

Journal Club Notes

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Part I

Real Analysis

Chapter 1

Natural Numbers

Numbers were built to count. A system for counting was made, and that system is the number system.

Definition 1.0.1

A natural number is an element of the set \mathbb{N} of the set

$$\mathbb{N} = \{0, 1, 2, 3 \dots\}$$

is obtained from 0 and counting forward indefinitely.

1.1 Peano Axioms

We start with axioms to help clarify this.

- Axiom 1 : $0 \in \mathbb{N}$
- Axiom 2: If $n \in \mathbb{N}$, then $n++ \in \mathbb{N}$
- Axiom 3: 0 is not an increment of any other natural number $n \in \mathbb{N}$
- Axiom 4: If $n \neq m$, $n++ \neq m++$
- Axiom 5: (Principle Of Mathematical Induction) Let $P(n)$ be any property pertaining to a natural number n . Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number.

We then make an assumption: That the set \mathbb{N} which satisfies these five axioms is called the set of natural numbers. With these 5 axioms, we can construct sequences

1.2 Recursive Definitions

Proposition 1.2.1 (Recursive Definitions). Suppose for each natural number n , we have some function $f_n : \mathbb{N} \rightarrow \mathbb{N}$ from the natural numbers to the natural numbers. Then we can assign a unique natural number a_n to each natural number n , such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n .

1.3 Addition

Definition 1.3.1: Addition Of Natural Numbers

Let n be a natural number. ($n \in \mathbb{N}$). To add zero to m , we define $0 + m := m$. Now suppose inductively that we have defined how to add n to m . Then we can add $n++$ to m by defining $(n++) + m := (n+m)++$

Lemma 1.3.1. For any natural number $n + 0 = n$

Proof. We use induction,

The base case, $n = 0$,

$$n = 0, 0 + 0 = 0$$

$$n + 0 = n$$

$$(n++) + 0 = (n + 0)++ = (n++)$$

Suppose inductively, that $n + 0 = n$,

For $n = n++$,

$$(n++) + 0 = (n + 0)++$$

We know that $n + 0 = n$

$$(n++) + 0 = (n++)$$

□

Lemma 1.3.2. For any natural numbers n and m ,

$$n + (m++) = (n + m)++$$

Proof. Inducting on n while keeping m fixed,

$$n = 0,$$

$$0 + (m++) = (0 + m)++$$

$$0 + (m++) = (m++)$$

This we know is true from the definition of addition ($0 + m := m$)

Suppose inductively, that $n + (m++) = (n + m)++$ is true. For $n = (n++)$,

$$\begin{aligned} (n++) + (m++) &= ((n++) + m)++ \\ &= (n + (m++))++ \\ &= ((n + m)++)++ \end{aligned}$$

From the definition of addition

□

Putting $m = 0$, we get $n + 1 = n++$

Proposition 1.3.1 (Addition is commutative). For any natural numbers n and m , $n + m = m + n$

Proof. We induct over n . For the base case, $n = 0$,

We must show that $m + 0 = 0 + m$. From the definition of addition, we have

$$0 + m = m$$

As shown earlier, we have

$$m + 0 = m$$

This is clearly true for $n = 0$.

Now suppose inductively that $m + n = n + m$

For $n = n + +$, we must show that $m + (n + +) = (n + +) + m$

We know from the definition of addition that,

$$(n + +) + m := (m + n) + +$$

And we proved earlier that,

$$m + (n + +) = (m + n) + +$$

Therefore,

$$m + (n + +) = (n + +) + m$$

□

Proposition 1.3.2 (Addition is associative). For any natural numbers, a, b and c , we have $(a + b) + c = a + (b + c)$

