Modelling, Simulation, and Analysis

Subject Code: 23AID201

Session 2024-25 (Third Semester)

Section F: AI&DS

Lecture Contents after Mid-Semester

“Topic: Frequency Response and Bode Plots”

Read the from Ogata**, Chapter 7 from pages 398 to 427.**

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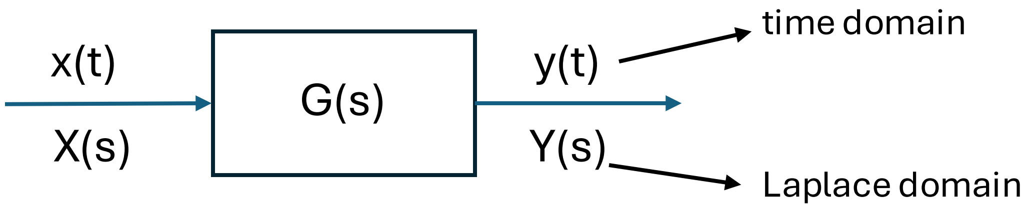
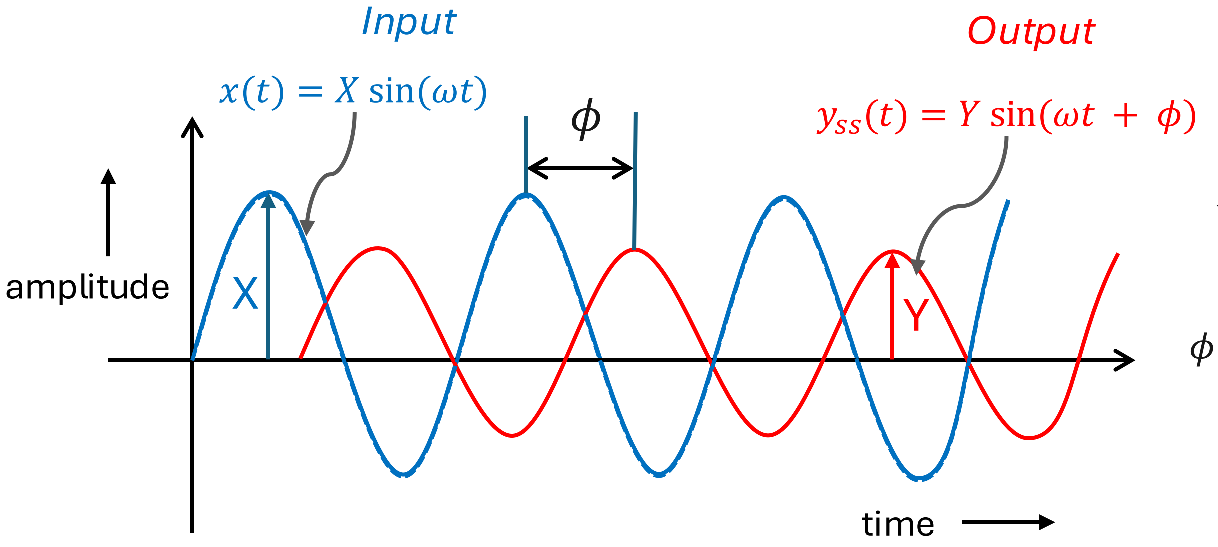
Specialization in Robotics in Healthcare and Rehabilitation

Frequency Response



System dynamics embedded in G(s) is

- Linear
- Time-invariant
- Stable



$$Y = X * |G(j\omega)|$$

$$\phi = tan^{-1}\left(\frac{Im(G(j\omega))}{Re(G(j\omega))}\right)$$

The output y_{ss}(t) has three characteristics:

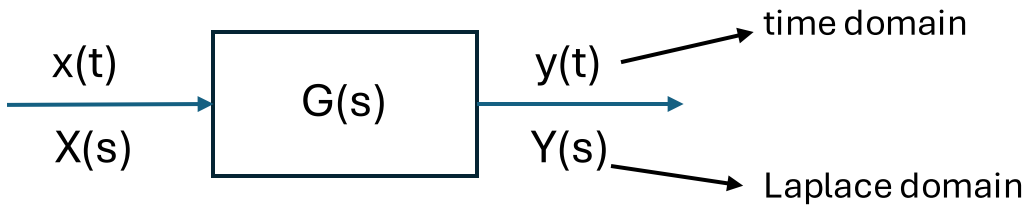
- It has the **same frequency** as the input sinusoid.
- The amplitude of the y_{ss}(t), i.e., **Y is 'scaled' factor of the amplitude if input (X)**. The scaling factor depends on $|G(j\omega)|$.
- The output y_{ss}(t) is phase shifted relative to the input x(t) given by ϕ.

y_{ss} : steady state response/output

Read derivation of y_{ss} for a sinusoidal input (Chp 7, page 398-402).

Frequency Response

Example 7-1 (Ogata):



$$G(s) = \frac{K}{Ts + 1}$$

Q. Find the steady state response for a sinusoidal input?

$x(t) = X \sin(\omega t)$ → *Input*

$y_{ss}(t) = Y \sin(\omega t + \phi)$ → *Output*

$$\begin{cases} Y = X * |G(j\omega)| \\ \phi = tan^{-1}\left(\dfrac{Im(G(j\omega))}{Re(G(j\omega))}\right) \end{cases}$$

$$G(j\omega) = \frac{K}{T(j\omega) + 1} = \frac{K}{1 + j(\omega T)}$$

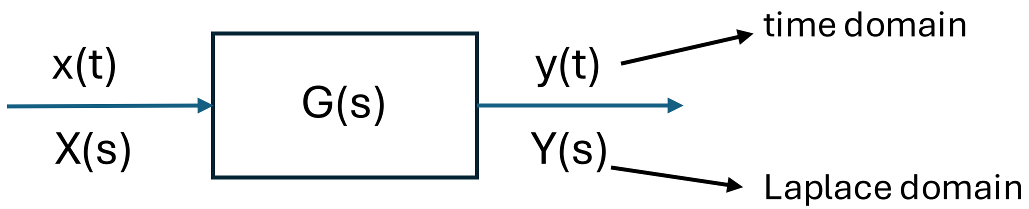
$$|G(j\omega)| = \frac{K}{\sqrt{1 + \omega^2 T^2}}$$

$$\emptyset = ang\big(G(j\omega)\big) = \tan^{-1}\frac{Im(num)}{Re(num)} - \tan^{-1}\frac{Im(den)}{Re(den)}$$

$$\emptyset = \tan^{-1}\frac{0}{K} - \tan^{-1}\frac{\omega T}{1} = -\tan^{-1}\omega T$$

$$y_{ss}(t) = \frac{XK}{\sqrt{1 + \omega^2 T^2}} \sin(\omega t - \tan^{-1}\omega T)$$

Bode Plots



System modelling transfer function, G(s)

Substitute ‘$j\omega$’ in place of ‘s’

$G(j\omega)$

$x(t) = X \sin(\omega t)$ → *Input*

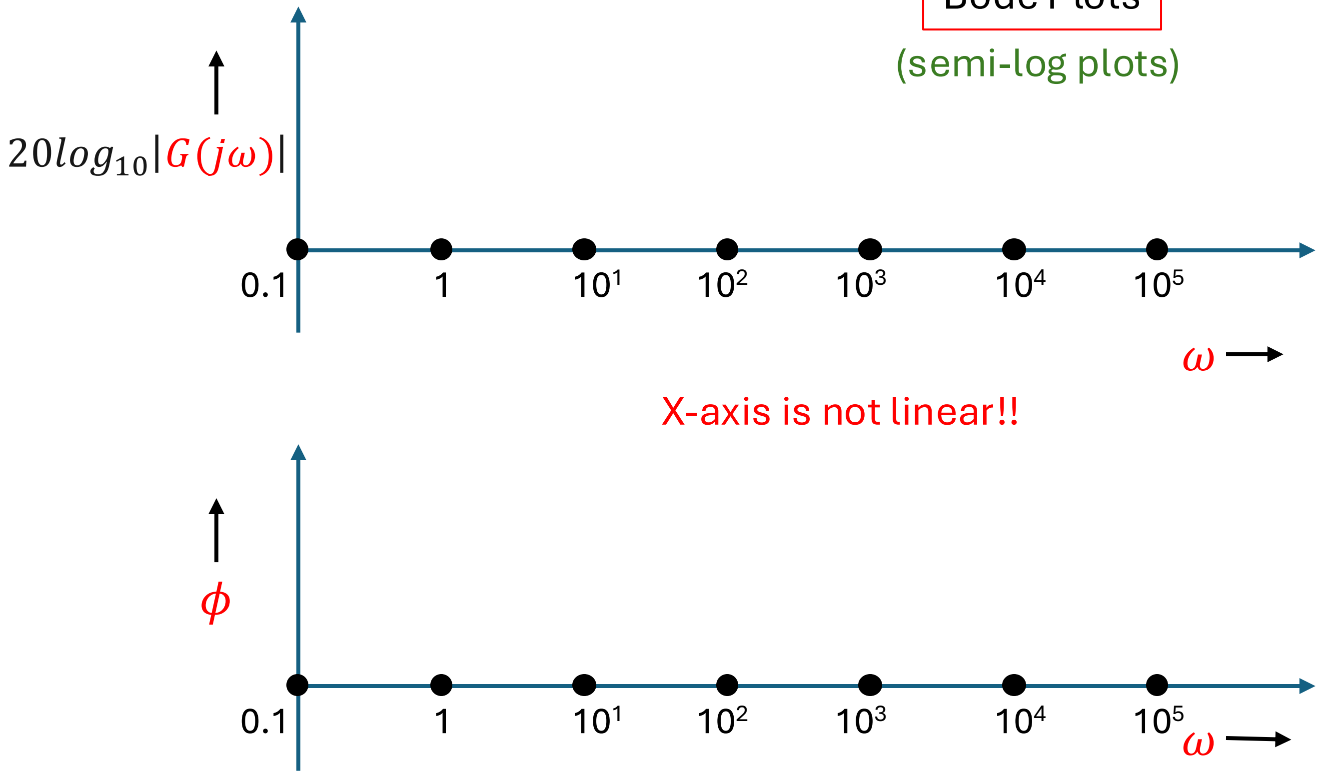
$y_{ss}(t) = Y \sin(\omega t + \phi)$ → *Output*

