Journal Club

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August 8, 2024

Contents

The textbook is Terrence Tao's Analysis 1.

Part I

Real Analysis

Chapter 1

Introduction

Real Analysis is useful for engineers and physicists. It's said to have a bad reputation, due to its rigour. This textbook offsets that by starting at the very base, the numbers themselves. It starts by defining natural numbers, then building integers, rational, real, complex and so on. We have the continuous medium which is expressed by the PDEs, Dynamics expressed by ODEs. Computer scientists aren't taught this, they're taught Discrete Math.

Chapter 2

Natural Numbers

Numbers were built to count. A system for counting was made, and that system is the number system.

Definition 2.0.1

A natural number is an element of the set $\mathbb{N}$ of the set

$$\mathbb{N} = \{0, 1, 2, 3 \cdots \}$$

is obtained from 0 and counting forward indefinitely.

2.1 Peano Axioms

We start with axioms to help clarify this.

- Axiom 1 : $0 \in \mathbb{N}$
- Axiom 2: If $n \in \mathbb{N}$,, then $n++ \in \mathbb{N}$

This means that we have 0, 0++, ((0++)++)++ . . . We can then give these numbers symbols for ease, 0,1,2,3. . . NOTE: They do not hold any quantity as yet. They simply exist as representations of 0, 0++ and so on.

- Axiom 3: 0 is not an increment of any other natural number $n \in \mathbb{N}$
- Axiom 4: If $n \neq m$, $n++ \neq m++$

We need to remove the rogue elements from the set, such as fractions and half-integers.

- Axiom 5: (Principle Of Mathematical Induction) Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number.

We then make an assumption: That the set $\mathbb{N}$ which satisfies these five axioms is called the set of natural numbers. With these 5 axioms, we can construct sequences

2.2 Sequences

For $n \in \mathbb{N}$

$$\begin{aligned}
a_0 &= c, c \in \mathbb{N} \\
a_1 &= f_0(a_0), \\
a_2 &= f_1(a_1), \\
&\vdots \\
a_{n++} &= f_n(a_n),
\end{aligned}$$

Proposition 2.2.1. An operation f which operates on any number n in N

$$\begin{aligned}
f_n &: \mathbb{N} \to \mathbb{N} \\
&\forall n \in \mathbb{N}, \exists!\ a_n \text{ such that} \\
a_0 &= c \\
a_{n++} &= f_n(a_n)
\end{aligned}$$

Definition 2.2.1: Addition Of Natural Numbers

Let n be a natural number. $(n \in N)$. To add zero to m, we define $0 + m := m$ Now suppose inductively that we have defined how to add n to m. Then we can add n++ to m by defining(n++) + m := (n+m)++

Lemma 2.2.1. For any natural number $n + 0 = n$

Proof.

$$\begin{aligned}
n &= 0, 0 + 0 = 0 \\
n + 0 &= n \\
(n + +) + 0 &= (n + 0) + + = n + +
\end{aligned}$$

$\square$

Lemma 2.2.2.

$$n + (m + +) = (n + m) + +$$

Proof.

$$
\begin{aligned}
n &= 0, \\
0 + m{+}{+} &= (0+m){+}{+} \\
For (n{+}{+}) + (m{+}{+}) &= ((n{+}{+}) + m){+}{+} \\
&= (n + (m{+}{+})){+}{+} \\
&= ((n+m){+}{+})){+}{+}
\end{aligned}
$$

□

Putting m = 0, we get $n + 1 = n{+}{+}$

Definition 2.2.2: Positive natural number

All numbers where,

$$n \neq 0, n \in \mathbb{N}$$

Lemma 2.2.3. For every a, there exists a b such that $b{+}{+} = a$

Definition 2.2.3: Order

Let n and m be natural numbers we say that n is greater than or equal to m, and write $n \geq m$ iff we have $n = m + a$ for some natural number a. We say that $n > m$ when $n \geq m$ and $n \neq m$

2.3 Strong Induction

Theorem 2.3.1. Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true.(In particular this means that $P(m_0$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

Proof. For a property $Q(n)$, which is the property that $P(m')$ is true for $m_0 \leq m' < n$, then $P(n)$ is true.

For $Q(0)$, 0 is either lesser than or equal to m_0.
When 0 is lesser than m_0,
This is vacuously true.

When $0 = m_0$,

□

2.4 Multiplication

Definition 2.4.1

Let m be a natural number. To multiply zero to m, we define $0 \times m := 0$. Now suppose inductively that we have defined how to multiply n to m. Then we can multiply $n{+}{+}$ to m by defining $(n{+}{+}) \times m := (n \times m) + m$

Lemma 2.4.1. Prove that multiplication is commutative