23MAT204

Mathematics for Intelligent Systems - 3

CONSTRAINED OPTIMIZATION

FORMULATION OF FEW CONSTRAINED OPTIMIZATION PROBLEMS

Some Optimization Problems

Multivariable, Constrained, Linear, Discrete, Deterministic

- Two Products - Bulbs and Batteries
- Each requires two raw materials - P and Q
- One unit of bulb needs
  - 2 units of P, 4 units of Q
- One unit of battery needs
  - 3 units of P, 2 units of Q
- Raw material availability is limited
  - **24 units of P, 32 units of Q**
- Each product yields different profit/unit
  - bulb: $4 per unit
  - battery: $5 per unit

Variables:
X units of bulbs
Y units of batteries
Objective: Maximize $4x+5y$

	P	Q
Bulb	2	4
Battery	3	2
Avail	24	32

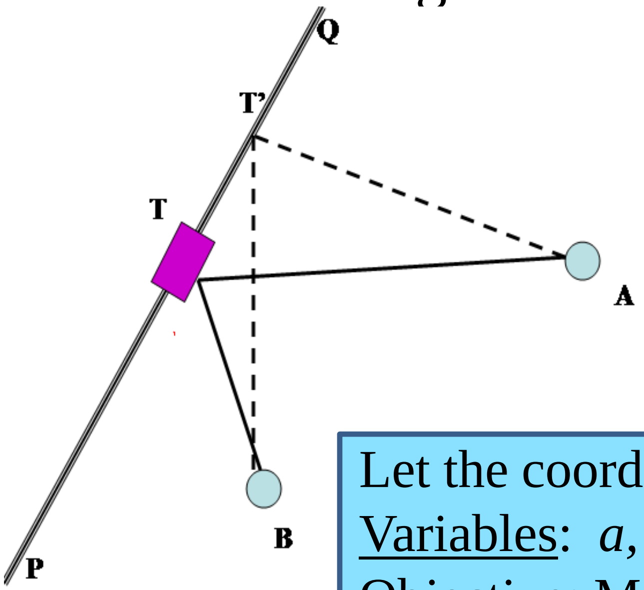
Constraints:
Usage of P : $2x+3y \leq 24$
Usage of Q : $4x+2y \leq 32$

$x \geq 0, y \geq 0$, x,y positive integers

Problem: How much of each product should be produced so that total Profit is maximized?

Some Optimization Problems Contd…

- **Bus Terminus Location Problem**: Find the location of the bus terminus T on the road segment PQ such that the lengths of the roads linking T with the two cities A and B is minimum.



Find the coordinates of the point T on the line PQ, x + y = 5, such that the lengths of the line segments AT and BT is minimum, Given, coordinate of A is (1,2) and B is (3,1).

Let the coordinates of the point T be (a, b)

Variables: a, b

Objective: Minimize the distance AT + BT

i.e. Minimize $\sqrt{(a-1)^2+(b-2)^2} + \sqrt{(a-3)^2+(b-1)^2}$

subject to $a + b = 5$

- Multivariable, Constrained, Non-linear, Continuous, Deterministic

Some Optimization Problems Contd…

- **Diet Problem:** Propose a diet containing at least 2,000 (Kcal), at least 55 grams of protein and 800 (mg) of calcium with reference to the given table and additionally at minimum cost.

Food	Portion Size	Energy (Kcal)	Proteins (grams)	Calcium (mg)	Price ($/portion)	Limit (portions/day)
Oats	28 g	110	4	2	30	4
Chicken	100 g	205	32	12	240	3
Eggs	2 big ones	160	13	54	130	2
Milk	237 cc	160	8	285	90	8
Kuchen	170 g	420	4	22	200	2
Beans	260g	260	14	80	60	2

- **Variables:** Let X_i be the portion of food i to eat during a day

  (i=1 for oats, i=2 for chicken, i=3 for egg, i=4 for milk, i=5 for kuchen, i=6 for beans)

Diet Problem continued…

Food	Portion Size	Energy (Kcal)	Proteins (grams)	Calcium (mg)	Price ($/portion)	Limit (portions/day)
Oats	28 g	110	4	2	30	4
Chicken	100 g	205	32	12	240	3
Eggs	2 big ones	160	13	54	130	2
Milk	237 cc	160	8	285	90	8
Kuchen	170 g	420	4	22	200	2
Beans	260g	260	14	80	60	2

Minimize $30X_1 + 240\,X_2 + 130\,X_3 + 90\,X_4 + 200\,X_5 + 60\,X_6$

s.t. $110\,X_1 + 205\,X_2 + 160\,X_3 + 160\,X_4 + 420\,X_5 + 260\,X_6 \geq 2000$

$4\,X_1 + 32\,X_2 + 13\,X_3 + 8\,X_4 + 4\,X_5 + 14\,X_6 \geq 55$

$2\,X_1 + 12\,X_2 + 54\,X_3 + 285\,X_4 + 22\,X_5 + 80\,X_6 \geq 800$

$X_1 \leq 4$;

$X_2 \leq 3$;

$X_3 \leq 2$;

$X_4 \leq 8$;

$X_5 \leq 2$;

$X_6 \leq 2$;

$X_i \geq 0$ for i = 1, 2, 3, 4, 5, 6

Multivariate, Constrained, Linear, Continuous, Deterministic

Some Optimization Problems Contd…

A gear manufacturing company received an order for three specific types of gears for regular supply. The management is considering to devote the available excess capacity to one or more of the types, say A, B and C. The available capacity on the machines which might limit the output and the number of machines hours required for each unit of the respective gear is also given below:

M/C type	Available M/C hours/week	Productivity in M/C hours/unit		
		Gear A	Gear B	Gear C
Gear Hobbling M/C	250	8	2	3
Gear Shaping M/C	150	4	3	0
Gear Grinding M/C	50	2	4	1

The unit profit would be Rs. 20, Rs. 6 and Rs. 8 respectively for the gears A, B and C. Formulate the model to find how much of the gear the company should produce in order to maximize the profit?

Some Optimization Problems Contd…

- **Portfolio Optimization**
  - **Variables:** amounts to be invested in different assets
  - **Objective:** Minimize the overall risk or return variance
  - **Constraints:** budget, max/min investment per asset, minimum return

- **Development of device in electronic circuit**
  - **Variables:** device width and length
  - **Objective:** Minimize power consumption
  - **Constraints:** manufacturing limits, timing requirements, maximum area

Formulation Exercise:

1. Write a mathematical model to find the dimensions of a cylindrical tin (with top and bottom) made up of sheet metal to maximize its volume such that the total surface area is equal to 50 sqm. Classify the model based on all five classifications.
2. A firm manufactures two products A and B on which the profit earned per unit are Rs.3 and Rs.4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and two minutes on M_2 , while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 7 hours and 30minutes, while machine M_2 is available for 10 hours. Formulate the problem as a mathematical model, if the objective is to maximize the profit.

Formulation Exercise:

3. A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of Aspirin, 5 grains of bicarbonate and 1 grain of codein. Size B contains 1 grain of aspirin, 8 grains of bicarbonate and 6 grains of codein. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codein for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem.
4. The final product of a firm has a requirement that it must weigh exactly 150 kg. The two raw materials used are product A with cost of Rs.2/unit and B with a cost of Rs.8/unit. At least 14 units of B and not more than 20 units of A must be used. Each unit of A weighs 5kg and that of B weighs 10kg. Formulate the problem so as to know how much of each type of raw materials should be used for each unit if the cost is to be minimized.

Formulation Exercise:

5. A balanced diet for a five year old boy should contain at least 1400 Kcal of energy, 75 grams of proteins and 800 mg of calcium per day. With reference to the given table, formulate a mathematical model to decide the diet of a five year old boy, with cost minimization as the objective.

Food	Portion size	Energy (Kcal)	Proteins (gram)	Calcium (mg)	Price (Rs./portion)	Limit portion/ day
Rice	150 gram	380	14	19	25	3
Wheat	150 gram	295	21	33	42	2
Milk	250 ml	160	11	285	20	4
Dal	250 gram	180	23	52	165	2

Methods of Solving Optimization Problems

- Graphical method
- Analytical methods (or classical methods)
- Numerical Methods

Graphical method

– Unconstrained or Constrained
– Optimization problems with one and two variables only

Multivariable Constrained Optimization Problems

- Graphical method
- Analytical methods
- Numerical methods (Next semester-ADMM)

Graphical Method for solving constrained multivariable optimization problems

Steps involved:

1. Obtain the feasible region – the region consisting of all the points that satisfy all the constraints.
2. Using Objective contours find the point(s) in the feasible region that gives the minimum or maximum value for the objective function

Graphical Method – Linear Optimization Problems

1. **Maximize x + y**

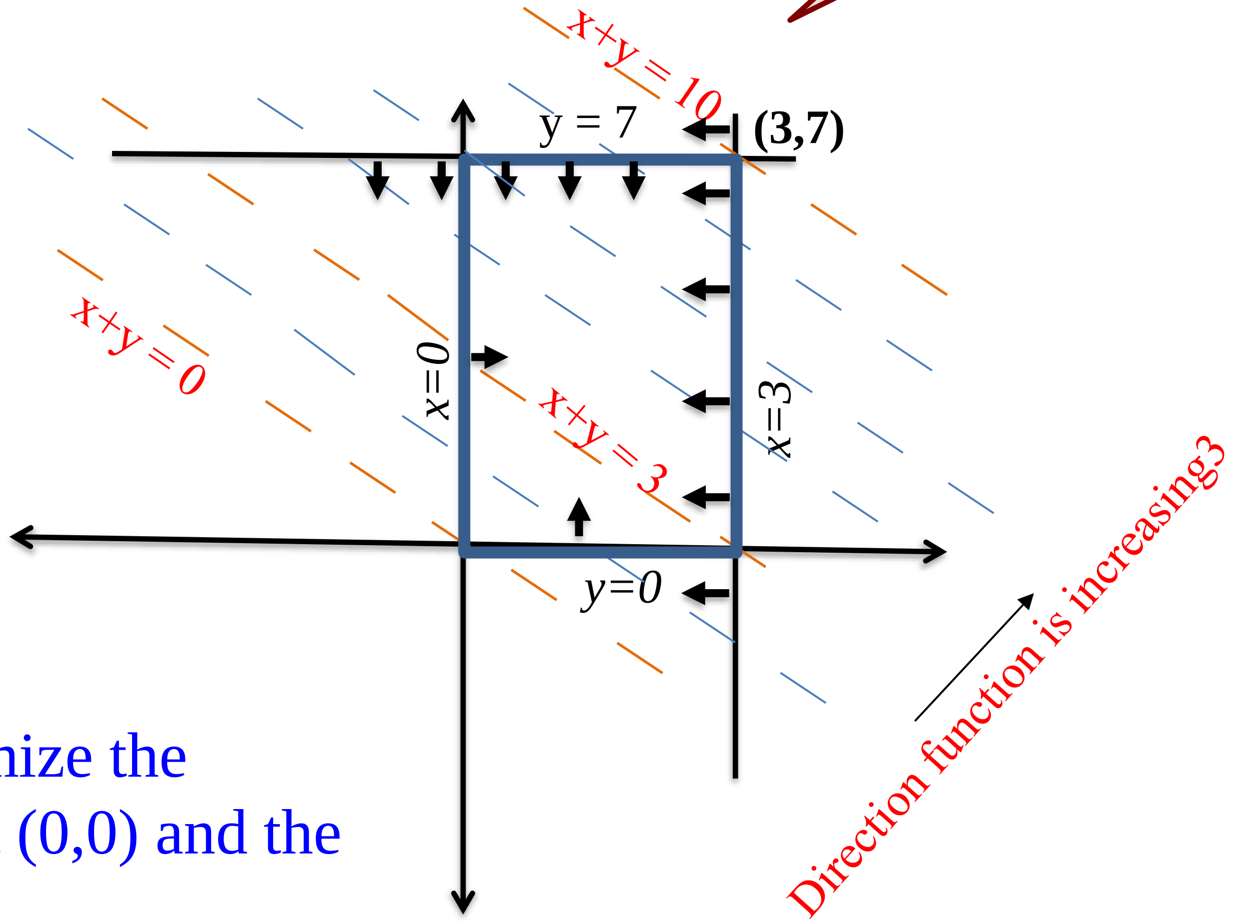
subject to $x \le 3$,

$y \le 7$,

$x \ge 0$,

$y \ge 0$.

