22MAT220

Mathematics for Computing - 3

Multiple Joint Distributions

- **Multiple Discrete Joint Distributions**

  - Multinomial Distribution

- **Multiple Continuous Joint Distributions**

  - Multinormal Distribution

Multiple Discrete Random Variables

The **joint probability mass function** of $X_1, X_2, \ldots, X_p$ is

$$f_{X_1 X_2 \ldots X_p}(x_1, x_2, \ldots, x_p) = P(X_1 = x_1, X_2 = x_2, \ldots, X_p = x_p)$$

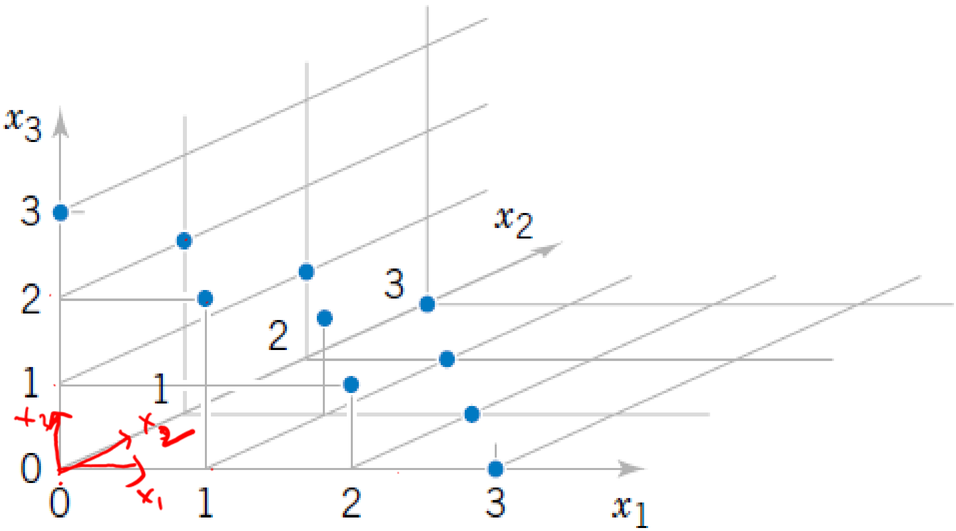
for all points $(x_1, x_2, \ldots, x_p)$ in the range of $X_1, X_2, \ldots, X_p$.

Example: Points that have positive probability in the joint probability distribution of three random variables X_1, X_2, X_3, the range of which is the nonnegative integers with $x_1 + x_2 + x_3 = 3$.

$x_1 = 0, 1, 2, 3$

x_2

x_3



$$\sum_{x_1} \sum_{x_2} \sum_{x_3} f_{x_1,x_2,x_3}(x_1, x_2, x_3) = 1$$

$$0 \le f_{x_1,x_2,x_3}(x_1, x_2, x_3) \le 1$$

Multiple Discrete Random Variables

If $X_1, X_2, X_3, \ldots, X_p$ are discrete random variables with joint probability mass function $f_{X_1X_2\ldots X_p}(x_1, x_2, \ldots, x_p)$, the **marginal probability mass function** of any X_i is

$$f_{X_i}(x_i) = P(X_i = x_i) = \sum_{R_{x_i}} f_{X_1X_2\ldots X_p}(x_1, x_2, \ldots, x_p)$$

where R_{x_i} denotes the set of points in the range of $(X_1, X_2, \ldots, X_p)$ for which $X_i = x_i$.

Points that have positive probability in the joint probability distribution of three random variables X_1, X_2, X_3, the range of which is the nonnegative integers with $x_1 + x_2 + x_3 = 3$. The marginal distribution of X_2 is:

$$P(X_2 = 0) = f_{X_1X_2X_3}(3, 0, 0) + f_{X_1X_2X_3}(0, 0, 3) + f_{X_1X_2X_3}(1, 0, 2) + f_{X_1X_2X_3}(2, 0, 1)$$
$$P(X_2 = 1) = f_{X_1X_2X_3}(2, \underline{1}, 0) + f_{X_1X_2X_3}(0, \underline{1}, 2) + f_{X_1X_2X_3}(1, \underline{1}, 1)$$
$$P(X_2 = 2) = f_{X_1X_2X_3}(1, 2, 0) + f_{X_1X_2X_3}(0, 2, 1)$$
$$P(X_2 = 3) = f_{X_1X_2X_3}(0, 3, 0)$$

$f_{x_2}(x_2=0) =$

x_2	0	1	2	3
$f_{x_2}(x_2)$	$P(x_2=0)$	$P(x_2=1)$	$P(x_2=2)$	$P(x_2=3)$

Mean and Variance of each RV from joint pmf.

$$E(X_i) = \sum_R x_i f_{X_1 X_2 \ldots X_p}(x_1, x_2, \ldots, x_p)$$

X_i | $f_{x_i}(x_i)$

$E(x) = \sum x_i f_{x_i}(x_1, \ldots x_n)$

and

$$V(X_i) = \sum_R (x_i - \mu_{X_i})^2 f_{X_1 X_2 \ldots X_p}(x_1, x_2, \ldots, x_p) = E(X_i^2) - [E(X_i)]^2$$

$= \sum_{x_i} x_i^2 f_{x_i}(x_1 \ldots x_p) - [E(X_i)]^2$

where R is the set of all points in the range of $X_1, X_2, \ldots, X_p$.

