23MAT204

Mathematics for Intelligent Systems - 3

Joint Probability

- **Joint Probability Mass function (Discrete Random Variables)**
- **Joint Probability Density function (Continuous Random Variables)**

Joint Probability Mass function (Discrete Random Variables)

The function $f(x, y)$ is a **joint probability distribution** or **probability mass function** of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ for all (x, y),
2. $\sum_x \sum_y f(x, y) = 1$,
3. $P(X = x, Y = y) = f(x, y)$.

For any region A in the xy plane, $P[(X, Y) \in A] = \sum \sum_A f(x, y)$.

J.P 2-Dim. Discrete R.Vs

Eg. 1 $X_1 \rightarrow$ outcome in 1^{st} die, $x_1 = 1, 2, 3, 4, 5, 6$

$X_2 \rightarrow$ " " 2^{nd} die, $x_2 = 1, 2, 3, 4, 5, 6$

$f(1,1) = P(X_1 = 1, X_2 = 1)$

For 2 dis. R.Vs X_1 & X_2

$P(X_1 = a, X_2 = b) = f(a,b)$

i.e prob. of intersection of $X_1 = a$ & $X_2 = b$.

$X_2 \backslash X_1$	1	2	3	4	5	6	$f_2(x_2) = P(X_2 = x_2)$
1	$f(1,1) = 1/36$	1/36	1/36	1/36	1/36	1/36	1/6
2	1/36	1/36	1/36	1/36	1/36	1/36	1/6
3	1/36	1/36	1/36	1/36	1/36	1/36	1/6
4	1/36	1/36	1/36	1/36	1/36	1/36	1/6
5	1/36	1/36	1/36	1/36	1/36	1/36	1/6
6	1/36	1/36	1/36	1/36	1/36	1/36	1/6
$f_1(x_1) = P(X_1 = x_1)$	1/6	1/6	1/6	1/6	1/6	1/6	

Dis. 2-D R.Vs

Prop. of $f(x_1, x_2)$

1. $0 \le f(x_1, x_2) \le 1$
2. $\sum_{x_1} \sum_{x_2} f(x_1, x_2) = 1$

Marginal distn.

$P(X_1 = 1) = f_1(x_1 = 1)$

$P(X_1 = 2) = f_1(x_1 = 2)$

$P(X_2 = 1) = f_2(x_2 = 1) = 1/6$

$P(X_2 = 5) = f_2(x_2 = 5) = 1/6$

Marginal prob distn. $= f_1(x_1) = P(X_1 = x_1) = \sum_{\text{values of } x_2} f(x_1, x_2)$

Conditional probability distn.

$$P(X_1=a / X_2=b) = \frac{f(x_1=a, x_2=b)}{f_2(x_2=b)} = \frac{f(x_1=a, x_2=b)}{\sum_{\text{values of } x_1} f(x_1, x_2=b)}$$

$P(A/B) = \frac{P(A\cap B)}{P(B)}$

If $P(A\cap B) = P(A)P(B)$

A, B are indpt.

Independence of R.V. X_1 & R.V X_2

$$\underbrace{f(x_1, x_2)}_{\text{n; jt prob.}} = \underbrace{f_1(x_1)}_{\substack{P(X_1=x_1) \\ \text{m. distr}}} \underbrace{f_2(x_2)}_{\substack{P(X_2=x_2) \\ \text{m. distr.}}}$$ for all values of X_1 & X_2

In previous eg:-, for all x_1 & x_2

$$\left.\begin{aligned} f(x_1, x_2) &= 1/36 \\ f_1(x_1)\cdot f_2(x_2) &= 1/36 \end{aligned}\right\} \Rightarrow$$ X_1, X_2 are indpt.

① Let X_1 & X_2 have joint prob. distr. as given in the table:-

$X_2 \backslash X_1$	0	1	2	$f_2(x_2)$
0	0.1	0.4	0.1	0.6
1	0.2	0.2	0	0.4
$f_1(x_1)$	0.3	0.6	0.1	

(a) Find $P(X_1+X_2>1)$

$$P(X_1+X_2>1) = f(1,1)+f(2,0)+f(2,1)$$
$$= 0.2+0.1+0$$
$$= 0.3$$

(b) $P(X_1>X_2)$

$$= f(1,0)+f(2,0)+f(2,1)$$
$$= 0.4+0.1+0$$
$$= 0.5$$

(c) Find all marginal probabilities.

$f_1(x_1=0)=0.3$ $\quad f_2(x_2=0)=0.6$

$f_1(x_1=1)=0.6$ $\quad f_2(x_2=1)=0.4$

$f_1(x_1=2)=0.1$

$$\sum_{x_1} f_1(x_1)=1$$
$$\sum_{x_2} f_2(x_2)=1$$

(d) Find conditional prob. distr. of X_1 given $X_2=1$.

$$f(X_1=0/X_2=1)=\frac{f(0,1)}{f_2(x_2=1)}=\frac{0.2}{0.4}=0.5$$

$$f(X_1=1/X_2=1)=\frac{f(1,1)}{f_2(x_2=1)}=\frac{0.2}{0.4}=0.5$$

$$f(X_1=2/X_2=1)=\frac{f(2,1)}{f_2(x_2=1)}=\frac{0}{0.4}=0$$

(e) Are X_1 & X_2 independent?

Is $f(x_1,x_2)=f_1(x_1)f_2(x_2)$ for all x_1,x_2

$f(0,0)=0.1$, $f_1(x_1=0)=0.3$, $f_2(x_2=0)=0.6$

$f(0,0)\neq f_1(x_1=0)\cdot f_2(x_2=0) \Rightarrow X_1$ & X_2 are not indept.

② The jt. prob. mass fn. of X_1 & X_2 is given by
$f(x_1, x_2) = k(2x_1 + 3x_2)$, $x_1 = 0, 1, 2$, $x_2 = 1, 2, 3$.
ⓐ Find all marginal distributions. ⓑ Find all condl. distrns. of X_2 given $X_1 = 1$
ⓒ Is X_1 & X_2 indept.