$$Y = X * |G(j\omega)|$$

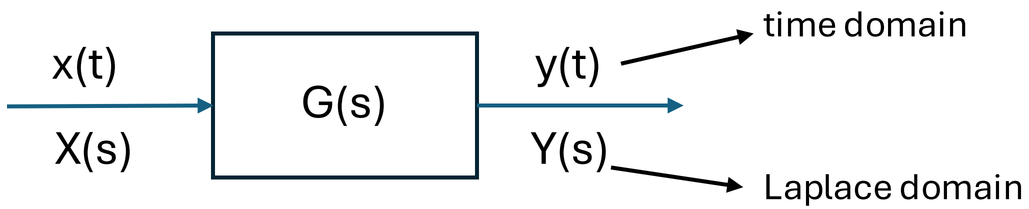
$$\phi = tan^{-1}\left(\frac{Im(G(j\omega))}{Re(G(j\omega))}\right)$$

Bode Plots

(semi-log plots)



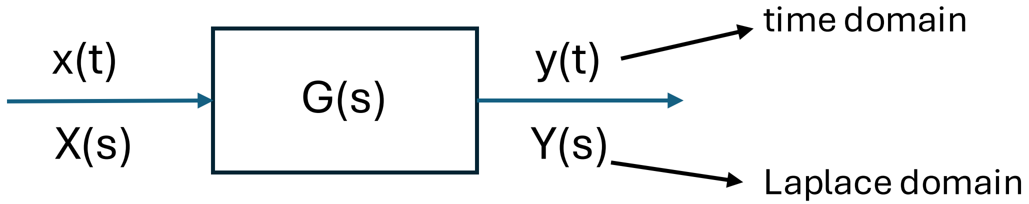
Bode Plots



Standard Forms of G(s)

1. K **Gain**

Bode Plots

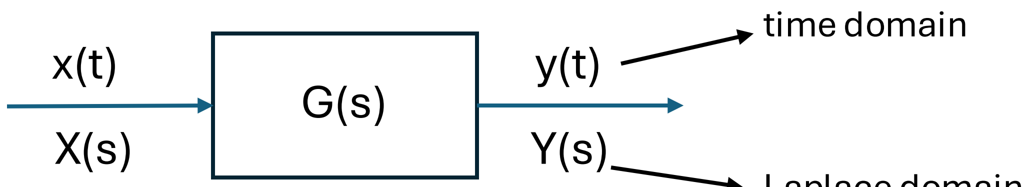


Standard Forms of G(s)

1 $\boldsymbol{K}$ **Gain**

2 $\boldsymbol{\frac{1}{s}}$, $\boldsymbol{s}$ **Integrator, Differentiator**

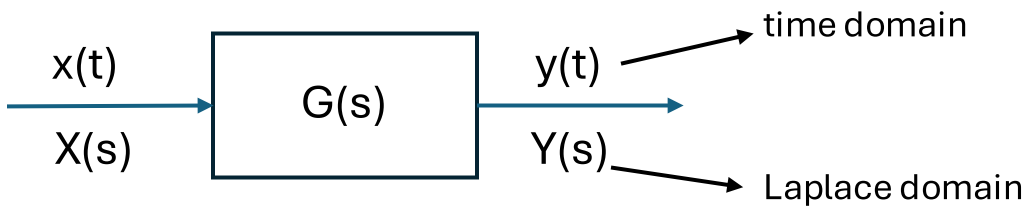
Bode Plots



Standard Forms of G(s)

1. K **Gain**
2. $\frac{1}{s}$, s **Integrator, Differentiator**
3. $\frac{1}{1+sT}$, $1+sT$ **Generic 1st Order System**

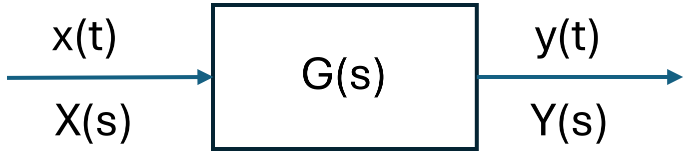
Bode Plots



Standard Forms of G(s)

1. K **Gain**
2. $\frac{1}{s}, s$ **Integrator, Differentiator**
3. $\frac{1}{1+sT}, 1+sT$ **Generic 1st Order System**
4. $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}, s^2+2\xi\omega_n s+\omega_n^2$ **Generic 2nd Order System**

Bode Plots



Standard Forms of G(s)

1 $\quad K \quad$ **Gain**

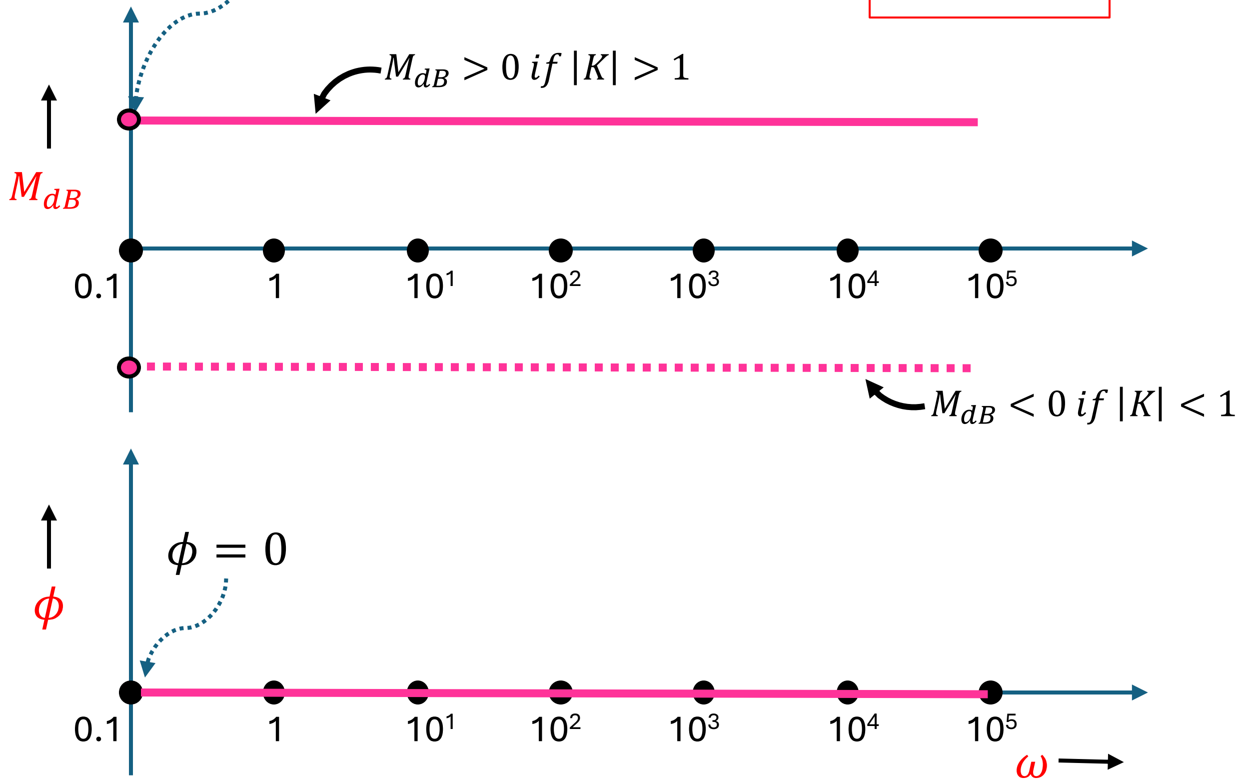
Case 1: G(s) = K

$G(j\omega) = K \leftarrow$ *Substitute s = $j\omega$*

$$M_{dB} = 20\ log_{10}|G(j\omega)|$$
$$= 20\ log_{10}|K|$$

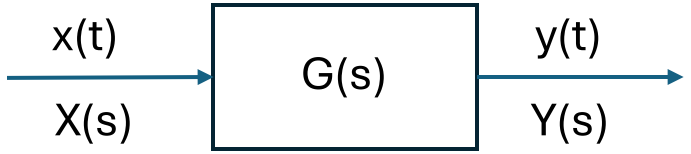
$$\phi = \tan^{-1}\frac{0}{K} = 0$$

Bode Plots



Bode Plots

Case 2: $G(s) = s$ or $G(s) = \frac{1}{s}$



Standard Forms of G(s)

