On Chapter 7 of *Laws of Form*

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1 Introduction

In chapter 7 of *Laws of Form* Spencer-Brown extends the scope of his basic equations to expressions with any finite number of variables. Some of his arguments, when he provides them, are rigorous; others are mere sketches, and some possible generalizations are left unmentioned. This paper will present fully rigorous proofs of the propositions.

Below is a list of axioms and theorems referenced in subsequent proofs:

$$\overline{\overline{pr}|\;\overline{qr}|}| = \overline{\overline{p}|\;\overline{q}|}|\,r \tag{J2}$$

$$\overline{pr}|\;\overline{qr}| = \overline{\overline{\overline{p}|\;\overline{q}|}|\,r}| \tag{J2.1}$$

$$\overline{\overline{a}|}| = a \tag{C1}$$

$$\overline{ab}|\,b = \overline{a}|\,b \tag{C2}$$

$$\overline{\overline{\overline{a}|\,b}|\;c}| = \overline{ac}|\;\overline{\overline{b}|\,c}| \tag{C7}$$

$$\overline{\overline{a}|\;\overline{br}|\;\overline{cr}|}| = \overline{\overline{a}|\;\overline{b}|\;\overline{c}|}|\;\overline{\overline{a}|\;\overline{r}|}| \tag{C8}$$

$$\overline{\overline{\overline{a}|\;\overline{r}|}|\;\overline{\overline{b}|\;\overline{r}|}|\;\overline{\overline{x}|\,r}|\;\overline{\overline{y}|\,r}|}| = \overline{\overline{r}|\,ab}|\;\overline{rxy}| \tag{C9}$$

$$\overline{\overline{\overline{a}|\;\overline{r}|}|\;\overline{\overline{x}|\;\overline{r}|}|}| = \overline{\overline{r}|\;a}|\;\overline{rx}| \tag{C9.1}$$

2 General theorems

Spencer-Brown begins the chapter by sketching an inductive generalization of J2. Here is the proof in full.

Theorem (J2*).

$$\overline{\overline{a_1}|\;\overline{a_2}|\;\ldots\;\overline{a_n}|}|\,r = \overline{\overline{a_1 r}|\;\overline{a_2 r}|\;\ldots\;\overline{a_n r}|}|$$

Proof. The proof proceeds by induction on n. The base case is J2, where $n = 2$. Let the induction hypothesis (J2h) be:

$$\overline{\overline{a_1}|\;\overline{a_2}|\;\ldots\;\overline{a_n}|}|\,r = \overline{\overline{a_1 r}|\;\overline{a_2 r}|\;\ldots\;\overline{a_n r}|}|$$

The induction step:

$$
\begin{aligned}
&\overline{\overline{a_1|}\ \overline{a_2|}\ \dots\ \overline{a_n|}\ \overline{a_{n+1}|}\,|}\ r \\
&= \overline{\overline{\overline{\overline{a_1|}\ \overline{a_2|}\ \dots\ \overline{a_n|}\,|}\,|}\ \overline{a_{n+1}|}\,|}\ r && \text{(C1)} \\
&= \overline{\overline{\overline{\overline{a_1|}\ \overline{a_2|}\ \dots\ \overline{a_n|}\,|}\ r\,|}\ \overline{a_{n+1}r|}\,|} && \text{(J2)} \\
&= \overline{\overline{\overline{\overline{a_1r|}\ \overline{a_2r|}\ \dots\ \overline{a_nr|}\,|}\,|}\ \overline{a_{n+1}r|}\,|} && \text{(J2h)} \\
&= \overline{\overline{a_1r|}\ \overline{a_2r|}\ \dots\ \overline{a_nr|}\ \overline{a_{n+1}r|}\,|} && \text{(C1)}
\end{aligned}
$$