Proof. We take $(a + b) + n = a + (b + n)$

Inducting over n ,

For $n = 0$,

We have in the LHS,

$$\begin{aligned} &= (a + b) + 0 \\ &= a + b \end{aligned}$$

$$\text{Since } n + 0 = n$$

On the RHS,

$$\begin{aligned} &= a + (b + 0) \\ &= a + b \end{aligned}$$

$$\text{Since } n + 0 = n$$

Suppose inductively that $(a + b) + n = a + (b + n)$,

For $n = n + +$, We have to show that $(a + b) + (n + +) = a + (b + (n + +))$

On the LHS we have,

$$\begin{aligned}
&= (a + b) + (n + +) \\
&= (a + b + n) + + \quad (\text{From the lemma } m + (n + +) = (m + n) + +)
\end{aligned}$$

On the RHS we have,

$$\begin{aligned}
&= a + (b + (n + +)) \\
&= a + (b + n) + + \quad (\text{From the lemma } m + (n + +) = (m + n) + +) \\
&= (a + b + n) + +
\end{aligned}$$

LHS = RHS

□

Proposition 1.3.3 (Cancellation Law). Let a, b, c be natural numbers such that $a + b = a + c$. Then we have $b = c$.

Proof. We have,

$$n + b = n + c$$

Inducting over n , For the base case, $n = 0$

$$\begin{aligned}
0 + b &= 0 + c \\
b &= c
\end{aligned}$$

Suppose inductively that $n + b = n + c$ For $n = n + +$,

$$(n + +) + b = (n + +) + c$$

On the LHS

$$\begin{aligned}
&= (n + +) + b \\
&= (n + b) + +
\end{aligned}$$

On the RHS

$$\begin{aligned}
&= (n + +) + c \\
&= (n + c) + +
\end{aligned}$$

We know from the inductive hypothesis that,

$$\text{If } n + b = n + c, \text{ then } b = c$$

Thus we have,

$$b + + = c + +$$

□

Definition 1.3.2: Positive natural number

All numbers where,

$$n \neq 0, n \in \mathbb{N}$$

Proposition 1.3.4. If a is a positive natural number and b is a natural number, then $a + b$ is positive.

Proof. Inducting over b ,

For $b = 0$,

$$a + 0 = a$$

This proves the base case, since we know a is positive.

Now, suppose inductively, that $(a + b)$ is positive.

For $(a + (n + +))$,

$$a + (n + +) = (a + n) + +$$

We know from Axiom 3 that $n + + \neq 0$. Thus we close the inductive loop. \square

Lemma 1.3.3. For every a , there exists a unique b such that $b + + = a$

Proof. Proof by contradiction, Suppose that there are two different increments, $m + +$, $n + +$ that equal to a ,

We have,

$$m + + = a$$

$$n + + = a$$

Then we can say,

$$m + + = n + +$$

$$m + 1 = n + 1$$

$$m = n$$

(By Cancellation Law)

But we said that m and n are different numbers which increment to a .

Therefore, we can conclude that there is only one number b which increments to a \square

1.4 Order

Definition 1.4.1: Order

Let n and m be natural numbers we say that n is greater than or equal to m , and write $n \geq m$ iff we have $n = m + a$ for some natural number a . We say that $n > m$ when $n \geq m$ and $n \neq m$

Proposition 1.4.1 (Basic properties of order for natural numbers). Let a, b, c be natural numbers then

1. (Order is reflexive) $a \geq a$
2. (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$
3. (Order is antisymmetric) If $a \geq b$ and $b \geq a$ then $a = b$
4. (Addition preserves order) $a \geq b$ if and only if $a + c \geq b + c$
5. $a < b$ if and only if $a + + \leq b$
6. $a < b$ if and only if $b = a + d$ for some positive number d .

Proof. 1. Proving order is reflexive, $a \geq a$

We know that,

$$a = a + 0$$

From the definition of order, We can write that $a \geq b$ when $a = b + d$ where $d \in \mathbb{N}$

Thus $a \geq a$.

