 & -6 \\ -6 & 9 \end{pmatrix}}{20+50-1}$$

$$= \begin{pmatrix} 23.32 & -5.72 \\ -5.72 & 7.60 \end{pmatrix} \left(\frac{1}{20} + \frac{1}{50}\right)$$

$$\left[\left(\frac{1}{n_A} + \frac{1}{n_B}\right) S\right]^{-1} = \begin{pmatrix} 0.75 & 0.57 \\ 0.57 & 2.36 \end{pmatrix} \quad ?$$

Sampling distribution: $\frac{(n_A + n_B - 2)\, p}{(n_A + n_B) - p - 1} F_{p,\, n_A + n_B - p - 1}$

$$\text{CR}: P\left\{ 0.75\,(10-\theta_A)^2 + 2 \times 0.57\,(10-\theta_A)(-2-\theta_B) + 2.30\,(-2-\theta_B)^2 \leq F_{2,67}(0.05) = 3.15 \right\} = 0.95.$$