23MAT230 – Mathematics for Intelligent Systems - 3
Practise Sheet-2
(Vector Norms, Matrix Norms)

Vector norms:

- **norm(X,2) returns the 2-norm of X.**
    - **norm(X) is the same as norm(X,2).**
- **norm(X,1) returns the 1-norm of X.**
- **norm(X,p) returns the p-norm of X.**
- **norm(X, inf) returns the infinity norm of X.**

Example:

$\bar{v} = [4, 2, 6, -5, -8]$

$\|v\|_1 = |4| + |2| + |6| + |-5| + |-8| = 25$

$\|v\|_2 = \sqrt{4^2 + 2^2 + 6^2 + (-5)^2 + (-8)^2} = \sqrt{145} = 12.0415$

$\|v\|_3 = \sqrt[3]{4^3 + 2^3 + 6^3 + (-5)^3 + (-8)^3} = (\quad)^{1/3} = 9.7435$

$\|v\|_\infty = \max\{|4|, |2|, |6|, |-5|, |-8|\} = 8$

```
>> v=[4,2,6,-5,-8];
>> nv1=norm(v,1)
nv1 =
   25
>> nv2=norm(v,2)
nv2 =
  12.0416
>> nv3=norm(v,3)
nv3 =
   9.7435
>> nvinf=norm(v,inf)
nvinf =
    8
```

- S-norm of a vector **v:**
  Given a symmetrix positive definte matrix S, the S-norm of a vector **v** is defined as $\mathbf{v}^T S \mathbf{v}$.
  >> A=randi([0,9],3,3); S=A*A' (generation of symmetric positive definite matrix)
  >> v=[3;8;2]; Snormv=v'*S*v

Matrix Norms:

- **Frobenius norm** of a matrix $M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $\|M\| = \sqrt{a_{11}{}^2 + a_{12}{}^2 + a_{21}{}^2 + a_{22}{}^2}$. It is same as L2 norm of the vector of all singular values of the matrix.

```
A =
    1    2    8    2    9    4    9
    4    7    9    8    3    3    2
    9    2    5    2    1    8    7
    3    5    1    8    2    5    7
    5    6    1    2    6    5    3
>> Fnorm=norm(A,'fro')
Fnorm =
   32
>> s=svd(A)
s =
  27.9866
   9.3020
   8.8838
   6.5575
   5.6832
>> norm(s,2)
ans =
  32.0000
```

Proof:

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}; \quad A^T A = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}^2 + a_{21}^2 & a_{11}a_{12} + a_{21}a_{22} \\ a_{12}a_{11} + a_{22}a_{21} & a_{12}^2 + a_{22}^2 \end{bmatrix}$

$\boxed{\mathrm{Trace}(A^T A) = a_{11}^2 + a_{21}^2 + a_{12}^2 + a_{22}^2} = (\text{Frobenius norm})^2$

$\mathrm{Trace}(A^T A) = \text{Sum of all eigenvalues of } A^T A$
$= \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2$

For $A_{m \times n}$, $\boxed{\sigma_1^2 + \sigma_2^2 + \dots = (\text{Frobenius norm})^2}$

Hence, Frobenius norm is same as L2 norm of the vector of all singular values of the matrix.

Spectral norm $||A||_2 = \max \frac{||Ax||}{||x||} = \sigma_1$ (often called the ℓ^2 **norm**)

Frobenius norm $||A||_F = \sqrt{\sigma_1^2 + \cdots + \sigma_r^2}$ (12) and (13) also define $||A||_F$

Nuclear norm $||A||_N = \sigma_1 + \sigma_2 + \cdots + \sigma_r$ (the trace norm) .

Nuclear norm:

```
>> nucnormA = norm(svd(A),1) %L1 norm of vector of all singular values
```

Spectral norm:

```
>> Sv_A=svd(A);
>> specnormA = Sv_A(1) %Linf norm of vector of all singular values (max singular value)
```

Practice Questions:

1. Generate a random 5 by 4 integer matrix and find the spectral norm, Frobenius norm and Nuclear norm of this matrix.
2. Generate a random integer matrix A of size 7 by 9. Verify the Frobenius norm is same as the L2 norm of the vector s=svd(A), Nuclear norm is same as the L1 norm of the vector s=svd(A) and Spectral norm is same as the L infinity norm of the vector s=svd(A)
3. Generate a random vector of dimension 7 and evaluate the L1, L2, L3 and L-infinity norms of it.
4. Generate a random vector of dimension 15 with integer values and evaluate the L infinity norm of it.
5. Consider the vector $x = [2,1,8]$.
   (a) Find the L2 norm of the vector x.
   (b) Generate an orthogonal matrix Q of order 3 and find y = Qx.
   (c) Find the L2 norm of y = Qx.
   (d) What do you conclude from results in (a) and (c)?
6. Consider the vector $x = [2,9,4,2]$. [6 marks]
   (a) Find the L3 norm of the vector.
   (b) Generate an orthogonal matrix Q of order 4 and find y = Qx. (Hint: Use QR decomposition)
   (c) Find the L3 norm of y = Qx.
   (d) What do you conclude from results in (a) and (c)?
7. Let u=[2,9,-1], v=[3,0,8] and $S = \begin{bmatrix} 9 & 2 & 2 \\ 2 & 9 & 2 \\ 2 & 2 & 9 \end{bmatrix}$.
   (a) Is S a positive definite matrix? (Hint: Check using eigenvalues)
   (b) Find the S-norm of u and v.
   (c) Find L2 norm of u and v.
   (d) Find the Frobenious norm of S.
8. Generate a random vector v of dimension 7 and a symmetric positive definite matrix S of order 7.
   (a) Find the S-norm of vector v
   (b) Find the Spectral and nuclear norms of S.
9. Generate a random vector v of dimension 4. Find the L1, L2, L3 and L4 norms for it. Which norm is the largest?