2 $\frac{1}{s}, s$ **Integrator, Differentiator**

$$G(s) = s \implies G(j\omega) = j\omega$$

$$M_{dB} = 20\, log_{10}|j\omega| = \mathbf{20 log_{10}\omega}$$

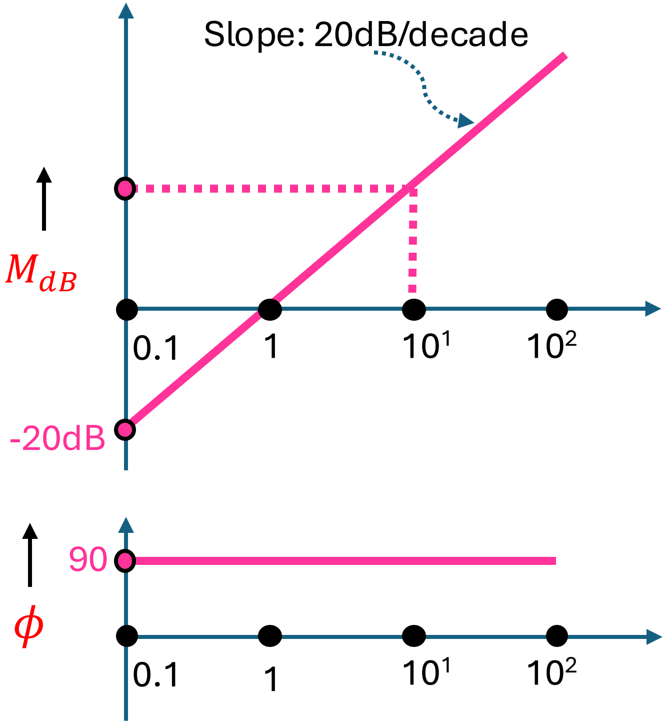
$$\phi = \tan^{-1}\frac{\omega}{0} = \mathbf{90°}$$

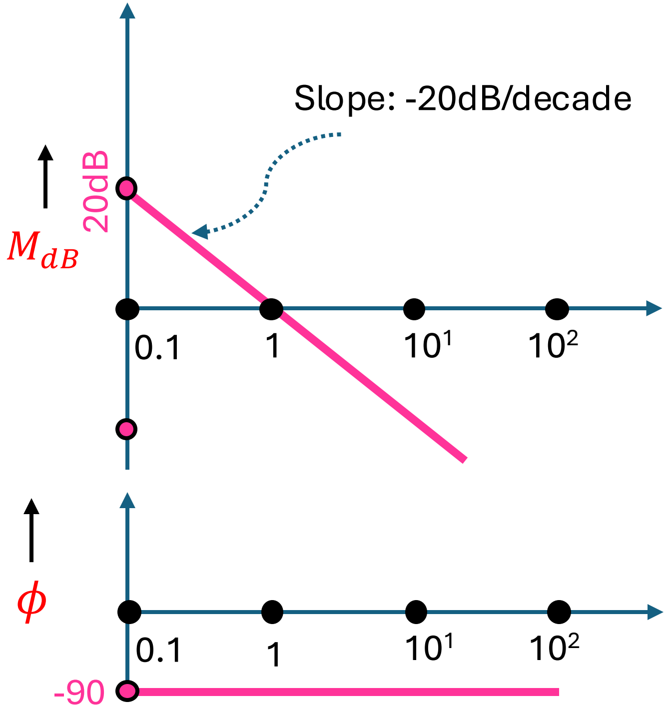
$$G(s) = \frac{1}{s} \implies G(j\omega) = \frac{1}{j\omega} = -\frac{1}{\omega}j$$

$$M_{dB} = 20\, log_{10}\left|\frac{1}{j\omega}\right| = \mathbf{-20\, log_{10}\omega}$$

$$\phi = \tan^{-1}\frac{-1/\omega}{0} = \mathbf{-90°}$$

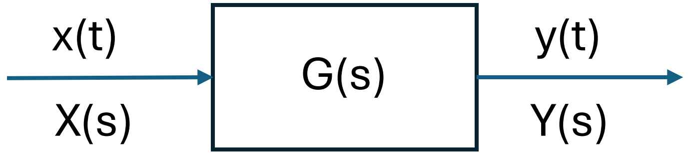
Bode Plots





Bode Plots

Case 3a: $G(s) = \dfrac{1}{1+sT}$



Standard Forms of G(s)

3

Generic 1st Order System

$$\frac{1}{1+sT}\ ,\ 1+sT$$

$$G(s) = \frac{1}{1+\ sT} \implies G(j\omega) = \frac{1}{1+\ j\omega T}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\ \omega^2T^2}}$$

$$M_{dB} = -20\ log_{10}\sqrt{1+\omega^2T^2}$$

$$\phi = -\tan^{-1}\omega T$$

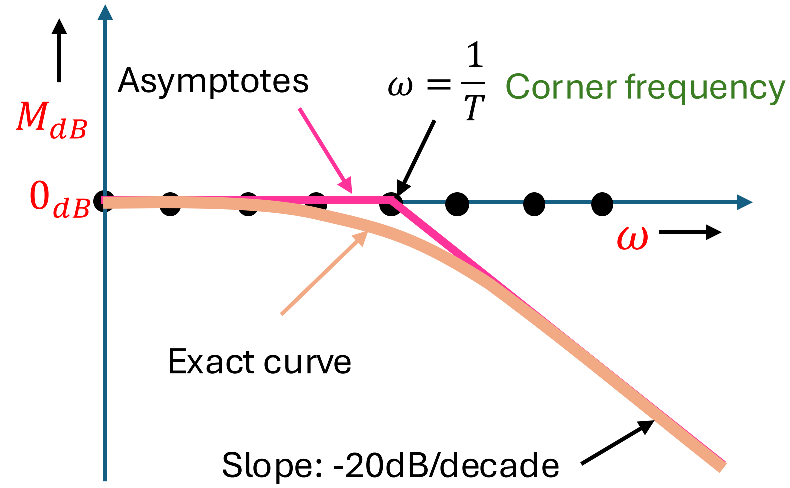
$if\ \omega \lll 1/T$

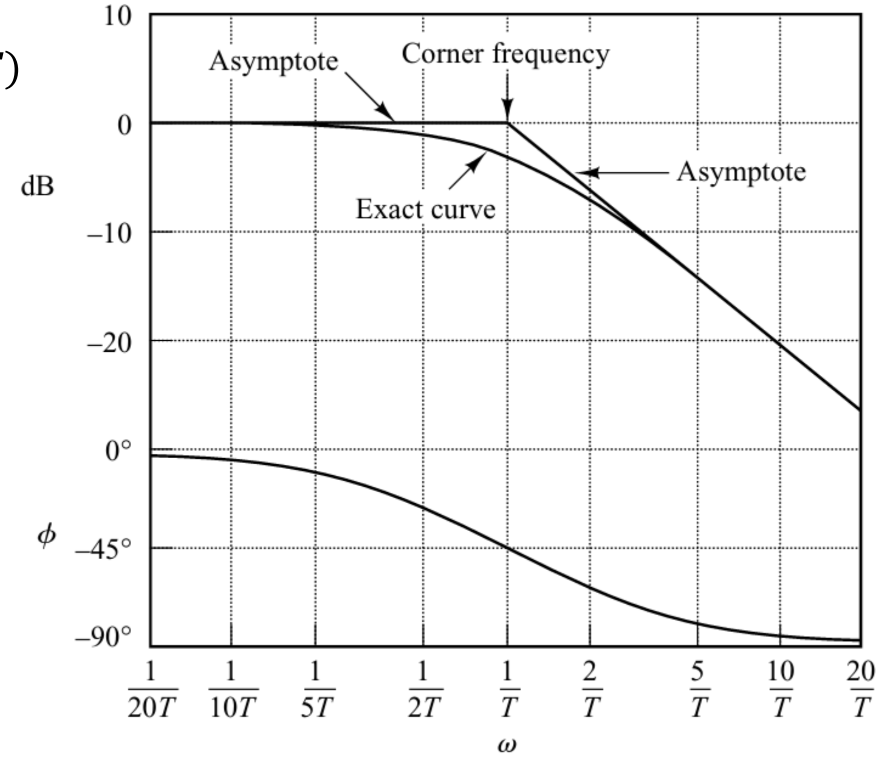
$$M_{dB} = -20\ log_{10}1$$

$$M_{dB} = 0$$

$if\ \omega \ggg 1/T$

$$M_{dB} = -20\ log_{10}(\omega T)$$





Bode Plots

Case 3a: G(s) = $\frac{1}{1+sT}$

$$G(s) = \frac{1}{1+sT} \Rightarrow G(j\omega) = \frac{1}{1+j\omega T}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\omega^2T^2}}$$

