Journal Club

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Contents

Part I

Real Analysis

TEXTBOOK: Analysis 1 by Terence Tao

Chapter 1

Natural Numbers

Numbers were built to count. A system for counting was made, and that system is the number system.

Definition 1.0.1

A natural number is an element of the set $\mathbb{N}$ of the set

$$\mathbb{N} = \{0, 1, 2, 3 \cdots\}$$

is obtained from 0 and counting forward indefinitely.

1.1 Peano Axioms

We start with axioms to help clarify this.

- Axiom 1 : $0 \in \mathbb{N}$
- Axiom 2: If $n \in \mathbb{N}$,, then $n++ \in \mathbb{N}$
- Axiom 3: 0 is not an increment of any other natural number $n \in \mathbb{N}$
- Axiom 4: If $n \neq m$, $n++ \neq m++$
- Axiom 5: (Principle Of Mathematical Induction) Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number.

We then make an assumption: That the set $\mathbb{N}$ which satisfies these five axioms is called the set of natural numbers. With these 5 axioms, we can construct sequences

1.2 Recursive Definitions

Proposition 1.2.1 (Recursive Definitions). Suppose for each natural number n, we have some function $f_n : \mathbb{N} \to \mathbb{N}$ from the natural numbers to the natural numbers. Then we can assign a unique natural number a_n to each natural number n, such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n.

1.3 Addition

Definition 1.3.1: Addition Of Natural Numbers

Let n be a natural number. $(n \in N)$. To add zero to m, we define $0 + m := m$ Now suppose inductively that we have defined how to add n to m. Then we can add n++ to m by defining(n++) + m := (n+m)++

Lemma 1.3.1. For any natural number $n + 0 = n$

Proof. We use induction,
The base case, n = 0,

$$\begin{aligned} n = 0, 0 + 0 = 0 \\ n + 0 = n \\ (n++) + 0 = (n + 0) ++ = (n++) \end{aligned}$$

Suppose inductively, that $n + 0 = n$,
For $n = n++$,

$$\begin{aligned} (n++) + 0 &= (n + 0)++ \\ \text{We know that } n + 0 &= n \\ (n++) + 0 &= (n++) \end{aligned}$$

□

Lemma 1.3.2. For any natural numbers n and m,

$$n + (m++) = (n + m)++$$

Proof. Inducting on n while keeping m fixed,

$$\begin{aligned} n = 0, \\ 0 + (m++) = (0 + m)++ \\ 0 + (m++) = (m++) \end{aligned}$$

This we know is true from the definition of addition $(0 + m := m)$
Suppose inductively, that $n + (m++) = (n + m)++$ is true. For $n = (n++)$,

$$\begin{aligned} (n++) + (m++) &= ((n++) + m)++ && \text{From the definition of addition} \\ &= (n + (m++))++ \\ &= ((n + m)++))++ \end{aligned}$$

□

Putting m = 0, we get $n + 1 = n++$

Proposition 1.3.1 (Addition is commutative). For any natural numbers n and m, $n + m = m + n$

Proof. We induct over n, For the base case, $n = 0$,

We must show that $m + 0 = 0 + m$ From the definition of addition, we have

$$0 + m = m$$

As shown earlier, we have

$$m + 0 = m$$

This is clearly true for $n = 0$.

Now suppose inductively that $m + n = n + m$

For $n = n++$, we must show that $m + (n++) = (n++) + m$

We know from the definition of addition that,

$$(n++) + m := (m + n)++$$

And we proved earlier that,

$$m + (n++) = (m + n)++$$

Therefore,

$$m + (n++) = (n++) + m$$

□

Proposition 1.3.2 (Addition is associative). For any natural numbers, a, b and c, we have $(a+b)+c = a + (b + c)$

Proof. We take $(a + b) + n = a + (b + n)$

Inducting over n,

For $n = 0$,

We have in the LHS,

$$\begin{aligned} &= (a + b) + 0 && \text{Since } n + 0 = n \\ &= a + b \end{aligned}$$