$x_2 \backslash x_1$	0	1	2	$f_2(x_2)$
1	3k	5k	7k	15/72
2	6k	8k	10k	24/72
3	9k	11k	13k	33/72
$f_1(x_1)$	$\frac{18}{72}$	$\frac{24}{72}$	$\frac{30}{72}$	

$f(x_1=0, x_2=1) = k(0+3) = 3k$, $f(x_1=1, x_2=1) = k(2+3) = 5k$,
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$\sum_{x_1}\sum_{x_2} f(x_1, x_2) = 1 \Longrightarrow 3k + 5k + 7k + 6k + 8k + 10k + 9k + 11k + 13k = 1$

$\Longrightarrow k = 1/72$

(a) $f_1(x_1=0) = \frac{18}{72} = \frac{2}{8} = 1/4$

$f_1(x_1=1) = \frac{24}{72} = 1/3$

$f_1(x_1=2) = \frac{30}{72} = \frac{5}{12}$

$f_2(x_2=1) = 5/24$

$f_2(x_2=2) = 1/3$

$f_2(x_2=3) = 11/24$

(b) $P(X_2 = 1/X_1 = 1) = \frac{f(1,1)}{f_1(x_1=1)} = \frac{5}{24}$

$P(X_2 = 2/X_1 = 1) = \frac{f(1,2)}{f_1(x_1=1)} = \frac{8}{24}$

$P(X_2 = 3/X_1 = 1) = \frac{f(1,3)}{f_1(x_1=1)} = \frac{11}{24}$

(c) $f(0,1) = \frac{3}{72}$, $f_1(x_1=0) \cdot f_2(x_2=1) = \frac{18}{72} \times \frac{15}{72} = \frac{5}{96}$

$f(0,1) \neq f_1(x_1=0) \times f_2(x_2=1)$
$\Longrightarrow X_1$ & X_2 are not indpt.

③ Two scanners are needed for an experiment. Of the five available, two have electronic defects, another one has a defect in the memory and two are in good working condition. Two units are selected in random. Find the joint probability distribution of X=number of scanners picked with electronic defect and Y=number of scanners picked with memory defect

2G, 2ED, 1MD → 2

X can be 0, 1, 2 , Y can be 0, 1

Y \ X	0	1	2	$f_y(y)$
0	1/10	2/5	1/10	3/5
1	1/5	1/5	0	2/5
$f_x(x)$	3/10	3/5	1/10	

$f(0,0) = P(X=0, Y=0) = P(2 \text{ Good condition})$
$= \frac{2}{5} \times \frac{1}{4} \left(\text{or } \frac{2C_2}{5C_2}\right)$
$= \frac{1}{10}$

$f(0,1) = P(1MD, 1GC)$
$= \frac{1}{5} \times \frac{2}{4} \times 2 \left(\text{or } \frac{2C_1 \cdot 1C_1}{5C_2}\right)$
$= 1/5$

$f(1,0) = P(1ED, 1GC) = \frac{2}{5} \times \frac{2}{4} \times 2 = 2/5$

$f(1,1) = P(1ED, 1MD) = \frac{2}{5} \times \frac{1}{4} \times 2 = 1/5$

$f(2,0) = P(2ED) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10} \left(\frac{2C_2}{5C_2}\right)$

$f(2,1) \quad 0$

(a) Find $P(X>Y) = f(1,0) + f(2,0) + f(2,1)$
$= \frac{2}{5} + \frac{1}{10} = \frac{1}{2}$

(b) Find the average of X & Y.
$E(X), E(Y)$

$E(X) = \sum x f_x(x)$, $E(Y) = \sum y f_y(y)$

x	0	1	2
$f_x(x)$	$\frac{3}{10}$	$\frac{3}{5}$	$\frac{1}{10}$

$\Rightarrow E(X) = \sum x f_x(x)$
$= 0 \times \frac{3}{10} + 1 \times \frac{3}{5} + 2 \times \frac{1}{10}$
$= 4/5$

Y	0	1
$f_y(y)$	3/5	2/5

$\Rightarrow E(Y) = \sum y f_y(y)$
$= 0 \times \frac{3}{5} + 1 \times \frac{2}{5}$
$= \frac{2}{5}$

HW ④ Two RVs are independent & each has a binomial distribution with $p = 0.3$ and $n = 2$. Find the joint pmf. of the 2 R.Vs.

Let X & Y be the R.Vs.

X	0	1	2
$f_X(x)$	0.49	0.42	0.09

Y	0	1	2
$f_Y(y)$	0.49	0.42	0.09

$f_X(0) = f_Y(0)$
$= 2C_0 (0.3)^0 (0.7)^2$
$= 0.49$

$f_X(1) = f_Y(1) =$
$= 2C_1 (0.3)^1 (0.7)^1$
$= 0.42$

$f_X(2) = f_Y(2)$
$= 2C_2 (0.3)^2 (0.7)^0$
$= 0.09$

$Y \backslash X$	0	1	2
0	$(0.49)(0.49) = 0.2401$	$(0.49)(0.42) = 0.2058$	$(0.49)(0.09) = 0.0441$
1	$(0.49)(0.42) = 0.2058$	$(0.42)(0.42) = 0.1764$	$(0.42)(0.09) = 0.0378$
2	$(0.49)(0.09) = 0.0441$	$(0.42)(0.09) = 0.0378$	$(0.09)(0.09) = 0.0081$

$f(0,0) = f_X(0) \cdot f_Y(0)$
$= (0.49)(0.49)$ → Since they are indpt.

Similarly

$f(1,0) = f_X(1) f_Y(0) = (0.42)(0.49)$

$f(2,0) = f_X(2) f_Y(0) = (0.09)(0.49)$

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Home Work:

1.Given is the joint probabilities of X and Y,

(a) Obtain the marginal distributins.

(b) Obtain the conditional distributions of X, given Y = 0.

Y / X	0	1	2
0	0.02	0.08	0.10
1	0.05	0.20	0.25
2	0.03	0.12	0.15

2. Suppose the joint probability mass function of a bivariate random variable (X,Y) is given by: $p(x,y) = 1/3$ for (0,1) , (1,0) , (2,0) ; 0 , otherwise. Are X and Y independent ?

3. The input to a binary communication system, denoted by a random variable X, takes on one of two values 0 or 1 with probabilities ¾ and ¼ respectively. Because of errors caused by noise in the system, the output Y differs from the input occasionally. The behaviour of the communication system is modeled by the conditional probabilities: $P(Y=1/X=1) = 3/4$ and $P(Y=0/X=0) = 7/8$. Find $P(Y=1)$, $P(Y=0)$ and $P(X=1/Y=1)$.

4. Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

   (a) the joint probability function $f(x, y)$,

   (b) $P[(X, Y) \in A]$, where A is the region $\{(x, y) | x + y \leq 1\}$.

   (c) Find all marginal probabilities.

   d) Find $E(X)$ and $E(Y)$.