2. Proving order is transitive, $a \geq b$ and $b \geq c$ then $a \geq c$

We write,

$$a = b + d$$

$$b = c + e$$

$$a = c + e + d$$

We can say that since $(e + d) \in \mathbb{N}$

We define $f := (e + d)$ Where $f \in \mathbb{N}$

$$a = c + (f)$$

Thus we can say,

$$\text{If } a \geq b, b \geq c \text{ then } a \geq c$$

3. Proving order is antisymmetric, If $a \geq b$ and $b \geq a$ then $a = b$ We can say,

$$a = b + d$$

$$b = a + e$$

Where $d, e \in \mathbb{N}$

$$a = (a + e) + d$$

$$b = (b + d) + e$$

Then we can write,

$$\begin{aligned}a &= a + (e + d) \\ b &= b + (d + e)\end{aligned}$$

Then we can say that $(e + d)$ and $(d + e)$ are 0.

We know that if $a + b = 0$ then $a, b = 0$

Thus d and e are 0.

$$\begin{aligned}a &= b + d \\ a &= b\end{aligned}$$

4. Proving $a < b$ if and only if $b = a + d$ for some positive number d If $b = a + d$ where d is a positive natural number, $d \neq 0$

Which means that $b \neq a + 0$ or $b \neq a$

This means that b is strictly greater than a

If $a < b$ then $a \geq b$ and $a \neq b$

So if $a \geq b$ Then,

$$a = b + d$$

But,

$$\begin{aligned}a &\neq b \\ a &\neq b + 0 \\ d &\neq 0\end{aligned}$$

Thus d cannot be 0. d can only be a positive natural number.

5. Proving addition preserves order, $a \geq b$ if and only if $a + c \geq b + c$ Proving $a \geq b$ if $a + c \geq b + c$

Where $d \in \mathbb{N}$

$$\begin{aligned}a + c &= b + c + d && \text{By definition} \\ a + c &= (b + d) + c \\ a &= (b + d) && \text{By cancellation law} \\ a &\geq b\end{aligned}$$

Proving $a + c \geq b + c$ if $a \geq b$

We know,

$$a = b + d$$

Where $d \in \mathbb{N}$

We write $a+c$ using what we know from above,

$$\begin{aligned} a + c &= b + d + c \\ a + c &= b + c + d \\ (a + c) &= (b + c) + d \\ a + c &\geq b + c \end{aligned}$$

6. Proving $a < b$ if and only if $a + + \leq b$ Proving $a < b$ if $a + + \leq b$

We can write,

$$\begin{aligned} a + + &= b + d && \text{Where } d \in \mathbb{N} \\ a + + + d &= b \\ a + (d + +) &= b \end{aligned}$$

Since from Axiom 3, we know that 0 is not an increment of any natural number, $(d + + \neq 0)$
Therefore,

$$a < b$$

□

Proposition 1.4.2 (Trichotomy of order for natural numbers). Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b, a = b$ or $a > b$

Proof. First we show that no more than one of the statements is true. If $a < b$ then $a \neq b$ by definition. If $a > b$ then $a \neq b$ by definition. If $a > b$ and $a < b$ then $a = b$, which we proved earlier.

Now to show that exactly one of these statements are true. We induct on a ,

When $a = 0$, We know that,

$$\begin{aligned} b &= 0 + b && (\forall b \in \mathbb{N}) \\ b &\geq 0 \end{aligned}$$

Suppose inductively that exactly one of the above statements are true for a and b . For $a + +$, We take each statement. First for $a > b$

$$\begin{aligned} a &> b \\ a &= b + d \\ (a + +) &= (b + d) + + && \text{Incrementing both sides} \\ (a + +) &= b + d + + && \text{From Lemma 1.3.2} \\ (a + +) &> b && \text{If } d \in \mathbb{N} \text{ then } d + + \in \mathbb{N} \end{aligned}$$

For $a = b$

$$\begin{aligned}a &= b \\(a++) &= (b)++ \\(a++) &= b+1 \\a &> b\end{aligned}$$

For $a < b$

$$\begin{aligned}a &< b \\a + d &= b \\(a + d)++ &= b++ \\(a++) + d &= b++ \\(a++) + d &= b+1\end{aligned}$$