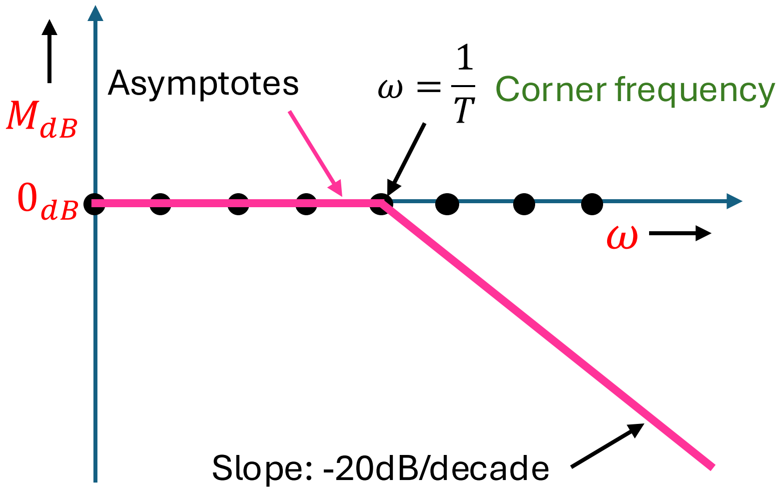
$$M_{dB} = -20\log_{10}\sqrt{1+\omega^2T^2}$$

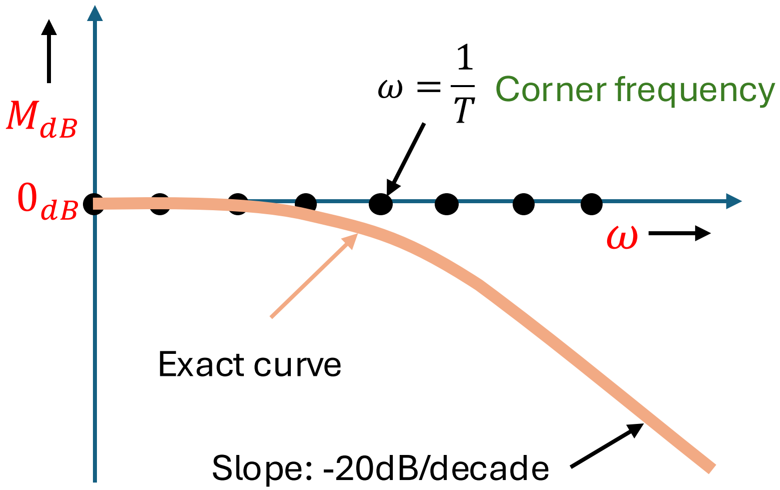
$$\phi = -\tan^{-1}\omega T$$

Standard Forms of G(s)

Generic 1st Order System

3 $\quad \frac{1}{1+sT}, \; 1+sT$





1. $M_{dB} = 0 \Rightarrow |G(j\omega)| = 1 \Rightarrow \quad Y = X$
2. $M_{dB} < 0 \Rightarrow 0 < |G(j\omega)| < 1 \Rightarrow Y \lll X$

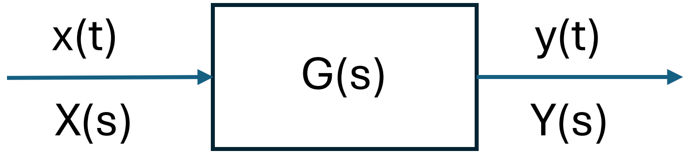
$Y = X * |G(j\omega)|$

Amp of O/P   Amp of I/P

G(s) = $\frac{1}{1+sT}$ *Has the characteristics of a* ***"low pass filter"***

Bode Plots

Case 3a: $G(s) = \dfrac{1}{1+sT}$



$$G(s) = \frac{1}{1+\ sT} \Longrightarrow G(j\omega) = \frac{1}{1+\ j\omega T}$$

$$|G(j\omega)| = \frac{1}{\sqrt{1+\ \omega^2 T^2}}$$

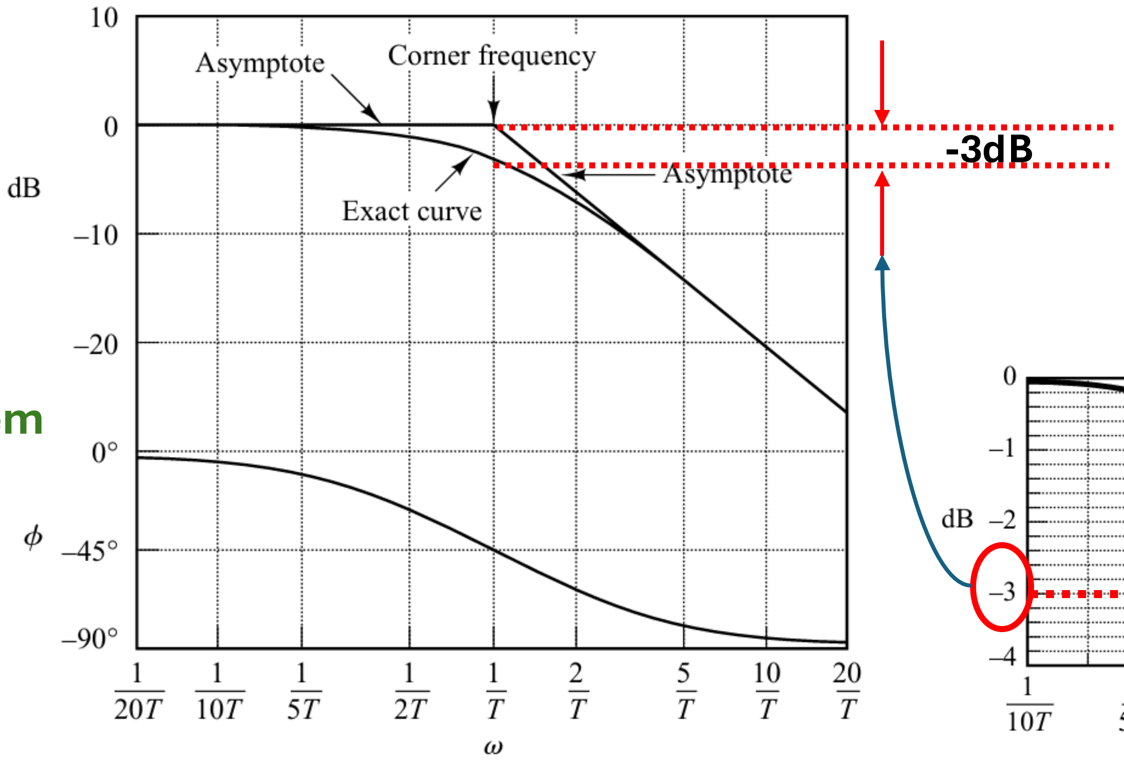
$$M_{dB} = -20\ log_{10}\sqrt{1+\omega^2 T^2}$$

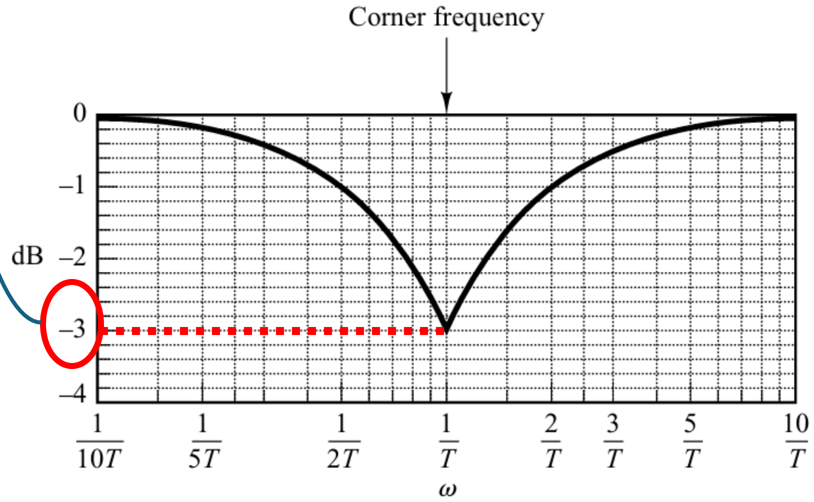
$$\phi = -\tan^{-1}\omega T$$

Standard Forms of G(s)

3 **Generic 1st Order System**

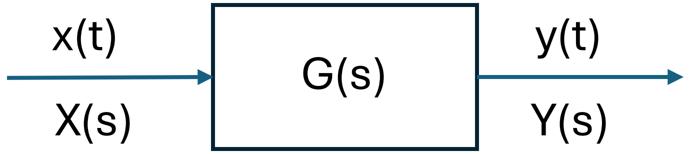
$$\frac{1}{1+sT},\ 1+sT$$





Bode Plots

Case 3b: G(s) = $1 + sT$



Standard Forms of G(s)