Independence of RVs

Discrete variables $X_1, X_2, \ldots, X_p$ are **independent** if and only if

$$f_{X_1 X_2 \ldots X_p}(x_1, x_2, \ldots, x_p) = f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_p}(x_p)$$

for *all* $x_1, x_2, \ldots, x_p$.

Suppose the random variables X, Y, and Z have the following joint probability distribution

x	y	z	$f(x, y, z)$
1	1	1	0.05
1	1	2	0.10
1	2	1	0.15
1	2	2	0.20
2	1	1	0.20
2	1	2	0.15
2	2	1	0.10
2	2	2	0.05

Determine the following:

(a) $P(X = 2)$ (b) $P(X = 1, Y = 2)$

(c) $P(Z < 1.5)$ (d) $P(X = 1 \quad \text{or} \quad Z = 2)$

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1	2	1	0.15
1	2	2	0.20
2	1	1	0.20
2	1	2	0.15
2	2	1	0.10
2	2	2	0.05

Determine the following:

(a) $P(X = 2)$ (b) $P(X = 1, Y = 2)$

(c) $P(Z < 1.5)$ (d) $P(X = 1 \quad \text{or} \quad Z = 2)$

$X = 1, 2$; $Y = 1, 2$; $Z = 1, 2$

(a) $P(X=2) = f(2,1,1) + f(2,1,2) + f(2,2,1) + f(2,2,2)$

$= 0.20 + 0.15 + 0.10 + 0.05$

$= \underline{\underline{0.5}}$

(b) $P(X=1, Y=2) = f(1,2,1) + f(1,2,2) =$

$= 0.15 + 0.20 = \underline{\underline{0.35}}$

(c) $P(Z < 1.5) = P(Z=1) = f(1,1,1) + f(1,2,1) + f(2,1,1) + f(2,2,1) = 0.5 //$

(d) $P(X=1 \underset{\text{union}}{\underset{\downarrow}{\text{or}}} Z=2) = f(1,1,1) + f(1,1,2) + f(1,2,1) + f(1,2,2)$

$+ f(2,1,2) + f(2,2,2)$

$= 0.05 + 0.1 + 0.15 + 0.2 + 0.15 + 0.05$

$= \underline{\underline{0.7}}$

Suppose the random variables *X*, *Y*, and *Z* have the following joint probability distribution

x	y	z	$f(x, y, z)$
1	1	1	0.05
1	1	2	0.10
1	2	1	0.15
1	2	2	0.20
2	1	1	0.20
2	1	2	0.15
2	2	1	0.10
2	2	2	0.05

Find average and variance of all the variables. i.e., E(X), E(Y), E(Z), Var(X) , Var(Y), Var(Z)

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x	y	z	$f(x, y, z)$
1	1	1	0.05
1	1	2	0.10
1	2	1	0.15
1	2	2	0.20
2	1	1	0.20
2	1	2	0.15
2	2	1	0.10
2	2	2	0.05

Find average and variance of all the variables. i.e., E(X), E(Y), E(Z), Var(X) , Var(Y), Var(Z)

Marginal distributions:-

For X:-

x	1	2
$f_x(x)$	$P(x=1) = 0.5$	$P(x=2) = 0.5$

$\left(\sum_x f_x(x) = 1\right)$

$$E(X) = \sum_x x\, f_x(x) = 1(0.5) + 2(0.5) = 1.5$$

$$Var(X) = E(X^2) - (E(X))^2$$
$$= \sum_x x^2 f_x(x) - (1.5)^2 = 1^2(0.5) + 2^2(0.5) - (1.5)^2$$
$$= 0.25$$

For Y:-

Marginal distr. is

y	1	2
$f_y(y)$	0.5	0.5

$$E(Y) = \sum_y y f(y) = 1.5$$

$$Var(Y) = \sum_y y^2 f(y) - [E(Y)]^2 = 0.25$$

For Z:-

Marginal distr.

z	1	2
$f_z(z)$	0.5	0.5

$$E(Z) = 1.5$$

$$Var(Z) = 0.25$$

Suppose the random variables X, Y, and Z have the following joint probability distribution

x	y	z	$f(x, y, z)$
1	1	1	0.05
1	1	2	0.10
1	2	1	0.15
1	2	2	0.20
2	1	1	0.20
2	1	2	0.15
2	2	1	0.10
2	2	2	0.05

Are the RVs independent?