Unique solution



Maximum value is 10 at the point (3,7)

If the question was minimize the minimum point will be at (0,0) and the minimum value is 0

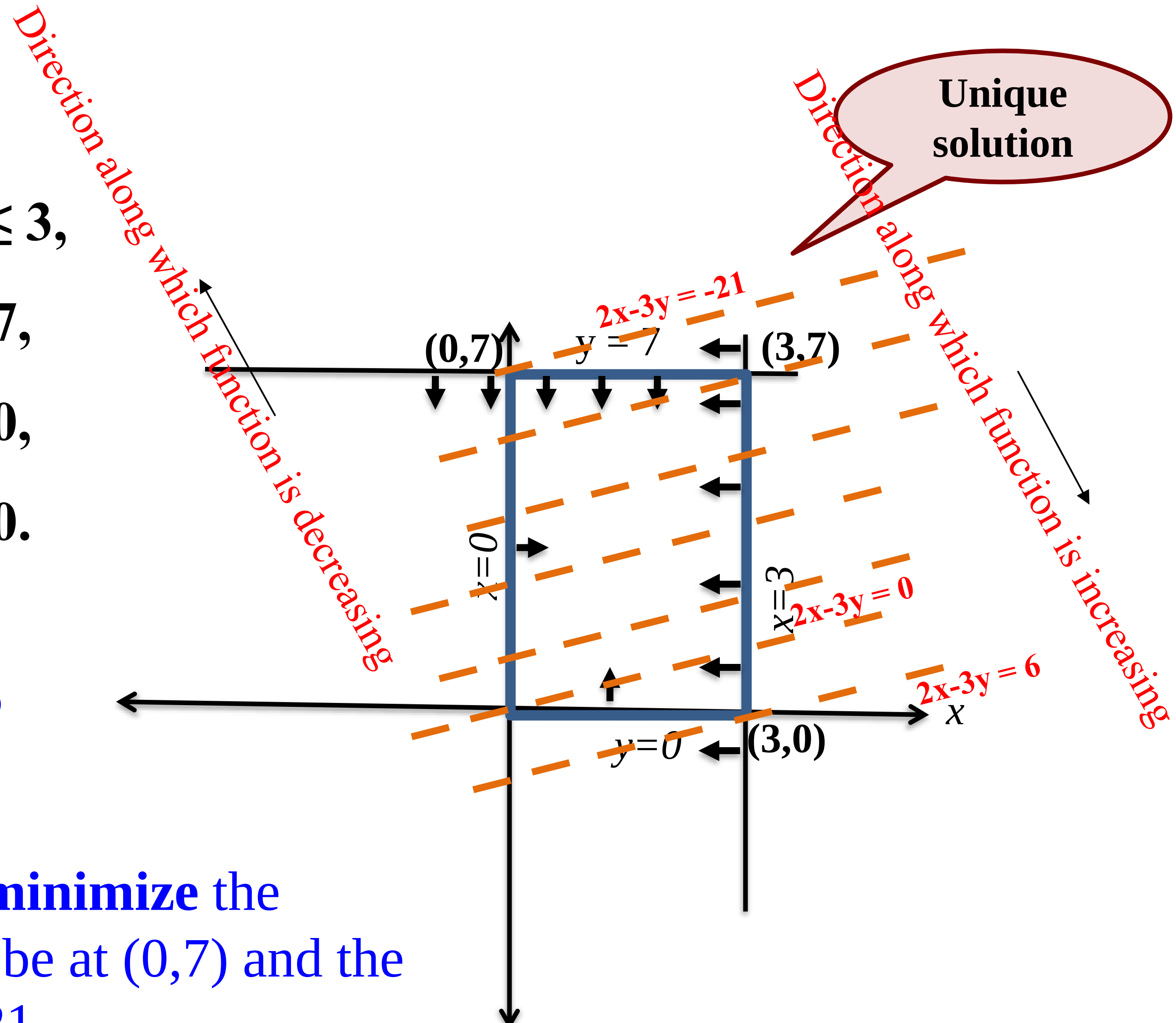
2. **Maximize 2x - 3y**

subject to $x \leq 3$,

$y \leq 7$,

$x \geq 0$,

$y \geq 0$.



Maximum value is 6
at the point (3,0)

If the question was **minimize** the minimum point will be at (0,7) and the minimum value is -21

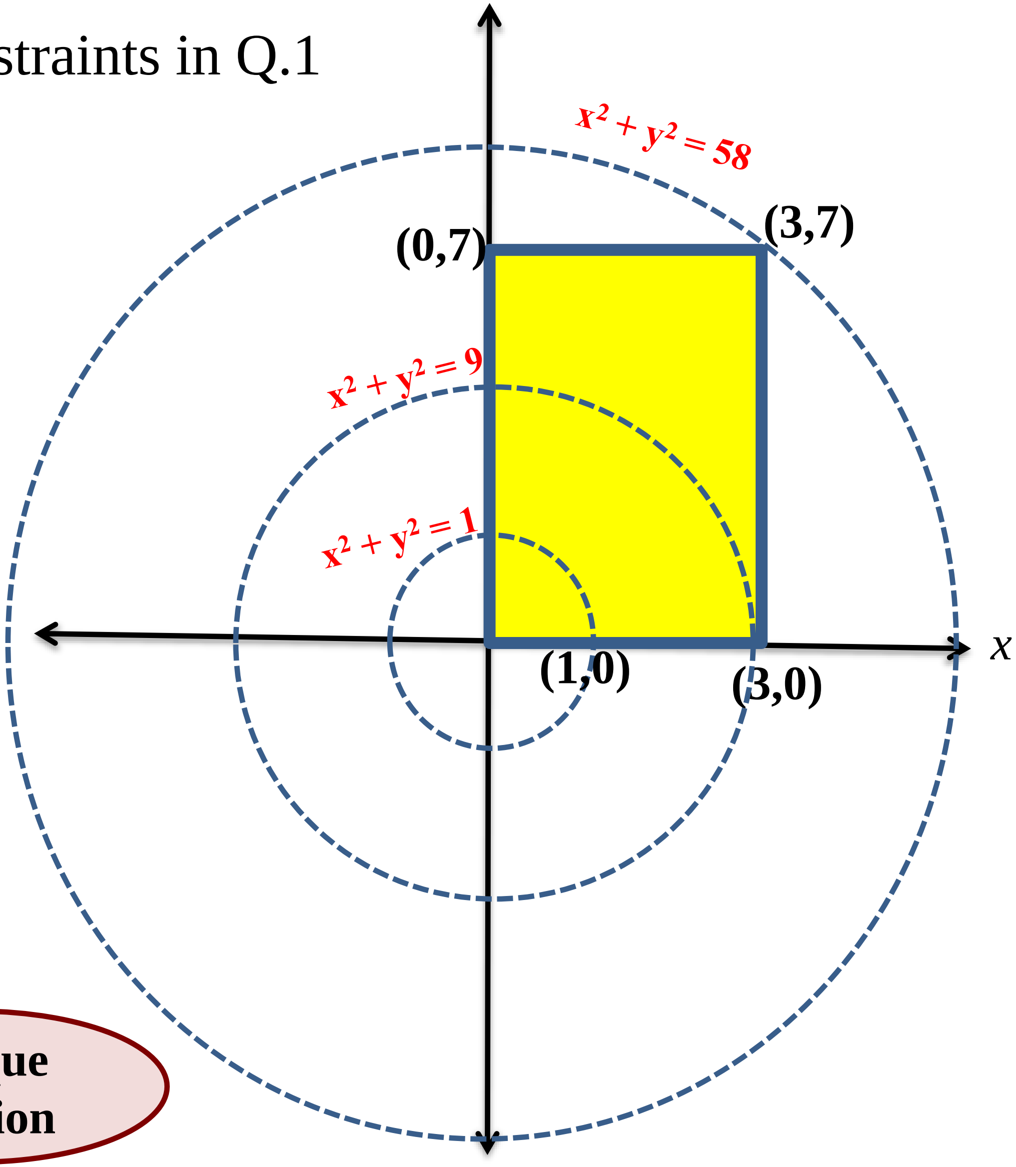
Graphical Method

3. Maximize $x^2 + y^2$ for constraints in Q.1

Maximum value is 58
at the point (3,7)

If the question was **minimize** the minimum point will be at (0,0) and the minimum value is 0

Unique solution



Graphical Method

4. Maximize 3y

subject to $x \le 3$,

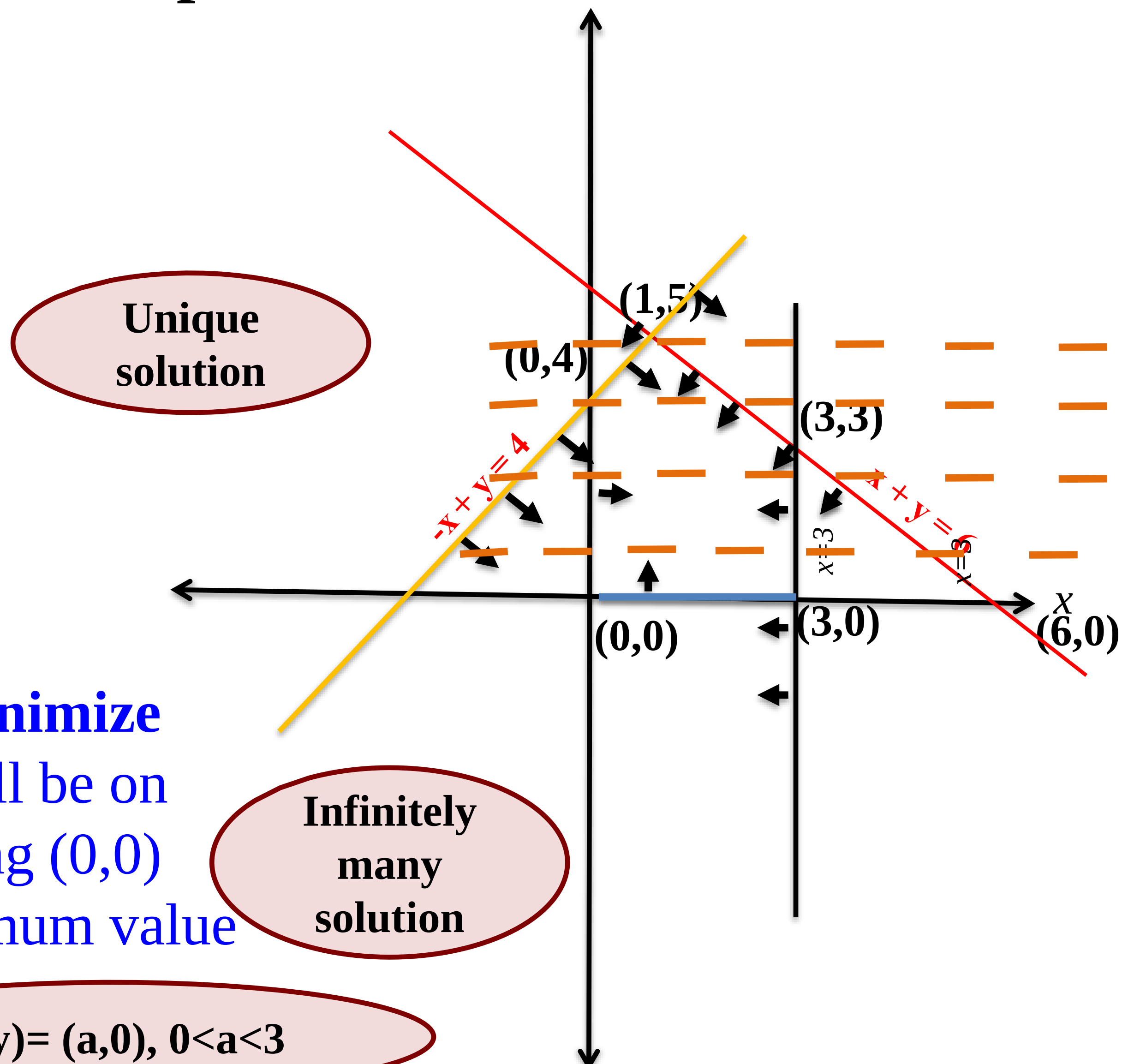
$-x+y \le 4$,

$x+y \le 6$

$x \ge 0, \ y \ge 0$.

