

23PHY114  
Class Notes

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# Chapter 1

## Solids

### 1.1 Moment Of Area

The moment of inertia is used to help find the "resistance" to the force, given a specific axis or direction.

### 1.2 Resisting Force And Moments From Supports

There are three main kinds of supports -

1. Pin/Hinge - Fixes linear motion but leaves rotation free.
2. Roller - Fixes rotation but leaves motion free.
3. Clamp - Fixes both linear and rotational motion.

Take this case, with a bunch of forces being applied to the given load.

There are three main things we need to find for this figure.

1. The resultant force acting on this bar fixed to a hinge.
2. The support reaction force and moment.
3. The moment on the object (maximum)

The way to approach the problem is as always,

1. Free Body Diagram first.
2. Assuming  $\Sigma f = 0$ , because the object has no acceleration currently, because it is fixed.
3. Assuming  $\Sigma M = 0$
4. The point at which the resultant force is applied is found by,

$$\frac{\int |r| dm}{\int dm}$$

### 1.3 Derivation Of The Uniaxial Formula

### 1.4 Code For Uniaxial Deformation

Main Subroutines For Uniaxial Deformation

### 1.4.1 Finding The Local Stiffness Matrix

```

1 function stiffnessLocal = localStiffnessGenerator(E,A,l,theta);
2     stiffnessConstant = E*A/l;
3     R = [cos(theta) -sin(theta); sin(theta) cos(theta)];
4     stiffnessMatrix = [stiffnessConstant 0 -stiffnessConstant 0; zeros(1,4); -stiffnessConstant 0 stiffnessConstant
5                       0; zeros(1,4)];
6     R4 = [R zeros(2,2); zeros(2,2) R];
7     stiffnessLocal = R4*stiffnessMatrix*R4';
8 end

```

### 1.4.2 Converting The Local Stiffness To A Global Stiffness Matrix

```

8 function stiffnessLocalGlobal = local2Global(stiffnessLocal,node1,node2,nodeCount)
9     stiffnessLocalGlobal = zeros(2*nodeCount,2*nodeCount);
10    i = 2*node1 - 1;
11    j = 2*node2 - 1;
12    stiffnessLocalGlobal(i:(i+1),i:(i+1)) = stiffnessLocal(1:2,1:2);
13    stiffnessLocalGlobal(i:(i+1),j:(j+1)) = stiffnessLocal(1:2,3:4);
14    stiffnessLocalGlobal(j:(j+1),i:(i+1)) = stiffnessLocal(3:4,1:2);
15    stiffnessLocalGlobal(j:(j+1),j:(j+1)) = stiffnessLocal(3:4,3:4);
16 end

```

### 1.4.3 Main Loop Through Evaluating The Global Stiffness Matrix

```

17 A = 0; theta = 0; l = 0; stiffnessLocal = zeros(4,4); nodeAxialForces = zeros(nodeCount,1);
18 for element = 1:elementCount % For first three bars
19     A = areaVector(element);
20     theta = angleVector(element);
21     l = lengthVector(element);
22     stiffnessLocal = localStiffnessGenerator(E,A,l,theta);
23     nodeCounter = element*2 - 1;
24     stiffnessLocalGlobal = local2Global(stiffnessLocal,nodeVector(nodeCounter),nodeVector(nodeCounter+1),nodeCount
25 );
26     stiffnessLocal
27     stiffnessLocalGlobal
28     stiffnessGlobal += stiffnessLocalGlobal;
29 end

```

### 1.4.4 Applying Boundary Conditions

```

29 selectedVector = [3 5:end]
30 forceEval = forceVector(selectedVector);
31 displacementEval = displacementVector(selectedVector);
32 stiffnessEval = stiffnessGlobal(selectedVector);
33 displacementEval = stiffnessEval\forceEval;

```

## 1.5 Deriving For Bending Deformation

Taking this example of a bar fixed to the wall, and examining small components of the bar and examining the forces at work on the component, we get a Tension, Moment and Stress.