Suppose the random variables X, Y, and Z have the following joint probability distribution

x	y	z	$f(x, y, z)$
1	1	1	0.05
1	1	2	0.10
1	2	1	0.15
1	2	2	0.20
2	1	1	0.20
2	1	2	0.15
2	2	1	0.10
2	2	2	0.05

Are the RVs independent?

X, Y, Z will be independent if and only if

$$f_{xyz}(x, y, z) = f_x(x)\, f_y(y)\, f_z(z)$$

for all values of x, y, z

Check for $x=1, y=1, z=1$

$f_{xyz}(x,y,z) = 0.05$.

$f_x(x=1)\, f_y(y=1)\, f_z(z=1) = (0.5)(0.5)(0.5)$

$= 0.125$.

For $(1,1,1)$ $\quad f_{xyz}(x,y,z) \neq f_x(x)\, f_y(y)\, f_z(z)$

$\therefore$ X, Y, Z are not independent

Multinomial Distribution (Extension of Binomial Distributiom)

Suppose a random experiment consists of a series of n trials. Assume that

(1) The result of each trial is classified into one of k classes.

(2) The probability of a trial generating a result in class 1, class 2, … , class k is constant over the trials and equal to $p_1, p_2, \ldots, p_k$, respectively.

(3) The trials are independent.

The random variables $X_1, X_2, \ldots, X_k$ that denote the number of trials that result in class 1, class 2, … , class k, respectively, have a **multinomial distribution** and the joint probability mass function is

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) = \frac{n!}{x_1! x_2! \cdots x_k!} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$$

for $x_1 + x_2 + \cdots + x_k = n$ and $p_1 + p_2 + \cdots + p_k = 1$.

$$\frac{n!}{\underbrace{x!}_{x_1}\,\underbrace{(n-x)!}_{x_2}}\;\underbrace{p}_{p_1}{}^{x}\;\underbrace{(1-p)}_{p_2}{}^{n-x}.$$

Multinomial Distribution (Extension of Binomial Distributiom)

Of the 20 bits received, what is the probability that 14 are excellent, 3 are good, 2 are fair, and 1 is poor? Assume that the classifications of individual bits are independent events and that the probabilities of *E*, *G*, *F*, and *P* are 0.6, 0.3, 0.08, and 0.02, respectively.

Multinomial Distribution (Extension of Binomial Distributiom)

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follows multinomial

$P(E=\underset{x_1}{14}, G=\underset{x_2}{3}, F=\underset{x_3}{2}, P=\underset{x_4}{1}) = ?$

$P(E) = p_1 = 0.6$

$P(G) = p_2 = 0.3$

$P(F) = p_3 = 0.08$

$P(P) = p_4 = 0.02$

Check: $\sum x_i = n \qquad \sum p_i = 1$

$n = 20$

$n=3$, 2 class

HHH
HHT
HTH
THH
HTT
THT
TTH
TTT

$n = 20$, 4 classes.

$E' - - -$

$$P(E=14, G=3, F=2, P=1) = \frac{n!}{x_1!\, x_2!\, x_3!\, x_4!} p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4}$$

$$= \frac{20!}{14!\,3!\,2!\,1!}(0.6)^{14}(0.3)^{3}(0.08)^{2}(0.02)^{1}$$

$$= 0.00629$$

Find

(b) $P(E=10, G=3, F=5, P=2)$

>> x= [10, 3, 5, 2]

>> p= [0.6, 0.3, 0.08, 0.02]

>> mnpdf(x,p)

Output = 9.963×10^{-5}

(c) P(all bits are of excellent quality) =

>> x=[20,0,0,0]

multinomial pdf

MATLAB Command – 'mnpdf(x,p)'

>> x=[14,3,2,1];p=[0.6,0.3,0.08,0.02];mnpdf(x,p)

Mean and Variance of each RV in multinomial distribution

If $X_1, X_2, \ldots, X_k$ have a multinomial distribution, the marginal probability distribution of X_i is binomial with

$$E(X_i) = np_i \quad \text{and} \quad V(X_i) = np_i(1 - p_i)$$

Binomial RV:-
Mean $= np$
Var $= np(1-p)$

Consider the previous example of transmission of bits. Assume that the classifications of individual bits are independent events and that the probabilities of *E*, *G*, *F*, and *P* are 0.6, 0.3, 0.08, and 0.02, respectively. Find the mean and variance of each of E, G, F and P.

$n = 20, \quad p_e = 0.6, \quad p_g = 0.3, \quad p_f = 0.08, \quad p_p = 0.02$

$E(E) = np_e = 20(0.6) = 12, \quad Var(E) = np_e(1-p_e) = 12(0.4) = 4.8$

$E(G) = np_g = 20(0.3) = 6, \quad Var(G) = np_g(1-p_g) = 6(1-0.3) = 4.2$

$E(F) = np_f = 20(0.08) = 1.6, \quad Var(F) = np_f(1-p_f) = 20(0.08)(1-0.08) = 1.472$

$E(P) = np_p = 20(0.02) = 0.4, \quad Var(P) = np_p(1-p_p) = 0.4(0.98) = 0.392$

Four electronic ovens that were dropped during shipment are inspected and classified as containing either a major, a minor, or no defect. In the past, 60% of dropped ovens had a major defect, 30% had a minor defect, and 10% had no defect. Assume that the defects on the four ovens occur independently.