Maximum value is 15
at the point (1,5)

If the question was **minimize** the minimum point will be on the line segment joining (0,0) and(3,0) and the minimum value is 0.



(x,y)= (a,0), 0<a<3

Graphical Method

5. Maximize x+y

subject to $x \le 3$,

$-x+y \le 4$,

$x+y \le 6$

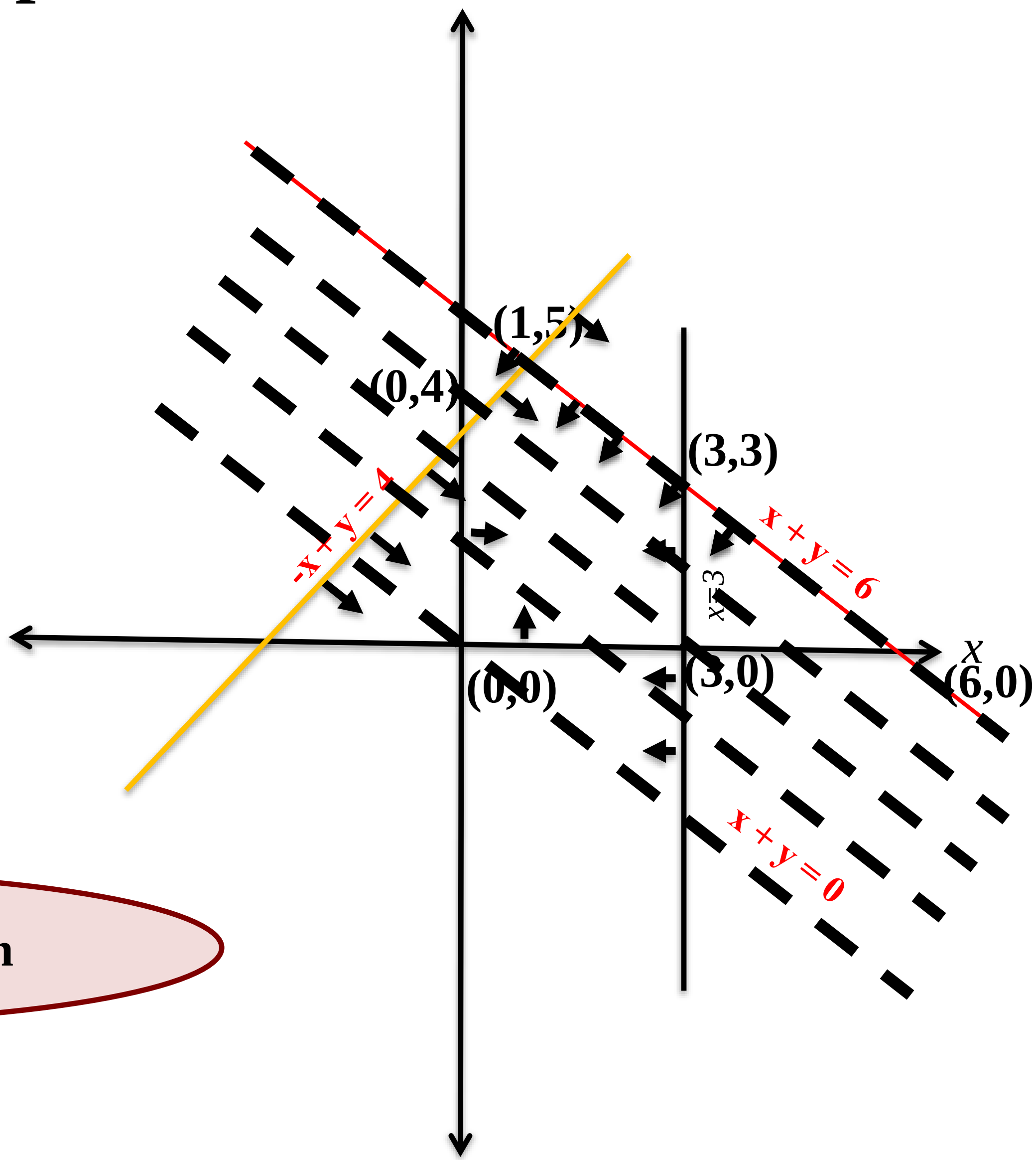
$x \ge 0,\ y \ge 0$.

Maximum value is 6 at every point on the line segment joining (1,5) and (3,3)

$(x,y)=(1-a)(1,5)+a(3,3)$

$=(1+2a,\ 5-2a),\ 0<a<1$

Infinitely many solution

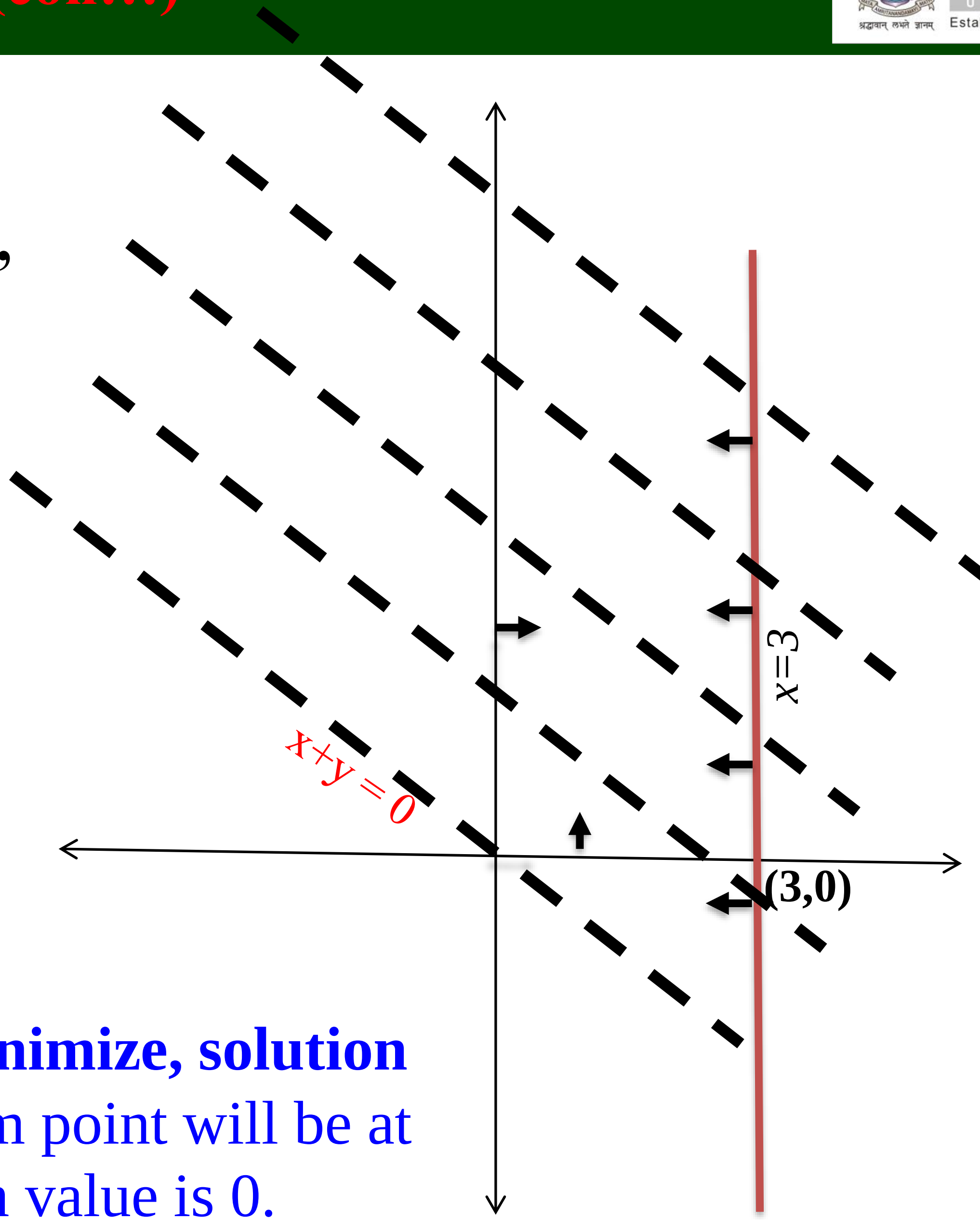


6. Maximize x + y

subject to $x \leq 3$,

$x \geq 0$,

$y \geq 0$.



Unbounded Solution

If the question was **minimize, solution is unique** the minimum point will be at (0,0) and the minimum value is 0.

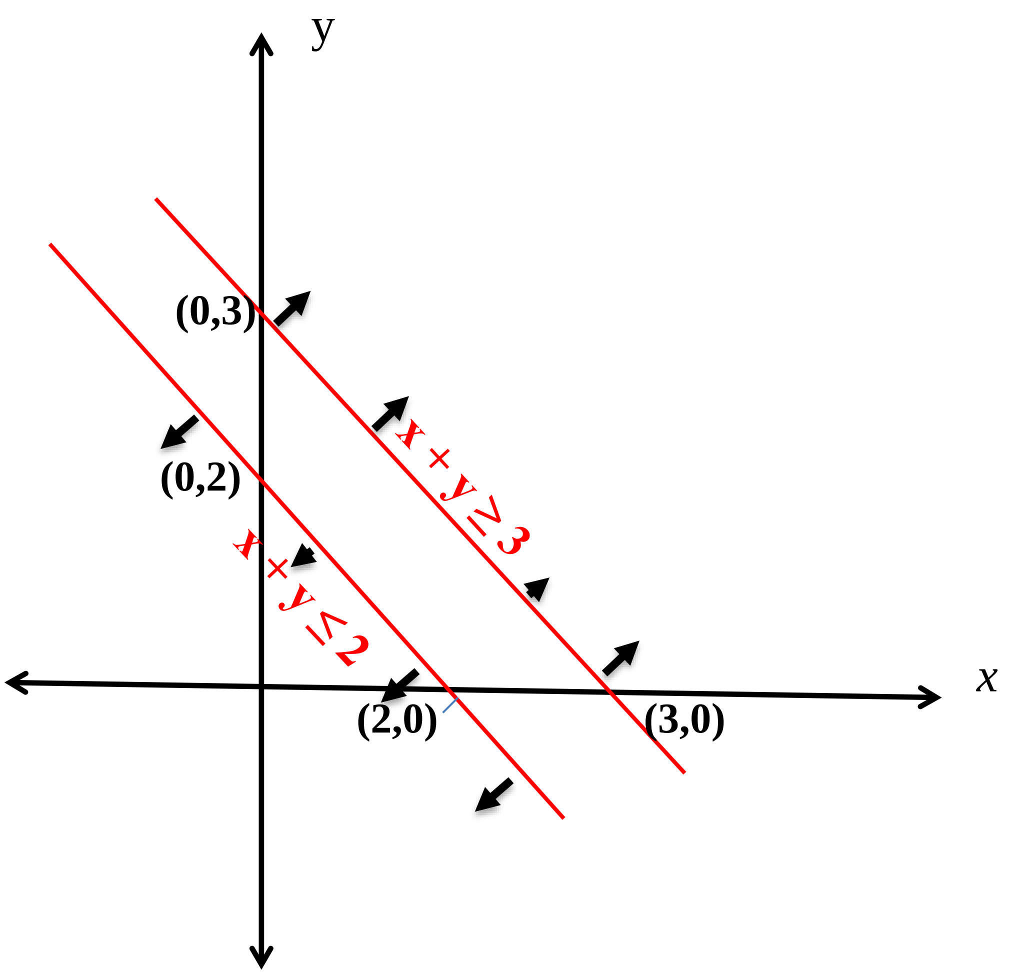
Graphical Method (Con…)