Taking things along the  $\hat{i}$  direction, we get

$$-T + T + \Delta T = 0$$

What does this tell us? Tension is constant.

While  $-S + S + \Delta S - w\Delta x = 0$

$$\frac{dS}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta S}{\Delta x} = w(x)$$

We get an expression for the effects of the distributed force on the shear force of an individual object.

Looking at the angular momentum balance,

$$\begin{aligned} \sum M/P &= 0 \\ -M + (M + \Delta M)\hat{k} + (-\Delta x\hat{i} \times (-T\hat{i} - S\hat{j})) + \left(\frac{-\Delta x}{2}\hat{i} \times (-w(x)\Delta x\hat{j})\right) \\ \Rightarrow \frac{\Delta M}{\Delta x} + S - w\frac{\Delta x}{2} &= 0 \end{aligned}$$

Taking the limit on the equations, we get,

$$\frac{dM}{dx} + S = 0$$

Thus we get,

$$\begin{aligned} \frac{dT}{dx} &= 0 \\ \frac{dS}{dx} &= w(x) \\ \frac{dM}{dx} &= -S \end{aligned}$$

Now we get to the Finite Element Method for bending.

$$\theta(x) = v'(x)$$

We find by simplification,

$$\begin{aligned} c_1 &= v_l \\ c_2 &= \theta_l \\ c_3 &= \frac{3}{l^2}(v_r - v_l) - \frac{1}{l}(2\theta_l + \theta_r) \\ c_4 &= \frac{2}{l^3}(v_l - v_r) + \frac{\theta_l + \theta_r}{l^2} \end{aligned}$$

$$v(x) = H_1(x)v_l + H_2(x)\theta_l + H_3(x)v_r + H_4(x)\theta_r$$

$$H_1 = 2\left(\frac{x}{l}\right)^3 - 3\frac{x^2}{l} + 1$$

$$H_2 = x - 2\frac{x^2}{l} + \frac{x^3}{l^2}$$

$$H_3 = 3\left(\frac{x}{l}\right)^2 - 2\frac{x^3}{l}$$

$$H_4 = \frac{x^3}{l^2} - \frac{x^2}{l}$$

Since we know that,

$$\begin{aligned} \int_0^l EI q'' v'' dx \\ v &= \underline{H}(x)^T v \\ q &= H(x) \end{aligned}$$

## 1.6 Quick Reference Notes

### 1.6.1 Derivations

The governing differential equation is

$$EIv^{(4)} = w(x)$$

Then we can form a linear relationship

$$v(x) - \text{deflection}$$

$$v^{(2)}(x) = \theta(x) - \text{slope}$$

$$EIv^{(2)}(x) = M(x) - \text{bending moment related to curvature}$$

$$EIv^{(3)}(x) = \frac{dM}{dx} = V(x) - \text{transverse shear}$$

$$EIv^{(4)}(x) = \frac{dV}{dx} = w(x) - \text{load}$$

Where, v is the deflection or displacement of the beam. E is the Young's Modulus and I is

### 1.6.2 Deriving Stiffness Matrix

Use Hermite's interpolation formula to derive cubic shape functions for the deflection of beams.

## 1.7 Final Equation

For a bar, with axial, bending and shear deformation... The final equation is, where R, S and M are Axial, Shear and Moment.

$$F = k\Delta x$$

$$\begin{bmatrix} R_1 \\ S_1 \\ M_1 \\ R_2 \\ S_2 \\ M_2 \end{bmatrix} = \begin{bmatrix} C_1 & 0 & 0 & -C_1 & 0 & 0 \\ 0 & 12C_2 & 6C_2L & 0 & -12C_2 & 6C_2L \\ 0 & 6C_2L & 4C_2L^2 & 0 & -6C_2L & 2C_2L^2 \\ -C_1 & 0 & 0 & C_1 & 0 & 0 \\ 0 & -12C_2 & -6C_2L & 0 & 12C_2 & -6C_2L \\ 0 & 6C_2L & 2C_2L^2 & 0 & -6C_2L & 2C_2L^2 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{bmatrix}$$