Generic 1st Order System

3 $\frac{1}{1+sT}$, $1 + sT$

$G(s) = 1 + sT \Rightarrow G(j\omega) = 1 + j\omega T$

$|G(j\omega)| = \sqrt{1 + \omega^2 T^2}$

$\phi = \tan^{-1} \omega T$

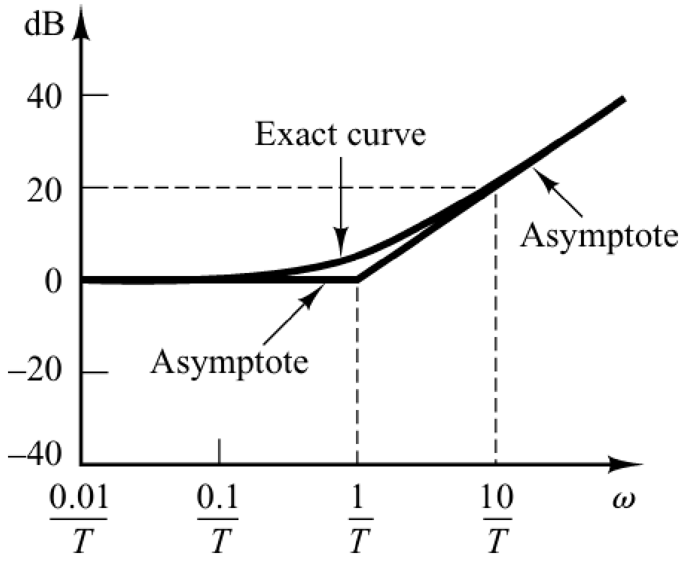
$M_{dB} = 20\, log_{10} \sqrt{1 + \omega^2 T^2}$

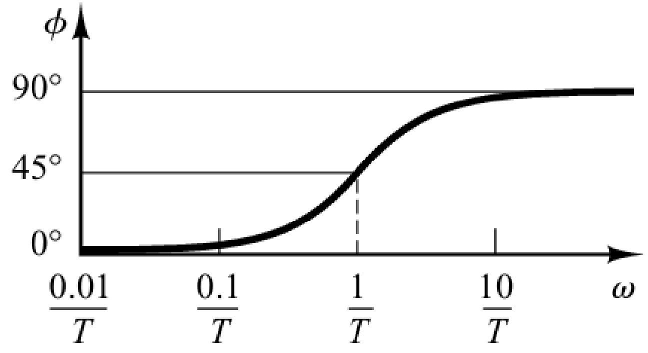
$if\ \omega \lll 1/T$

$M_{dB} = 0$

$if\ \omega \ggg 1/T$

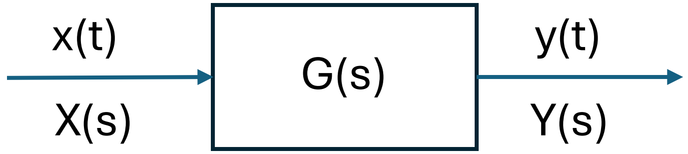
$M_{dB} = 20\, log_{10}(\omega T)$





Bode Plots

Case 3a: $G(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$



Standard Forms of G(s)

Generic 2nd Order System

4

$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2},\ s^2+2\xi\omega_n s+\omega_n^2$$

$$G(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$G(s) = \frac{1}{1+2\xi\frac{s}{\omega_n}+\left(\frac{s}{\omega_n}\right)^2}$$

$$G(j\omega) = \frac{1}{1+2\xi\frac{j\omega}{\omega_n}+\left(\frac{j\omega}{\omega_n}\right)^2}$$

$$G(j\omega) = \frac{1}{1-\left(\frac{\omega^2}{\omega_n^2}\right)+j\left(2\xi\frac{\omega}{\omega_n}\right)}$$

$$|G(j\omega)| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

$$M_{dB} = -20\,log_{10}\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}$$

$\boldsymbol{if\ \omega \lll \omega_n}$

$$\Longrightarrow M_{dB} = 0$$

$\boldsymbol{if\ \omega \ggg \omega_n}$

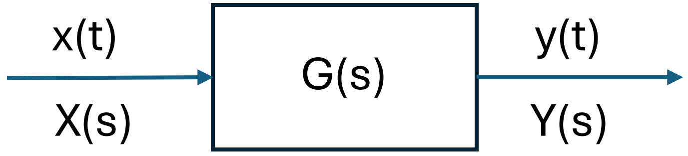
$$M_{dB} = -20\,log_{10}\frac{\omega^2}{\omega_n^2}$$

$$\Longrightarrow M_{dB} = -40\,log_{10}\frac{\omega}{\omega_n}$$

$$\phi = -\tan^{-1}\left(\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$

Bode Plots

Case 3a: $G(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$



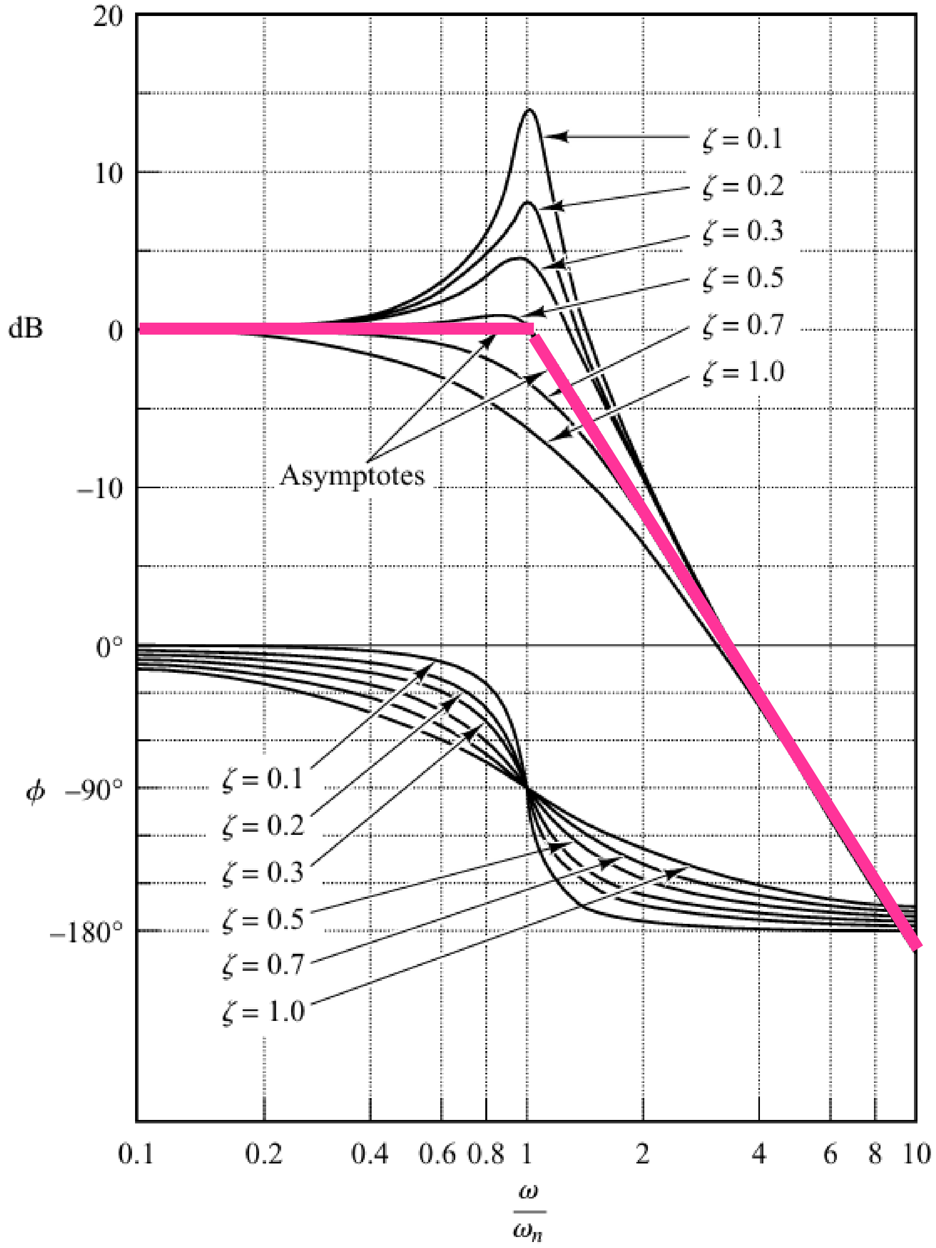
$$M_{dB} = -\underbrace{20\log_{10}}_{\substack{\text{if } \omega \lll \omega_n \\ \Rightarrow M_{dB}=0 \quad\quad \text{if } \omega \ggg \omega_n \\ \Rightarrow M_{dB} = -40\log_{10}\frac{\omega}{\omega_n}}} \sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2 + \left(2\xi\frac{\omega}{\omega_n}\right)^2}$$

$$\phi = -\tan^{-1}\left(\frac{2\xi\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$$

Standard Forms of G(s)

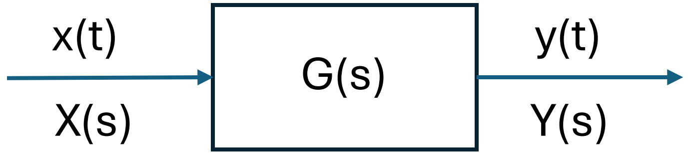
Generic 2nd Order System

4 $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$, $s^2+2\xi\omega_n s+\omega_n^2$



Bode Plots

Case 3a: G(s) = $\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$



Standard Forms of G(s)