7. Minimize 3x + 5y

subject to the constraints,

$x + y \leq 2,$

$x + y \geq 3,$



No Feasible Region
No Solution /
Infeasible Solution

AMRITA VISHWA VIDYAPEETHAM UNIVERSITY Established u/s 3 of UGC Act 1956

8. Maximize x + y

subject to $y \le 2$,

$x^2 - y \le 0$,

$y \ge 0$.

Steps

Draw feasible region using constraints.

$y=0 \rightarrow y \geq 0$

$y=2 \rightarrow y \leq 2$

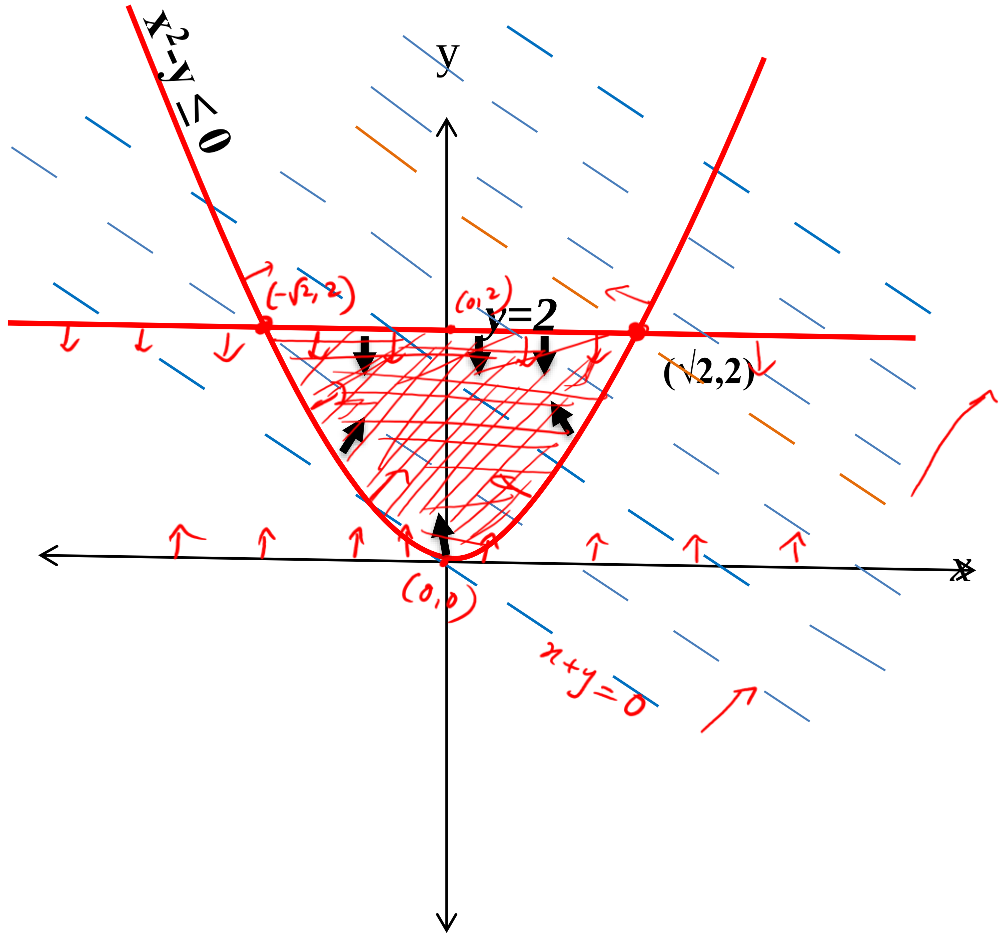
$x^2-y=0 \rightarrow x^2-y \leq 0$

Substitute (0,2)

$0-4 \leq 0 \rightarrow$ True

$y=2$ & $x^2-y=0$

$x^2-2=0 \Rightarrow x=\sqrt{2}, -\sqrt{2}$



Maximum value is $2+\sqrt{2}$

at the point $(\sqrt{2},2)$

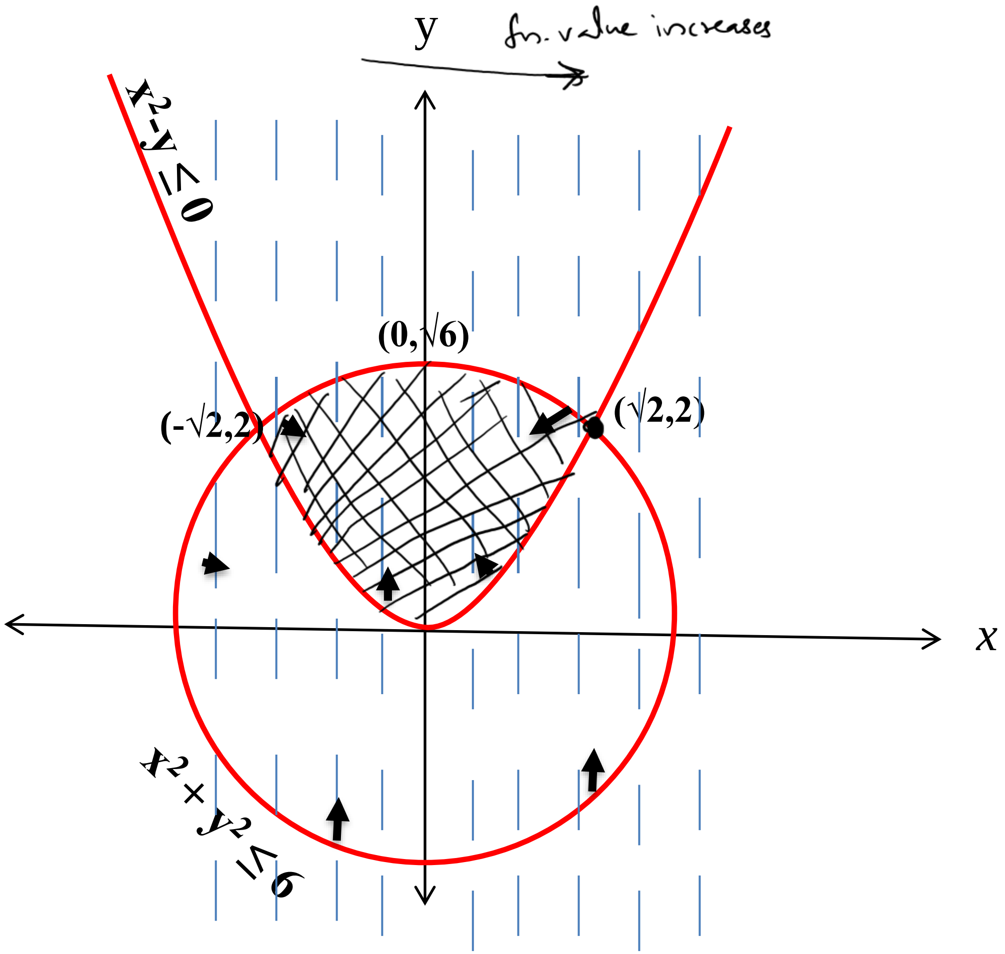
9. **Maximize 6x**

subject to

$$x^2 + y^2 \leq 6,$$

$$y - x^2 \geq 0,$$

$$y \geq 0.$$



Maximum value is $6\sqrt{2}$
at the point $(\sqrt{2},2)$

Graphical Method

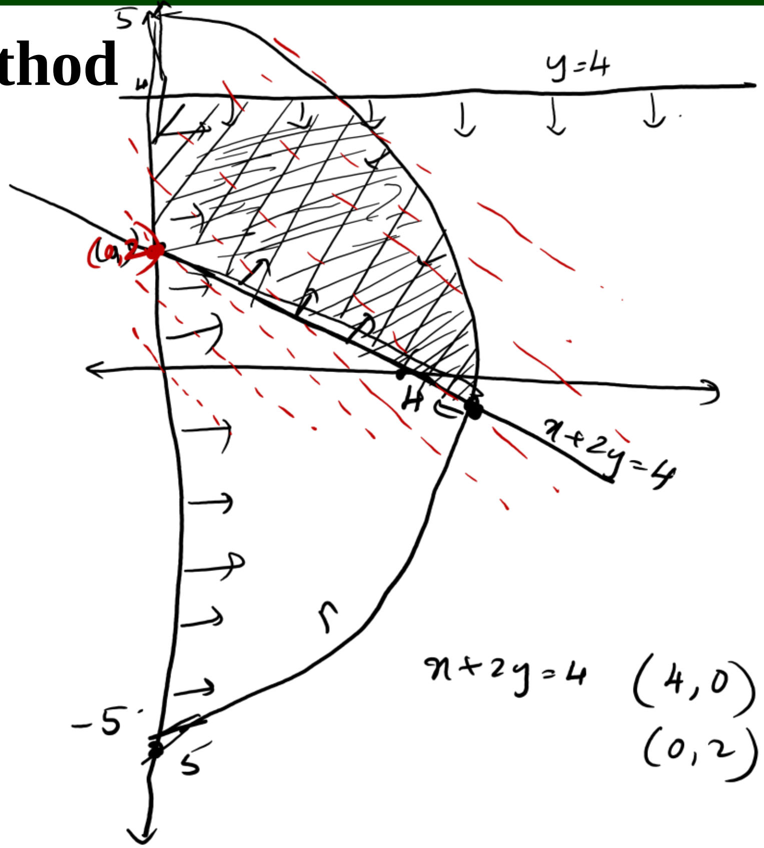
10. Minimize x + y

subject to

$$x^2 + y^2 \leq 25,$$

$$x + 2y \geq 4,$$

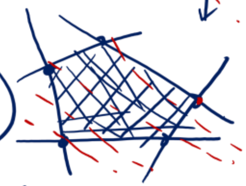
$$x \geq 0\ ,\ y \leq 4.$$



Graphical Method

Linear Optimization

F.R → Polygon, Polyhedra.
(constr. are linear)



Obj. fn:- linear → contour is a line (curve)

Soln. is always in the boundary

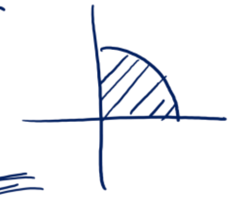
$\boxed{\text{Inside the F R optimum will not occur}}$

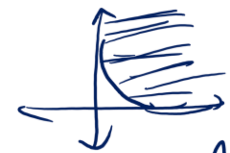
Type of soln:-
1) Unique
2) Inf. many solns. (line segment)
3) Unbounded soln.
4) No solution. (No Feasible region)

$\boxed{\text{Two solns or 3 solns. will never occur for linear OP}}$

Non-linear Optimization

F.R can be anything < bounded / not bdd.