(a) What is the probability that all the four ovens dropped are having major defect?

(b) What is the probability that 2 ovens have no defect, I has a major defect and one has a minor defect?

(c) What is the probability that, of the four dropped ovens, two have a major defect and two have a minor defect?

(d) What is the probability that no oven has a defect?

$n = 4$, 3 classes $\rightarrow M_j, M_i, N$ (major df, min dif, No Defect), $p_1 = P(M_j) = 0.6$; $p_2 = P(M_i) = 0.3$; $p_3 = P(N) = 0.1$

$(p_1 + p_2 + p_3 = 1)$

$\Rightarrow$ Also defects occur independently $\Rightarrow$ M_j, M_i, N follows multinomial distr.

(a) $P(M_j = 4, M_i = 0, N = 0) = 0.1296$ $\rightarrow$ `>> x=[4,0,0]; p=[0.6,0.3,0.1]; mnpdf(x,p)`

(b) $P[M_j = 1, M_i = 1, N = 2) = 0.0216$

(c) $P[M_j = 2, M_i = 2, N = 0) = 0.1944$

(d) $P[M_j = 0, M_i = 0, N = 4] = 0.0001$

(e) Find mean and variance of M_j, M_i & N.

$E(M_j) = np_1 = 4 \times 0.6 = 2.4$

$E(M_i) = np_2 = 4 \times 0.3 = 1.2$

$E(N) = np_3 = 4 \times 0.1 = 0.4$

$Var(M_j) = np_1(1-p_1) = 0.96$

$Var(M_i) = np_2(1-p_2) = 0.84$

$Var(N) = np_3(1-p_3) = 0.36$

Homework:

1. In the transmission of digital information, the probability that a bit has high, moderate, or low distortion is 0.01, 0.04, and 0.95, respectively. Suppose that three bits are transmitted and that the amount of distortion of each bit is assumed to be independent.

(a) What is the probability that two bits have high distortion and one has moderate distortion?

(b) What is the probability that all three bits have low distortion?

(c) What is the probability that all three bits have moderate distortion?

(d) What is the probability that all three bits have high distortion?

(e) If X and Y denote the number of bits with high and moderate distortion, find the mean and variance of X and Y.

Multiple Continuous Random Variables

A **joint probability density function** for the continuous random variables $X_1, X_2, X_3, \ldots, X_p$, denoted as $f_{X_1X_2\ldots X_p}(x_1, x_2, \ldots, x_p)$, satisfies the following properties:

(1) $f_{X_1X_2\ldots X_p}(x_1, x_2, \ldots, x_p) \geq 0$

(2) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_{X_1X_2\ldots X_p}(x_1, x_2, \ldots, x_p)\, dx_1\, dx_2 \ldots dx_p = 1$

(3) For any region B of p-dimensional space

$$P[(X_1, X_2, \ldots, X_p) \in B] = \int\int \cdots \int f_{X_1X_2\ldots X_p}(x_1, x_2, \ldots, x_p)\, dx_1\, dx_2 \ldots dx_p \quad (5\text{-}22)$$

In an electronic assembly, let the random variables X_1, X_2, X_3, X_4 denote the lifetimes of four components in hours. Suppose that the joint probability density function of these variables is

$$f_{X_1X_2X_3X_4}(x_1, x_2, x_3, x_4) = 9 \times 10^{-2} e^{-0.001x_1 - 0.002x_2 - 0.0015x_3 - 0.003x_4}$$
$$\text{for } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0$$

Find the probability that X1 < 1000 and X2 < 2000, X3<1000, X4<2000

$$P(X_1 < 1000, X_2 < 2000, X_3 < 1000, X_4 < 2000)$$
$$= \int_{x_4=0}^{2000} \int_{x_3=0}^{1000} \int_{x_2=0}^{2000} \int_{x_1=0}^{1000} f(x_1, x_2, x_3, x_4)\, dx_1\, dx_2\, dx_3\, dx_4 =$$

Find marginal density of X_1 :- $f_{x_1}(x_1, x_2, x_3, x_4) = \int_{x_4=0}^{\infty} \int_{x_3=0}^{\infty} \int_{x_2=0}^{\infty} 9 \times 10^{-2} \times e^{-0.001x_1 - 0.002x_2 - 0.0015x_3 - 0.003x_4}$

$=$

If the joint probability density function of continuous random variables $X_1, X_2, \ldots, X_p$ is $f_{X_1 X_2 \ldots X_p}(x_1, x_2 \ldots, x_p)$ the **marginal probability density function** of X_i is

$$f_{X_i}(x_i) = \iint\limits_{R_{x_i}} \ldots \int f_{X_1 X_2 \ldots X_p}(x_1, x_2, \ldots, x_p)\, d\underline{x_1}\, d\underline{x_2} \ldots d\underline{x_{i-1}}\, d\underline{x_{i+1}} \ldots dx_p$$

where R_{x_i} denotes the set of all points in the range of $X_1, X_2, \ldots, X_p$ for which $X_i = x_i$.