We have two cases, If $d = 1$, Then by cancellation law

$$a++ = b$$

If $d \neq 1$ Then

$$a++ < b$$

But never both, which concludes the inductive loop. \square

1.5 Special Forms Of Induction

Proposition 1.5.1 (Strong Principle Of Induction). Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m . Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true. (In particular this means that $P(m_0)$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

Proof. For a property $Q(n)$, which is the property that $P(m')$ is true for $m_0 \leq m < n$, then $P(n)$ is true... Then it is true $\forall m \geq m_0$

For $Q(0)$, we can say that the statement is vacuous since the conditions are not satisfied for both when $m_0 = 0$ and when $m_0 < 0$

Suppose inductively that $Q(n)$ is true.

Which means that,

$P(m)$ is true for $m_0 \leq m \leq n$

Then for $Q(n++)$,

We know that the property $P(m)$ is true for $m_0 \leq m \leq n$ Take the upper limit of the range,

$$m_0 \leq n$$

We can rewrite this as,

$$m_0 < n++$$

Since $a < b$ if and only if $a++ \leq b$

So we have the property is true for $m_0 \leq m < n++$, the property is true for $P(n++)$, since that is how we chose $Q(n)$

Thus $Q(n++)$ is true, closing the inductive loop.

Since we know that $P(m)$ is true for any n larger than m , we can then say that $m \geq m_0$ \square

Proposition 1.5.2 (Backward Induction). Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m++)$ is true, then $P(m)$. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$.

Proposition 1.5.3 (Induction starting from the base case n). Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m)$ is true, $P(m++)$ is true. Show that if $P(n)$ is true, then $P(m)$ is true for all $m \geq n$. (This principle is sometimes referred to as the principle of induction starting from the base case n .)

Proof. Take a property $P(n)$, $m \geq n$

Inducting over n , \square

1.6 Multiplication

Definition 1.6.1: Multiplication

Let m be a natural number. To multiply zero to m , we define $0 \times m := 0$. Now suppose inductively that we have defined how to multiply n to m . Then we can multiply $n++$ to m by defining $(n++) \times m := (n \times m) + m$

We can say $0 \times m = 0$, $1 \times m = 0 + m$, $2 \times m = 0 + m + m$ and so on.

Lemma 1.6.1. Prove that multiplication is commutative

Proof. We use the way we proved that addition is commutative as a blueprint. There are two things we need to prove first.

1. For any natural number, n , $n \times 0 = 0$
2. For any natural numbers, n and m , $n \times (m++) = (n \times m) + m$

First we prove,

1. For any natural number, n , $n \times 0 = 0$ We induct over n , For $n = 0$,

$$0 \times 0 = 0$$

Which is true from the definition

Now suppose inductively, that $n \times 0 = 0$, For $(n++) \times 0$, From the definition we can write

this as,

$$(n++) \times 0 = (n \times 0) + 0$$

We know that $n \times 0 = 0(n++) \times 0 = 0 + 0$

$$(n++) \times 0 = 0$$

Therefore,

$$n \times 0 = n$$

2. For any natural numbers, n and m , $n \times (m++) = (n \times m) + m$ We induct over n , (keeping m fixed)

For $n = 0$, We know from the definition for multiplication with zero that,

$$0 \times (m++) = 0$$

We also know that

$$(m++) \times 0 = (m \times 0) + 0$$

$$(m++) \times 0 = 0$$

$$(m++) \times 0 = 0 \times (m++) = (0 \times m) + m$$

Suppose inductively that $n \times (m++) = (n \times m) + m$ For $n = (n++)$ To prove $(n++) \times (m++) = ((n++) \times m) + m$,

$$(n++) \times (m++) = (n \times (m++)) + m++$$

We can rewrite RHS using the inductive hypothesis

$$(n++) \times (m++) = ((n \times m) + m) + m++$$

Taking the LHS, we write

$$\begin{aligned} (n++) \times (m++) &= (n \times (m++)) + m++ && \text{From the definition} \\ &= (n \times m) + m) + m++ && \text{From the inductive hypothesis} \\ &= (n \times m) + m + (m++) && \text{Associativity of addition} \\ &= (n \times m) + (m++) + m && \text{Commutativity of addition} \end{aligned}$$