   (e) What are the conditional probabilities of Y given $X = 1$.

Joint Probability Density function (Continuous Random Variables)

Since this requires the knowledge of Double Integrals, first we shall learn about double integration, before learning joint pdf

Double Integrals

Double Integrals

Evaluate the following double integrals:

(a) $\int_{-1}^{2}\int_{1}^{3}(x+y)dx\,dy$

$$= \int_{y=-1}^{2}\int_{x=1}^{3}(x+y)dx\,dy$$

$$= \int_{y=-1}^{2}\left(\frac{x^2}{2}+xy\right)_{x=1}^{x=3}dy$$

$$= \int_{y=-1}^{2}\left(\frac{9}{2}+3y-\frac{1}{2}-y\right)dy$$

$$= \int_{y=-1}^{2}(4+2y)dy$$

$$= \int_{y=-1}^{2}(4+2y)dy = 15$$

(b) $\int_{0}^{1}\int_{0}^{x}(x+y)dy\,dx$

$$= \int_{x=0}^{1}\left(xy+\frac{y^2}{2}\right)_{y=0}^{y=x}dx$$

$$= \int_{x=0}^{1}\left(x^2+\frac{x^2}{2}\right)dx$$

$$= \left(\frac{x^3}{3}+\frac{x^3}{6}\right)_{x=0}^{x=1}$$

$$= \frac{1}{3}+\frac{1}{6}=\frac{1}{2}$$

(c) $\int_{0}^{2}\int_{0}^{1}(x^2+xy)\,dy\,dx = \frac{11}{3}$

(d) $\int_{0}^{1}\int_{y}^{1}(x+y)\,dx\,dy = \frac{1}{2}$

Double Integrals

$$\int_{-1}^{2}\int_{1}^{3}(x+y)dx\,dy = \int_{y=-1}^{2}\int_{x=1}^{3}(x+y)dx\,dy$$

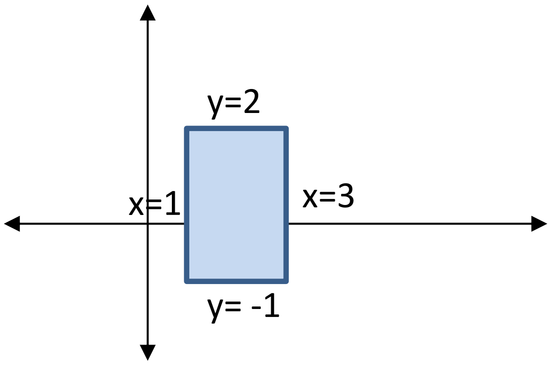
$$= \int_{y=-1}^{2}\left(\frac{x^2}{2}+xy\right)_{x=1}^{x=3} dy$$

$$= \int_{y=-1}^{2}\left(\frac{9}{2}+3y-\frac{1}{2}-y\right) dy$$

$$= \int_{y=-1}^{2}(4+2y)dy$$

$$= \int_{y=-1}^{2}(4y+y^2)dy$$

$$= \int_{y=-1}^{2}(4+2y)dy = 15$$



$$\int_{1}^{3}\int_{-1}^{2}(x+y)dy\,dx = \int_{x=1}^{3}\int_{y=-1}^{2}(x+y)dy\,dx$$

$$= \int_{x=1}^{3}\left(xy+\frac{y^2}{2}\right)_{y=-1}^{y=2} dx$$

$$= \int_{x=1}^{3}\left(2x+\frac{4}{2}-\left(-x+\frac{1}{2}\right)\right) dx$$

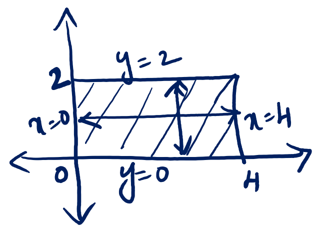
$$= \int_{x=1}^{3}\left(3x+\frac{3}{2}\right)dx$$

$$= \left(\frac{3x^2}{2}+\frac{3x}{2}\right)_{x=1}^{x=3} = 15$$

When sides of the region of integration are parallel to X axis or Y axis, the limits of integration will be constants as in this problem

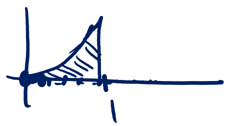
Double Integrals

Eg. 1

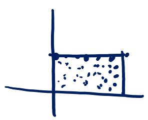


$$\int_{x=0}^{4}\int_{y=0}^{2} xy\, dy\, dx \quad \text{OR} \quad \int_{y=0}^{2}\left(\int_{x=0}^{4} xy\, dx\right) dy$$

region of integration



$$\int_0^1 \int^{x^2} f(x)\, dx.$$



$f(x,y) = xy$ $\quad z = xy$

$$\int_0^4 \left(\int_{y=0}^{2} xy\, dy\right) dx =$$

$$= \int_0^4 \left.\frac{xy^2}{2}\right)_{y=0}^{2} dx$$

$$= \int_0^4 \left(\frac{4x}{2} - 0\right) dx$$

$$= \left. x^2 \right)_0^4$$

$$= \underline{\underline{16}}$$

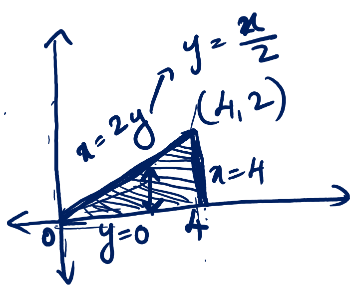
$$= \int_{y=0}^{2} \left.\frac{x^2}{2} y\right)_0^4 dy$$

$$= \int_{y=0}^{2} (8y - 0)\, dy$$

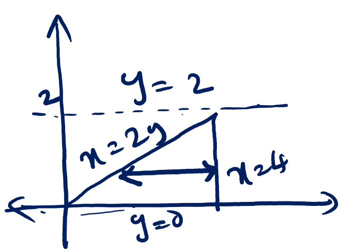
$$= \left.\frac{8y^2}{2}\right)_0^2$$

$$= \underline{\underline{16}}$$

Eg. 2



$$\int_{x=0}^{4}\int_{y=0}^{x/2} f(x,y)\, dy\, dx \quad \text{OR} \quad \int_{y=0}^{2}\int_{x=2y}^{4} f(x,y)\, dx\, dy.$$



Rule:- for dy dx

① Fix x limits (min of x ≤ max of x in fig.)