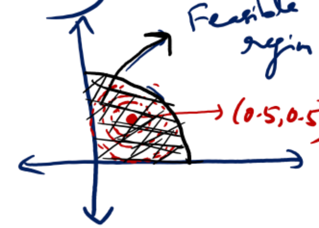


optimum can be inside F.R also (unlike linear OPs)

(1) Minimize $(x-0.5)^2+(y-0.5)^2$

s.t. $x^2+y^2 \leq 1$

$x \geq 0,\ y \geq 0$



Minimum at (0.5, 0.5)

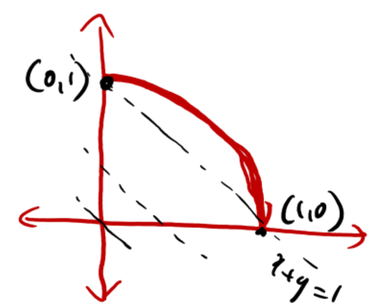
↳ A pt. inside Feasible region.

(2) Minimize $x+y$

s.t. $x^2+y^2=1$

$x \geq 0,\ y \geq 0$

Minimum at 2 points at (1,0) & (0,1)



Solve the following Linear Optimization Problems graphically

1. Maximize $3x + 2y$

   subject to $x + y \le 5$,

   $x - y \le 2$,

   $x \ge 0, \; y \ge 0$.

2. Maximize $x + y$

   subject to $y \le 5$,

   $x + y \le 10$.

   $x \ge 0, y \ge 0$.

3. Maximize $2x + y$

   subject to $x + y \ge 3$,

   $x - y \le 2$.

   $x \ge 0, y \ge 0$.

4. Minimize $x-9y$

   subject to $x + y \ge 3$,

   $x + y \le 2$.

Solve the following Non-Linear Optimization Problems graphically

5. Minimize $x^2 + y^2$

subject to $x + y \geq 1$

$x, y \geq 0$

6. Consider the optimization problem,

Minimize $x_1 + x_2$

subject to $x_1{}^2 + x_2{}^2 = 1$

$x_1 \geq 0, \quad x_2 \geq 0$

(a) Solve this problem graphically.

(b) Is the solution unique?

7. Maximize $x + y$

subject to $x + y \geq 5, \quad x^2 + y^2 \leq 25, y \geq 0, \quad x \geq 0$

8. Minimize the objective in Q.7.

9. Minimize $x^2 + y^2$

subject to $x - y \geq 1, x \leq 3, y \geq 0$

Plotting the 2D feasible region in MATLAB

Plot the feasible Region meeting the following constraints

$$x + y - 4 \le 0 \quad Or \quad \frac{x}{4} + \frac{y}{4} \le 1;$$

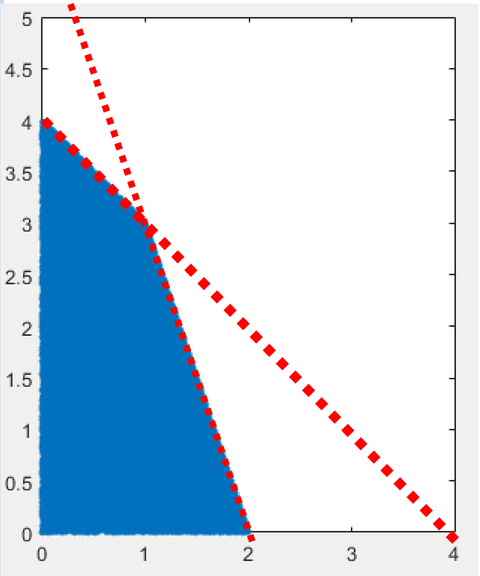
$$3x + y - 6 \le 0 \quad Or \quad \frac{x}{2} + \frac{y}{6} \le 1;$$

$$x \ge 0;$$
$$y \ge 0.$$

Code is not generic.

Depending on the y -intercept of constraints, you need to change Range of x and y in the code

```
N=200000; % number (x,y) points in domain
x = rand(N,1)*6;
y = rand(N,1)*6;
f1 = x+y-4;  % An array of N fn values in domain
ind1 = (f1<0); % An N_array of logical 0s and 1s
f2 = 3*x+y-6; % An array of N fn values in domain
ind2 = (f2<0); % An N_array of logical 0s and 1s
ind3=and(ind1,ind2); % An N_array of logical 0s and 1s
a = [x(ind3),y(ind3)];% points which are in feasible region
figure
plot(a(:,1),a(:,2),'.','MarkerSize',10);
axis equal
xlim([0 6])
ylim([0 6])
```

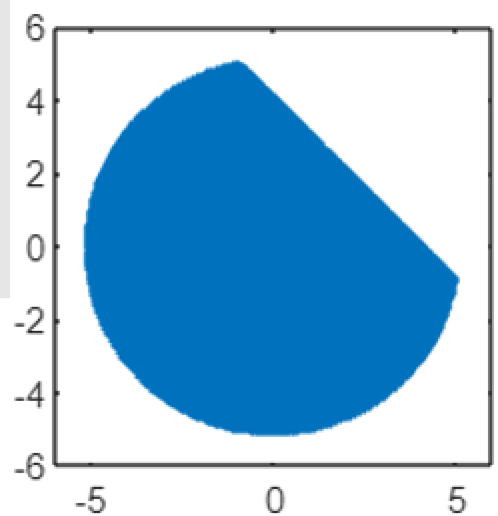


Try plotting the feasible regions of all optimization problems in the last two slides in MATLAB

Plotting the 2D feasible region in MATLAB

Plot the region: $x + y \leq 4,\ \ x^2 + y^2 \leq 25$

```
N=200000; % number (x,y) points in domain
x = -6+12*rand(N,1); % generates values in (-6,6)
y = -6+12*rand(N,1); % generates values in (-6,6)
f1 = x+y-4;  % An array of N fn values in domain
ind1 = (f1<0); % An N array of logical 0s and 1s
f2 = x.^2+y.^2-25; % An array of N fn values in domain
ind2 = (f2<0); % An N  array of logical 0s and 1s
ind3=and(ind1,ind2); % An N  array of logical 0s and 1s
a = [x(ind3),y(ind3)];% points which are in feasible region
figure
plot(a(:,1),a(:,2),'.','MarkerSize',10);
axis equal
xlim([-6 6])
ylim([-6 6])
```



Analytical solution for Constrained Optimization Problems

Getting analytical solutions for all Optimization problems is practically not possible.

Constrained Optimization problems which has analytical solutions are

Quadratic opt. pbms

- Least squares Problems → Data fitting pbms. Minimize $\|A x - b\|$

  analytical soln. of L.S.P is

  soln. of $A^T A \bar{x} = A^T b \rightarrow \boxed{\bar{x} = (A^T A)^{-1} (A^T b)}$

  $\rightarrow$ pinv(A)

- Linear Optimization Problems → Simplex Method

Subset

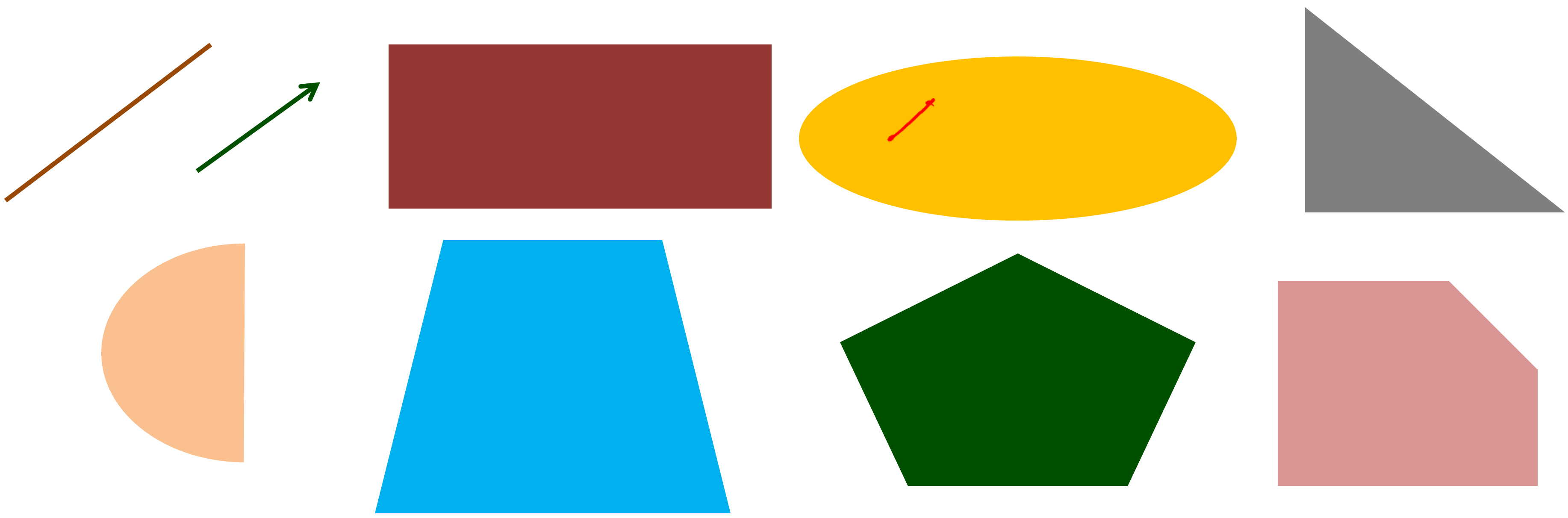
- **Convex Optimization Problems**

Convex Sets

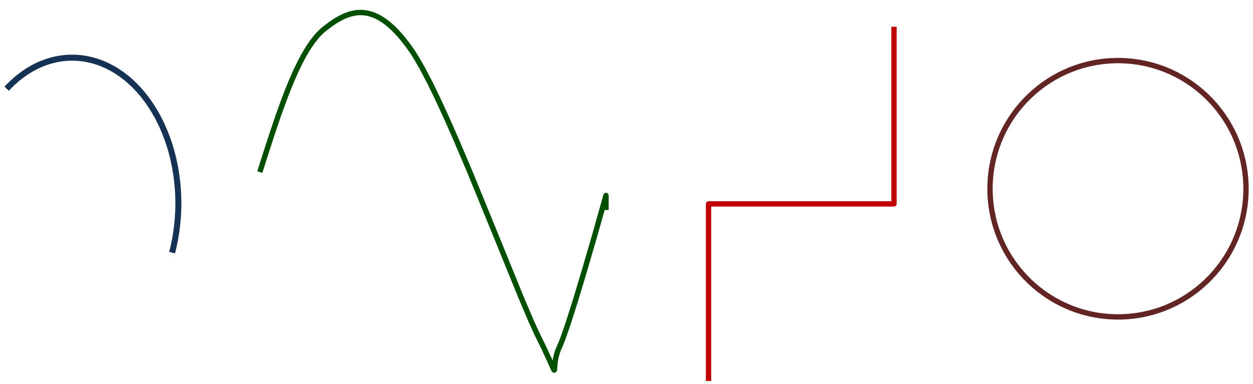
- Let S be a set and let $\mathbf{x_1}$ and $\mathbf{x_2}$ be elements of the set. If the line segment joining $\mathbf{x_1}$ and $\mathbf{x_2}$ is also an element of S, then we say that S is convex
- Mathematically:

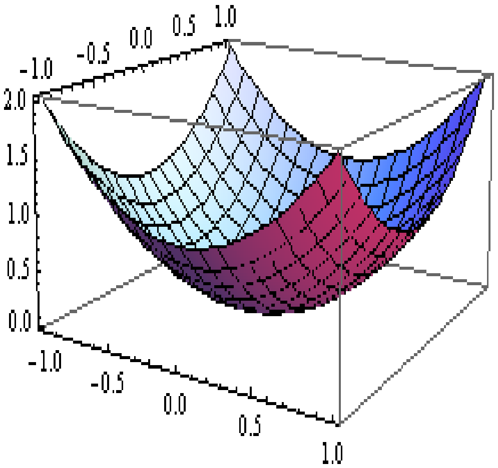
  Let $\boldsymbol{x_1}, \boldsymbol{x_2} \epsilon S$

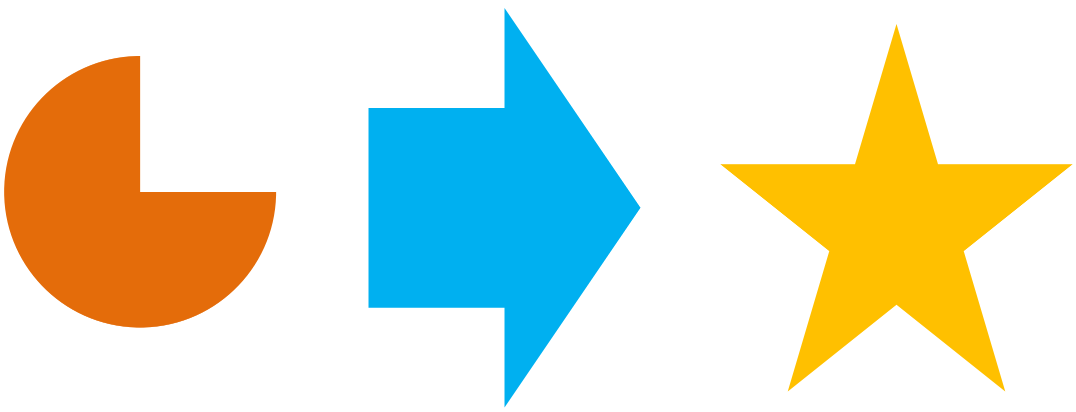
  If $\theta \boldsymbol{x_1} + (1-\theta)\boldsymbol{x_2} \epsilon S$ for $0 < \theta < 1$, then S is a Convex set.



Examples of Non-Convex Sets







Any curve or curved surface is non-convex

Examples of Convex Sets

- The empty set Φ is a convex set.
- A single point (singleton), i.e., $\{x_0\}$ is a convex set.
- $\mathbf{R}$, $\mathbf{R}^2$, $\mathbf{R}^3$, …, $\mathbf{R}^n$ are convex sets.
- **X axis**, **Y axis** and any other **line** is convex.
- Any line segment and any ray is convex.
- Any vector space and subspace is convex.
- A **hyperplane** is convex.
  - Hyperplane is a solution set of a single linear equation.
  - In $\mathbf{R}^2$ it is a line, in $\mathbf{R}^3$ it is a plane.
- Solution set of linear equations, $C = \{\boldsymbol{x} | \, A\boldsymbol{x} = \boldsymbol{b}\}$ is convex

  (point/ line/ plane/ hyperplane)

More Examples of Convex Sets

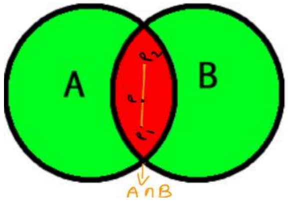
- A **halfspace** is convex.
  - Halfspace is the solution set of a single linear inequality.

    $\mathrm{H} = \{\boldsymbol{x} \mid \boldsymbol{a}^T\boldsymbol{x} \leq b\},\ \boldsymbol{a} \neq \mathbf{0}$
  - Hyperplane divides the domain into 2 halfspaces.
- Every **quadrant**(n=2), **octant**(n=3) and **orthant**(n) is convex.
- A **polyhedron** and a **polytope** is convex.
  - Polyhedron is a solution set of finitely many linear inequalities.
  - A bounded polyhedron is often called a polytope.
- **Feasible region** of a linear optimization problem is always convex.

Operations that preserve convexity

- Intersection of convex sets is convex

  (Union of convex sets need not be convex)

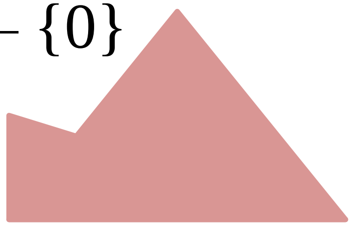


Exercise

1. Which of the following sets are convex?

(a) $\mathbf{R} - \{0\}$

(b)



(c) All the points in an interval, [a, b]

(d) All the points in an interval, (a, b)

(e) All points in the XY plane except the origin

(f) All points in the parabola, $y = x^2$

Answers1. Only (c) and (d) are convex

2. Find the domain of the following functions and mention if it is convex or not

(a) $f(x) = \frac{1}{x-1}$ (b) $g(x) = \sqrt{4 - x^2}$ (c) $h(x) = \frac{\sqrt{x-1}}{x}$

Answers2. (a) Domain of f is $\mathbf{R} - \{1\}$, which is not convex

(b) Domain of g is [-2,2], which is convex

(c) Domain of h is $[1,\infty)$, which is convex

Convex functions

$f(x)$

$f(\bar{x})$, $f(x)$
$f(x,y)$
$f(x,y,z)$
$f(x_1, \ldots x_n)$

- A function $f: \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ is a convex function if:

  (i) the domain of f is convex and ✓

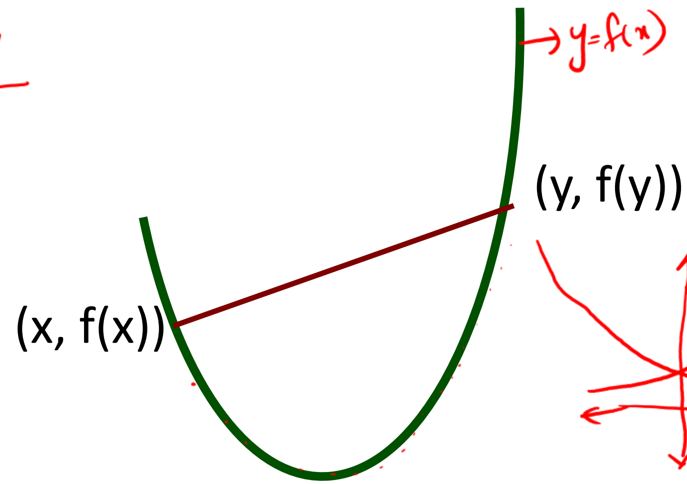
  $\boxed{y = f(x)}$

  (ii) $f(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y}) \leq \theta f(\boldsymbol{x}) + (1-\theta) f(\boldsymbol{y})$ ✓

  for all $x, y \in$ domain f and $0 \leq \theta \leq 1$

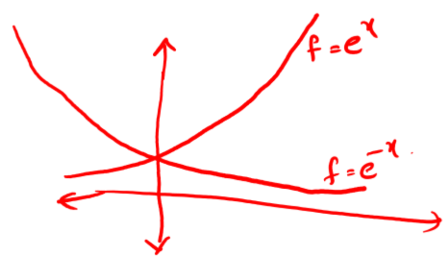
Geometricall
$y = f(x)$
$z = f(x,y)$ convex.

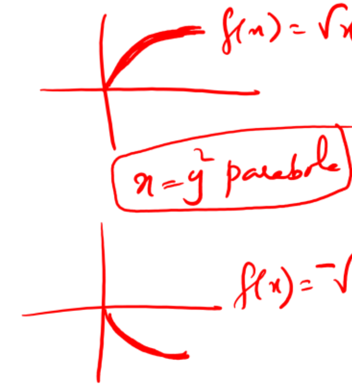
- Graphically, a function y=f(x) is convex if the curve y = f(x) lies below the line segment joining any two points on the curve.



$\boxed{f(x)}$

Egs:- $f(x) = x^2$
$f(x) = x^4$
$f(x) = e^x$
$f(x) = e^{-x}$





- The function is strictly convex if the inequality is strictly less (<)

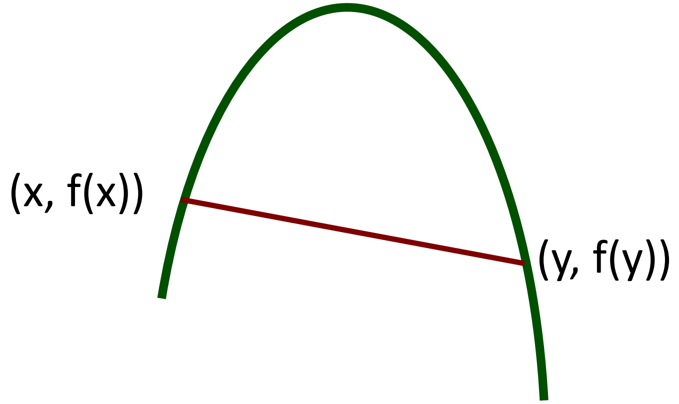
Concave functions

- A function $f: \boldsymbol{R}^n \rightarrow \boldsymbol{R}$ is a concave function if $-f$ is convex, i.e., when,

$$f(\theta \boldsymbol{x} + (1-\theta)\boldsymbol{y}) \geq \theta f(\boldsymbol{x}) + (1-\theta) f(\boldsymbol{y})$$

for all $x, y \in$ domain f and $0 \leq \theta \leq 1$

- Graphically, a function y=f(x) is concave if the curve y = f(x) lies above the line segment joining any two points on the curve.

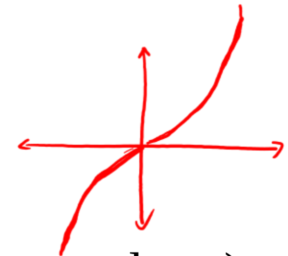


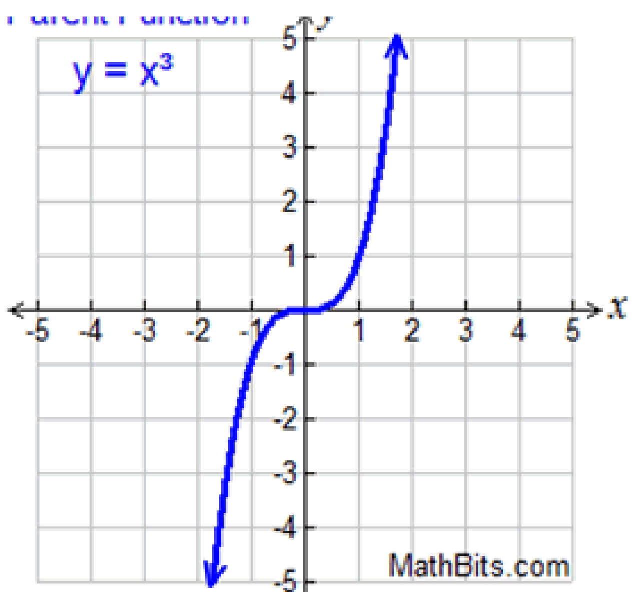
- The function is strictly concave if the inequality is strictly greater (>)

