Nuclear Reactor Criticality Analysis

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1 Introduction

The stability and behavior of a nuclear reactor are determined by its criticality—the precise balance between neutron production from fission and neutron loss from absorption and leakage. This analysis demonstrates a foundational method in reactor physics for calculating criticality by modeling the system's neutron balance as a matrix eigenvalue problem. The largest eigenvalue of the system directly corresponds to the effective multiplication factor (k_{eff}), which indicates whether the neutron population is stable, growing, or diminishing over time.

2 Theoretical Framework: Neutron Balance in Infinite Space

2.1 Derivation of the Multi-Group Neutron Balance in Infinite Space

2.1.1 Energy-Group Approximation

In reactor-physics (neutron-transport) analysis, the most complete treatment resolves the neutron flux $\phi(\mathbf{r}, E, t)$ as a function of position, energy, and time. Here, we simplify by neglecting spatial variation and collapsing the continuous spectrum into two energy groups - fast (group 1) and thermal (group 2). Within each group we balance sources (fission + in-scattering) against losses (absorption + out-scattering) to obtain the multi-group equations.

2.1.2 Neutron Balance for Each Group

In an infinite reactor, the neutron balance for each group equates losses to gains, adjusted by a factor k (the criticality factor, where $k = 1$ means the reactor is critical, i.e. the number of neutrons stays constant over time). For **group 1 (fast neutrons)** with flux ϕ_1:

Losses	$\Sigma_{a1}\,\phi_1$ (absorption), $\Sigma_{1\to 2}\,\phi_1$ (scattering to group 2)
Gains	$\Sigma_{2\to 1}\,\phi_2$ (scattering from group 2), $\frac{1}{k}\big(\nu\Sigma_{f1}\,\phi_1 + \nu\Sigma_{f2}\,\phi_2\big)$ (fission source)

Table 1: Neutron balance for group 1: losses and gains

The balance equation is:

$$\Sigma_{a1}\phi_1 + \Sigma_{1\to 2}\phi_1 = \Sigma_{2\to 1}\phi_2 + \frac{1}{k}(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2) \tag{1}$$

Losses	$\Sigma_{a2}\,\phi_2$ (absorption), $\Sigma_{2\to1}\,\phi_2$ (scattering to group 1)
Gains	$\Sigma_{1\to2}\,\phi_1$ (scattering from group 1), 0 (no fission source in group 2)

Table 2: Neutron balance for group 2: losses and gains

For **group 2 (slow neutrons)** with flux ϕ_2:

The balance equation is:

$$\Sigma_{a2}\phi_2 + \Sigma_{2\to1}\phi_2 = \Sigma_{1\to2}\phi_1 \tag{2}$$

2.1.3 Converting to a Matrix Equation

We rewrite eq (1) and (2) as a matrix equation by moving terms involving ϕ_1 and ϕ_2 to the left:

- **Group 1:** $(\Sigma_{a1} + \Sigma_{1\to2})\phi_1 - \Sigma_{2\to1}\phi_2 = \frac{1}{k}(\nu\Sigma_{f1}\phi_1 + \nu\Sigma_{f2}\phi_2)$
- **Group 2:** $-\Sigma_{1\to2}\phi_1 + (\Sigma_{a2} + \Sigma_{2\to1})\phi_2 = 0$

In matrix form:

$$\begin{bmatrix} \Sigma_{a1} + \Sigma_{1\to2} & -\Sigma_{2\to1} \\ -\Sigma_{1\to2} & \Sigma_{a2} + \Sigma_{2\to1} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \frac{1}{k} \begin{bmatrix} \nu\Sigma_{f1} & \nu\Sigma_{f2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{3}$$

This is in the form:

$$\mathbf{M}\phi = \frac{1}{k}\mathbf{F}\phi \tag{4}$$

The migration matrix **M** describes neutron losses and gains due to absorption and scattering. It's defined as $\mathbf{M} = \mathbf{A} + \mathbf{S}_{out} - \mathbf{S}_{in}$, where, in this example:

A (Absorption):

$$\mathbf{A} = \begin{bmatrix} \Sigma_{a1} & 0 \\ 0 & \Sigma_{a2} \end{bmatrix} \tag{5}$$

$\mathbf{S}_{out}$ (Outscattering):

$$\mathbf{S}_{out} = \begin{bmatrix} \Sigma_{1\to2} & 0 \\ 0 & \Sigma_{2\to1} \end{bmatrix} \tag{6}$$

$\mathbf{S}_{in}$ (Inscattering):

$$\mathbf{S}_{in} = \begin{bmatrix} 0 & \Sigma_{2\to1} \\ \Sigma_{1\to2} & 0 \end{bmatrix} \tag{7}$$

So computing **M**, we get:

$$\mathbf{M} = \mathbf{A} + \mathbf{S}_{out} - \mathbf{S}_{in} = \begin{bmatrix} \Sigma_{a1} + \Sigma_{1\to2} & -\Sigma_{2\to1} \\ -\Sigma_{1\to2} & \Sigma_{a2} + \Sigma_{2\to1} \end{bmatrix} \tag{8}$$

F is the Fission Matrix, which handles fission production.

$$\mathbf{F} = \begin{bmatrix} \chi_1\nu\Sigma_{f1} & \chi_1\nu\Sigma_{f2} \\ \chi_2\nu\Sigma_{f1} & \chi_2\nu\Sigma_{f2} \end{bmatrix} \tag{9}$$

With $\chi_1 = 1$, $\chi_2 = 0$:

$$\mathbf{F} = \begin{bmatrix} \nu\Sigma_{f1} & \nu\Sigma_{f2} \\ 0 & 0 \end{bmatrix} \tag{10}$$

The Migration matrix $\mathbf{M}$ and Fission matrix $\mathbf{F}$ help us find k (criticality) and ϕ (flux).

2.1.4 Eigenvalue Problem

Starting from $\mathbf{M}\phi = \frac{1}{k}\mathbf{F}\phi$, multiply by $\mathbf{M}^{-1}$:

$$\phi = \frac{1}{k}\mathbf{M}^{-1}\mathbf{F}\phi \tag{11}$$

Multiply by k:

$$k\phi = \mathbf{M}^{-1}\mathbf{F}\phi \tag{12}$$

Define $\mathbf{B} = \mathbf{M}^{-1}\mathbf{F}$. The eigenvalue problem then reads:

$$\mathbf{B}\phi = k\phi \tag{13}$$

To apply the theoretical framework described in section 2, the analysis will proceed in two parts. First, a simplified two-group model is solved. Its small scale is ideal for a thorough validation, as the results from computational methods can be directly compared against a full analytical derivation performed by hand. Second, having established the validity of the numerical approach, the analysis is extended to a more complex eight-group model. This will demonstrate the scalability of the computational methods on a more physically realistic system where a by-hand solution is no longer feasible.

3 Two-Group eigenvalue/eigenvector problem

We determine the criticality and the flux shape three ways: by hand, computationally (using python), and through power iteration. Let's start by showing the migration, and fission matrices with numerical values. Using sample 2-group data and eqs (8) and (10):

$$\mathbf{M} = \mathbf{A} + \mathbf{S}_{out} - \mathbf{S}_{in} = \begin{bmatrix} 0.02931 & 0 \\ -0.0202 & 0.09323 \end{bmatrix} \tag{14}$$

$$\mathbf{F} = \begin{bmatrix} \nu\Sigma_{f1} & \nu\Sigma_{f2} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.0046 & 0.1139 \\ 0 & 0 \end{bmatrix} \tag{15}$$

Now let's calculate all eigenvalues and eigenvectors analytically by hand.

