Journal Club

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Contents

Part I

Real Analysis

TEXTBOOK: Analysis 1 by Terence Tao

Chapter 1

Natural Numbers

Numbers were built to count. A system for counting was made, and that system is the number system.

Definition 1.0.1

A natural number is an element of the set $\mathbb{N}$ of the set

$$\mathbb{N} = \{0, 1, 2, 3 \cdots\}$$

is obtained from 0 and counting forward indefinitely.

1.1 Peano Axioms

We start with axioms to help clarify this.

- Axiom 1 : $0 \in \mathbb{N}$
- Axiom 2: If $n \in \mathbb{N}$,, then $n++ \in \mathbb{N}$
- Axiom 3: 0 is not an increment of any other natural number $n \in \mathbb{N}$
- Axiom 4: If $n \neq m$, $n++ \neq m++$
- Axiom 5: (Principle Of Mathematical Induction) Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number.

We then make an assumption: That the set $\mathbb{N}$ which satisfies these five axioms is called the set of natural numbers. With these 5 axioms, we can construct sequences

1.2 Recursive Definitions

Proposition 1.2.1 (Recursive Definitions). Suppose for each natural number n, we have some function $f_n : \mathbb{N} \to \mathbb{N}$ from the natural numbers to the natural numbers. Then we can assign a unique natural number a_n to each natural number n, such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n.

1.3 Addition

Definition 1.3.1: Addition Of Natural Numbers

Let n be a natural number. $(n \in N)$. To add zero to m, we define $0 + m := m$ Now suppose inductively that we have defined how to add n to m. Then we can add n++ to m by defining(n++) + m := (n+m)++

Lemma 1.3.1. For any natural number $n + 0 = n$

Proof. We use induction,

The base case, n = 0,

$$\begin{aligned} n = 0, 0 + 0 = 0 \\ n + 0 = n \\ (n++) + 0 = (n+0)++ = (n++) \end{aligned}$$

Suppose inductively, that $n + 0 = n$,

For $n = n++$,

$$\begin{aligned} (n++) + 0 = (n+0)++ \\ \text{We know that } n + 0 = n \\ (n++) + 0 = (n++) \end{aligned}$$

□

Lemma 1.3.2. For any natural numbers n and m,

$$n + (m++) = (n+m)++$$

Proof. Inducting on n while keeping m fixed,

$$\begin{aligned} n = 0, \\ 0 + (m++) = (0+m)++ \\ 0 + (m++) = (m++) \end{aligned}$$

This we know is true from the definition of addition $(0 + m := m)$

Suppose inductively, that $n + (m++) = (n+m)++$ is true. For $n = (n++)$,

$$\begin{aligned} (n++) + (m++) &= ((n++) + m)++ && \text{From the definition of addition} \\ &= (n + (m++))++ \\ &= ((n+m)++))++ \end{aligned}$$

□

Putting m = 0, we get $n + 1 = n++$

Proposition 1.3.1 (Addition is commutative). For any natural numbers n and m, $n + m = m + n$

Proof. We induct over n, For the base case, $n = 0$,
We must show that $m + 0 = 0 + m$ From the definition of addition, we have

$$0 + m = m$$

As shown earlier, we have

$$m + 0 = m$$

This is clearly true for $n = 0$.
Now suppose inductively that $m + n = n + m$
For $n = n++$, we must show that $m + (n++) = (n++) + m$
We know from the definition of addition that,

$$(n++) + m := (m + n)++$$

And we proved earlier that,

$$m + (n++) = (m + n)++$$

Therefore,

$$m + (n++) = (n++) + m$$

□

Proposition 1.3.2 (Addition is associative). For any natural numbers, a, b and c, we have $(a+b)+c = a + (b + c)$

Proof. We take $(a + b) + n = a + (b + n)$
Inducting over n,
For $n = 0$,
We have in the LHS,

$$\begin{aligned} &= (a + b) + 0 && \text{Since } n + 0 = n \\ &= a + b \end{aligned}$$

On the RHS,

$$\begin{aligned} &= a + (b + 0) && \text{Since } n + 0 = n \\ &= a + b \end{aligned}$$

Suppose inductively that $(a + b) + n = a + (b + n)$,
For $n = n++$, We have to show that $(a + b) + (n++) = a + (b + (n++))$
On the LHS we have,

$$\begin{aligned} &= (a + b) + (n++) \\ &= (a + b + n)++ && \text{(From the lemma } m + (n++) = (m + n)++) \end{aligned}$$

On the RHS we have,

$$
\begin{aligned}
&= a + (b + (n++)) \\
&= a + (b + n)++ && \text{(From the lemma } m + (n++) = (m+n)++\text{)} \\
&= (a + b + n)++
\end{aligned}
$$

LHS = RHS □

Proposition 1.3.3 (Cancellation Law). Let a, b, c be natural numbers such that $a + b = a + c$. Then we have $b = c$.