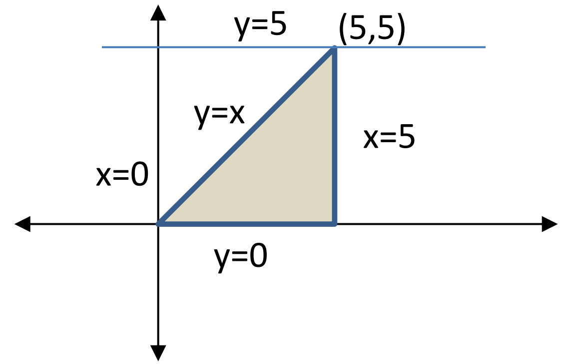
② bottom curve ≤ Y ≤ top curve

Rule for dx dy

① Fix Y limits (b/w min of Y ≤ max Y)

② left curve ≤ X ≤ right curve

Double Integrals



What will be the limits of integration to integrate a function f(x,y) over the given triangle?

Limits of integration: $0 \leq x \leq 5;\quad 0 \leq y \leq x$

$$\int_0^5 \int_0^x f(x,y)\,dy\,dx$$

OR

Limits of integration: $0 \leq y \leq 5;\quad y \leq x \leq 5$

$$\int_0^5 \int_y^5 f(x,y)\,dx\,dy$$

Method1: $0 \leq x \leq 5;\quad 0 \leq y \leq x$
bottom curve: $y = 0$,
top curve: $y = x$

Method2: $0 \leq y \leq 5;\quad y \leq x \leq 5$
left curve: $x = y$,
right curve: $x = 5$

Methods to find the limits of integration:

Method1:
Fix the limits of X between its least and highest values: a < x < b
Limits of Y will be Y from the bottom curve to that in the top curve, bottom curve < y < top curve

Method2:
Fix the limits of Y between its least and highest values: c < y < d
Limits of X will be X from the left curve to that in the right curve, left curve < y < right curve

Double Integrals

Method1:

$a \le x \le b;$

$b.c. \le y \le t.c.$

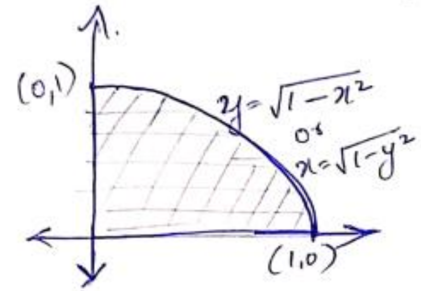
Method2:

$c \le y \le d;$

$l.c. \le x \le r.c.$

Write the limits of integration to integrate $f(x,y)$ over the given regions.

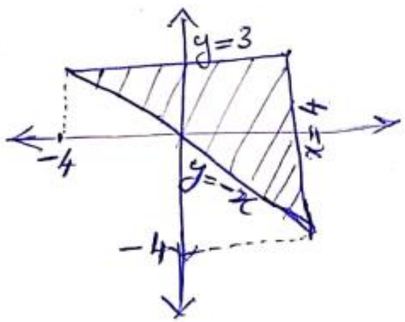
(a)



M1:- $\int_0^1\int_0^{\sqrt{1-x^2}} f(x,y)\,dy\,dx.$

M2:- $\int_0^1\int_0^{\sqrt{1-y^2}} f(x,y)\,dx\,dy.$

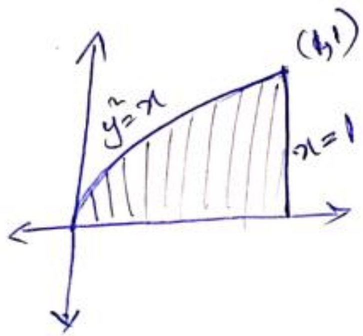
(b)



M1:- $\int_{-4}^{4}\int_{-x}^{3} f(x,y)\,dy\,dx$

M2:- $\int_{-4}^{3}\int_{-y}^{4} f(x,y)\,dx\,dy.$

(c)



M1:- $\int_0^1\int_0^{\sqrt{x}} f(x,y)\,dy\,dx$

M2:- $\int_0^1\int_{y^2}^{1} f(x,y)\,dx\,dy.$

Double Integrals

Method1:

$a \le x \le$ b;

$bottom\ curve \le y \le top\ curve$

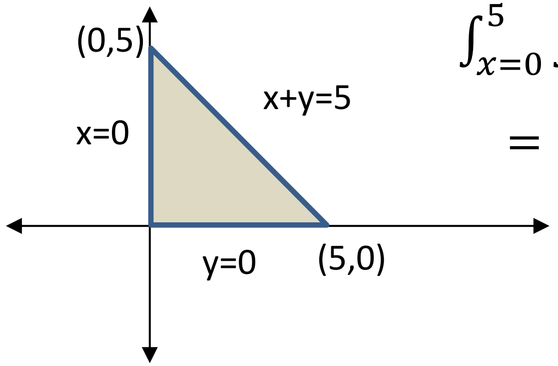
Method2:

$c \le y \le$ d;

$left\ curve \le x \le right\ curve$

Write the limits of integration over the given regions and integrate the function f(x,y) = x + y over those regions

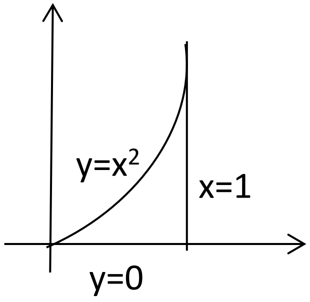
1.



$$\int_{x=0}^{5}\int_{y=0}^{5-x}(x+y)dy\,dx$$

$$= \int_{y=0}^{5}\int_{x=0}^{5-y}(x+y)dx\,dy = \frac{125}{3}$$

2. Over the region bounded by y = x^2, x=1 and the X axis in the first quadrant.



$$\int_{x=0}^{1}\int_{y=0}^{x^2}(x+y)dy\,dx$$

$$= \int_{y=0}^{1}\int_{x=\sqrt{y}}^{1}(x+y)dx\,dy = \frac{7}{20}$$

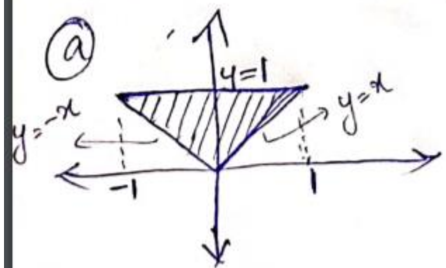
Double Integrals

Method1:
$a \leq x \leq$ b;
$b.c. \leq y \leq t.c.$

Method2:
$c \leq y \leq$ d;
$l.c. \leq x \leq r.c.$

Write the double integral for $f(x,y)$ over the given region in both orders. ($dxdy$ and $dydx$)