Where,  $C_1 = \frac{EA}{L}$  and  $C_2 = \frac{EI}{L^3}$

Rotating the matrix by angle  $\theta$

$$R = \begin{bmatrix} c & s & 0 & 0 & 0 & 0 \\ -s & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

# Chapter 2

## Fluids

### 2.1 Main Contents

#### Computational Fluids Dynamics

1. Flow in channels and pipes
2. Flow over objects

##### Definition 2.1.1: Laminar Region

The region in a fluid where the flow is in a straight line is known as the laminar region

##### Definition 2.1.2: Turbulent Region

The region in a fluid where the flow is not smooth is known as the turbulent region

#### The motivation behind learning CFD

Computational Fluid Dynamics has been around as long as computers have been around. Using computation to calculate and visualize the flow of fluids is the basis behind wind tunnels. Trials can be done to help optimize solutions for the best aerodynamics.

Examples of data used is the velocity profile, the force and the acceleration. These fluid simulations can take months and be used to simulate the effects of fluids on a given shape.

### 2.2 Terminology

In engineering terms, fluids are deformable bodies. They take up the shape of its container. Viscosity, is defined as the friction to flow -  $\mu$ . It dictates the rate of flow of a fluid.

The counterintuitive picture. Imagine a stack of papers on your palm. On tilting your palm, the first stack of paper goes out then the next then the next. It is dependent on the friction of the piece of paper below it. The last piece of paper has zero friction, and flows out the easiest. But a fluid does not work that way, when some water is placed on your palm, even after flowing out, some imprint of the water remains.

Think of fluids as layered.

Taking a co-ordinate system, plotting the horizontal and vertical velocity of a fluid,  $u$  and  $v$ .

##### Definition 2.2.1: No Slip Condition

A condition where the part of the fluid at the base or in contact with the surface has no horizontal velocity.

$$U(x, 0) = 0$$

Unlike solids, fluids have a profile that is counterintuitive to the way we imagine physics.

Counterintuitively, a fluid's profile is parabolic, in a pipe. Closed on all sides, the middlemost component of the fluid moves with

**Note:-**

**Lagrangian Method** - The method of following a single point and discovering interesting properties about it is known as the Lagrangian Method

**Eulerian Method** - The method of following a single location and performing analyses on that location.

The study of fluids is performed using the Eulerian Method.

## 2.3 Derivation Of Equations

Taking the fluid flowing through the figure given. In the Eulerian Method, we focus on a given location. This location here is the outlined box.

The mass flow rate for a given location is

$$\begin{aligned}\dot{m}_{in} - \dot{m}_{out} &= \frac{dm}{dt} & (\dot{m} = \frac{dm}{dt}) \\ \dot{m} &= \frac{d\rho V}{dt} \\ \dot{m} &= \rho \frac{dV}{dt} \\ &= \rho Q & (Q - \text{volume flow rate})\end{aligned}$$

$$\rho \frac{dV}{dt} = \Delta y \Delta z \cdot u$$

$$\begin{aligned}(\dot{m}_{intx} + \dot{m}_{inty}) - (\dot{m}_{outx} + \dot{m}_{outy}) &= \frac{dm_{inside}}{dt} \\ (\rho \Delta Y \Delta Z u + \rho \Delta X \Delta Z v) - (\rho \Delta Y \Delta Z (U + \Delta U) + \rho \Delta X \Delta Z (v + \Delta v)) &= \frac{d(\rho \Delta X \Delta Y \Delta Z)}{dt} \\ (\rho \Delta Y \Delta Z u + \rho \Delta X \Delta Z v) - (\rho \Delta Y \Delta Z (U + \Delta U) + \rho \Delta X \Delta Z (v + \Delta v)) &= \frac{d(\rho \Delta X \Delta Y \Delta Z)}{dt} \\ -\rho \Delta Y \Delta Z \frac{\Delta U}{\Delta X} - \rho \Delta X \Delta Z \frac{\Delta V}{\Delta Y} &= \Delta X \Delta Y \Delta Z \frac{d\rho}{dt} \\ -\rho \frac{\partial U}{\partial x} - \rho \frac{\partial v}{\partial y} &= \frac{d\rho}{dt}\end{aligned}$$

