Journal Club

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Contents

I Real Analysis **2**

1 Natural Numbers **4**

1.1 Peano Axioms . . . 4
1.2 Recursive Definitions . . . 4
1.3 Addition . . . 5
1.4 Order . . . 8
1.5 Special Forms Of Induction . . . 12
1.6 Multiplication . . . 13

2 Set Theory **15**

Part I

Real Analysis

TEXTBOOK: Analysis 1 by Terence Tao

Chapter 1

Natural Numbers

Numbers were built to count. A system for counting was made, and that system is the number system.

Definition 1.0.1

A natural number is an element of the set $\mathbb{N}$ of the set

$$\mathbb{N} = \{0, 1, 2, 3 \cdots\}$$

is obtained from 0 and counting forward indefinitely.

1.1 Peano Axioms

We start with axioms to help clarify this.

- Axiom 1 : $0 \in \mathbb{N}$
- Axiom 2: If $n \in \mathbb{N}$,, then $n++ \in \mathbb{N}$
- Axiom 3: 0 is not an increment of any other natural number $n \in \mathbb{N}$
- Axiom 4: If $n \neq m$, $n++ \neq m++$
- Axiom 5: (Principle Of Mathematical Induction) Let $P(n)$ be any property pertaining to a natural number n. Suppose that $P(0)$ is true, and suppose that whenever $P(n)$ is true, $P(n++)$ is also true. Then $P(n)$ is true for every natural number.

We then make an assumption: That the set $\mathbb{N}$ which satisfies these five axioms is called the set of natural numbers. With these 5 axioms, we can construct sequences

1.2 Recursive Definitions

Proposition 1.2.1 (Recursive Definitions). Suppose for each natural number n, we have some function $f_n : \mathbb{N} \to \mathbb{N}$ from the natural numbers to the natural numbers. Then we can assign a unique natural number a_n to each natural number n, such that $a_0 = c$ and $a_{n++} = f_n(a_n)$ for each natural number n.

1.3 Addition

Definition 1.3.1: Addition Of Natural Numbers

Let n be a natural number. $(n \in N)$. To add zero to m, we define $0 + m := m$ Now suppose inductively that we have defined how to add n to m. Then we can add n++ to m by defining(n++) + m := (n+m)++

Lemma 1.3.1. For any natural number $n + 0 = n$

Proof. We use induction,
The base case, n = 0,

$$n = 0, 0 + 0 = 0$$
$$n + 0 = n$$
$$(n++) + 0 = (n + 0)++ = (n++)$$

Suppose inductively, that $n + 0 = n$,
For $n = n++$,

$$(n++) + 0 = (n + 0)++$$
$$\text{We know that } n + 0 = n$$
$$(n++) + 0 = (n++)$$

□

Lemma 1.3.2. For any natural numbers n and m,

$$n + (m++) = (n + m)++$$

Proof. Inducting on n while keeping m fixed,

$$n = 0,$$
$$0 + (m++) = (0 + m)++$$
$$0 + (m++) = (m++)$$

This we know is true from the definition of addition $(0 + m := m)$
Suppose inductively, that $n + (m++) = (n + m)++$ is true. For $n = (n++)$,

$$\begin{aligned} (n++) + (m++) &= ((n++) + m)++ \qquad \text{From the definition of addition} \\ &= (n + (m++))++ \\ &= ((n + m)++))++ \end{aligned}$$

□

Putting m = 0, we get $n + 1 = n++$

Proposition 1.3.1 (Addition is commutative). For any natural numbers n and m, $n + m = m + n$

Proof. We induct over n, For the base case, $n = 0$,
We must show that $m + 0 = 0 + m$ From the definition of addition, we have

$$0 + m = m$$

As shown earlier, we have

$$m + 0 = m$$

This is clearly true for $n = 0$.
Now suppose inductively that $m + n = n + m$
For $n = n++$, we must show that $m + (n++) = (n++) + m$
We know from the definition of addition that,

$$(n++) + m := (m + n)++$$

And we proved earlier that,

$$m + (n++) = (m + n)++$$

Therefore,

$$m + (n++) = (n++) + m$$

$\square$

Proposition 1.3.2 (Addition is associative). For any natural numbers, a, b and c, we have $(a+b)+c = a + (b + c)$