$= \int_{x_i} x_i f_{x_i}(x_1, \ldots x_p) \longrightarrow$ if marginal density is found

$$E(X_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} x_i f_{X_1 X_2 \ldots X_p}(x_1, x_2, \ldots, x_p)\, dx_1\, dx_2 \ldots dx_p$$

$$V(X_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} (x_i - \mu_{X_i})^2 f_{X_1 X_2 \ldots X_p}(x_1, x_2, \ldots, x_p)\, dx_1\, dx_2 \ldots dx_p$$

Continuous random variables $X_1, X_2, \ldots, X_p$ are **independent** if and only if

$$f_{X_1 X_2 \ldots X_p}(x_1, x_2 \ldots, x_p) = f_{X_1}(x_1) f_{X_2}(x_2) \ldots f_{X_p}(x_p) \quad \text{for } \textit{all } x_1, x_2, \ldots, x_p$$

1. Suppose the random variables *X*, *Y*, and *Z* have the joint probability density function: $f(x,y,z) = 8xyz$, for $0 < x < 1$, $0 < y < 1$, and $0 < z < 1$. Determine the following:

(a) $P(X < 0.5)$ (b) $P(X < 0.5, Y < 0.5)$

(c) $P(Z < 2)$ (d) $P(X < 0.5 \text{ or } Z < 2)$

(e) $E(X)$

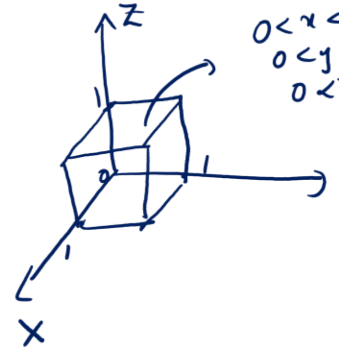
1. Suppose the random variables X, Y, and Z have the joint probability density function:
$f(x,y,z) = 8xyz$, for $0 < x < 1$, $0 < y < 1$, and $0 < z < 1$. Determine the following:

(i) Verify $f(x,y,z)$ is a joint pdf.

(a) $P(X < 0.5)$ (b) $P(X < 0.5, Y < 0.5)$

(c) $P(Z < 2)$ (d) $P(X < 0.5$ or $Z < 2)$

(e) $E(X)$ (f) $P(X < 0.5$ and $Z < 2)$



$0<x<1$, $0<y<1$, $0<z<1$ } Region inside the cube.

(i) $\iiint f(x,y,z)\,dx\,dy\,dz = 1$ (To prove) Also $f(x,y,z) \geq 0$ for $0<x<1$, $0<y<1$, $0<z<1$

$$\int_0^1\int_0^1\underbrace{\int_0^1 8xyz\,dx}\,dy\,dz$$

$$= \int_0^1\int_0^1 \left(4x^2yz\right)_0^1 dy\,dz = \int_0^1\underbrace{\int_0^1 4yz\,dy}\,dz$$

$$= \int_0^1 \left(2y^2z\right)_0^1 dz = \int_0^1 2z\,dz = z^2\Big)_0^1 = \underline{\underline{1}}$$

$\Longrightarrow$ Given $f(x,y,z)$ is a joint pdf of X, Y & Z.

(a) $P(X<0.5) = P(0<X<0.5,\ 0<Y<1,\ 0<Z<1)$

$$= \int_{x=0}^{0.5}\int_{y=0}^{1}\int_{z=0}^{1} 8xyz\,dz\,dy\,dx = \int_0^{0.5} 2x\,dx = x^2\Big)_0^{0.5} = \underline{\underline{0.25}}$$

(b) $P(X<0.5,\ Y<0.5) = P(0<X<0.5,\ 0<Y<0.5,\ 0<Z<1)$

$$= \int_{x=0}^{0.5}\int_{y=0}^{0.5}\int_{z=0}^{1} 8xyz\, dz\, dy\, dx = \int_0^{0.5}\int_0^{0.5} 4yx\, dy\, dx.$$

$$= \int_{x=0}^{0.5} \left(2y^2 x\right)_0^{0.5} dx = \int_0^{0.5} 0.5x\, dx = \left.\frac{0.5\, x^2}{2}\right)_0^{0.5} = \frac{1}{4}\times\frac{1}{4} = 0.0625$$