## 2.4 Recall

$$\begin{aligned}\dot{m}_{in} - \dot{m}_{out} &= \dot{m}_{within} & (\text{Continuity Equation}) \\ \frac{\partial \rho}{\partial t} &= -\rho \frac{\partial u}{\partial x} - \rho \frac{\partial v}{\partial y} & (\text{Mass Continuity Equation}) \text{or} \\ \frac{\partial \rho}{\partial t} &= -\rho \frac{\partial u}{\partial x} - \rho \frac{\partial v}{\partial y} - \rho \frac{\partial w}{\partial z} & (\text{in 3 dimensions})\end{aligned}$$

Density is a function of position and time, then it can be expressed as,  $\rho(x, y, z, t)$ . Instantaneously, density can be observed to change. Similarly with viscosity,

But the question might come about that the density in the mass continuity equation, is taken as constant, since they move outside the derivative. We assume that the density does not change with position rather than changing with time.



## 2.5 Density As A Function Of Time And Position

$$\begin{aligned}
1) \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial y} + \frac{\partial(\rho w)}{\partial z} &= 0 \\
2) \frac{\partial \rho}{\partial t} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + \frac{\partial \rho}{\partial x} u + \frac{\partial \rho}{\partial y} v + \frac{\partial \rho}{\partial z} w & \quad (\text{Applying chain rule}) \\
3) \frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial \rho}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial \rho}{\partial z} \frac{\partial z}{\partial t} & \quad (\text{General form of partial derivatives to total derivatives}) \\
4) \frac{d\rho}{dt} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) & \quad (\text{Final expression}) \\
5) \vec{\nabla} = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} & \quad (\text{Differential Operator}) \\
6) \vec{v} = u\hat{i} + v\hat{j} + w\hat{k} & \quad (\text{Velocity Vector}) \\
7) \frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v} = 0 & \quad \nabla \cdot v (\text{Divergence of } \vec{v})
\end{aligned}$$

## 2.6 External Forces On Fluid

$$\vec{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Given the fact that a fluid's smallest element does not rotate. Then,

$$\Sigma M/O = 0$$

Then looking at any given face for the element, the shear forces are equal and opposite to each other, from this we get.

$$\begin{aligned}
\sigma_{xy} &= \sigma_{yx} \\
\sigma_{xz} &= \sigma_{zx} \\
\sigma_{yz} &= \sigma_{zy}
\end{aligned}$$

## 2.7 Variation of stress across a small distance

insert diagram here(of cube with all forces measured)

$$\begin{aligned}
\Sigma \vec{f} &= m\vec{a} \\
\hat{i}(-\sigma_{xx}\Delta y\Delta z + (\sigma_{xx} + \Delta\sigma_{xx})\Delta y\Delta z + (\sigma_{xy} + \Delta\sigma_{xy})\Delta x\Delta z - \sigma_{xy}\Delta x\Delta z) + \\
\hat{j}(-\sigma_{yy}\Delta x\Delta z + (\sigma_{yy} + \Delta\sigma_{yy})\Delta x\Delta z + (\sigma_{xy} + \Delta\sigma_{xy})\Delta y\Delta z - \sigma_{xy}\Delta y\Delta z) - (\rho\Delta x\Delta y\Delta z)g\hat{j} &= (\rho\Delta x\Delta y\Delta z)\frac{d\vec{v}}{dt} \\
\vec{v} &= u\hat{i} + v\hat{j} + w\hat{k}
\end{aligned}$$

Dotting along  $\hat{i}$  and  $\hat{j}$ , when  $\Delta x \rightarrow 0, \Delta y \rightarrow 0, \Delta z \rightarrow 0$

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= \rho \frac{du}{dt} \\
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} - \rho g &= \rho \frac{dv}{dt}
\end{aligned}$$

There are certain equations that cannot be derived, they are simply observed and a relation is formed. Such equations are known as the constituent equation. Such as Hooke's Law.