Proof. We take $(a + b) + n = a + (b + n)$
Inducting over n,
For $n = 0$,
We have in the LHS,

$$\begin{aligned} &= (a + b) + 0 && \text{Since } n + 0 = n \\ &= a + b \end{aligned}$$

On the RHS,

$$\begin{aligned} &= a + (b + 0) && \text{Since } n + 0 = n \\ &= a + b \end{aligned}$$

Suppose inductively that $(a + b) + n = a + (b + n)$,
For $n = n++$, We have to show that $(a + b) + (n++) = a + (b + (n++))$
On the LHS we have,

$$\begin{aligned} &= (a + b) + (n++) \\ &= (a + b + n)++ && \text{(From the lemma } m + (n++) = (m + n)++\text{)} \end{aligned}$$

On the RHS we have,

$$\begin{aligned}
&= a + (b + (n++)) \\
&= a + (b + n)++ && \text{(From the lemma } m + (n++) = (m + n)++\text{)} \\
&= (a + b + n)++
\end{aligned}$$

LHS = RHS ☐

Proposition 1.3.3 (Cancellation Law). Let a, b, c be natural numbers such that $a + b = a + c$. Then we have $b = c$.

Proof. We have,

$$n + b = n + c$$

Inducting over n, For the base case, $n = 0$

$$\begin{aligned}
0 + b &= 0 + c \\
b &= c
\end{aligned}$$

Suppose inductively that $n + b = n + c$ For $n = n++$,

$$(n++) + b = (n++) + c$$

On the LHS

$$\begin{aligned}
&= (n++) + b \\
&= (n + b)++
\end{aligned}$$

On the RHS

$$\begin{aligned}
&= (n++) + c \\
&= (n + c)++
\end{aligned}$$

We know from the inductive hypothesis that,

$$\text{If} n + b = n + c, \text{then} b = c$$

Thus we have,

$$b++ = c++$$

☐

Definition 1.3.2: Positive natural number

All numbers where,

$$n \neq 0, n \in \mathbb{N}$$

Proposition 1.3.4. If a is a positive natural number and b is a natural number, then $a+b$ is positive.

Proof. Inducting over b,

For $b = 0$,

$$a + 0 = a$$

This proves the base case, since we know a is positive.

Now, suppose inductively, that $(a + b)$ is positive.

For $(a + (n++))$,

$$a + (n++) = (a + n)++$$

We know from Axiom 3 that $n++ \neq 0$. Thus we close the inductive loop. □

Lemma 1.3.3. For every a, there exists a unique b such that $b++ = a$

Proof. Proof by contradiction, Suppose that there are two different increments, $m++$, $n++$ that equal to a,

We have,

$$m++ = a$$
$$n++ = a$$

Then we can say,

$$\begin{aligned} m++ &= n++ \\ m+1 &= n+1 \\ m &= n && \text{(By Cancellation Law)} \end{aligned}$$

But we said that m and n are different numbers which increment to a.

Therefore, we can conclude that there is only one number b which increments to a □

1.4 Order

Definition 1.4.1: Order

Let n and m be natural numbers we say that n is greater than or equal to m, and write $n \geq m$ iff we have $n = m + a$ for some natural number a. We say that $n > m$ when $n \geq m$ and $n \neq m$

Proposition 1.4.1 (Basic properties of order for natural numbers). Let a, b, c be natural numbers then

1. (Order is reflexive) $a \geq a$
2. (Order is transitive) If $a \geq b$ and $b \geq c$, then $a \geq c$
3. (Order is antisymmetric) If $a \geq b$ and $b \geq a$ then $a = b$
4. (Addition preserves order) $a \geq b$ if and only if $a + c \geq b + c$
5. $a < b$ if and only if $a++ \leq b$

6. $a < b$ if and only if $b = a + d$ for some positive number d.