Proof. We have,

$$n + b = n + c$$

Inducting over n, For the base case, $n = 0$

$$
\begin{aligned}
0 + b &= 0 + c \\
b &= c
\end{aligned}
$$

Suppose inductively that $n + b = n + c$ For $n = n++$,

$$(n++) + b = (n++) + c$$

On the LHS

$$
\begin{aligned}
&= (n++) + b \\
&= (n + b)++
\end{aligned}
$$

On the RHS

$$
\begin{aligned}
&= (n++) + c \\
&= (n + c)++
\end{aligned}
$$

We know from the inductive hypothesis that,

$$\text{If} n + b = n + c, \text{then} b = c$$

Thus we have,

$$b++ = c++$$

□

Definition 1.3.2: Positive natural number

All numbers where,

$$n \neq 0, n \in \mathbb{N}$$

Proposition 1.3.4. If a is a positive natural number and b is a natural number, then $a+b$ is positive.

Proof. Inducting over b,

For $b = 0$,

$$a + 0 = a$$

This proves the base case, since we know a is positive.

Now, suppose inductively, that $(a + b)$ is positive.

For $(a + (n++))$,

$$a + (n++) = (a + n)++$$

We know from Axiom 3 that $n++ \neq 0$. Thus we close the inductive loop. □

Lemma 1.3.3. For every a, there exists a unique b such that $b++ = a$

Proof. Proof by contradiction, Suppose that there are two different increments, $m++$, $n++$ that equal to a,

We have,

$$m++ = a$$
$$n++ = a$$

Then we can say,

$$\begin{aligned} m++ &= n++ \\ m+1 &= n+1 \\ m &= n && \text{(By Cancellation Law)} \end{aligned}$$

But we said that m and n are different numbers which increment to a.

Therefore, we can conclude that there is only one number b which increments to a □

1.4 Order

Definition 1.4.1: Order

Let n and m be natural numbers we say that n is greater than or equal to m, and write $n \geq m$ iff we have $n = m + a$ for some natural number a. We say that $n > m$ when $n \geq m$ and $n \neq m$

Proposition 1.4.1 (Basic properties of order for natural numbers). Let a, b, c be natural numbers then

1. (Order is reflexive) $a \geq a$
2. (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$
3. (Order is antisymmetric) If $a \geq b$ and $b \geq a$ then $a = b$
4. (Addition preserves order) $a \geq b$ if and only if $a + c \geq b + c$
5. $a < b$ if and only if $a++ \leq b$

6. $a < b$ if and only if $b = a + d$ for some positive number d.

Proof. 1. Proving order is reflexive, $a \geq a$

We know that,

$a = a + 0$

From the definition of order, We can write that $a \geq b$ when $a = b + d$ where $d \in \mathbb{N}$

Thus $a \geq a$.

2. Proving order is transitive, $a \geq b$ and $b \geq c$ then $a \geq c$

We write,

$$
\begin{aligned}
a &= b + d \\
b &= c + e \\
a &= c + e + d
\end{aligned}
$$

We can say that since $(e + d) \in \mathbb{N}$

We define $f := (e + d)$ Where $f \in \mathbb{N}$

$$a = c + (f)$$

Thus we can say,

$$\text{If } a \geq b, b \geq c \text{ then } a \geq c$$

3. Proving order is antisymmetric, If $a \geq b$ and $b \geq a$ then $a = b$ We can say,

$$
\begin{aligned}
a &= b + d \\
b &= a + e
\end{aligned}
$$

Where $d, e \in \mathbb{N}$

$$
\begin{aligned}
a &= (a + e) + d \\
b &= (b + d) + e
\end{aligned}
$$

Then we can write,

$$
\begin{aligned}
a &= a + (e + d) \\
b &= b + (d + e)
\end{aligned}
$$

Then we can say that $(e + d)$ and $(d + e)$ are 0.

We know that if $a + b = 0$ then $a, b = 0$

Thus d and e are 0.

$$a = b + d$$
$$a = b$$

4. Proving $a < b$ if and only if $b = a + d$ for some positive number d If $b = a + d$ where d is a positive natural number, $d \neq 0$

   Which means that $b \neq a + 0$ or $b \neq a$

   This means that b is strictly greater than a

   If $a < b$ then $a \geq b$ and $a \neq b$

   So if $a \geq b$ Then,

$$a = b + d$$

   But,

$$a \neq b$$
$$a \neq b + 0$$
$$d \neq 0$$

   Thus d cannot be 0. d can only be a positive natural number.