4

Generic 2nd Order System

$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\ ,\ s^2+2\xi\omega_n s+\omega_n^2$$

Resonant Frequency of a 2nd Order System

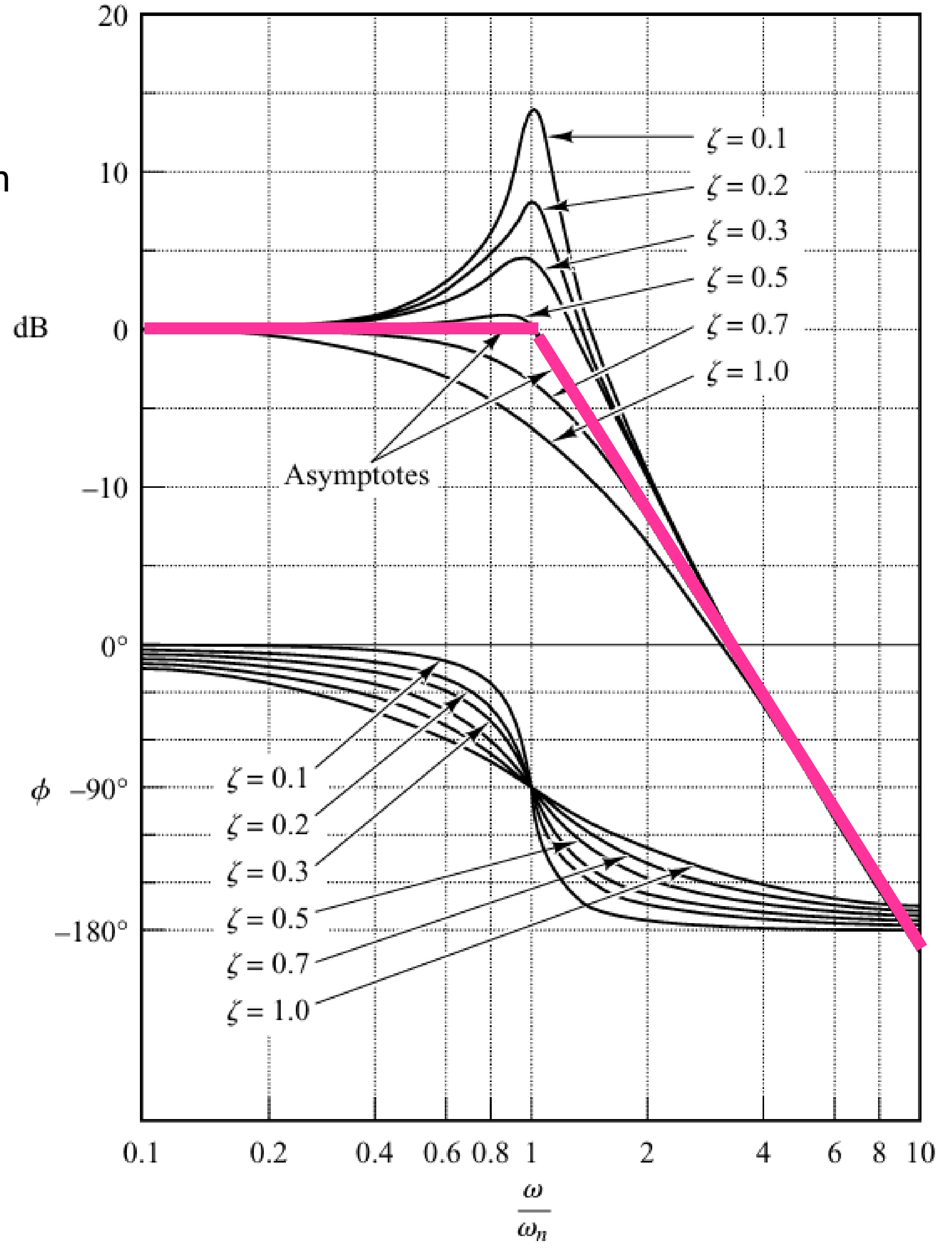
Frequency at which $|G(j\omega)|$ is **maximum**

$$|G(j\omega)| = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_n^2}\right)^2+\left(2\xi\frac{\omega}{\omega_n}\right)^2}}$$

For max $|G(j\omega)|$, denominator should be minimum

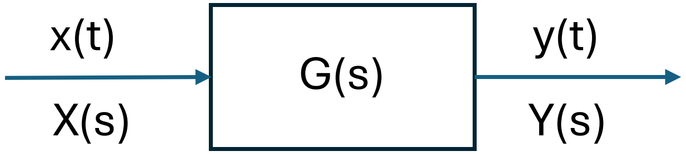
$$\omega = \omega_n\sqrt{1-2\xi^2}$$

Resonant frequency (ω_r)



Bode Plots

Case 3a: $G(s) = \dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$



Standard Forms of G(s)

Resonant Frequency

$$\omega_r = \omega_n\sqrt{1-2\xi^2}$$

$$at\ \underbrace{\xi = 0}_{\text{No damping}};\ \omega_r = \omega_n$$

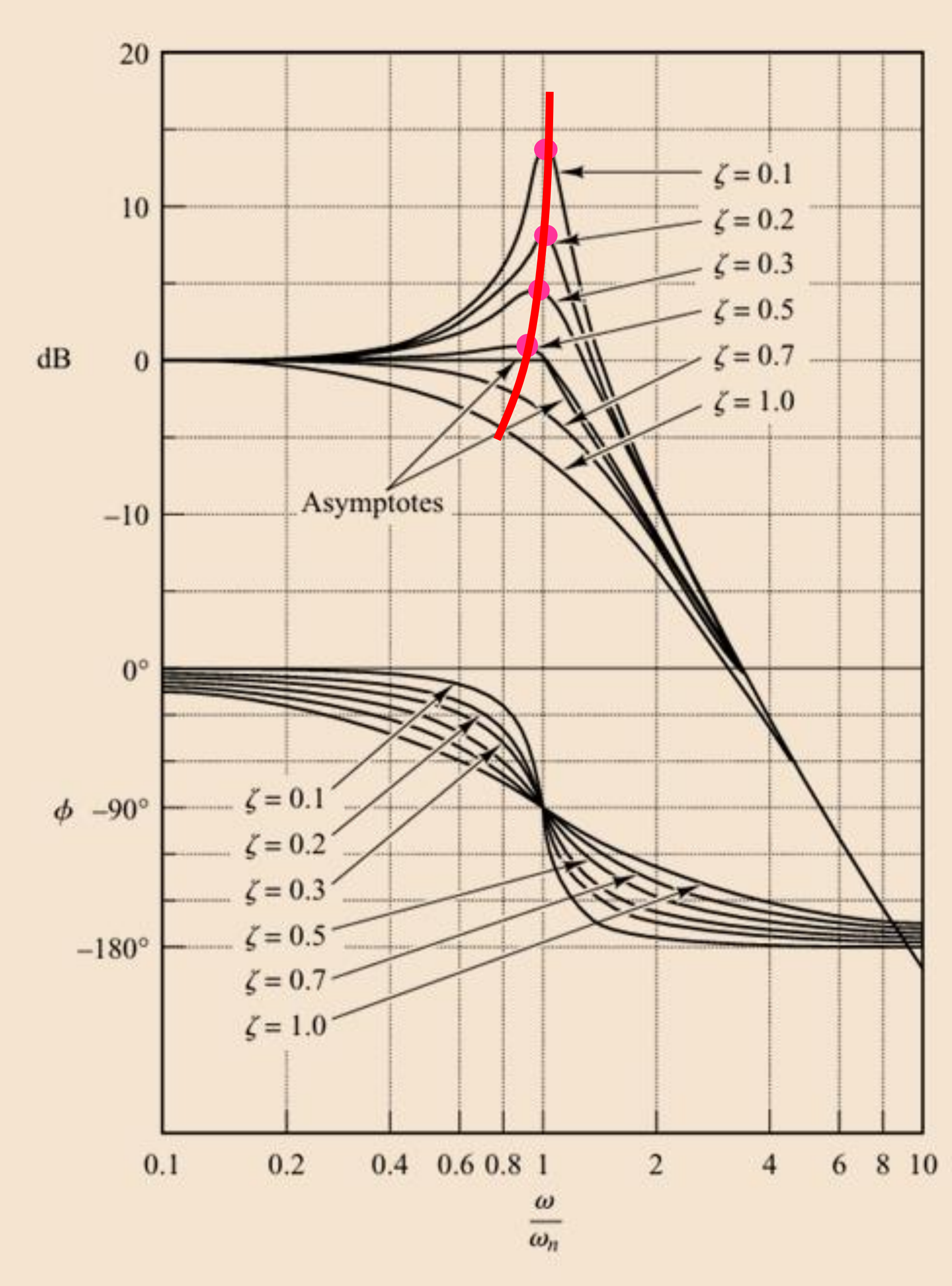
Valid for $0 \le \xi \le 0.707$

For $\xi > 0.707$

- No resonant peak occurs
- $|G(j\omega)|$ decreases monotonically with increasing ω