(c) $P(Z<2) = P(0<X<1,\ 0<Y<1,\ 0<Z<1) = 1$

(d) $P(\underbrace{X<0.5 \text{ or } Z<2}) = P(\underbrace{0<X<1,\ 0<Y<1,\ 0<Z<1}) = 1$

Union will be $\longrightarrow$

(f) $P(X<0.5 \underset{\cap}{\text{and}} Z<2) = P(X<0.5) = 0.25$ (in (a))

(e) $E(X) = \int x\, f_X(x)\, dx$

To find marginal density $f_x(x)$

$$f_x(x) = \int_{y=0}^{1}\int_{z=0}^{1} f(x,y,z)\,dz\,dy = \int_{y=0}^{1}\int_{z=0}^{1} 8xyz\,dz\,dy$$

$$= \int_{y=0}^{1} 4xy\,dy = 2xy^2\Big)_0^1 = 2x$$

$$E(X) = \int_{x=0}^{1} x f_x(x)\,dx = \int_{x=0}^{1} x(2x)\,dx = \frac{2}{3}$$

(h) Are X, Y, Z independent?

$$f_x(x) f_y(y) f_z(z)$$
$$= 2x \times 2y \times 2z$$
$$= 8xyz$$
$$= f(x,y,z)$$

$\therefore X, Y$ & Z are independent.

(g) Also find $E(Y)$ and $E(Z)$.

$$f_y(y) = \int_{x=0}^{1}\int_{z=0}^{1} 8xyz\,dz\,dy = 2y$$

$$E(Y) = \int_{y=0}^{1} y f_y(y)\,dy = \frac{2}{3}$$

$$f_z(z) = \int_{x=0}^{1}\int_{y=0}^{1} 8xyz\,dy\,dx = 2z$$

$$E(Z) = \int_{z=0}^{1} z f_z(z) = \frac{2}{3}$$

Homework

Let X, Y and Z be three jointly continuous random variables with joint PDF

$$f_{XYZ}(x,y,z) = \begin{cases} \frac{1}{3}(x+2y+3z) & 0 \leq x,y,z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the joint PDF of X and Y, $f_{XY}(x,y)$.

Solution

$$\begin{aligned} f_{XY}(x,y) &= \int_{-\infty}^{\infty} f_{XYZ}(x,y,z)dz \\ &= \int_0^1 \frac{1}{3}(x+2y+3z)dz \\ &= \frac{1}{3}\left[(x+2y)z + \frac{3}{2}z^2\right]_0^1 \\ &= \frac{1}{3}\left(x+2y+\frac{3}{2}\right), \qquad \text{for} \quad 0 \leq x,y \leq 1. \end{aligned}$$

Thus,

$$f_{XY}(x,y) = \begin{cases} \frac{1}{3}\left(x+2y+\frac{3}{2}\right) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Homework

1. Let X, Y and Z be three jointly continuous random variables with joint PDF

$$f_{XYZ}(x,y,z) = \begin{cases} c(x+2y+3z) & 0 \leq x,y,z \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

1. Find the constant c.
2. Find the marginal PDF of X.

Multinormal Distribution

Normal Distribution

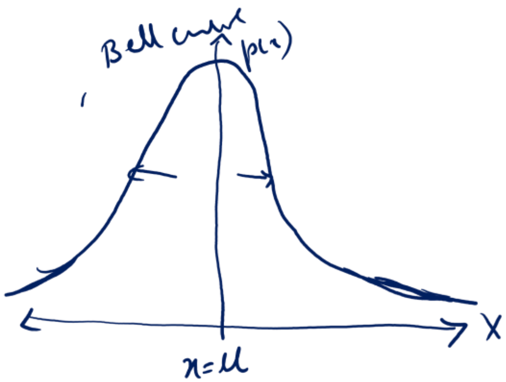
The normal probability density $p(x)$ (the Gaussian) depends on only two numbers

Mean $\overset{\mu}{m}$ **and variance** σ^2

$$p(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-(x-\overset{\mu}{m})^2/2\sigma^2}$$

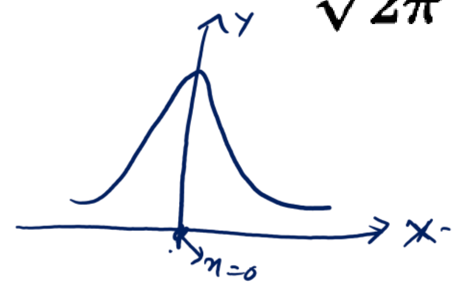
$-\infty < x < \infty$

The standard normal distribution $N(\overset{\mu}{0}, \overset{\sigma}{1})$ **has** $p(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$



$X \sim N(\mu, \sigma)$

$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$



$\boxed{P(x<5) = \text{normcdf}(5, \mu, \sigma)}$

Two-Dimensional Gaussian/Normal Distribution

When X and Y are independent Normal RVs

$\mu_x = E(X)$
$\sigma_x^2 = Var(X)$
$\mu_y = E(Y)$
$\sigma_y^2 = Var(Y)$

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{\frac{-(x-\mu_x)^2}{2\sigma_x^2}} \quad ; \quad f_y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{\frac{-(y-\mu_y)^2}{2\sigma_y^2}}$$

The joint density of X, Y is :-

$$f(x,y) = f_x(x) \cdot f_y(y)$$

$$\Rightarrow f(x,y) = \frac{1}{(\sqrt{2\pi})^2} \cdot \frac{1}{(\sigma_x^2 \sigma_y^2)^{1/2}} e^{\frac{-1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]} \quad \text{——— (1)}$$

Let V be the covariance matrix of X and Y.