□

Alternate proof. A very similar and equally short proof, using the same induction hypothesis as above. The induction step:

$$
\begin{aligned}
&\overline{\overline{a_1|}\ \overline{a_2|}\ \dots\ \overline{a_n|}\ \overline{a_{n+1}|}\,|}\ r \\
&= \overline{\overline{a_1|}\ \overline{a_2|}\ \dots\ \overline{\overline{\overline{\overline{a_n|}\ \overline{a_{n+1}|}\,|}\,|}\,|}\ r && \text{(C1)} \\
&= \overline{\overline{a_1r|}\ \overline{a_2r|}\ \dots\ \overline{\overline{\overline{a_n|}\ \overline{a_{n+1}|}\,|}\ r\,|}\,|} && \text{(J2h)} \\
&= \overline{\overline{a_1r|}\ \overline{a_2r|}\ \dots\ \overline{\overline{\overline{a_nr|}\ \overline{a_{n+1}r|}\,|}\,|}\,|} && \text{(J2)} \\
&= \overline{\overline{a_1r|}\ \overline{a_2r|}\ \dots\ \overline{a_nr|}\ \overline{a_{n+1}r|}\,|} && \text{(C1)}
\end{aligned}
$$

□

Before continuing, I prove a useful generalization of corollary J2.1.

Theorem (J2.1*).

$$\overline{a_1r|}\ \overline{a_2r|}\ \dots\ \overline{a_nr|} = \overline{\overline{\overline{a_1|}\ \overline{a_2|}\ \dots\ \overline{a_n|}\,|}\ r\,|}$$

Proof.

$$
\begin{aligned}
&\overline{a_1r|}\ \overline{a_2r|}\ \dots\ \overline{a_nr|} \\
&= \overline{\overline{\overline{a_1r|}\ \overline{a_2r|}\ \dots\ \overline{a_nr|}\,|}\,|} && \text{(C1)} \\
&= \overline{\overline{\overline{a_1|}\ \overline{a_2|}\ \dots\ \overline{a_n|}\,|}\ r\,|} && \text{(J2*)}
\end{aligned}
$$

□

Spencer-Brown states the generalizations of C8 and C9 but omits the proofs, merely noting that they are similar to J2*.

Theorem (C8*).

$$\overline{\overline{a|}\ \overline{b_1r|}\ \overline{b_2r|}\ \dots\ \overline{b_nr|}\,|} = \overline{\overline{a|}\ \overline{b_1|}\ \overline{b_2|}\ \dots\ \overline{b_n|}\,|}\ \overline{\overline{a|}\ \overline{r|}\,|}$$

Proof. The proof proceeds by induction on n. The base case is C8, where $n = 2$. Let the induction hypothesis (C8h) be:

$$\overline{\overline{a\rvert}\ \overline{b_1 r\rvert}\ \overline{b_2 r\rvert}\ \dots\ \overline{b_n r\rvert}\,\rvert} = \overline{\overline{a\rvert}\ \overline{b_1\rvert}\ \overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\,\rvert}\ \overline{\overline{a\rvert}\ \overline{r\rvert}\,\rvert}$$

The induction step:

$$\begin{aligned}
&\overline{\overline{a\rvert}\ \overline{b_1 r\rvert}\ \overline{b_2 r\rvert}\ \dots\ \overline{b_n r\rvert}\ \overline{b_{n+1} r\rvert}\,\rvert} \\
&= \overline{\overline{\overline{\overline{a\rvert}\ \overline{b_1 r\rvert}\,\rvert}\,\rvert}\ \overline{b_2 r\rvert}\ \dots\ \overline{b_n r\rvert}\ \overline{b_{n+1} r\rvert}\,\rvert} && \text{(C1)} \\
&= \overline{\overline{\overline{\overline{a\rvert}\ \overline{b_1 r\rvert}\,\rvert}\,\rvert}\ \overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\ \overline{b_{n+1}\rvert}\,\rvert}\ \overline{\overline{\overline{\overline{a\rvert}\ \overline{b_1 r\rvert}\,\rvert}\,\rvert}\ \overline{r\rvert}\,\rvert} && \text{(C8h)} \\
&= \overline{\overline{\overline{\overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\ \overline{b_{n+1}\rvert}\,\rvert}\ \overline{\overline{r\rvert}\,\rvert}\,\rvert}\ \overline{\overline{\overline{a\rvert}\ \overline{b_1 r\rvert}\,\rvert}\,\rvert}\,\rvert} && \text{(J2.1)} \\
&= \overline{\overline{\overline{\overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\ \overline{b_{n+1}\rvert}\,\rvert}\ r\rvert}\ \overline{a\rvert}\ \overline{b_1 r\rvert}\,\rvert} && \text{(C1 twice)} \\
&= \overline{\overline{\overline{\overline{\overline{\overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\ \overline{b_{n+1}\rvert}\,\rvert}\,\rvert}\ \overline{b_1\rvert}\,\rvert}\ r\rvert}\ \overline{a\rvert}\,\rvert} && \text{(J2.1)} \\
&= \overline{\overline{\overline{\overline{b_1\rvert}\ \overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\ \overline{b_{n+1}\rvert}\,\rvert}\ r\rvert}\ \overline{a\rvert}\,\rvert} && \text{(C1)} \\
&= \overline{\overline{\overline{\overline{b_1\rvert}\ \overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\ \overline{b_{n+1}\rvert}\,\rvert}\ \overline{\overline{r\rvert}\,\rvert}\,\rvert}\ \overline{a\rvert}\,\rvert} && \text{(C1)} \\
&= \overline{\overline{\overline{\overline{a\rvert}\ \overline{b_1\rvert}\ \overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\ \overline{b_{n+1}\rvert}\,\rvert}\ \overline{\overline{a\rvert}\ \overline{r\rvert}\,\rvert}\,\rvert}\,\rvert} && \text{(J2)} \\
&= \overline{\overline{a\rvert}\ \overline{b_1\rvert}\ \overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\ \overline{b_{n+1}\rvert}\,\rvert}\ \overline{\overline{a\rvert}\ \overline{r\rvert}\,\rvert} && \text{(C1)}
\end{aligned}$$