5. Proving addition preserves order, $a \geq b$ if and only if $a+c \geq b+c$ Proving $a \geq b$ if $a+c \geq b+C$

   Where $d \in \mathbb{N}$

$$\begin{aligned} a + c &= b + c + d && \text{By definition} \\ a + c &= (b + d) + c \\ a &= (b + d) && \text{By cancellation law} \\ a &\geq b \end{aligned}$$

   Proving $a + c \geq b + c$ if $a \geq b$

   We know,

$$a = b + d$$

   Where $d \in \mathbb{N}$

   We write a+c using what we know from above,

$$a + c = b + d + c$$
$$a + c = b + c + d$$
$$(a + c) = (b + c) + d$$
$$a + c \geq b + c$$

6. Proving $a < b$ if and only if $a++ \leq b$ Proving $a < b$ if $a++ \leq b$

   We can write,

$$\begin{aligned} a++ &= b + d && \text{Where } d \in \mathbb{N} \\ a+++d &= b \\ a+(d++) &= b \end{aligned}$$

   Since from Axiom 3, we know that 0 is not an increment of any natural number, $(d++ \neq 0)$ Therefore,

$$a < b$$

□

Proposition 1.4.2 (Trichotomy of order for natural numbers). Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b, a = b or a > b$

Proof. First we show that no more than one of the statements is true. If $a < b$ then $a \neq b$ by definition. If $a > b$ then $a \neq b$ by definition. If $a > b$ and $a < b$ then $a = b$, which we proved earlier.

Now to show that exactly one of these statements are true. We induct on a,

When a $=$ 0, We know that,

$$\begin{aligned} b &= 0 + b (\forall b \in \mathbb{N}) \\ b &\geq 0 \end{aligned}$$

Suppose inductively that exactly one of the above statements are true for a and b. For a$++$, We take each statement. First for $a > b$

$$\begin{aligned} a &> b \\ a &= b + d \\ (a++) &= (b+d)++ \\ (a++) &= b + d++ \\ (a++) &> b && \text{If } d \in \mathbb{N} \text{ then } d++ \in \mathbb{N} \end{aligned}$$

For $a = b$

$$\begin{aligned} a &= b \\ (a++) &= (b)++ \\ (a++) &= b + 1 \\ a &> b \end{aligned}$$

For $a < b$

$$
\begin{aligned}
a &< b \\
a + d &= b \\
(a + d) + + &= b + + \\
(a + +) + d &= b + + \\
(a + +) + d &= b + 1
\end{aligned}
$$

We have two cases, If $d = 1$, Then by cancellation law

$$a + + = b$$

If $d \neq 1$ Then

$$a + + < b$$

But never both, which concludes the inductive loop. □

1.5 Special Forms Of Induction

1. Strong Induction

Theorem 1.5.1. Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true.(In particular this means that $P(m_0$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

Proof. For a property $Q(n)$, which is the property that $P(m')$ is true for $m_0 \leq m' < n$, then $P(n)$ is true.

For $Q(0)$, 0 is either lesser than or equal to m_0.

When 0 is lesser than m_0,

This is vacuously true.

When $0 = m_0$, □

2. Backward Induction

3. Induction Starting From The Base Case n Let n be a natural number, and let P(m) be a property pertaining to the natural numbers such that whenever P(m) is true, P(m++) is true. Show that if P(n) is true, then P(m) is true for all m n. (This principle is sometimes referred to as the principle of induction starting from the base case n.)

Proof. Take a property $P(n)$, $m \geq n$

Inducting over n, □

1.6 Multiplication

Definition 1.6.1

Let m be a natural number. To multiply zero to m, we define $0 \times m := 0$. Now suppose inductively that we have defined how to multiply n to m. Then we can multiply $n++$ to m by defining $(n++) \times m := (n \times m) + m$

Lemma 1.6.1. Prove that multiplication is commutative

1.7 Exercise

1. Prove the identity $(a+b)^2 = a^2 + 2ab + b^2$
2. (Euclid's division lemma)
3. Backward Induction $m \in \mathbb{N}, P(m), P(m++) \Rightarrow P(m)$, Suppose $P(n)$ is true, then $P(m) \forall m \leq n$ For the base case, n = 0, P(0) $\Rightarrow P(0), soit's true.$

For the inductive step, supposing $Q(n)$ is true,

4. Strong induction
5. Distributive Law
6. Multiplication
   (a) Cancellation Law
   (b) Associativity
   (c) If a<b, and c is positive then ac<bc