(a)



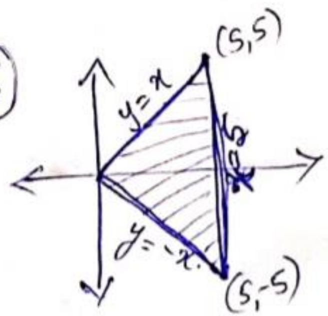
Method1:-

$$\int_{-1}^{0}\int_{-x}^{1} f(x,y)\,dy\,dx + \int_{0}^{1}\int_{x}^{1} f(x,y)\,dy\,dx$$

Method2:-

$$\int_{0}^{1}\int_{-y}^{y} f(x,y)\,dx\,dy,$$

(b)



Method1:-

$$\int_{0}^{5}\int_{-x}^{x} f(x,y)\,dy\,dx.$$

Method2:-

$$\int_{-5}^{0}\int_{-y}^{5} f(x,y)\,dx\,dy + \int_{0}^{5}\int_{y}^{5} f(x,y)\,dx\,dy$$

Double Integrals

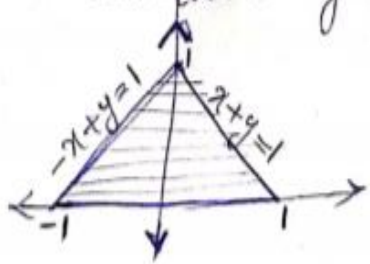
Method1:
$a \le x \le$ b;
$b.c. \le y \le t.c.$

Method2:
$c \le y \le$ d;
$l.c. \le x \le r.c.$

When f(x,y) = 1, double integral gives the area of the region of integration

Area of a region R = $\iint_R 1\, dxdy = \iint_R 1\, dydx$

Evaluate the integral of $f(x,y)=1$ over the given region:-



Method 1:- $\int_{-1}^{0}\int_{0}^{1+x} f(x,y)\,dy\,dx + \int_{0}^{1}\int_{0}^{1-x} f(x,y)\,dy\,dx$

$= \int_{-1}^{0}\int_{0}^{1+x} 1\,dy\,dx + \int_{0}^{1}\int_{0}^{1-x} 1\,dy\,dx$

$= \int_{-1}^{0}(1+x)\,dx + \int_{0}^{1}(1-x)\,dx$

$= \left(x+\frac{x^2}{2}\right)_{-1}^{0} + \left(x-\frac{x^2}{2}\right)_{0}^{1}$

$= 0-\left(-1+\frac{1}{2}\right)+\left(1-\frac{1}{2}\right)-0$

$= \frac{1}{2}+\frac{1}{2} = 1$

OR

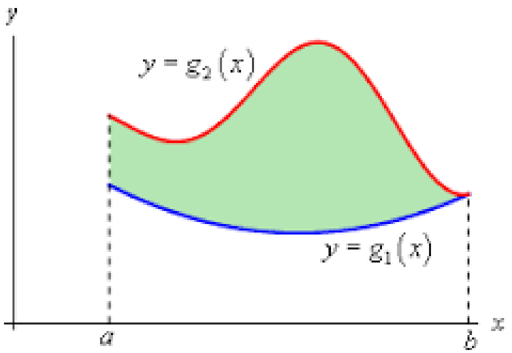
Method 2:- $\int_{0}^{1}\int_{y-1}^{1-y} f(x,y)\,dx\,dy$

$= \int_{0}^{1}\int_{y-1}^{1-y} 1\,dx\,dy$

$= \int_{0}^{1}[(1-y)-(y-1)]\,dy$

$= \int_{0}^{1} 2(1-y)\,dy = 2\left(y-\frac{y^2}{2}\right)_{0}^{1} = 1$

Area using Double Integrals



Area of the shaded region is:

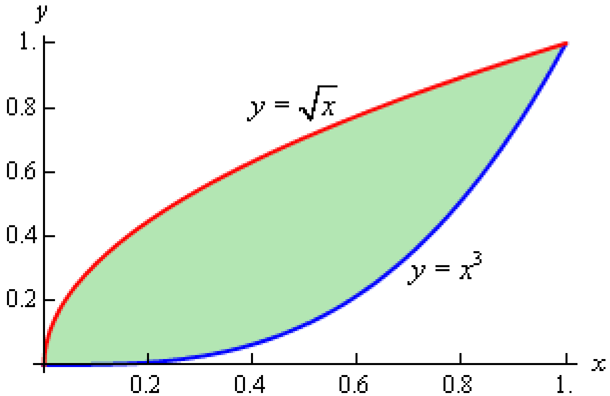
$$\int_a^b \int_{g_1(x)}^{g_2(x)} 1\, dy\, dx$$

Method1:
$a \le x \le$ b;
$b.c. \le y \le t.c.$

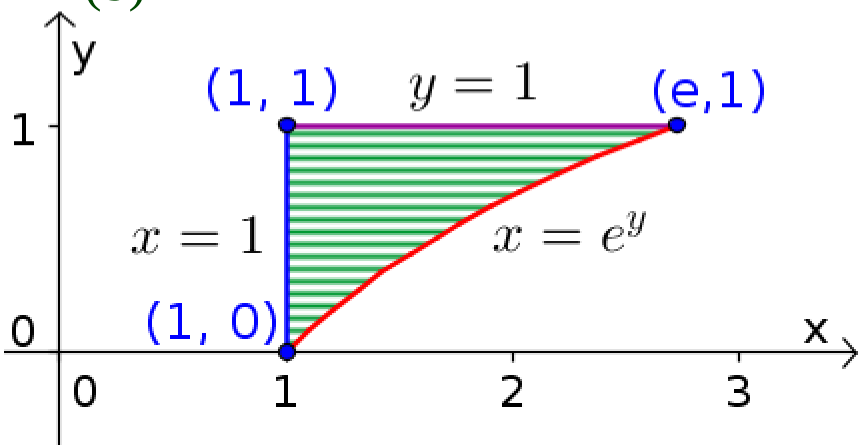
Method2:
$c \le y \le$ d;
$l.c. \le x \le r.c.$

Find the area of the given regions:

(a)