3.1 Calculation By-Hand

For $\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0.02931 & 0 \\ -0.0202 & 0.09323 \end{bmatrix}$, the inverse is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, which means:

$$\mathbf{M}^{-1} = \frac{1}{0.0027319713}\begin{bmatrix}0.09323 & 0\\ 0.0202 & 0.02931\end{bmatrix} \approx \begin{bmatrix}34.1219 & 0\\ 7.3935 & 10.7266\end{bmatrix} \tag{16}$$

$$\mathbf{B} = \mathbf{M}^{-1}\mathbf{F} = \begin{bmatrix}34.1219 & 0\\ 7.3935 & 10.7266\end{bmatrix}\begin{bmatrix}0.0046 & 0.1139\\ 0 & 0\end{bmatrix} = \begin{bmatrix}0.156961 & 3.8855\\ 0.03401 & 0.8421\end{bmatrix} \tag{17}$$

To find eigenvalues, we solve $\det(\mathbf{B} - \lambda\mathbf{I}) = 0$:

$$\mathbf{B} - \lambda\mathbf{I} = \begin{bmatrix}0.156961 - \lambda & 3.8855\\ 0.03401 & 0.8421 - \lambda\end{bmatrix}$$

This gives a quadratic equation that can be solved for the eigenvalues λ.

$$\lambda = \frac{0.999061 \pm \sqrt{(0.999061)^2 - 4 \cdot 0.00004}}{2}$$

$$\lambda_{large} = \frac{0.999061 + 0.998981}{2} \approx 0.999021, \quad \lambda_{small} = \frac{0.999061 - 0.998981}{2} \approx 0.00004$$

We need to find the eigenvectors by solving $(\mathbf{B} - \lambda\mathbf{I})\phi = 0$ for each eigenvalue, where $\phi = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$.

- For the tiny root $\lambda_{small} = k_1 \approx 0.00004$, $\mathbf{B} - \lambda\mathbf{I} \approx \mathbf{B}$, so

$$0.156921\phi_1 + 3.8855\phi_2 = 0 \quad \Rightarrow \quad \phi^{(1)} \propto \begin{pmatrix}-24.764\\ 1\end{pmatrix} \tag{18}$$

- For the big root $\lambda_{large} = k_2 \approx 0.999021$:

$$-0.84206\phi_1 + 3.8855\phi_2 = 0 \quad \Rightarrow \quad \phi^{(2)} \propto \begin{pmatrix}4.6143\\ 1\end{pmatrix} \tag{19}$$

Results:

Mode	Eigenvalue k	Eigenvector ϕ (normalized so $\phi_2 = 1$)
k_1	0.00004	$\begin{bmatrix}-24.764\\ 1\end{bmatrix}$
k_2	0.99902	$\begin{bmatrix}4.6143\\ 1\end{bmatrix}$

Table 3: Calculation by-hand 2G eigenvalues and eigenvectors.

3.2 Python

Using python we assembled the balance matrix:

$$B = M^{-1}F = \begin{bmatrix} 0.15694302 & 3.88604572 \\ 0.03400460 & 0.84198352 \end{bmatrix} \tag{20}$$

Eq (23) matches B in our hand-calculation to the third decimal place. We went on to produce:

Mode	Eigenvalue k	Eigenvector ϕ (normalized so $\phi_2 = 1$)
k_1	-1.11×10^{-16}	$\begin{bmatrix} -0.99918547 \\ 0.04035341 \end{bmatrix}$
k_2	0.99892654	$\begin{bmatrix} -0.97732278 \\ -0.21175502 \end{bmatrix}$

Table 4: Python produced 2G eigenvalues and eigenvectors.

The first eigenvalue is effectively zero (machine noise). The second, $k_2 = 0.99892654$, agrees with our by-hand result to within 2×10^{-3}. Scaling the second eigenvector so its second component is 1 gives $\phi^{(2)} \propto [-0.9773, -0.2118] \approx [4.614, 1]$, which matches the hand solution perfectly.