If the fluid is Newtonian(which means that stress is directly proportional to strain) and isotropic(properties are same in all directions)

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} + \lambda \vec{\nabla} \cdot \vec{v}$$

$\mu$ -viscosity,  $\lambda = -\frac{2\mu}{3}$  This constant is related to  $\mu$  for many fluids(including air).

## 2.8 Navier-Stokes Equation For Newtonian Isotropic Fluids

$$\begin{aligned} -\frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) &= \rho \frac{du}{dt} \\ -\frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) - \rho g &= \rho \frac{dv}{dt} \\ -\frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) &= \rho \frac{dw}{dt} \\ \frac{du}{dt} &= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \end{aligned}$$

Analyzing the given equations, here's a breakdown of what the individual terms mean,

$$\begin{aligned} \frac{\partial P}{\partial x} & \quad \text{(The force due to the pressure on the fluid)} \\ \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) & \quad \text{(Viscous Forces)} \\ -\rho g & \quad \text{(Gravitational forces)} \end{aligned}$$

Simplying using the gradient or "nabla" operator,

$$\begin{aligned} -\frac{\partial P}{\partial x} \hat{i} - \frac{\partial P}{\partial y} \hat{j} - \frac{\partial P}{\partial z} \hat{k} &= -\vec{\nabla} p \\ \mu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u &= \mu \vec{\nabla} \cdot \vec{\nabla} u \\ \rho \frac{d\vec{v}}{dt} &= \rho \left( \frac{\partial \vec{v}}{\partial t} + u \frac{\partial \vec{v}}{\partial x} + v \frac{\partial \vec{v}}{\partial y} + w \frac{\partial \vec{v}}{\partial z} \right) = \rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right) \end{aligned}$$

Resulting with,

$$-\vec{\nabla} p + \mu \nabla^2 \vec{v} - \rho g \hat{j} = \rho \frac{d\vec{v}}{dt} = \rho \left( \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} \right)$$

The continuity equation is,

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = \frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{v} = 0$$

Refer to *Schaum's Fluid Dynamics Chapter 5* for derivations

## 2.9 Special Flows

### 1. Bernoulli's Equation

$$\frac{d}{dx} \left( p + \rho g + \frac{v^2}{2} \right) = 0$$

2. **Flow between parallel plates** - in this case,  $\vec{v} = u\hat{i}$  only, since the water only flows horizontally, which means  $\frac{\partial u}{\partial x} = 0$ .

$$\begin{aligned}\frac{d\rho}{dt} + \rho(\vec{\nabla} \cdot \vec{v}) &= 0 \\ \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) &= 0 \\ \frac{\partial u}{\partial x} &= 0\end{aligned}$$

The second thing to look at is,

$$-\frac{dp}{dx} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right) = \rho\frac{d\rho}{dt} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

These are fully developed flows

$$\begin{aligned}&= -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} = 0 \\ &u \text{ is a function of } y \\ &-\frac{dp}{dy} + \mu\left(\frac{d^2\rho}{dx^2} + \frac{d^2\rho}{dy^2} + \frac{d^2\rho}{dz^2} - \rho g = \rho\rho\frac{d\rho}{dt} - \rho g\right) = \rho\frac{dv}{dt} \\ &\therefore -\frac{dP}{dy} - \rho g = 0 \\ &\quad -\frac{dP}{dy} = \rho g \\ &\int dp = -\int \rho g dy \\ &p = -\rho gy + c\end{aligned}$$

p is a function of y, but we don't know whether p is a function of x or not. Now,

$$\mu\frac{d^2 u}{dy^2} = \frac{dp}{dx}$$

Since u is a function of y, and pressure is a partial derivative of x, the only possible value for both of them to have is a constant C.