(b)



$$\int_0^1 \int_{x^3}^{\sqrt{x}} 1\, dy\, dx = \int_0^1 \int_{y^2}^{\sqrt[3]{y}} 1\, dx\, dy = \frac{5}{12}$$

$$\int_1^e \int_{\ln x}^{1} 1\, dy\, dx = \int_0^1 \int_1^{e^y} 1\, dx\, dy = e - 2$$

Double Integrals - Change of order

Draw the region of integration given in the following integral:

$$\int_1^3 \int_1^x f(x,y)dy\,dx$$

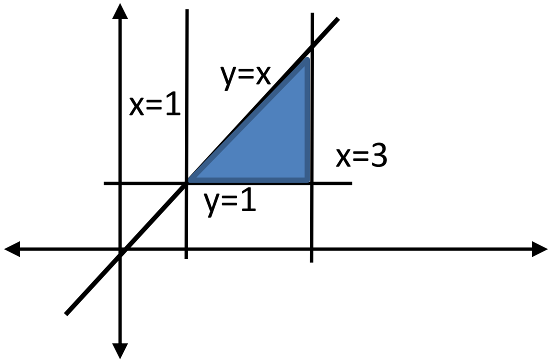
Method1:

$1 \le x \le 3;$

$1 \le y \le x$

Bottom curve: y=1

Top curve: y=x



Method1:

$a \le x \le \mathrm{b};$

$b.c. \le y \le t.c.$

Method2:

$c \le y \le \mathrm{d};$

$l.c. \le x \le r.c.$

Method2:

$1 \le y \le 3;$

$\mathrm{y} \le x \le 3$

Left curve: x=y

Right curve: x=3

Change the order of integration for the integral $I = \int_1^3 \int_1^x f(x,y)dy\,dx$

$$\int_1^3 \int_1^x f(x,y)\,dy\,dx = \int_1^3 \int_y^3 f(x,y)\,dx\,dy$$

How to change the order of integration of an integral?

1. Sketch the region of integration using the given limits of integration.
2. Using the alternate method write the integral limits.(*dxdy* to *dydx* or viceversa)

Double Integrals – Change of Order

Change the order of integration and write the integral:

Method1:
$a \le x \le$ b;
$b.c. \le y \le t.c.$

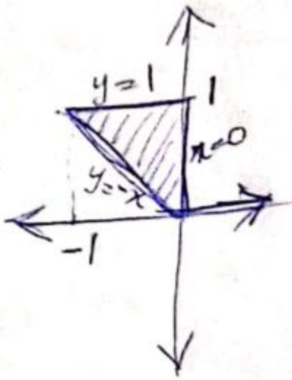
Method2:
$c \le y \le$ d;
$l.c. \le x \le r.c.$

① $I = \int_{-1}^{0}\int_{-x}^{1} f(x,y)\,dy\,dx$

$-1 \le x \le 0$

$\underbrace{-x \le y \le 1}$

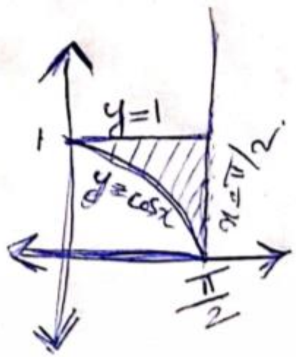
b.c $y = -x$ t.c $y = 1$



So after changing order we get,

$I = \int_{0}^{1}\int_{-y}^{0} f(x,y)\,dx\,dy.$

② $\int_{0}^{\frac{\pi}{2}}\int_{\cos x}^{1} f(x,y)\,dy\,dx$



$0 \le x \le \frac{\pi}{2}$

$\cos x \le y \le 1 \implies$ b.c $\to y = \cos x$
t.c $\to y = 1$

$\to$ So after order change

$I = \int_{0}^{1}\int_{\cos^{-1} y}^{\frac{\pi}{2}} f(x,y)\,dx\,dy$

Double Integrals - Change of order

Integration may not be possible in some integrals in the given order, but by changing the order of integration the integral can be easily evaluated.

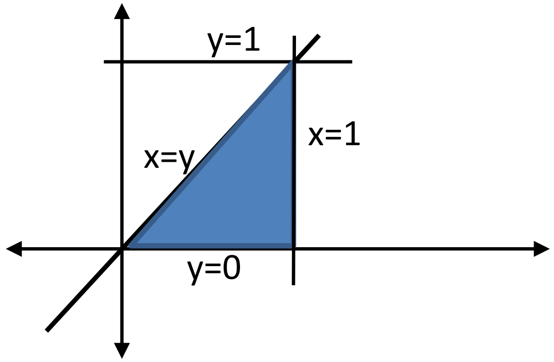
Method1:
$a \leq x \leq \mathrm{b};$
$b.c. \leq y \leq t.c.$

Method2:
$c \leq y \leq \mathrm{d};$
$l.c. \leq x \leq r.c.$

Change the order of integration and evaluate the following integral:

(a)

$$\int_0^1 \int_y^1 \frac{sinx}{x} dx\, dy$$



$$I = \int_0^1 \int_y^1 \frac{sinx}{x} dx\, dy = \int_0^1 \int_0^x \frac{sinx}{x} dy\, dx = \int_0^1 \frac{sinx}{x}(x) dx$$

$$= (-\cos x)_{x=0}^{x=1} = 1 - \cos 1$$

2–9 DOUBLE INTEGRALS

Describe the region of integration and evaluate. (Show the details.)

2. $\int_0^1 \int_x^{2x} (x + y)^2 \, dy \, dx$

3. $\int_0^1 \int_{x^2}^{x} (1 - 2xy) \, dy \, dx$

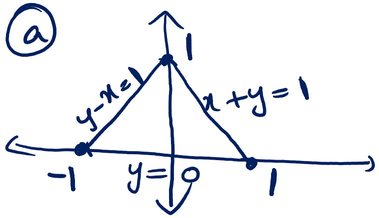
4. As Prob. 3, order reversed

5. $\int_0^3 \int_0^y \cosh (x + y) \, dx \, dy$

6. As Prob. 5, order reversed

7. $\int_0^4 \int_{-x}^{x} e^{x+2y} \, dy \, dx$

3. (a)



Integrate $f(x,y)=1$ over the given regions using any order of integration.

$$I = \int_{y=0}^{1}\int_{x=y-1}^{1-y} 1 \, dx \, dy = 1 \qquad \left(= \text{Area of } \Delta = \tfrac{1}{2} \times 2 \times 1 = 1 \right)$$

OR

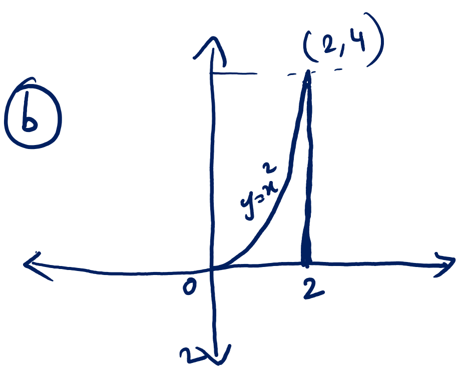
If you fix x limits first there are 2 top curves,

$$I = \int_{x=-1}^{0}\int_{y=0}^{1+x} 1 \, dy \, dx + \int_{x=0}^{1}\int_{y=0}^{1-x} 1 \, dy \, dx = 1$$

$$\underbrace{\int_{x=-1}^{1}\int_{y=1+x}^{1-x}}$$

Not the correct region.