Proof. 1. Proving order is reflexive, $a \geq a$

We know that,

$a = a + 0$

From the definition of order, We can write that $a \geq b$ when $a = b + d$ where $d \in \mathbb{N}$

Thus $a \geq a$.

2. Proving order is transitive, $a \geq b$ and $b \geq c$ then $a \geq c$

We write,

$$a = b + d$$
$$b = c + e$$
$$a = c + e + d$$

We can say that since $(e + d) \in \mathbb{N}$

We define $f := (e + d)$ Where $f \in \mathbb{N}$

$$a = c + (f)$$

Thus we can say,

$$\text{If } a \geq b, b \geq c \text{ then } a \geq c$$

3. Proving order is antisymmetric, If $a \geq b$ and $b \geq a$ then $a = b$ We can say,

$$a = b + d$$
$$b = a + e$$

Where $d, e \in \mathbb{N}$

$$a = (a + e) + d$$
$$b = (b + d) + e$$

Then we can write,

$$a = a + (e + d)$$
$$b = b + (d + e)$$

Then we can say that $(e + d)$ and $(d + e)$ are 0.

We know that if $a + b = 0$ then $a, b = 0$

Thus d and e are 0.

$$a = b + d$$
$$a = b$$

4. Proving $a < b$ if and only if $b = a + d$ for some positive number d If $b = a + d$ where d is a positive natural number, $d \neq 0$

   Which means that $b \neq a + 0$ or $b \neq a$

   This means that b is strictly greater than a

   If $a < b$ then $a \geq b$ and $a \neq b$

   So if $a \geq b$ Then,

$$a = b + d$$

   But,

$$a \neq b$$
$$a \neq b + 0$$
$$d \neq 0$$

   Thus d cannot be 0. d can only be a positive natural number.

5. Proving addition preserves order, $a \geq b$ if and only if $a+c \geq b+c$ Proving $a \geq b$ if $a+c \geq b+C$

   Where $d \in \mathbb{N}$

$$\begin{aligned} a + c &= b + c + d && \text{By definition} \\ a + c &= (b + d) + c && \\ a &= (b + d) && \text{By cancellation law} \\ a &\geq b && \end{aligned}$$

   Proving $a + c \geq b + c$ if $a \geq b$

   We know,

$$a = b + d$$

   Where $d \in \mathbb{N}$

   We write a+c using what we know from above,

$$a + c = b + d + c$$
$$a + c = b + c + d$$
$$(a + c) = (b + c) + d$$
$$a + c \geq b + c$$

6. Proving $a < b$ if and only if $a++ \leq b$ Proving $a < b$ if $a++ \leq b$

   We can write,

$$
\begin{aligned}
a++ &= b + d && \text{Where } d \in \mathbb{N} \\
a++ + d &= b \\
a + (d++) &= b
\end{aligned}
$$

   Since from Axiom 3, we know that 0 is not an increment of any natural number, $(d++ \neq 0)$ Therefore,

$$a < b$$

□

Proposition 1.4.2 (Trichotomy of order for natural numbers). Let a and b be natural numbers. Then exactly one of the following statements is true: $a < b, a = b or a > b$

Proof. First we show that no more than one of the statements is true. If $a < b$ then $a \neq b$ by definition. If $a > b$ then $a \neq b$ by definition. If $a > b$ and $a < b$ then $a = b$, which we proved earlier.