□

J2.1* allows for a quicker direct proof.

Alternate proof.

$$\begin{aligned}
&\overline{\overline{a\rvert}\ \overline{b_1 r\rvert}\ \overline{b_2 r\rvert}\ \dots\ \overline{b_n r\rvert}\,\rvert} \\
&= \overline{\overline{a\rvert}\ \overline{\overline{\overline{b_1\rvert}\ \overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\,\rvert}\ r\rvert}\,\rvert} && \text{(J2.1*)} \\
&= \overline{\overline{a\rvert}\ \overline{\overline{\overline{b_1\rvert}\ \overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\,\rvert}\ \overline{\overline{r\rvert}\,\rvert}\,\rvert}\,\rvert} && \text{(C1)} \\
&= \overline{\overline{\overline{\overline{a\rvert}\ \overline{b_1\rvert}\ \overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\,\rvert}\ \overline{\overline{a\rvert}\ \overline{r\rvert}\,\rvert}\,\rvert}\,\rvert} && \text{(J2)} \\
&= \overline{\overline{a\rvert}\ \overline{b_1\rvert}\ \overline{b_2\rvert}\ \dots\ \overline{b_n\rvert}\,\rvert}\ \overline{\overline{a\rvert}\ \overline{r\rvert}\,\rvert} && \text{(C1)}
\end{aligned}$$

□

Theorem (C9*).

$$\overline{\overline{\overline{a_1}\rvert\,\overline{r}\rvert}\rvert\;\overline{\overline{a_2}\rvert\,\overline{r}\rvert}\rvert\;\dots\;\overline{\overline{a_n}\rvert\,\overline{r}\rvert}\rvert\;\;\overline{\overline{x_1}\rvert\,r}\rvert\;\overline{\overline{x_2}\rvert\,r}\rvert\dots\overline{\overline{x_m}\rvert\,r}\rvert}\rvert$$
$$=\;\overline{\overline{r}\rvert\,a_1a_2\dots a_n}\rvert\;\overline{rx_1x_2\dots x_m}\rvert$$

Proof.

$$\overline{\overline{\overline{a_1}\rvert\,\overline{r}\rvert}\rvert\;\overline{\overline{a_2}\rvert\,\overline{r}\rvert}\rvert\;\dots\;\overline{\overline{a_n}\rvert\,\overline{r}\rvert}\rvert\;\;\overline{\overline{x_1}\rvert\,r}\rvert\;\overline{\overline{x_2}\rvert\,r}\rvert\dots\overline{\overline{x_m}\rvert\,r}\rvert}\rvert$$
$$=\;\overline{\overline{\overline{\overline{\overline{a_1}\rvert}\rvert\,\overline{\overline{a_2}\rvert}\rvert\,\dots\,\overline{\overline{a_n}\rvert}\rvert}\rvert\,\overline{r}\rvert}\rvert\;\overline{\overline{\overline{\overline{x_1}\rvert}\rvert\,\overline{\overline{x_2}\rvert}\rvert\dots\overline{\overline{x_m}\rvert}\rvert}\rvert\,r}\rvert}\rvert \qquad \text{(J2.1* twice)}$$
$$=\;\overline{\overline{\overline{a_1a_2\dots a_n}\rvert\,\overline{r}\rvert}\rvert\;\overline{\overline{x_1x_2\dots x_m}\rvert\,r}\rvert}\rvert \qquad \text{(C1 } n{+}m \text{ times)}$$
$$=\;\overline{\overline{r}\rvert\,a_1a_2\dots a_n}\rvert\;\overline{rx_1x_2\dots x_m}\rvert \qquad \text{(C9.1)}$$