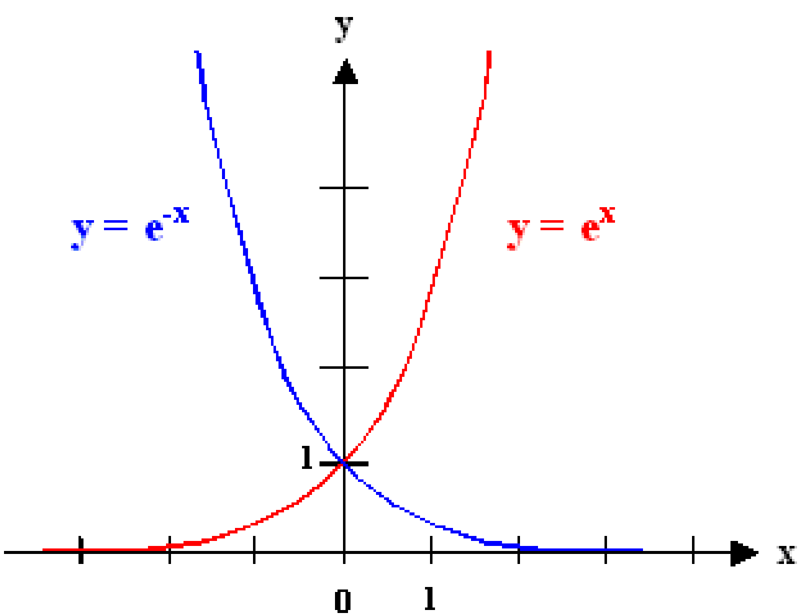
$f(x,y) = ax + by$ $\quad$ $f(x,y) = ax + by + c$

Examples

- Linear functions and affine functions are taken to be convex as well as concave as they don't have a curvature.
- $y = x^2$ is a convex function
- $y = x^4, y = x^6, y = x^8, \ldots$ are convex functions for all values of x.
- $y = x^3, y = x^5, y = x^7, \ldots$ are convex functions if $x > 0$ (in the first quadrant) and concave in the third quadrant (x<0).
- Exponential function, $y = e^{ax}$ for any $a \in \mathbf{R}$ is a convex function as it always curves up.







Second derivative test for convexity/concavity

- A twice differentiable function f(x) is convex if its Hessian $\nabla^2 f(x)$ is positive definite.
- A twice differentiable function f(x) is concave if its Hessian $\nabla^2 f(x)$ is negative definite.
- A function f(x) is non-convex if its Hessian $\nabla^2 f(x)$ is indefinite

$f(x) = x^2 \rightarrow f' = 2x,\ f'' = 2 > 0$

$f(x,y) = x^2 + y^2 \rightarrow H_f = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow$ +ve definite

$\nabla f = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$

$f(x,y) = -x^2 - y^2 \rightarrow H_f = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \rightarrow$ -ve definite.

$x^T A x \geq 0 \ \forall \bar{x}$

principal minors ≥ 0

$f(x,y,z)$

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{bmatrix}$$

Symmetric

Principal minors $\rightarrow$ +ve defn $\rightarrow$ + + + + ...

$\rightarrow$ -ve definite $\rightarrow$ − + − + ...

Eigenvalues $\rightarrow$ +ve def $\rightarrow$ > 0

-ve def $\rightarrow$ < 0

Second derivative test for convexity/concavity

Examples:

1. f(x) = x + 1/x is convex in x > 0

   (Since $\nabla^2 f(x) = \frac{d^2 f}{dx^2} = \frac{2}{x^3} > 0$ for $x > 0$)

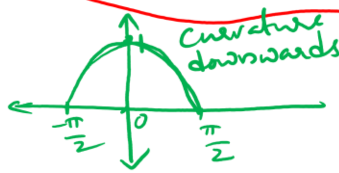
Domain:- $\mathbb{R}-\{0\}$. f is not convex in $\mathbb{R}-\{0\}$ since domain is not a convex set.

$f' = 1 - \frac{1}{x^2}$, $x > 0$

$f'' = 2x^{-3} = 2/x^3 > 0$ in $x > 0$

f is convex in $x > 0$

2. f(x) = cosx is concave in $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



$f' = -\sin x$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$

$f'' = -\cos x$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$ 4th & 1st Quad

i.e $f'' < 0$ in $(-\frac{\pi}{2}, \frac{\pi}{2})$

$\Rightarrow$ f is concave.

3. f(x,y) = 5x^2 + 6xy + 7y^2 is convex

$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \begin{pmatrix} 10x+6y \\ 6x+14y \end{pmatrix} \Rightarrow H_f = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 14 \end{bmatrix}$ $\rightarrow$ λs or principal minors (10, 104) $\rightarrow$ H_f is +ve definite $\Rightarrow$ f is a convex fn

4. f(x,y) = -2x^2 + xy -7y^2 is concave

$H_f = \begin{bmatrix} -4 & 1 \\ 1 & -14 \end{bmatrix}$; Principal minors:- −4, 55 $\Rightarrow$ H_f is −ve definite $\Rightarrow$ f is concave

PLOT THE SURFACES AND VERIFY

5. f(x,y) = 5x^2 + 19xy + 7y^2 is non-convex

$H_f = \begin{bmatrix} 10 & 19 \\ 19 & 14 \end{bmatrix}$; Principal minors $\rightarrow$ +, − $\Rightarrow$ H_f is neither convex nor concave

$\frac{\partial f}{\partial x} = \frac{2x}{y}$

$\frac{\partial f}{\partial y} = \frac{-x^2}{y^2}$

6. f(x,y) = x^2/y has a semidefinite Hessian matrix, so cannot mention about the convexity status

D = $\mathbb{R}^2$ − X axis

$H_f = \begin{bmatrix} 2/y & -2x/y^2 \\ -2x/y^2 & 2x^2/y^3 \end{bmatrix}$ $\rightarrow$ $\lambda = 0$, H_f semi +ve defn.

Operations that preserve convexity

f(x) convex
-f(x) concave

- If *f(x)* is a convex function, then $\alpha f(x)$ is also convex for $\alpha > 0$
  - $f(x) = 8e^x$ is convex
- The sum of convex functions is also convex
  - $f(x,y) = x^4 + y^4 + e^x$ is convex
- Non-negative weighted sum of convex functions is convex
  - $f(x,y,z) = 4x^2 + 5y^2 + 3e^z$ is convex

$$\nabla f = \begin{pmatrix} 8x \\ 10y \\ 3e^z \end{pmatrix}$$

$$H_f = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 3e^z \end{bmatrix}$$

$\lambda s \rightarrow 8, 10, 3e^z > 0 \ \forall z$

Principal minors $\rightarrow 8, 80, 240e^z > 0 \ \forall z$ (8 > 0, 80 > 0)

H_f is +ve definite $\Rightarrow f$ is convex

Practical methods for establishing convexity of a function

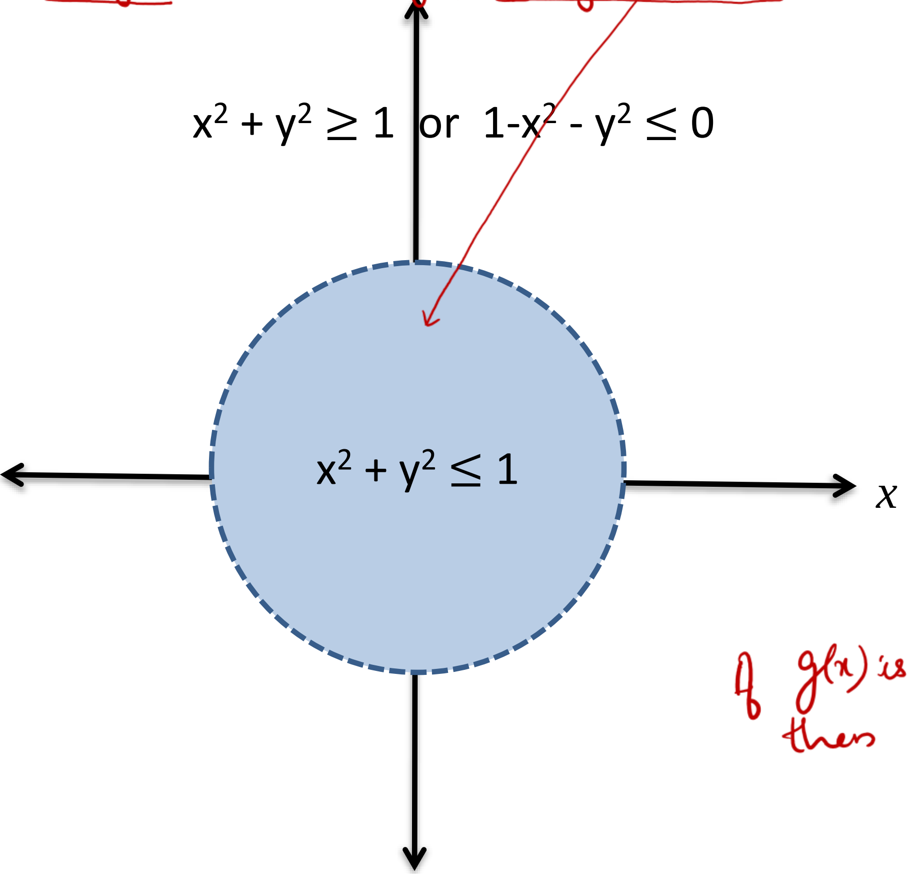
- Verify definition
- For twice differentiable functions, check whether hessian matrix is positive definite
- Show that f is obtained from simple convex functions by operations that preserve convexity

Relation between convex sets and convex functions

The set of all points in the region g(x) ≤ 0 is a convex set if the function g(x) is a convex function.

x^2+y^2-1 is a convex fn., $\boxed{x^2+y^2-1 \le 0}$

$1-x^2-y^2$ is not a convex fn.
$1-x^2-y^2 \le 0$
$x^2+y^2 \geq 1$
Not a convex set.



- g(x)= $x^2 + y^2$ -1 is a convex function so g(x) ≤ 0 is a convex set.
- f(x)= 1-x^2 - y^2 is not a convex function, so f (x) ≤ 0 is not a convex set

If g(x) is concave (i.e −g(x) is convex), then g(x) ≥ 0 is a convex set (i.e., −g(x) ≤ 0 is a convex set)

Optimization Problem in standard form

Minimize $f(\boldsymbol{x})$

subject to $g_i(\boldsymbol{x}) \leq 0, i = 1, 2, \ldots, m$

$h_i(\boldsymbol{x}) = 0, i = 1, 2, \ldots, p$

x : decision variable

$f: \boldsymbol{R}^n \to \boldsymbol{R}$ is the objective function (or cost fn. if obj. is min.)

$g_i: \boldsymbol{R}^n \to \boldsymbol{R}$ are the inequality constraints

$h_i: \boldsymbol{R}^n \to \boldsymbol{R}$ are the equality constraints

Maximize $3x - y^2$

s.t $6x + 2y^2 \geq 5$

$9x^2 - 6y \leq 3$

$7x + 6y = 6$

$x \geq 0, y \geq 0$

Std. O.P Minimize $y^2 - 3x$

$5 - 6x - 2y^2 \leq 0$

$9x^2 - 6y - 3 \leq 0$

$-x \leq 0$

$-y \leq 0$

$7x + 6y - 6 = 0$

Domain of the standard optimization problem:

$$\boldsymbol{D} = \cap_{i=1}^{m} dom\ g_i \ \cap \ \cap_{i=1}^{p} dom\ h_i$$

Convex Optimization Problem

An optimization problem in the standard form:

Minimize $f(x)$

subject to $g_i(x) \leq 0, i = 1, 2, \ldots, m$

$h_i(x) = 0, i = 1, 2, \ldots, p$

is said to be convex if

(i) the objective function is convex function.

(ii) the feasible region is convex set.