$$\frac{d^2 u}{dy^2} = \frac{C}{\mu}$$

Upon first integration,

$$\frac{du}{dy} = \frac{C}{\mu}y + C_1$$

Upon 2nd integration,

$$U(y) = \frac{c}{2\mu}y^2 + C_1y + C_2$$

Using boundary conditions,

$$U(0) = 0 \text{ (No slip condition)}$$

If the top part of the plates are fixed, then  $U(L) = 0$  but  $U(L) = u_0$  Since,

$$U(y) = \frac{c}{2\mu}y^2 + c_1y + c_2$$

$$U(0) = C_2 \Rightarrow C_2 = 0$$

$$U(L) = \frac{C}{2\mu}L^2 + C_1L + C_2$$

In no slip condition,

$$0 = \frac{C}{2\mu}L^2 + C_1L \Rightarrow \frac{C}{2\mu}L = -C_1$$

If plate is moving,

$$u_0 = \frac{c}{2\mu}L^2 + C_1L$$

$$\frac{u_0}{L} = \frac{c}{2\mu}L + c_1$$

$$c_1 = \frac{u_0}{L} - \frac{cL}{2\mu}$$

$$U(y) = \frac{c}{2\mu}(y^2) + c_1y + c_2 \text{ (Poiseuille flow)}$$

## 2.10 Computational Fluid Dynamics

This is a method to solve the Navier-Stokes equation, and all other fluids equations. This equation is a partial differential equation. We take finite difference approximations (using the Taylor series refer to Maths notes.) The beauty of this method is that every partial differential equation can be approximated to a Taylor series.

And we take this further,

$$A\vec{x} = \vec{b}$$

Where,  $x$  is a vector composed of  $u, v, w, p$ . These components are functions of  $x, y, z$  and  $t$

### 2.10.1 Taylor Series

Let's look at a case where  $U$  is just a function of  $y$ ,  $U(y)$ . What if I wanted to find what  $U(y + \Delta y)$  or some point later. The Taylor series of  $U(y + \Delta y)$  is,

$$U(y + \Delta y) = U(y) + U'(y)\frac{\Delta y}{1!} + U''(y)\frac{\Delta y^2}{2!} + U'''(y)\frac{\Delta y^3}{3!} + \dots$$

For  $U(y - \Delta y)$

$$U(y - \Delta y) = U(y) - U'(y)\frac{\Delta y}{1!} + U''(y)\frac{\Delta y^2}{2!} - U'''(y)\frac{\Delta y^3}{3!} + \dots$$

Let's say I want to find  $U'(y)$

$$U'(y) = \frac{\partial u}{\partial y} = \left[ \frac{U(y + \Delta y) - U(y)}{\Delta y} \right] - \Delta y \left( \frac{U''(y)}{2!} - U'''(y)\frac{\Delta y}{3!} + \dots \right)$$

I can assume the remaining terms after  $\left[ \frac{U(y + \Delta y) - U(y)}{\Delta y} \right]$  to be an error of the order of  $\Delta y$ . What this means is that, the further away I get from  $y$ , the higher the error is going to be. It is denoted by  $O(\Delta y)$

We can find another way to express  $U'(y)$  using the second equation.

$$U'(y) = \frac{U(y) + U(y + \Delta y)}{\Delta y} + O(\Delta y)$$

This is termed as the forward difference approximation

$$U'(y) = \frac{U(y) - U(y - \Delta y)}{\Delta y} - O(\Delta y)$$

This is termed as the backward difference approximation

On subtracting  $U(y - \Delta y)$  from  $U(y + \Delta y)$

$$U(y + \Delta y) - U(y - \Delta y) = 2U'(y)\frac{\Delta y}{1} + 2U'''(y)\frac{\Delta y^3}{3!}$$

An expression for  $U'(y)$  with this equation

$$U'(y) = \frac{U(y + \Delta y) - U(y - \Delta y)}{2\Delta y} - O(\Delta y^2)$$

This is termed as the central difference approximation. With these three approximations, we will solve the problem from earlier. For  $U''(y)$ ,

$$U''(y) = \frac{U(y + \Delta y) - 2U(y) + U(y - \Delta y)}{\Delta y^2}$$

## 2.11 For Unidirectional Flow

Let's look at two cases and solve,

1.  $\frac{\partial p}{\partial x}$ , the pressure gradient is given. We assume, to be general, that the plates are moving with a speed,  $U_h$  and  $U_0$  respectively for the top and bottom.