Now to show that exactly one of these statements are true. We induct on a,

When a $=$ 0, We know that,

$$
\begin{aligned}
b &= 0 + b (\forall b \in \mathbb{N}) \\
b &\geq 0
\end{aligned}
$$

Suppose inductively that exactly one of the above statements are true for a and b. For a$++$, We take each statement. First for $a > b$

$$
\begin{aligned}
a &> b \\
a &= b + d \\
(a++) &= (b + d)++ \\
(a++) &= b + d++ \\
(a++) &> b && \text{If } d \in \mathbb{N} \text{ then } d++ \in \mathbb{N}
\end{aligned}
$$

For $a = b$

$$
\begin{aligned}
a &= b \\
(a++) &= (b)++ \\
(a++) &= b + 1 \\
a &> b
\end{aligned}
$$

For $a < b$

$$\begin{aligned} a &< b \\ a + d &= b \\ (a + d) + + &= b + + \\ (a + +) + d &= b + + \\ (a + +) + d &= b + 1 \end{aligned}$$

We have two cases, If $d = 1$, Then by cancellation law

$$a + + = b$$

If $d \neq 1$ Then

$$a + + < b$$

But never both, which concludes the inductive loop. □

1.5 Special Forms Of Induction

Proposition 1.5.1 (Strong Principle Of Induction). Let m_0 be a natural number, and let $P(m)$ be a property pertaining to an arbitrary natural number m. Suppose that for each $m \geq m_0$, we have the following implication: if $P(m')$ is true for all natural numbers $m_0 \leq m' < m$, then $P(m)$ is also true.(In particular this means that $P(m_0$ is true, since in this case the hypothesis is vacuous.) Then we can conclude that $P(m)$ is true for all natural numbers $m \geq m_0$.

Proof. For a property $Q(n)$, which is the property that $P(m')$ is true for $m_0 \leq m < n$, then $P(n)$ is true... Then it is true $\forall m \geq m_0$

For $Q(0)$, we can say that the statement is vacuous since the conditions are not satisfied for both when $m_0 = 0$ and when $m_0 < 0$

Suppose inductively that $Q(n)$ is true. Which means that

Then for Q(n++) □

Proposition 1.5.2 (Backward Induction). Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever $P(m++)$ is true, then $P(m)$. Suppose that $P(n)$ is also true. Prove that $P(m)$ is true for all natural numbers $m \leq n$.

Proposition 1.5.3 (Induction starting from the base case n). Let n be a natural number, and let $P(m)$ be a property pertaining to the natural numbers such that whenever P(m) is true, P(m++) is true. Show that if P(n) is true, then P(m) is true for all m n. (This principle is sometimes referred to as the principle of induction starting from the base case n.)

Proof. Take a property $P(n)$, $m \geq n$

Inducting over n, □

1.6 Multiplication

Definition 1.6.1: Multiplication

Let m be a natural number. To multiply zero to m, we define $0 \times m := 0$. Now suppose inductively that we have defined how to multiply n to m. Then we can multiply $n++$ to m by defining $(n++) \times m := (n \times m) + m$

We can say $0 \times m = 0$, $1 \times m = 0 + m$, $2 \times m = 0 + m + m$ and so on.

Lemma 1.6.1. Prove that multiplication is commutative

Proof. We use the way we proved that addition is commutative as a blueprint. There are two things we need to prove first.

1. For any natural number, n, $n \times 0 = 0$
2. For any natural numbers, n and m, $n \times (m++) = (n \times m) + m$

First we prove, For any natural number, n, $n \times 0 = n$ We induct over n, For $n = 0$,

$$0 \times 0 = 0$$

Which is true from the definition

Now suppose inductively, that $n \times 0 = 0$, For $(n++) \times 0$, From the definition we can write this as,

$$(n++) \times 0 = (n \times 0) + 0$$
$$\text{We know that } n \times 0 = 0 (n++) \times 0 = 0 + 0$$
$$(n++) \times 0 = 0$$

Therefore,

$$n \times 0 = n$$

Now we prove, For any natural numbers, n and m, $n \times (m++) = (n \times m) + m$ We induct over n, (keeping m fixed)

For $n = 0$, We know from the definition for multiplication with zero that,

$$0 \times (m++) = 0$$
$$\text{We also know that}$$
$$(m++) \times 0 = (m \times 0) + 0$$
$$(m++) \times 0 = 0$$
$$(m++) \times 0 = 0 \times (m++) = (0 \times m) + m$$