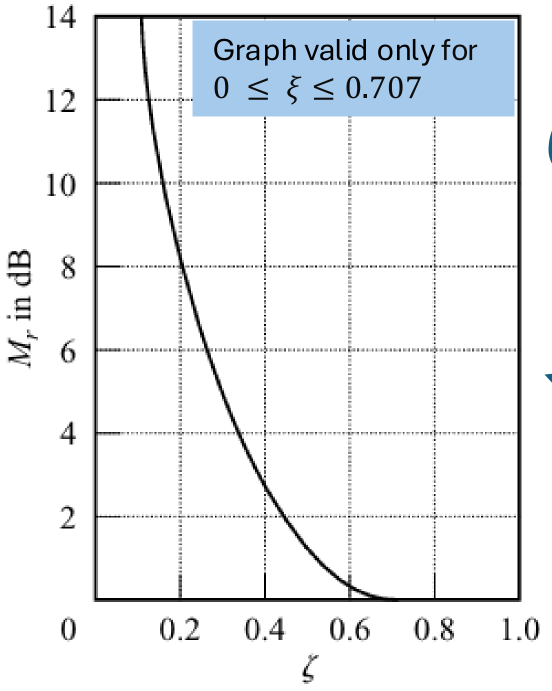
Generic 2nd Order System

4 $\dfrac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$, $s^2+2\xi\omega_n s+\omega_n^2$



Bode Plots

Case 3a: $G(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$



For $0 \le \xi \le 0.707$

$$At\ Resonant\ freq.:\ \omega = \omega_r = \omega_n\sqrt{1-2\xi^2};$$

$$Max.Magnitude:$$

$$|G(j\omega)|_{Max} = |G(j\omega_r)| = \frac{1}{2\xi\sqrt{1-\xi^2}}$$

$$M_r(in\ dB) = 20\ log\ \frac{1}{2\xi\sqrt{1-\xi^2}}$$

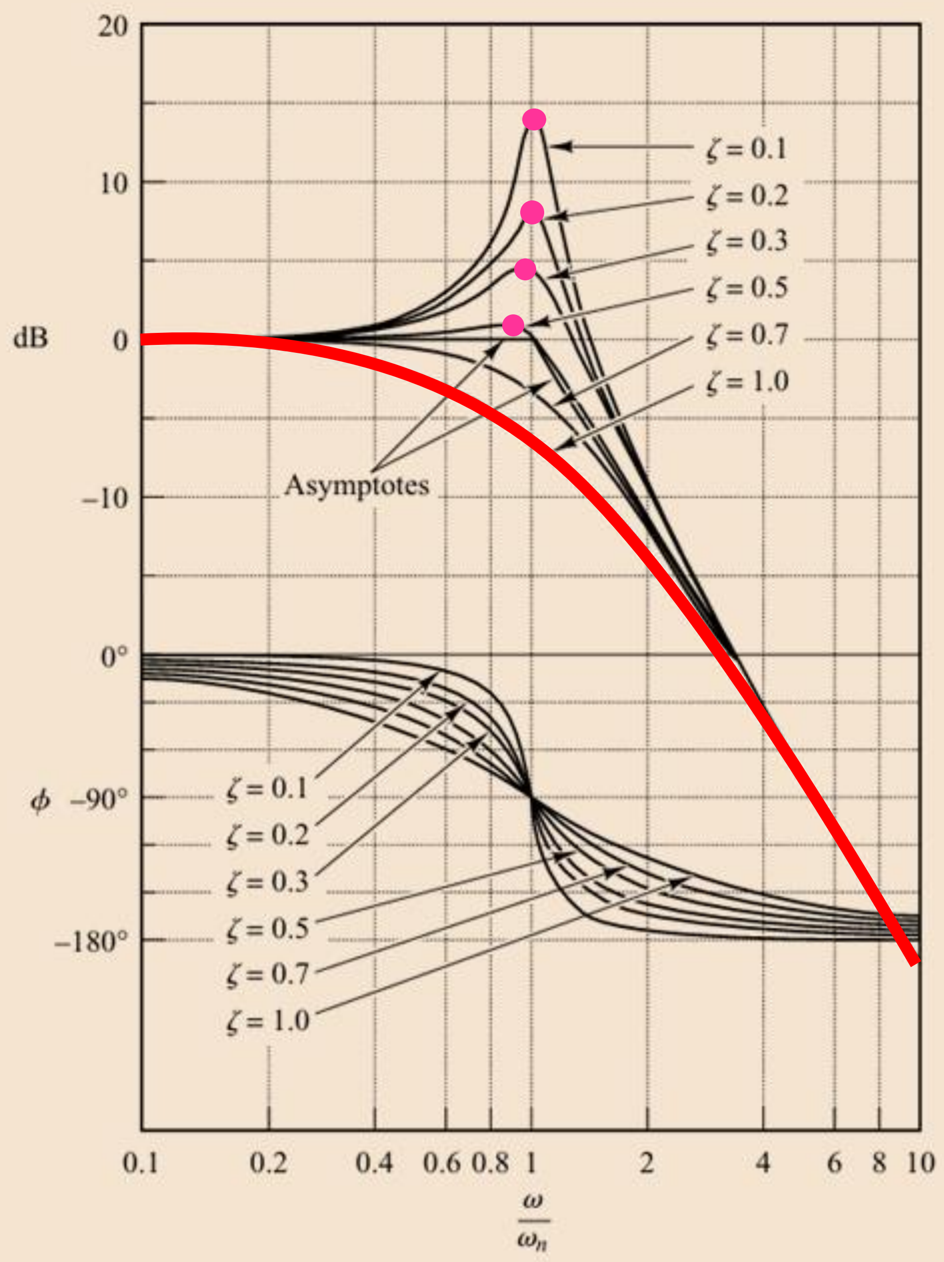
For $\xi > 0.707$

$$M_r(in\ dB) = 0$$

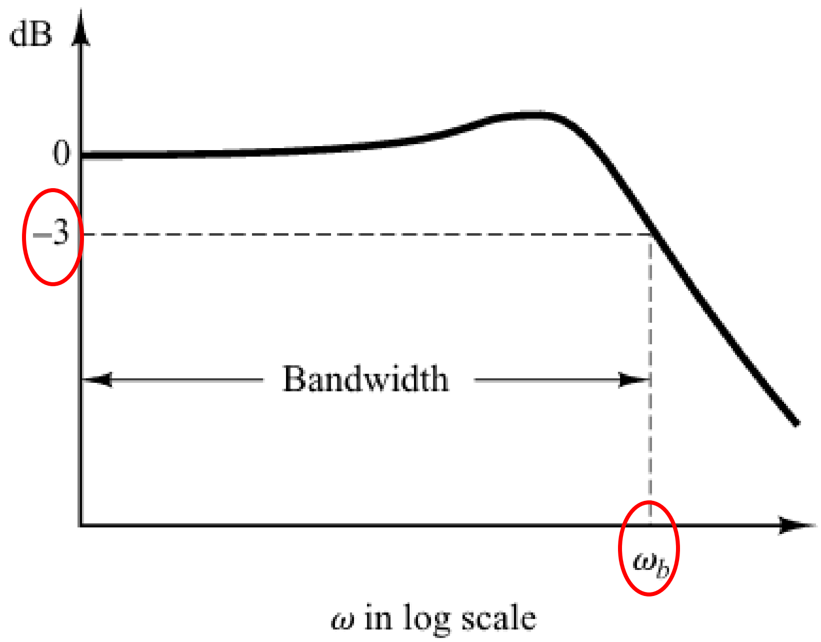
$$|G(j\omega)|_{Max} = 1$$

$$\boldsymbol{\phi\ at\ which\ resonant\ occurs\ (valid\ for\ 0 \le \xi \le 0.707}$$

$$\underline{/G(j\omega_r)} = -\tan^{-1}\frac{\sqrt{1-2\zeta^2}}{\zeta} = -90° + \sin^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}}$$



Bandwidth Frequency



$Frequency\ Range\!: \quad 0 \ \le\ \omega\ \le\ \boldsymbol{\omega_b}$

Bandwidth frequency

- Frequency after which the output amplitude starts to fall significantly
- Measure of how well the system, G(s), will track the input

$$20log_{10}|G(j\omega)| = -3$$

$$|G(j\omega)| = 10^{\frac{-3}{20}} = 0.707$$

At Bandwidth frequency

Y = 0.707 X

Output Amplitude

Input Amplitude

Example 7-3 (Ogata)

$$G(s) = \frac{10\,(s+3)}{s\,(s+2)(s^2+s+2)}$$

$$CF_1 = 3$$

$$\omega_n = \sqrt{2} = 1.414$$

$$CF_2 = 2$$

$$\frac{2\xi}{\omega_n} = \frac{1}{2} \quad \Rightarrow \xi = \frac{1}{2\sqrt{2}} = 0.35$$

$$G(s) = \frac{7.5\,\left(\frac{s}{3}+1\right)}{s\,\left(\frac{s}{2}+1\right)\left(\frac{s^2}{2}+\frac{s}{2}+1\right)}$$

