23MAT112
Class Notes

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Contents

Chapter 1

Eigenvalues And Eigenvectors

1.1 Introduction

Reference - **Learning from Data, Gilbert Strang as well as Chapter 6 of Introduction To Linear Algebra**

$$A\vec{x} = \lambda\vec{x}$$

This equation is a mathematical way of expressing the idea that there's some vector $\vec{x}$ that does not change in direction, but only changes in size or magnitude by some factor λ.

Definition 1.1.1

Eigenvector A vector that does not change in direction after a linear transformation.

Definition 1.1.2

Eigenvalue The scalar factor by which an eigenvector changes after a linear transformation

1.2 Example Matrices

1.2.1 Projection Matrices

A reminder, projection matrices are used to bring vectors outside the column space of a given matrix A to the column space of A.

$$Ax = b$$

3 Cases

When $\vec{x}$ is in the column space,

$$Px = \lambda x, where \lambda = 1$$

When $\vec{x}$ is out of the column space,

It is not an eigen vector

When $\vec{x}$ is orthogonal to the column space,

$$Px = \lambda x, where \lambda = 0$$

1.2.2 Permutation Matrix

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$$

The eigenvectors for this matrix is:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} and \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

1.2.3 Rotation matrix

$$R = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

For $\theta = \frac{\pi}{2}$

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The calculated eigenvalues for this matrix is $\pm i$

Note:-

A matrix is the representation of a linear transformation in a given basis

$$L(\alpha a + \beta b) = \alpha L(a) + \beta L(b)$$

Question 1

B is 3×3 matrix with eigenvalues $\lambda = 0, 1, 2$.

1. What is the rank of B?
2. What is the determinant of B?

Ans. 1. - The rank of B is 2. There is one distinct eigenvalue which is 0. This means that the nullspace is one-dimensional. The nullspace is (n-r) dimensions, therefore the rank of the matrix is 2.
Ans. 2. - The determinant is 0, since there exists an eigenvalue which is 0, which means that

$$det(A - \lambda I) = 0, \text{where } \lambda \text{ is } 0$$

1.3 Diagonalizable Matrices

Definition 1.3.1: Diagonalizable Matrices

Let A and B be two square matrices of size $n \times n$. We say that A and B are similar if there is an invertible matrix of the same size P such that:

$$A = PBP^{-1}$$

Then we can say that A is **diagonalizable** if A is similar to a diagonal matrix D

Lemma 1.3.1. Suppose that A and B are two $n \times n$ matrices and P is an invertible matrix, such that $A = PBP^{-1}$. Then, $A^n - PB^nP^{-1}$

Proof. Using the principle of mathematical induction.
We are given, $A = PBP^{-1}$
to show, $A^n = PB^nP^{-1}$
Base step - $n = 1, A^1 = P^1B^1P^{-1}$, which is true
Induction step - Supppose $A^n = PB^nP^{-1}$. We need to show that $A^{n+1} = PB^{n+1}P^{-1}$

$$A^{n+1} = A\dot{A}^n$$

$$= (PBP^{-1})(PB^nP^{-1})$$
$$= PBB^nP^{-1} = PB^{n+1}P^{-1}$$

□

Theorem 1.3.1. Let A be an $n \times n$ matrix and let $v_1, v_2, \ldots, v_k$ be eigenvectors of A with distinct eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_k$. Then $v_1, v_2 \ldots v_k$ are independent. In particular, if k = n, then $v_1, v_2 \ldots v_k$ are a basis of eigenvectors for $\mathbb{R}^n$

Proof. Suppose $v_1, v_2 \ldots v_n$ are dependent such that $\exists r_i$ such that,

$$\sum_{i=1}^{k} r_i v_i = 0 \tag{1.1}$$

Assume that k is minimal with this property and r_k is all non-zero,

$$A.0 = r_1 Av_1 + r_2 Av_2 \cdots + r_k Av_k$$

$$0 = r_1\lambda_1 v_1 + r_2\lambda_2 v_2 \ldots r_k\lambda_k v_k = 0 \tag{1.2}$$

$$\lambda_k \times 1.1 : r_1\lambda_k v_1 + \cdots + r_k\lambda_k v_k = 0 \tag{1.3}$$

1.3 - 1.2

$$r_1(\lambda_k - \lambda_1)v_1 + \cdots + r_{k-1}(\lambda_k - \lambda_{k-1})v_{k-1} = 0$$