(b)



$$I = \int_{x=0}^{2}\int_{y=0}^{x^2} 1 \, dy \, dx = 8/3 \qquad \text{OR} \qquad I = \int_{y=0}^{4}\int_{x=\sqrt{y}}^{2} 1 \, dx \, dy = 8/3$$

***

$$\boxed{\text{If } f(x,y)=1, \ \underbrace{\iint}_{R} f(x,y) \, \underbrace{dR}_{dxdy / dydx} = \text{Area of } R.}$$

Joint Probability Density function (Continuous Random Variables)

The function $f(x, y)$ is a **joint density function** of the continuous random variables X and Y if

$P(a \le x \le b, c \le Y \le d) = \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\, dy\, dx.$

1. $f(x, y) \ge 0$, for all (x, y),
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$,
3. $P[(X, Y) \in A] = \int \int_A f(x, y)\, dx\, dy$, for any region A in the xy plane.

$0 \le f(x,y) \le 1$

$\sum\sum f(x,y) = 1$

$f(x,y) = P(X=x, Y=y)$

$a \le x \le b$

$c \le y \le d$

Marginal Distributions of continuous RVs X and Y

The marginal distributions of X alone and of Y alone are

$$f_1(x_1) = / f_x(x) = g(x) = \sum_y f(x, y) \quad \text{and} \quad h(y) = \sum_x f(x, y)$$

$f_2(x_2)$

$f_y(y)$

$\to$ m.p.m.f of $f(x,y)$

for the discrete case, and

$$f_1(x_1) = f_x(x) = g(x) = \int_{y=-\infty}^{\infty} f(x, y)\, dy \quad \text{and} \quad h(y) = \int_{x=-\infty}^{\infty} f(x, y)\, dx$$

for the continuous case.

Conditional Distributions

Let X and Y be two random variables, discrete or continuous. The **conditional distribution** of the random variable Y given that $X = x$ is

$$f(y|x) = \frac{f(x,y)}{g(x)}, \text{ provided } g(x) > 0.$$

$g(x) = f_x(x)$

Similarly, the conditional distribution of X given that $Y = y$ is

$$f(x|y) = \frac{f(x,y)}{h(y)}, \text{ provided } h(y) > 0.$$

$h(y) = f_y(y)$

Statistical Independence

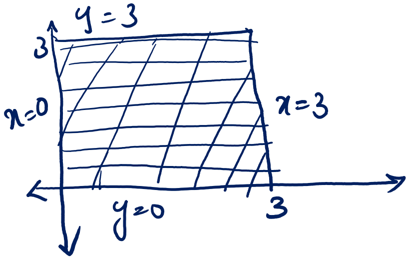
Let X and Y be two random variables, discrete or continuous, with joint probability distribution $f(x, y)$ and marginal distributions $g(x)$ and $h(y)$, respectively. The random variables X and Y are said to be **statistically independent** if and only if

$$f(x,y) = g(x)h(y)$$

$f(x,y) = f_x(x)\, f_y(y)$

for all (x, y) within their range.

1. Determine the value of c such that the function $f(x, y) = cxy$ for $0 < x < 3$ and $0 < y < 3$ satisfies the properties of a joint probability density function.



If $f(x,y)$ is joint pdf, then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$$

$$\int_{y=0}^{3}\int_{x=0}^{3} cxy\, dx\, dy = 1$$

$$\int_{y=0}^{3} cy \left.\frac{x^2}{2}\right)_0^3 dy = 1$$

$$\Rightarrow \int_{y=0}^{3} \frac{9c}{2} y\, dy = 1 \Rightarrow \frac{9c}{2} \left.\frac{y^2}{2}\right)_0^3 = 1$$

$$\Rightarrow \frac{9c}{4} \times 9 = 1 \Rightarrow c = 4/81$$

(2) $f(x,y) = \frac{4}{81}xy, \quad 0<x<3, \quad 0<y<3$

(a) Find the marginal density fn. of X.

(b) Find the marginal density fn. of Y.

(c) Is X & Y independent.

(d) Find condnal density fn. of X given Y.

(e) Find condnl densit fn. of Y given X $\rightarrow f(y/x)$

(a) $f_X(x) = \int_y f(x,y)\,dy = \int_{y=0}^{3} \frac{4}{81}xy\,dy = \frac{2x}{9} \Rightarrow$ M.D. of X is $\boxed{f_X(x) = \frac{2x}{9}, \ 0<x<3}$

(b) $f_Y(y) = \int_x f(x,y)\,dx = \int_{x=0}^{3} \frac{4}{81}xy\,dx = \frac{2y}{9} \Rightarrow$ Mar. densit of Y is $\boxed{f_Y(y) = \frac{2y}{9}, \ 0<y<3}$

(c) $f_X(x) \cdot f_Y(y) = \frac{2x}{9} \times \frac{2y}{9} = \frac{4xy}{81} = f(x,y)$

$\Rightarrow$ X & Y are independent random variables.

(d) $f(x/y) = \frac{f(x,y)}{f_Y(y)} = \frac{4/81\, xy}{2/9\, y} = \frac{2}{9}x$, $\quad f(x/y) = f_X(x)$

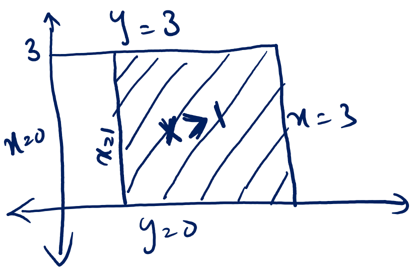
$(\because$ X & Y are independt.)