$\boxed{\Sigma \mid V}$ i.e, $V = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix}$ ($\because \sigma_{xy} = 0$ as X & Y are independent & hence covariance & correlation is 0)

$$|V| = \sigma_x^2 \sigma_y^2 \quad \text{——— (2)}$$

$$V^{-1} = \frac{1}{\sigma_x^2 \sigma_y^2} \begin{bmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_x^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma_x^2} & 0 \\ 0 & \frac{1}{\sigma_y^2} \end{bmatrix}$$

If $X = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$, then $(X-\mu)^T V^{-1} (X-\mu) = (x-\mu_x \quad y-\mu_y) \begin{bmatrix} 1/\sigma_x^2 & 0 \\ 0 & 1/\sigma_y^2 \end{bmatrix} \begin{pmatrix} x-\mu_x \\ y-\mu_y \end{pmatrix}$

$$= \left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2 \quad \text{——— (3)}$$

When X and Y are independent Normal RVs

Substituting (2) and (3) in (1), we get the joint pdf of independent x & y as :-

$$f(x,y) = \frac{1}{(\sqrt{2\pi})^2 |V|^{1/2}} e^{-\frac{1}{2}\left[(x-\mu)^T V^{-1} (x-\mu)\right]} , \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < y < \infty \end{array}$$

This can be extended to m variables $x_1, x_2, \ldots x_m$ to get the joint pdf as

$$f(x_1, x_2, \ldots x_m) = \frac{1}{(\sqrt{2\pi})^m |V|^{1/2}} e^{-\frac{1}{2}\left[(x-\mu)^T V^{-1} (x-\mu)\right]} //$$

where
$$V = \begin{bmatrix} \sigma_{x_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{x_2}^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \sigma_{x_m}^2 \end{bmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_{x_1} \\ \mu_{x_2} \\ \vdots \\ \mu_{x_m} \end{pmatrix}$$

Multivariate Gaussian probability distribution

$$p(x) = \frac{1}{(\sqrt{2\pi})^M \sqrt{\det V}} e^{-(x-\overset{\mu}{m})^T V^{-1} (x-\overset{\mu}{m})/2}$$

When we have **Dependent** Normal RVs $X_1, X_2, \ldots X_m$

Even in this case, the exponent term with V^{-1} is still correct when *V is* no longer a diagonal matrix and the joint pdf of the multi normal distribution is:

$$f(x_1, x_2, \ldots, x_m) = \frac{1}{\left(\sqrt{2\pi}\right)^m |V|^{\frac{1}{2}}} e^{-\frac{1}{2}(X-\mu)^T V^{-1}(X-\mu)}$$

This is proved using calculus in the given video: https://www.youtube.com/watch?v=YgExEVji7xs
Interested students can view this video.

The general formula for the n-dimensional normal density is

$$f_{\underline{X}}(x_1, \ldots, x_n) = \frac{1}{(\sqrt{2\pi})^n} \frac{1}{\sqrt{\det K}} \exp\left[-\frac{1}{2}(\underline{x} - \underline{\mu})' K^{-1} (\underline{x} - \underline{\mu})\right]$$

where $E(\underline{X}) = \underline{\mu}$ and K is the covariance matrix of $\underline{X}$. We specialize to the case $n = 2$:

$$V = K = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}, \quad \sigma_{12} = \mathrm{Cov}(X_1, X_2);$$

$$K^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix} = \frac{1}{1-\rho^2} \begin{bmatrix} 1/\sigma_1^2 & -\rho/\sigma_1\sigma_2 \\ -\rho/\sigma_1\sigma_2 & 1/\sigma_2^2 \end{bmatrix}.$$

Thus the joint density of X_1 and X_2 is

$$\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]\right\}.$$

It can also be understood with the help of Linear Algebra in the following manner

When $X_1, X_2, \ldots X_m$ are not independent, V is a non-diagonal matrix.
But V is symmetric and also positive definite. $\Longrightarrow$ $V = QDQ^T$ (spectral decomposition)
Q = Orthogonal
D = Diagonal.

$\therefore V^{-1} = (QDQ^T)^{-1} = QD^{-1}Q^T$ ($\because Q^T = Q^{-1}$ for orthogonal matrix Q).

Let $Z = X - \mu$.