□

Next we prove a generalizion of C2.

Theorem (C2*).

$$\overline{\overline{\overline{\overline{a_nb}\rvert\,\dots}\rvert\,a_2}\rvert\,a_1}\rvert\,b=\overline{\overline{\overline{\overline{a_n}\rvert\,\dots}\rvert\,a_2}\rvert\,a_1}\rvert\,b$$

Proof. The proof proceeds by induction on n. The base case is C2, where $n=1$. Let the induction hypothesis be:

$$\overline{\overline{\overline{\overline{a_nb}\rvert\,\dots}\rvert\,a_2}\rvert\,a_1}\rvert\,b=\overline{\overline{\overline{\overline{a_n}\rvert\,\dots}\rvert\,a_2}\rvert\,a_1}\rvert\,b$$

Substitute $\overline{a_{n+1}b}\rvert\,a_n$ for a_n . The induction step then follows immediately:

$$\overline{\overline{\overline{\overline{\overline{a_{n+1}b}\rvert\,a_nb}\rvert\,\dots}\rvert\,a_2}\rvert\,a_1}\rvert\,b=\overline{\overline{\overline{\overline{\overline{a_{n+1}b}\rvert\,a_n}\rvert\,\dots}\rvert\,a_2}\rvert\,a_1}\rvert\,b$$

□

Spencer-Brown does not mention a generalized C7. Here is one possible version.

Theorem (C7*). *Let n be a positive even number. Then for all such n the following pair of equations holds:*

(i) $\overline{\overline{\overline{\overline{a_n}\rvert\,\dots}\rvert\,a_2}\rvert\,a_1}\rvert \;=\; \overline{\overline{a_n}\rvert\,a_{n-1}\dots a_3a_1}\rvert\;\dots\;\overline{\overline{a_4}\rvert\,a_3a_1}\rvert\;\overline{\overline{a_2}\rvert\,a_1}\rvert$

(ii) $\overline{\overline{\overline{\overline{\overline{a_{n+1}}\rvert\,a_n}\rvert\,\dots}\rvert\,a_2}\rvert\,a_1}\rvert \;=\; \overline{a_{n+1}a_{n-1}\dots a_3a_1}\rvert\;\overline{\overline{a_n}\rvert\,a_{n-1}\dots a_3a_1}\rvert\;\dots\;\overline{\overline{a_4}\rvert\,a_3a_1}\rvert\;\overline{\overline{a_2}\rvert\,a_1}\rvert$

Proof. Let equation (i) be the induction hypothesis. The base case is the identity $\overline{\overline{a_2}\rceil\, a_1}\rceil = \overline{\overline{a_2}\rceil\, a_1}\rceil$, where $n = 2$. Now substitute $\overline{a_{n+1}}\rceil\, a_n$ for a_n. Then,

$$
\begin{aligned}
&\overline{\overline{\overline{\overline{\overline{a_{n+1}}\rceil\, a_n}\rceil \dots}\rceil\, a_2}\rceil\, a_1}\rceil \\
&= \overline{\overline{\overline{a_{n+1}}\rceil\, a_n}\rceil\ a_{n-1} \dots a_3a_1}\rceil \dots \overline{\overline{a_4}\rceil\, a_3a_1}\rceil\ \overline{\overline{a_2}\rceil\, a_1}\rceil && \text{(i)} \\
&= \overline{a_{n+1}a_{n-1} \dots a_3a_1}\rceil\ \overline{\overline{a_n}\rceil\, a_{n-1} \dots a_3a_1}\rceil \dots \overline{\overline{a_4}\rceil\, a_3a_1}\rceil\ \overline{\overline{a_2}\rceil\, a_1}\rceil && \text{(C7)}
\end{aligned}
$$