K

$1 + sT$

s

$$\frac{1}{1+\ sT}$$

$$\frac{1}{1+2\xi\frac{s}{\omega_n}+\left(\frac{s}{\omega_n}\right)^2}$$

Step: 1

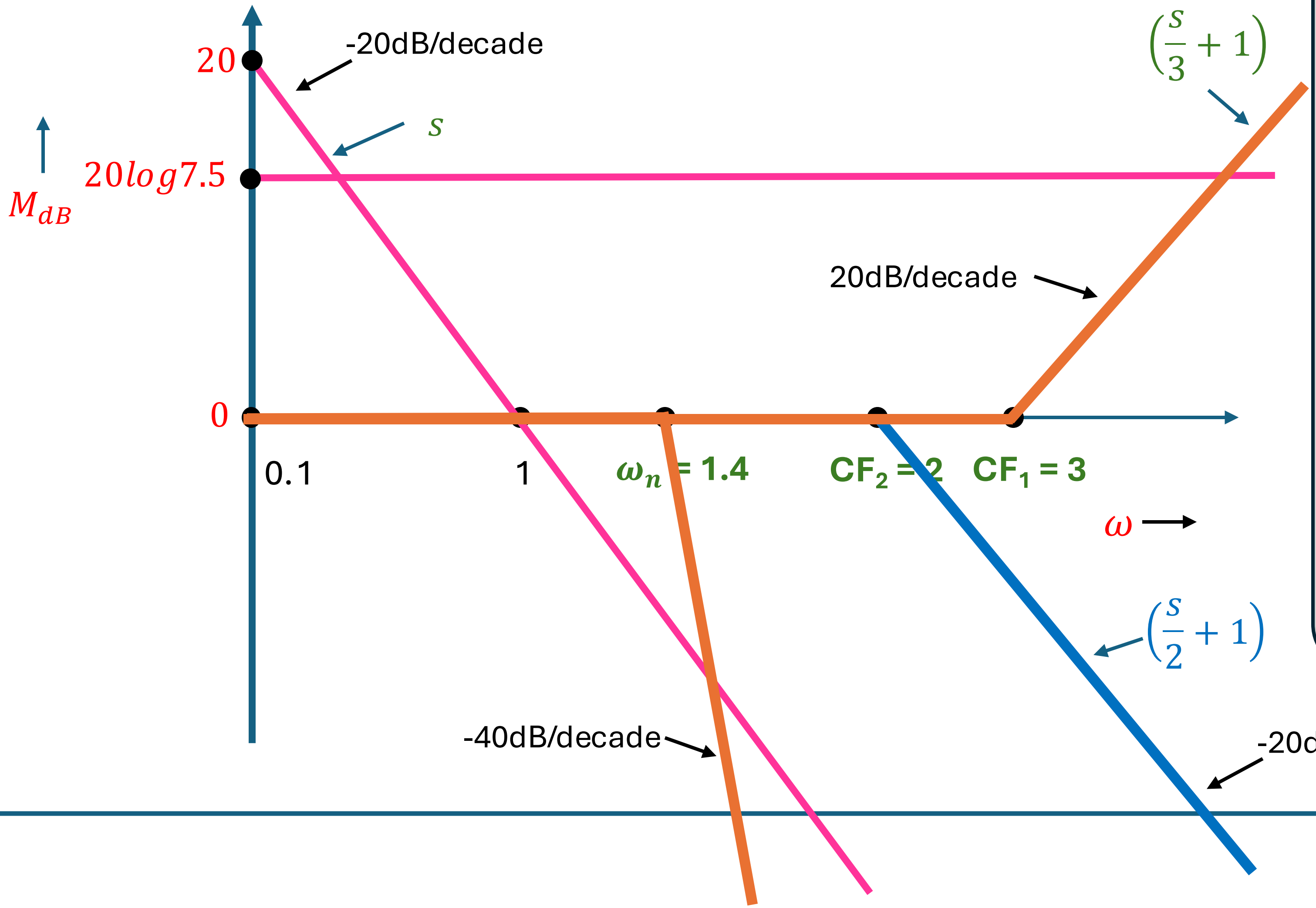
Simplify G(s) to combination of standard forms studied earlier.

Step: 2

Find the corner frequencies. Arrange them in ascending order.

Example 7-3 (Ogata)

$$G(s) = \frac{7.5\left(\frac{s}{3}+1\right)}{s\left(\frac{s}{2}+1\right)\left(\frac{s^2}{2}+\frac{s}{2}+1\right)}$$



Step: 1

Simplify G(s) to combination of standard forms studied earlier.

Step: 2

Find the corner frequencies. Arrange them in ascending order.

Step: 3

Make bode plots (magnitude and angle) for individual standard components.

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Write down slope contribution for every standard form

0.1 1 ω_n = 1.4 CF_2= 2 CF_1= 3 $\omega \longrightarrow$

7.5	0	0	0	0	0
s	-20	-20	-20	-20	-20
$\left(\frac{s^2}{2}+\frac{s}{2}+1\right)$	0	0	-40	-40	-40
s/2 + 1	0	0	0	-20	-20
s/3 + 1	0	0	0	0	20
Total	-20	-20	-60	-80	-60

Step: 1

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Make bode plots (magnitude and angle) for individual standard components.

Step: 4

Superimpose all the individual plots.

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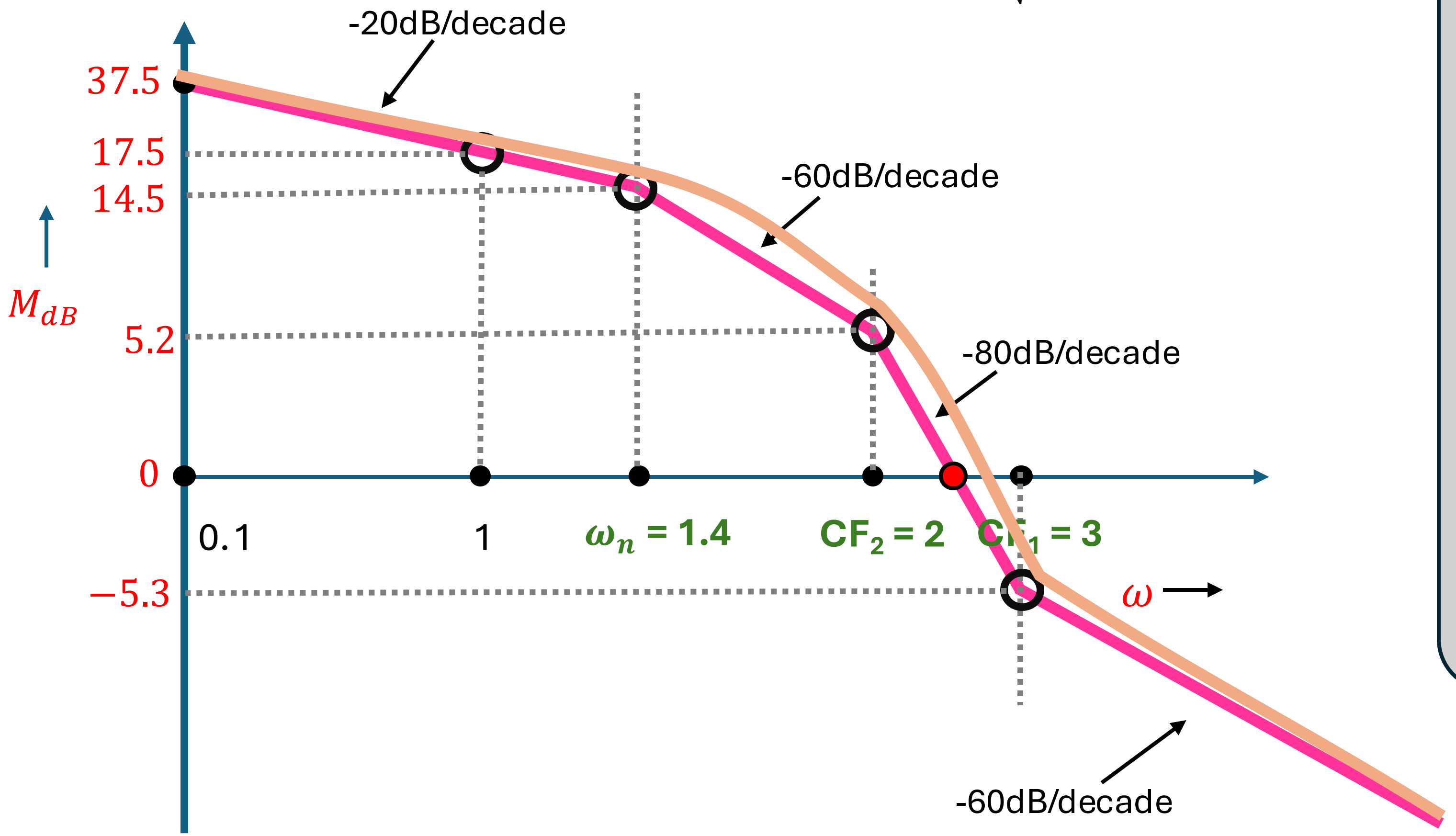
$$|G(j\omega)| = \frac{7.5\sqrt{\frac{\omega^2}{9}+1}}{\omega\sqrt{\frac{\omega^2}{4}+1}\sqrt{\frac{\omega^2}{4}+\left(1-\frac{\omega^2}{2}\right)^2}}$$



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