Then $(X-\mu)^T V^{-1}(X-\mu) = Z^T V^{-1} Z = Z^T(QD^{-1}Q^T)Z$
$= (Z^TQ)D^{-1}(Q^TZ)$
$= Y^T D^{-1} Y$ (by taking $Y = Q^TZ$)

The combinations $Y = Q^TZ = Q^T(X-\mu)$ are statistically independent (as Q is orthogonal) with D as the diagonal matrix.

This step of diagonalizing V by its eigenvector matrix Q is the same as "uncorrelating" the random variables. Covariances are zero for the new variables $Y_1, \ldots Y_M$. This is the point where linear algebra helps calculus to compute multidimensional integrals.

This step of diagonalizing V by its eigenvector matrix Q is the same as "uncorrelating" the random variables. Covariances are zero for the new variables $Y_1, \ldots Y_M$. This is the point where linear algebra helps calculus to compute multidimensional integrals.

$$\int \ldots \int e^{-\boldsymbol{Y}^{\mathbf{T}}\boldsymbol{\Lambda}^{-\mathbf{1}}\boldsymbol{Y}/\mathbf{2}}\, \boldsymbol{dY} = \left(\int_{-\infty}^{\infty} e^{-y_1^2/2\lambda_1}\, dy_1 \right) \ldots \left(\int_{-\infty}^{\infty} e^{-y_M^2/2\lambda_M}\, dy_M \right)$$

$$= \left(\sqrt{2\pi\lambda_1}\right) \ldots \left(\sqrt{2\pi\lambda_M}\right) = \left(\sqrt{2\pi}\right)^M \sqrt{\det V}. \qquad (7)$$

$$f(x_1, x_2, \ldots, x_m) = \frac{1}{\left(\sqrt{2\pi}\right)^m |V|^{\frac{1}{2}}} e^{-\frac{1}{2}(X-\mu)^T V^{-1} (X-\mu)}$$

MATLAB commands to find probabilities for multivariate normal distributions:

$F(x) = P(X \le x)$
$F(a,b,c) = P(X_1 \le a, X_2 \le b, X_3 \le c)$

>> Y = mvncdf(X,MU,SIGMA) returns the cumulative probability of the multivariate normal distribution with mean MU and covariance SIGMA

>>Y = mvncdf(XL,XU,MU,SIGMA) returns the multivariate normal cumulative probability evaluated over the rectangle (hyper-rectangle for $D>2$) with lower and upper limits defined by XL and XU, respectively.

Q1. If X, Y and Z are normal random variables with mean 2, 5 and -1, respectively and a covariance matrix, $V_{xyz} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. Evaluate:

$V \equiv$ Covariance matrix, $\Sigma \stackrel{?}{=}$ $= \begin{bmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{bmatrix}$

(a) $P(X<3, Y<4, Z<1)$

```
>> X=[3,4,1];Mu=[2,5,-1];V=[2,1,1;1,2,1;1,1,2];mvncdf(X, Mu, V)
ans =
   0.2223
```

(b) $P(-1<X<3, 1<Y<4, 0<Z<1)$

```
>> XL=[-1,1,0];XU=[3,4,1];Mu=[2,5,-1];V=[2,1,1;1,2,1;1,1,2];mvncdf(XL, XU, Mu, V)
ans =
   0.0106
```

Let $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ be a normal random vector with the following mean vector and covariance matrix

$$\mathbf{m} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$

Evaluate:

(a) $P(-1<X_1<1, 1<X_2<2)$, (b) $P(X_1<3)$

(Whitening/decorrelating transformation) Let $\mathbf{X}$ be an n-dimensional zero-mean random vector. Since C_X is a real symmetric matrix, we conclude that it can be diagonalized. That is, there exists an n by n matrix $\mathbf{Q}$ such that

$$QQ^T = I \quad (I \text{ is the identity matrix}),$$
$$C_X = QDQ^T,$$

where D is a diagonal matrix

$$D = \begin{bmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & d_{nn} \end{bmatrix}.$$

Now suppose we define a new random vector $\mathbf{Y}$ as $Y = Q^T X$, thus

$$X = QY.$$

Show that $\mathbf{Y}$ has a diagonal covariance matrix, and conclude that components of $\mathbf{Y}$ are uncorrelated, i.e., $\mathrm{Cov}(Y_i, Y_j) = 0$ if $i \neq j$.

Solution

$$\begin{aligned} C_Y &= E[(Y - E(Y))(Y - E(Y))^T] \\ &= E[(Q^T X - E(Q^T X))(Q^T X - E(Q^T X))^T] \\ &= E[Q^T (X - E(X))(X - E(X))^T]Q] \\ &= Q^T C_X Q \\ &= Q^T Q D Q^T Q \\ &= D \quad (\text{since} \quad Q^T Q = I). \end{aligned}$$

Therefore, $\mathbf{Y}$ has a diagonal covariance matrix, and $\mathrm{Cov}(Y_i, Y_j) = 0$ if $i \neq j$.