proving the implication from (i) to *(ii)*. In equation (ii) substitute $\overline{a_{n+2}}\rceil\, a_{n+1}$ for a_{n+1}. Then,

$$
\overline{\overline{\overline{\overline{\overline{a_{n+2}}\rceil\, a_{n+1}}\rceil \dots}\rceil\, a_2}\rceil\, a_1}\rceil = \overline{\overline{a_{n+2}}\rceil\, a_{n+1} \dots a_3a_1}\rceil \dots \overline{\overline{a_4}\rceil\, a_3a_1}\rceil\ \overline{\overline{a_2}\rceil\, a_1}\rceil \qquad \text{(ii)}
$$

proving (i) for the succeeding even number. This proves the proposition for all $n \geq 2$, and hence for all echelons of depth greater than or equal to 2. □

Theorem (T14). *Any expression can be reduced to an equivalent expression not more than two crosses deep. Specifically, any expression E is equivalent to $\overline{\overline{a_1}\rceil\, b_1}\rceil\ \overline{\overline{a_2}\rceil\, b_2}\rceil \dots \overline{\overline{a_n}\rceil\, b_n}\rceil\ \overline{c_1}\rceil\ \overline{c_2}\rceil \dots \overline{c_m}\rceil\ d$ where a_i, b_i, c_i, d are composed (at most) of juxtapositions of variables and the two constants, $\overline{\ \ }\rceil$ and .*

Proof. Repeated applications of **C7*** to any expression demonstrates the theorem. Spencer-Brown uses **C7** (not having proven a generalization), but it comes to the same thing. □

The final theorem follows Spencer-Brown closely.

Theorem (T15). *Given any expression E and any variable v, E can be reduced to an equivalent expression containing not more than two appearances of v.*

Proof. In the case where v is not in E, the theorem is trivially true, since $E = \overline{\overline{v}\rceil\, v}\rceil\, E$ by **J1**. So let us suppose that v appears in E. Using **C7*** as many times as necessary, we rewrite E:

$$
E = \overline{\overline{va_1}\rceil\, b_1}\rceil\ \overline{\overline{va_2}\rceil\, b_2}\rceil \dots \overline{\overline{va_n}\rceil\, b_n}\rceil\ \overline{vc_1}\rceil\ \overline{vc_2}\rceil \dots \overline{vc_m}\rceil\ d
$$

where a_i, b_i, c_i, and d are expressions free of v. Then, by n applications of **C8.1**,

$$
\begin{aligned}
E &= \overline{\overline{v}\rceil\, b_1}\rceil\ \overline{\overline{a_1}\rceil\, b_1}\rceil\ \overline{\overline{v}\rceil\, b_2}\rceil\ \overline{\overline{a_2}\rceil\, b_2}\rceil \dots \overline{\overline{v}\rceil\, b_n}\rceil\ \overline{\overline{a_n}\rceil\, b_n}\rceil\ \overline{vc_1}\rceil\ \overline{vc_2}\rceil \dots \overline{vc_m}\rceil\ d \\
&= \overline{\overline{v}\rceil\, b_1}\rceil\ \overline{\overline{v}\rceil\, b_2}\rceil \dots \overline{\overline{v}\rceil\, b_n}\rceil\ \overline{vc_1}\rceil\ \overline{vc_2}\rceil \dots \overline{vc_m}\rceil\ f \\
&\qquad \text{(where } f = \overline{\overline{a_1}\rceil\, b_1}\rceil\ \overline{\overline{a_2}\rceil\, b_2}\rceil \dots \overline{\overline{a_n}\rceil\, b_n}\rceil\ d \text{ is free of } v.) \\
&= \overline{\overline{\overline{b_1}\rceil\ \overline{b_2}\rceil \dots \overline{b_n}\rceil}\rceil\ \overline{v}\rceil}\rceil\ \overline{\overline{\overline{c_1}\rceil\ \overline{c_2}\rceil \dots \overline{c_m}\rceil}\rceil\ v}\rceil\ f && \text{(J2.1* twice)}
\end{aligned}
$$

□