We can say that LHS = RHS, closing the inductive loop

Now we can finally get on with proving that multiplication is commutative, armed with the lemmas we've proved here we can prove this similarly to how we proved addition is commutative We are proving, $m \times n = n \times m$ We fix m and induct over n

For $n = 0$,

$$0 \times m = 0$$

From the definition

$$m \times 0 = 0$$

Proved earlier

$$m \times 0 = 0 \times m$$

Suppose inductively that $m \times n = n \times m$ is true.

For proving $m \times (n++) = (n++) \times m$, Take the LHS,

$$m \times (n++) = (n \times m) + m \quad \text{Proved earlier}$$

Take the RHS,

$$(n++) \times m = (n \times m) + m \quad \text{From the definition}$$

We can see that LHS = RHS, closing the inductive loop

Thus multiplication is commutative \square

Lemma 1.6.2 (Positive natural numbers have no zero divisors). Let n, m be natural numbers, then $n \times m = 0$ if and only if at least one of n, m is equal to zero. In particular, if both n, m are both positive, then nm is positive

Proof. First we prove if both n, m are both positive, then nm is positive

We have, n and m such that $n, m > 0$ \square

Proposition 1.6.1 (Distributive Law). For any natural numbers, a, b, c We have, $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$

Proof. \square

Proposition 1.6.2 (Multiplication is associative). Prove,

$$(a \times b) \times c = a \times (b \times c)$$

Proof. \square

Proposition 1.6.3 (Multiplication preserves order). If a, b are natural numbers such that $a < b$ and c is positive, then $ac < bc$

Proof. \square

Corollary 1.6.1 (Cancellation Law). Let a, b, c be natural numbers such that $ac = bc$ and c is non-zero. Then $a = b$

Proof. \square

Proposition 1.6.4 (Euclid's division lemma). Let n be a natural number and let q be a positive number. Then there exists natural numbers m, r such that $0 \leq r < q$ and $n = mq + r$

Proof. Inducting over n ,

For $n = 0$,

$$0 = mq + r$$

□

Definition 1.6.2: Exponentiation For Natural Numbers

Let m be a natural number. To raise m to the power m^0 , we define $m^0 := 1$. Now suppose recursively that m^n has been defined for some natural number n then we define $m^{n++} = m^n \times m$

Thus we can write, $m^0 = 1, m^1 = m^0 + m = m, m^2 = m^1 + m = 2m$ and so on

Proposition 1.6.5. Prove,

$$(a + b)^2 = a^2 + b^2 + 2ab$$

Proof. Take the RHS, We have,

$$\begin{aligned} (a + b)^2 &= (a + b)(a + b) \\ &= (a + b)a + (a + b)b \\ &= aa + ba + ab + bb \\ &= a^2 + ba + ab + b^2 \\ &= a^2 + ab + ab + b^2 \\ &= a^2 + 2ab + b^2 \end{aligned}$$

From the definition

Distributive Law

$$m^2 = m \times m$$

Commutativity of multiplication

Definition of multiplication = $a^2 + b^2 + 2ab$ Definition of multiplication

Thus we have LHS = RHS

□

Chapter 2

Set Theory

We define first what a set is:

Definition 2.0.1: Sets

We define set A to be any unordered collection of objects. If x is an object, we say that x is an element of A or $x \in A$ if x lies in the collection. Otherwise $x \notin A$

We start with some axioms:

1. (Sets are objects) If A is a set, then A is also an object. A side track about “Pure Set Theory” - This theory states that everything in the mathematical universe is a set. We can write 0 as \emptyset or an empty set, 1 can be written as $\{\emptyset\}$ and 2 as $\{\emptyset, \{\emptyset\}\}$ and so on. Terence Tao argues that they are the ‘cardinalities of the set.’
2. (Equality of sets) Two sets A and B are equal, $A = B$, iff every element of A is an element of B . $A = B$, if and only if every element x of A also belongs to B , and every element y of B belongs to A .
3. (Empty set) There exists a set \emptyset known as the empty set, which contains no elements. $x \notin \emptyset$

Proposition 2.0.1 (Partial Order). If $A \subseteq B, B \subseteq C \Rightarrow A \subseteq C$

Proof. If $x \in A$, then $x \in B$, If $x \in B$, then $x \in C$, Then $x \in A$, then $x \in C$

Thus, $A \subseteq C$ □

Lemma 2.0.1 (Single choice). Let A be a non-empty set. Then there exists an object x such that $x \in A$

Proof. Proving by contradiction, Suppose there is no object x that belongs to A . For all x , we have $x \notin A$. We know from Axiom 3, that $x \notin \emptyset$

For the statement,

$$x \in A \Leftrightarrow x \in \emptyset$$

Is false both ways, which gives us the result true, which is a contradiction.

Thus we also prove that \emptyset is unique. □

4. (Pairwise Union) $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Lemma 2.0.2. $A \cup (B \cup C) = (A \cup B) \cup C$

Proof. Taking the left hand side, We have $x \in A$ or $x \in (B \cup C)$. If we look to the right hand side, we have $x \in (A \cup B)$ or $x \in C$ If we break the statement down further. We have $x \in A$ or $x \in B$ or $x \in C$, and on the right $x \in A$ or $x \in B$ or $x \in C$

The two statements are equivalent. \square

5. (Axiom Of Specification) A, $x \in A$, let $P(x)$ be a property pertaining to x . Then there exists a set called $\{x \in A, P(x) \text{ is true}\}$ whose elements are precisely the elements x in A for which $P(x)$ is true.
6. (Replacement) Let A be a set, for any object $x \in A$, and any object y , suppose we have a statement $P(x, y)$ pertaining to x and y , such that for each $x \in A$ there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y) \text{ is true for some } x \in A\}$
7. (Infinity) There exists a set \mathbb{N} , whose elements are called natural numbers, as well as an object 0 in \mathbb{N} , and an object $n++$ assigned to every natural number $n \in \mathbb{N}$ such that the Peano axioms hold.
8. **Russel's Paradox** (Axiom Of Universal Specification) Suppose for every x we have a property $P(x)$ pertaining to x , Then there exists a set $\{x : P(x) \text{ is true}\}$ such that for every object y :

$$y \in \{x : P(x) \text{ is true}\} \Leftrightarrow P(y) \text{ is true.}$$

This is quite a handy axiom, we can even find all the axioms through this axiom, but there's an issue.

Let's say we defined the property $P(x) \Leftrightarrow x$ is a set, and $x \notin x$ And we defined a set Ω for which this property is true. A paradox is created, where we don't know whether Ω belongs in this set or not, it's simultaneously true and untrue. If Ω is not in the set, then it is a set that is not in its own set, thereby being contained in the set. If Ω is in the set, then it's a set that is in itself, therefore it cannot be in the set. To resolve this paradox we have, the next axiom.

Note:-

There is a similar paradox known as the "Grelling-Nelson paradox" with the words heterological and autological, the word autological means that the word is an example of itself, for example, the word 'word' is a word or 'pronounceable' is pronounceable or a 'noun' is a noun. The word heterological means that the word is not an example of itself. The word 'triangle' is clearly not a \triangle and so on. The question arises, is heterological autological or heterological?

9. (Axiom Of Regularity) If A is a non-empty set, then there is at least one element x of A which is either not a set or is disjoint from A . How exactly does this mitigate the issues that come in from Russel's Paradox?

Definition 2.0.2: Intersection Of Sets

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$