$\lambda_k - \lambda_i$ is non-zero(distinct eigenvalues)
This forms a linear combination of vectors $v_1, v_2 \ldots v_{k-1}$ that equate to zero.
This contradicts the assumption that $v_1, v_2 \ldots v_{k-1}$ is the minimal set which is dependent.
Therefore the eigenvectors are independent. □

Theorem 1.3.2. Let A be a n × n matrix, Then A is diagonalizable if and only if we can find a basis $v_1, \cdots, v_n$ of eigen vectors for R^n. In this case,

$$A = PDP^1 \tag{1.4}$$

where P is the matrix whose eigenvectors $v_1, \cdots, v_n$ and D is the diagonal matrix whose diagonal entries are the corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$

Proof.

$$Av_i = (PDP^{-1}v_i)$$

where P is the matrix with column vectors $v_1, v_2 \cdots v_n$ and D is a diagonal matrix with the entries $\lambda_1, \cdots, \lambda_n$

$$Av_i = (PDP^{-1})P\hat{e}_i$$

$$Av_i = PD\hat{e}_i$$

$$Av_i = P\lambda_i\hat{e}_i = \lambda_i P\hat{e}_i = \lambda_i v_i$$

This proves that v_i is the eigenvector of A, and that λ_i is the corresponding eigenvalue. Because P^{-1} exists, $v_1, \cdots, v_n$ are independent, based on the theorem proven earlier, this is the basis for R^n
Part b Suppose $v_1, \cdots, v_n$ are an eigenvector basis with corresponding eigenvalues $\lambda_1, \cdots, \lambda_n$ Suppose that P is the matrix with column vectors $v_1, \cdots, v_n$

$$\text{Let} D = P^{-1}AP$$

$$D\hat{e}_i = (P^{-1}AP)\hat{e}_i$$

$$= P^{-1}Av_i = \lambda_i P^{-1}v_i = \lambda_i \hat{e}_i$$

Thus D is the diagonal matrix with the diagonal entries λ_i □

1.4 Symmetric Matrices

$$A = A^T$$

A symmetric matrix has real eigenvalues and orthogonal eigenvectors.

Example.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

The eigenvalues for the matrix is 2 and 4 The eigenvectors are:

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} and \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

These eigenvectors are orthogonal.

Definition 1.4.1

Hermitian Matrices A complex matrix in which all the entries in the given matrix are equal to their corresponding conjugate transposes.

Theorem 1.4.1. The eigenvalues for a symmetric matrix are real.

Proof. The claim:

$$For Ax = \lambda x, A = A^T$$
$$\lambda \in R$$

Take:

$$Ax = \lambda x$$

And the complex conjugate:

$$\overline{A}\overline{x} = \overline{\lambda}\overline{x} \qquad (\overline{A} = A)(realmatrix)$$
$$(\overline{A}\overline{x})^T = (\overline{\lambda}\overline{x})^T \Leftrightarrow \overline{x}^T A^T = \overline{\lambda}\overline{x}^T \tag{1.5}$$

1.5 × x

$$\overline{x}^T A^T x = \lambda \overline{x^T} x \Leftrightarrow \overline{x}^T Ax = \overline{\lambda}\overline{x}^T x \Leftrightarrow \overline{x}^T \lambda x = \overline{\lambda}\overline{x}^T x \Leftrightarrow \lambda = \overline{\lambda}$$

Then λ is real, when $\overline{x}^T x \neq 0$ □

Lemma 1.4.1. Let A be a symmetric matrix. If v and w are eigenvectors with distinct eigenvalues $\lambda \& \mu$ then v & w are orthogonal.

Proof.

$$Av.w = (Av)^T w = v^T A^T w = v^T Aw = v.Aw$$
$$\Leftrightarrow \lambda v.w = \mu v.w \Leftrightarrow (\lambda - \mu)v.w = 0$$
$$\Leftrightarrow v.w = 0$$

□

What this means is that, when A is symmetric

$$A = PDP^{-1} = PDP^T \qquad (P^{-1} = P^T \text{for orthogonal matrices})$$

Theorem 1.4.2. Let A be a symmetric matrix. Then we can find a diagonal matrix D and an orthogonal matrix P such that,

$$A = PDP^T$$

In particular, every symmetric matrix is diagonalisable.