$-x \leq 0 \rightarrow$

$y \leq 0 \rightarrow$

FR $\rightarrow$

Pts on $h_i(x)=0$

$x+y=0 \rightarrow$ line

$x^2+y=0 \rightarrow$ parabola

$\underbrace{2x+3y+4z=0}_{h_i(x)}$ plane

$\underbrace{x^2-y=0}_{h_2(x)} \rightarrow$ non-linear curved $\rightarrow$ Not convex set

feasible region of a O.P is $\cap$ of regions in each constraint

FR is convex if region in each constraint is convex.

$g_i(x) \leq 0$ is convex only if $g_i(x)$ is a convex fn

$h_i(x) = 0$ is convex only if $h_i(x)$ is linear

Convex Feasible regions

- Feasible region of the optimization problem is the intersection of each constraint $g_i(\boldsymbol{x}) \leq 0, i = 1, 2, \dots, m$, $h_j(x) = 0, i = 1, 2, \dots, p$

▪ Since intersection of convex sets is convex, convexity of the feasible region can be checked by checking whether each constraint gives a convex set.

▪ $g_i(\boldsymbol{x}) \leq 0$ is a convex set if g_i is a convex function.

▪ $h_j(x) = 0$ will be a convex set only if $h_j(x)$ is linear. (Any linear equality constraint represents a line or a plane or a hyperplane which is a convex set. Any non-linear equality constraint represents a curve or a curved surface which cannot be a convex set).

Convex Optimization Problem

An optimization problem in the standard form:

Minimize $f(x)$

subject to $g_i(x) \leq 0, i = 1, 2, \dots, m$

$h_j(x) = 0, i = 1, 2, \dots, p$

is said to be convex if

(i) the objective function is convex

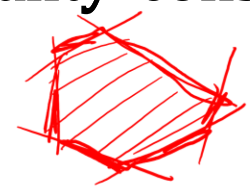
(ii) the inequality constraint functions, $g_i(x)$ are convex

(iii)the equality constraint functions, $h_j(x)$ are linear

Optimization problems with convex Feasible regions

- Optimization problems involving **linear inequality constraints** have convex feasible region.

  Eg: $-x + y \le 2, 2x + 3y \le 12, x \ge 0, y \ge 0$



  (Feasible region will be a null set, or a polyhedron)

- Optimization problems with **only linear equality constraints** have a convex feasible region.

  (Feasible region will be a null set, or a singleton or a line or a plane)

- Optimization problems with **linear equality and inequality constraints** have convex feasible region.

  Eg: $-x + 2y \le 2, 2x + 3y = 12, x \ge 0, y \ge 0$

- Optimization problems involving **convex inequality constraints** have a convex feasible region.

  $g_1 = x^2+y^2-1$; $g_2 = (x-1)^2+(y-1)^2-1$; $g_3 = -x$ linear ; $g_4 = -y$ linear. g_1, g_2 are convex; g_3, g_4 are linear $\Rightarrow$ convex.

  Eg: $x^2 + y^2 \le 1, (x-1)^2 + (y-1)^2 \le 1, x \ge 0, y \ge 0$

Feasible region for an optimization problem with **non-linear equality constraint(s) is never a convex set** regardless of whether the equality constraints is convex or not.

$$x^2 + y^2 = 1,$$
$$x \geq 0,$$
$$y \geq 0$$

Questions: 1. Minimize $5x + 9y^2$

subject to $x^2 + y^4 - 4y + 9x - 7 \leq 0$

$2x^2 + 3xy + 81y^2 - 5 \leq 0$

$f = 5x + 9y^2 \rightarrow f$ is convex

$g_1 = \underbrace{x^2}_{\text{convex}} + \underbrace{y^4}_{\text{convex}} \underbrace{- 4y + 9x - 7}_{\text{linear}}$

$g_2 = 2x^2 + 3xy + 81y^2 - 5 \leq 0$

x^2 convex, y^4 convex & $-4y+9x-7$ is linear (convex)

$\Longrightarrow g_1$ is convex.

$$\nabla g_2 = \begin{pmatrix} 4x+3y \\ 3x+162y \end{pmatrix}; \quad H_{g_2} = \begin{bmatrix} 4 & 3 \\ 3 & 162 \end{bmatrix}$$

is +ve def.

$\Longrightarrow g_2$ is convex.

$\Longrightarrow f$ is convex, g_1 & g_2 are convex

$\Longrightarrow$ Pbm is a convex Optimization pbm

2. Minimize $(x-1)^2+(y-1)^2+xy$

subject to $x+y \leq 4$

$$2x+x^2+y^2=16$$

3. Maximize x^2+y^2

subject to $x+y \leq 4$

$$2x+x^2+y^2 \leq 16$$

2. Minimize $\underbrace{(x-1)^2+(y-1)^2+xy}_{f}$

subject to $x+y \le 4$

$2x+x^2+y^2=16$

$f=(x-1)^2+(y-1)^2+xy$
$g_1 = x+y-4$
$h_1 = 2x+x^2+y^2-16$

$H_f = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ +ve definite

$\Rightarrow f$ is convex.
g_1 is linear $\Rightarrow g_1$ is convex
h_1 is not linear
$\therefore$ This is not a C.O.P.

Directly without checking any other you can say it is not a C.O.P as equality constraint is non-linear.

3. Maximize x^2+y^2

subject to $x+y \le 4$

$2x+x^2+y^2 \le 16$

Std. form:- Minimize $-x^2-y^2$
S.t. $x+y-4 \le 0$
$2x+x^2+y^2-16 \le 0$

$f=-x^2-y^2$
$g_1 = x+y-4$
$g_2 = 2x+x^2+y^2-16$

$\underbrace{f \text{ is a concave fn.}}$ as $H_f = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$

x^2 is convex
$\Rightarrow -x^2$ is concave
x^2+y^2 is convex
$\Rightarrow -x^2-y^2$ is concave.

$x^2+2x+y^2 \le 16$
$(x+1)^2-1+y^2 \le 16$
$(x+1)^2+y^2 \le 17$.
$(x+1)^2+y^2=17$

Since f is not a convex fn. this is not a convex optimization Problems

4. Maximize, $xy - (x-1)^2 - (y-1)^2$

subject to $2x + x^2 + y^2 \leq 16$

$3x - 7y = 9$

Std. form:- Minimize $-xy + (x-1)^2 + (y-1)^2 = f$

S.t $g_1 = 2x + x^2 + y^2 - 16 \leq 0$

$h_1 = 3x - 7y - 9 = 0$

$$\nabla f = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \end{pmatrix} = \begin{pmatrix} -y + 2(x-1) \\ -x + 2(y-1) \end{pmatrix}; \quad H_f = \begin{bmatrix} \partial^2 f/\partial x^2 & \partial^2 f/\partial x \partial y \\ \partial^2 f/\partial x \partial y & \partial^2 f/\partial y^2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

Pr. minors are 2, 3 > 0 $\Rightarrow H_f$ is +ve definite.

$\therefore f$ is convex

$g_1 = \underbrace{x^2}_{\text{convex}} + \underbrace{y^2}_{\text{convex}} + \underbrace{2x - 16}_{\text{linear} \Rightarrow \text{convex}}$; Sum of convex fns. are convex $\Rightarrow g_1$ is convex.

OR $H_{g_1} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is +ve def.

$h_1 = 3x - 7y - 9$ is linear.

$\therefore$ This is a Convex Optimization Plbm.

Is C.O.P. Since f(x) (after writing in standard form), $g_1(x)$ is convex and $g_2(x)$ is linear

Fundamental Property of Convex Optimization Problems

Any locally optimal point of a convex optimization problem is globally optimal

Practise questions

1. Consider the function $f(x_1, x_2) = x_1^2 + 2x_2^2 - 4x_1x_2 - 4x_2 + 7x_1 + 15$. Is this function convex?

$H_f = \begin{bmatrix} 2 & -4 \\ -4 & 4 \end{bmatrix} \Rightarrow$ H_f is not +ve definite nor negative defn. Neither convex nor concave.

If $f = -x_1^2 - 2x_2^2 - x_1x_2 - 4x_2 + 7x_1 + 14$ then $H_f = \begin{bmatrix} -2 & -1 \\ -1 & -4 \end{bmatrix}$ P. minors → + − P. Minors are −2, 7 ⇒ H_f is −ve definite ⇒ f is concave //

2. Is the given non-linear programming problem convex? Why or why not?

Maximize $z = 9x_1^2 - 4x_1x_2 + 7x_2^2$ f =

Subject to $x_1^2 + 2x_1x_2 + 3x_2^2 = 40$; $2x_1^2 - x_2 \le 80$; $x_1 \le 60$.

After converting to std. form, objective is Minimize $-9x_1^2 + 4x_1x_2 - 7x_2^2$

$H_f = \begin{bmatrix} -18 & 4 \\ 4 & -14 \end{bmatrix}$ f is concave. as H_f is −ve def.

⇒ Pbm. is not a C.O.P. //

3. Is the following statement true? Explain.

"Maximize $(x-3)^2 + (y-8)^2$ is not a convex optimization problem but Minimize $(x-3)^2 + (y-8)^2$ is a convex optimization problem"

TRUE

Practise questions

4. Determine if the following optimization problems are convex optimization problems. Use graphical methods to solve these problems.

(a) Maximize $-6x + 9y$

subject to $x - y \geq 2$

$3x + y \geq 1$

$2x - 3y \geq 3$

(b) Minimize $x^2 + 2y^2$

subject to $x + y \geq 1$; $x, y \geq 0$

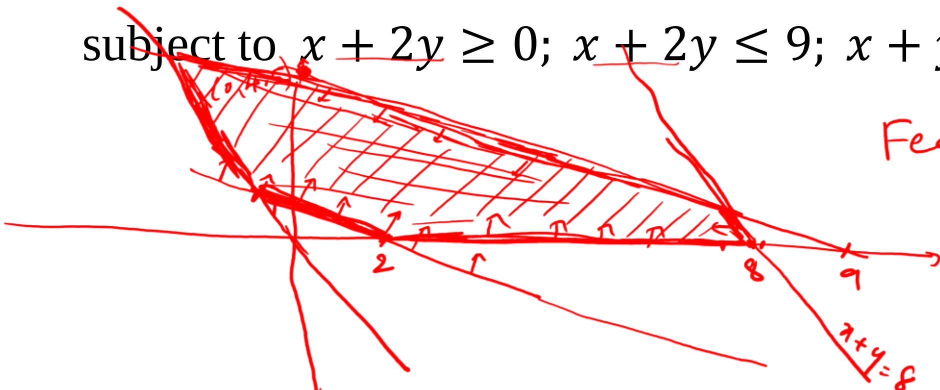
Tutorial questions

5. Draw the feasible region and determine the convexity status:

$x + 2y = 0$
$(2,0), (0,1)$

Minimize $x^2+2y^2-24x-20y = f$

subject to $x+2y \geq 0$; $x+2y \leq 9$; $x+y \leq 8$; $x+y \geq 0$



Feasible region is a polyhedron
$\Rightarrow$ Feasible region is a convex set. //

f is sum convex fns.
or $H_f = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ +ve def
$\Rightarrow$ f is convex fn.
$\Rightarrow$ It is C.O.P.

6. Explain why the given optimization problem is not convex.

Maximize $(x-2)^2+(y-10)^2$

subject to $x^2+y^2 = 50$

$x^2+y^2 + 2xy - x - y + 20 \geq 0; x, y \geq 0.$

7. Determine if the following optimization problem is convex optimization problems or not:

Minimize $\frac{100}{e^{2x+y}}$

subject to $e^x + e^y \leq 20$;

$x, y \geq 1.$