On the RHS,

$$\begin{aligned} &= a + (b + 0) && \text{Since } n + 0 = n \\ &= a + b \end{aligned}$$

Suppose inductively that $(a + b) + n = a + (b + n)$,

For $n = n++$, We have to show that $(a + b) + (n++) = a + (b + (n++))$

On the LHS we have,

$$\begin{aligned} &= (a + b) + (n++) \\ &= (a + b + n)++ && \text{(From the lemma } m + (n++) = (m + n)++) \end{aligned}$$

On the RHS we have,

$$\begin{aligned}
&= a + (b + (n++)) \\
&= a + (b + n)++ && \text{(From the lemma } m + (n++) = (m+n)++\text{)} \\
&= (a + b + n)++
\end{aligned}$$

LHS = RHS □

Proposition 1.3.3 (Cancellation Law). Let a, b, c be natural numbers such that $a + b = a + c$. Then we have $b = c$.

Proof. We have,

$$n + b = n + c$$

Inducting over n, For the base case, $n = 0$

$$\begin{aligned}
0 + b &= 0 + c \\
b &= c
\end{aligned}$$

Suppose inductively that $n + b = n + c$ For $n = n++$,

$$(n++) + b = (n++) + c$$

On the LHS

$$\begin{aligned}
&= (n++) + b \\
&= (n + b)++
\end{aligned}$$

On the RHS

$$\begin{aligned}
&= (n++) + c \\
&= (n + c)++
\end{aligned}$$

We know from the inductive hypothesis that,

$$\text{If} n + b = n + c, \text{then} b = c$$

Thus we have,

$$b++ = c++$$

□

Definition 1.3.2: Positive natural number

All numbers where,

$$n \neq 0, n \in \mathbb{N}$$

Lemma 1.3.3. For every a, there exists a b such that $b++ = a$

Definition 1.3.3: Order

Let n and m be natural numbers we say that n is greater than or equal to m, and write $n \geq m$ iff we have $n = m + a$ for some natural number a. We say that $n > m$ when $n \geq m$ and $n \neq m$

1.4 Strong Induction

Theorem 1.4.1. Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true.(In particular this means that $P(m_0$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

Proof. For a property $Q(n)$, which is the property that $P(m')$ is true for $m_0 \leq m' < n$, then $P(n)$ is true.

For $Q(0)$, 0 is either lesser than or equal to m_0.
When 0 is lesser than m_0,
This is vacuously true.
When $0 = m_0$, □

1.5 Induction Starting From The Base Case n

Let n be a natural number, and let P(m) be a property pertaining to the natural numbers such that whenever P(m) is true, P(m++) is true. Show that if P(n) is true, then P(m) is true for all m n. (This principle is sometimes referred to as the principle of induction starting from the base case n.)

Proof. Take a property $P(n)$, $m \geq n$
Inducting over n, □

1.6 Multiplication

Definition 1.6.1

Let m be a natural number. To multiply zero to m, we define $0 \times m := 0$. Now suppose inductively that we have defined how to multiply n to m. Then we can multiply $n++$ to m by defining $(n++) \times m := (n \times m) + m$

Lemma 1.6.1. Prove that multiplication is commutative

1.7 Exercise

1. Prove the identity $(a + b)^2 = a^2 + 2ab + b^2$

2. (Euclid's division lemma)

3. Backward Induction $m \in \mathbb{N}, P(m), P(m++) \Rightarrow P(m)$, Suppose $P(n)$ is true, then $P(m) \forall m \leq n$
   For the base case, n = 0, P(0) $\Rightarrow P(0), soit'strue.$

For the inductive step, supposing $Q(n)$ is true,

4. Strong induction
5. Distributive Law
6. Multiplication
    (a) Cancellation Law
    (b) Associativity
    (c) If a<b, and c is positive then ac<bc

Exam 1	Exam 2	Mean
12	19	15.5
14	13	13.5
19	19	19