Suppose inductively that $n \times (m++) = (n \times m) + m$ For $n = (n++)$ To prove $(n++) \times (m++) = ((n++) \times m) + m$,

$\square$

For simplicity we now write $a \times b$ as ab

Chapter 2

Set Theory

We define first what a set is:

Definition 2.0.1: Sets

We define set A to be any unordered collection of objects. If x is an object, we say that x is an element of A or $x \in A$ if x lies in the collection. Otherwise $x \in A$

We start with some axioms:

1. (Sets are objects) If A is a set, then A is also an object. A side track about "Pure Set Theory" - This theory states that everything in the mathematical universe is a set. We can write 0 as or an empty set, 1 can be written as 0 and 2 as 0,1 and so on. Terence Tao argues that they are the 'cardinalities of the set.'

2. (Equality of sets) Two sets A and B are equal, A $=$ B, iff every element of A is an element of B. A $=$ B, if and only if every element of x of A also belongs to B, and every element y of B belongs to A.

3. (Empty set) There exists a set $\emptyset$ known as the empty set, which contains no elements. $x \notin \emptyset$

   Proposition 2.0.1 (Partial Order). If A $\subseteq B, B \subseteq C \Rightarrow A \subseteq C$

   Proof. If $x \in A$, then $x \in B$, If $x \in B$, then $x \in C$, Then $x \in A$, then $x \in C$

   Thus, $A \subseteq C$ □

   Lemma 2.0.1 (Single choice). Let A be a non-empty set. Then there exists an object x such that $x \in A$

   Proof. Proving by contradiction, Suppose there is no object x that belongs to A. For all x, we have $x \notin A$. We know from Axiom 3, that $x \notin \emptyset$

   For the statement,

   $$x \in A \leftrightarrow x \in \emptyset$$

   Is false both ways, which gives us the result true, which is a contradiction.

   Thus we also prove that $\emptyset$ is unique. □

4. (Pairwise Union) $A \cup B = \{x : x \in A or x \in B\}$

Lemma 2.0.2. $A \cup (B \cup C) = (A \cup B) \cup C$

Proof. Taking the left hand side, We have $x \in A$ or $x \in (B \cup C)$. If we look to the right hand side, we have $x \in (A \cup B)$ or $x \in C$ If we break the statement down further. We have $x \in A$ or $x \in B$ or $x \in C$, and on the right $x \in A$ or $x \in B$ or $x \in C$

The two statements are equivalent. □

5. (Axiom Of Specification) A, $x \in A$, let P(x) be a property pertaining to x. Then there exists a set called $\{x \in A, P(x) \text{is true}\}$ whose elements are precisely the elements x in A for which $P(x)$ is true.

6. (Replacement) Let A be a set, for any object $x \in A$, and any object y, suppose we have a statement $P(x, y)$ pertaining to x and y, such that for each $x \in A$ there is at most one y for which $P(x, y)$ is true. Then there exists a set $\{y : P(x, y) \text{ is true for some } x \in A\}$

7. (Infinity) There exists a set $\mathbb{N}$, whose elements are called natural numbers, as well as an object 0 in $\mathbb{N}$, and an object $n++$ assigned to every natural number $n \in \mathbb{N}$ such that the Peano axioms hold.

8. **Russel's Paradox** (Axiom Of Universal Specification) Suppose for every x we have a property $P(x)$ pertaining to x, Then there exists a set $\{x : P(x) \text{ is true}\}$ such that for every object y:

   $$y \in \{x : P(x) istrue\} \Leftrightarrow P(y) \text{is true.}$$

   There is an issue, let's say we have a set, where the property of the objects is that they themselves are sets.

   Let's say we look at one of these sets,

   $$\begin{aligned} \Omega &= \{x : P(x) istrue\} \\ &= \{x : x \notin x\} \end{aligned}$$

9. (Regularity) If A is a non-empty set, then there is at least one element x of A which is either not a set or disjoint from A.

Definition 2.0.2: Intersection Of Sets

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$