(e) $f(y|x) = \frac{f(x,y)}{f(\ )} = \frac{2}{9}y$

(3) $f(x,y) = \frac{4}{81}xy, \quad 0<x<3, \quad 0<y<3$

(a) $P(X>1)$

$= P(1<X<3,\ 0<Y<3)$



$$P(X>1) = \int_{y=0}^{3}\int_{x=1}^{3} \frac{4}{81}xy\,dx\,dy$$

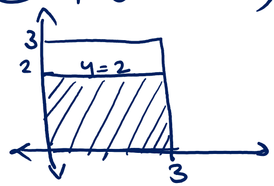
$$= \int_0^3 \left.\frac{4y}{81}\frac{x^2}{2}\right)_1^3 dy$$

$$= \int_0^3 \left.\frac{2y}{81}\frac{x^3}{3}\right)_1^3 dy$$

$$= \frac{2(27-1)}{81(3)}\left.\frac{y^2}{2}\right)_0^3$$

$=$

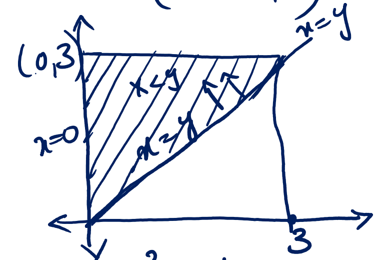
(b) $P(Y<2)$



$$P(Y<2) = \int_{y=0}^{2}\int_{x=0}^{3} \frac{4}{81}xy\,dx\,dy.$$

$$= \frac{4}{9}$$

(c) $P(X<Y)$



$$P(X<Y) = \int_{y=0}^{3}\int_{x=0}^{y} \frac{4}{81}xy\,dx\,dy.$$

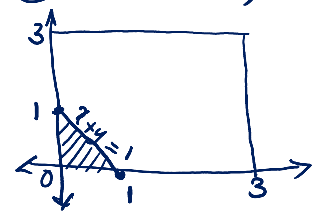
OR

$$= \int_{x=0}^{3}\int_{y=x}^{3} \frac{4}{81}xy\,dy\,dx.$$

$$= \frac{1}{2}$$

$$\boxed{\begin{array}{l} P(X+Y<1) \\ \quad = P(X+Y\le 1) \\ (\because \text{Conts.}\ X \,\&\, Y) \end{array}}$$

(d) $P(X+Y<1)$



$$P(X+Y<1) = \int_{y=0}^{1}\int_{x=0}^{1-y} \frac{4}{81}xy\,dx\,dy$$

$$= \int_{y=0}^{1} \frac{4}{81}y\frac{(1-y)^2}{2}\,dy.$$

$$= \int_0^1 \frac{2}{81}(y+y^3-2y^2)\,dy$$

$$= \frac{2}{81}\left(\frac{y^2}{2}+\frac{y^4}{4}-\frac{2y^3}{3}\right)_0^1$$

$$= \frac{2}{81}\left(\frac{1}{2}+\frac{1}{4}-\frac{2}{3}\right)$$

$$= \frac{2}{81}\times\frac{1}{12}$$

$$= \frac{1}{486} = P(X+Y\le 1)$$

(e) $P(X+Y\ge 1) = 1-P(X+Y\le 1)$

$$= 1-\frac{1}{486} = \frac{485}{486}$$

(4) Given 2 r.vs having the joint pdf

$$f(x_1, x_2) = \begin{cases} \frac{2}{3}(x_1 + 2x_2), & \text{for } 0 < x_1 < 1,\ 0 < x_2 < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal pdfs and check if X_1 & X_2 are independent.

(b) Find the conditional density of X_1 given X_2

(a) $f_1(x_1) = \int_{x_2} f(x_1, x_2)\,dx_2 = \int_0^1 \frac{2}{3}(x_1 + 2x_2)\,dx_2 = \frac{2}{3}\left[x_1 x_2\Big|_0^1 + x_2^2\Big|_0^1\right] = \frac{2}{3}(x_1 + 1),\ 0 < x_1 < 1$

$f_2(x_2) = \int_{x_1} f(x_1, x_2)\,dx_1 = \int_0^1 \frac{2}{3}(x_1 + 2x_2)\,dx_1 = \frac{2}{3}\left[\frac{x_1^2}{2}\Big|_0^1 + 2x_2 x_1\Big|_0^1\right] = \frac{2}{3}\left[\frac{1}{2} + 2x_2\right]$

$f_2(x_2) = \frac{1}{3} + \frac{4x_2}{3},\ 0 < x_2 < 1$

$f_1(x_1) \times f_2(x_2) \neq f(x_1, x_2) \Rightarrow X_1$ & X_2 are not independent.

(b) $f(X_1/X_2) = \frac{f(x_1, x_2)}{f_2(x_2)} = \frac{\frac{2}{3}(x_1 + 2x_2)}{\frac{2}{3}\left(\frac{1}{2} + 2x_2\right)} = \frac{2(x_1 + 2x_2)}{(1 + 4x_2)}$

$$\therefore f(x_1/x_2) = \begin{cases} \frac{2(x_1 + 2x_2)}{(1 + 4x_2)}, & 0 < x_1, x_2 < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

Homework:

1. If X and Y are two random variables having joint density function

   $f(x, y) = (1/8)(6 - x - y)$ for $0 < x < 2$ and $2 < y < 4$. Find $P(X < 1 \cap Y < 3)$

2. Find the value of 'k', if $f(x,y) = k(1-x)(1-y), 0 < x, y < 1$ is a joint pdf.

3. The joint pdf of two dimensional random variables (X,Y) is given by

$$f(x,y) = xy^2 + \frac{x^2}{8}, 0 \le x \le 2, 0 \le y \le 1$$

   Compute: (a) $P(X>1)$ (b) $P(Y<0.5)$ (c) $P(X>1/Y<0.5)$ (d) $P(Y<0.5/X>1)$ (e) $P(X>Y)$

4. If the joint pdf of 2 RVs is given by:

$$f(x,y) = 6e^{-2x-3y}, x > 0, y > 0$$

   Find the probabilities that:

   (a) the first RV takes on a value b/w 1 and 2 and second RV will take a value b/w 2 and 3.

   (b) the first RV will take a value less than 2 and the second RV will take a value greater than 2.

   (c) Are X and Y independent?

5. If the joint pdf of the random variable (X,Y) is given by $f(x,y)=k(x^3y+xy^3)$, $0 \le x,y \le 2$. Find the value of k and also the conditional and marginal densities of X and Y.