The equation is,

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial x^2} + \rho g_x = \rho \frac{\partial u}{\partial t}$$

Substituting the equations we found using the Taylor series.

$$-\frac{\partial p}{\partial x} + \mu \left( \frac{U(y + \Delta y) - 2U(y) + U(y - \Delta y)}{\Delta y^2} \right) + \rho g_y = \rho \left( \frac{U(y, t + \Delta t) - U(y, t - \Delta t)}{2\Delta t} \right)$$

Finding the partial derivative due to time,

$$U'' = \frac{U(y + \Delta y, t) - 2U(y, t) + U(y - \Delta y, t)}{\Delta y^2}$$

$$\dot{U} = \frac{U(y, t + \Delta t) - U(y, t - \Delta t)}{2\Delta t}$$

We finally arrive at the final method for solving this. We break each point of the fluid vertically into discrete locations. Termed as  $(U_1, U_2, \dots, U_n)$

$$U_i(t) = \frac{-\partial p}{\partial x} + \mu \left( \frac{U_{i+1} - 2U_i + U_{i-1}}{\Delta y} \right) + \rho g_y$$

As well as,

$$U_i(t) = \rho \left( \frac{U_i(t + \Delta t) - U_i(t - \Delta t)}{2\Delta t} \right)$$

So,

$$U_i(t + \Delta t) = 2\Delta t \left( -\frac{1}{\rho} \frac{\partial p}{\partial x} + g_y \right) + \frac{2\Delta t \mu}{\rho} \left( \frac{U_{i+1} - 2U_i + U_{i-1}}{\Delta y^2} \right) + U_{i-\Delta t}$$

These two equations form the basis of what we are trying to code.

The final equation to be is,

$$U_{new}[i] = \frac{2\Delta t}{\rho} \left( -\frac{\partial p}{\partial x} + \rho g_y \right) + \frac{2\Delta t \mu}{\rho \Delta y^2} (U_{now}[i + 1] - 2U_{now}[i] + U_{now}[i - 1]) + U_{old}[i]$$

What these terms mean,

$$\begin{aligned} \frac{\mu}{\rho} & \quad \text{(Kinematic viscosity)} - \nu \\ \frac{\Delta t \mu}{\rho \Delta y^2} & \quad \text{(Diffusion number)} - \alpha \end{aligned}$$

How do we put boundary conditions, we use the velocities of the plates.

$$U(0, t) = U_0 = U_1(t)$$

$$U(h, t) = U_h = U_{N+1}(t)$$

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34 % Defining constants
35 mu = 0.6;
36 rho = 100;
37 g = 0.4;
38 h = 1;
39 N = 32;
40 deltaY = h/N;
41
42 % Terms in the equation
43 nu = mu/rho; % Kinematic Viscosity
44 alpha = 0.51; % Diffusion Number
45 deltaT = alpha*rho*(deltaY^2)/mu; % Evaluating time step from alpha
46 dpdx = -2; % Pressure Gradient
47
48 % Velocities of the plates
49 U0 = 0;
50 Uh = 0;
51
52 % Time for code to run
53 T = 5;
54 U = zeros(floor(T/deltaT),N+1);
55 % Initial Conditions
56 y = 0:deltaY:h;
57
58 % Equation
59 for k = 2:floor(T/deltaT)
60     U(k,1) = U0;
61     U(k,N+1) = Uh;
62     for i = 2:N
63         U(k,i) = deltaT/rho*(-dpdx + rho*g) + alpha*(U(k-1,i+1)-2*U(k-1,i) + U(k-1,i-1)) + U(k-1,i);
64     end
65 end
66
67 % Plotting
68 % Y axis - y, x axis - U(y), For a given time t
69 plot(U(end,:),y,'blue');

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2.  $Q(t)$  or the flow rate is given.