3.3 Power Iteration

Power iteration can only find the eigenvalue of largest magnitude (and its eigenvector). In reactor criticality analysis the *fundamental mode* (i.e. the largest eigenvalue k and its flux shape ϕ) determines whether the system is subcritical, critical, or supercritical. We iterate over:

$$\phi^{(i+1)} = \frac{B\,\phi^{(i)}}{\|B\,\phi^{(i)}\|_2} \tag{21}$$

$$k^{(i+1)} = \frac{\phi^{(i+1)\,T}\,B\,\phi^{(i+1)}}{\phi^{(i+1)\,T}\,\phi^{(i+1)}}, \tag{22}$$

starting from $\phi^{(0)} = [1, 1]^T$. The first two iterations are:

iteration i	$\phi_1^{(i)}$	$\phi_2^{(i)}$	$k^{(i)}$
0	1.000000	1.000000	—
1	0.97732278	0.21175502	0.99892654
2	0.97732278	0.21175501	0.99892653

Table 5: Power iterated 2G eigenvalues and eigenvectors.

So by iteration two, the estimate has converged to:

$$k \approx 0.99893, \qquad \phi^{(2)} \propto \begin{bmatrix} 0.9773 \\ 0.2118 \end{bmatrix} \longrightarrow \text{group ratio } \phi_1/\phi_2 \approx 4.61$$

This matches the hand calculation.

3.4 Summary of Results

The results from all three methods are in excellent agreement and consistently demonstrate that the reactor is slightly subcritical:

- Hand Calculation: $k_2 \approx 0.999021$, $\phi^{(2)} \propto [4.6143,\ 1]^T$.
- NumPy (`linalg.eig`): $k_2 = 0.99892654$, $\phi^{(2)} \propto [4.614,\ 1]^T$.
- Power Iteration (2iter): $k^{(2)} \approx 0.99892653$, $\phi^{(2)} = [0.9773,\ 0.2118]^T \to \phi_1/\phi_2 \approx 4.61$.

The dominant eigenvalue $k_2 \approx 0.999 < 1$, indicating the system is subcritical and the neutron population will slowly decrease over successive generations. The excellent agreement between the analytical, numerical, and iterative methods for the two-group model provides high confidence in the computational approach. The analysis is now extended to an eight-group system to model the reactor with higher fidelity.

4 Eight-Group eigenvalue/eigenvector problem

Fission Matrix F

$$F = \begin{bmatrix} 4.70\times10^{-3} & 1.96\times10^{-3} & 3.86\times10^{-4} & 2.35\times10^{-3} & 7.72\times10^{-3} & 7.79\times10^{-3} & 0.0315 & 0.0751 \\ 5.50\times10^{-3} & 2.30\times10^{-3} & 4.52\times10^{-4} & 2.75\times10^{-3} & 9.03\times10^{-3} & 9.11\times10^{-3} & 0.0368 & 0.0879 \\ 3.20\times10^{-3} & 1.34\times10^{-3} & 2.63\times10^{-4} & 1.60\times10^{-3} & 5.25\times10^{-3} & 5.30\times10^{-3} & 0.0214 & 0.0511 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

Migration Matrix M

$$\mathbf{M} = \begin{bmatrix} 0.08880 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.05300 & 0.11930 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.03010 & -0.11590 & 0.08130 & 0 & 0 & 0 & 0 & 0 \\ -0.00010 & -0.00050 & -0.07690 & 0.21520 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.00190 & -0.19610 & 0.25290 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.00500 & -0.17370 & 0.34370 & -0.00230 & 0 \\ 0 & 0 & 0 & -0.00070 & -0.02460 & -0.27070 & 0.41690 & -0.02740 \\ 0 & 0 & 0 & -0.00010 & -0.00730 & -0.05500 & -0.35880 & 0.20720 \end{bmatrix} \tag{24}$$

4.1 Python

The eigenvalues and eigenvectors calculated on python are:

$$k_1 = 0, \quad k_2 = 1.090, \quad k_3 = -6.325\times10^{-17}, \quad k_4 = 1.677\times10^{-17},$$
$$k_5 = 1.498\times10^{-17}, \quad k_6 = 3.629\times10^{-18}, \quad k_7 = -4.694\times10^{-18} + 3.097\times10^{-18}i,$$

$$k_8 = -4.694 \times 10^{-18} - 3.097 \times 10^{-18} i.$$

$$\Phi_1 = \begin{bmatrix} -0.983 \\ 0.0660 \\ 0.1499 \\ 0.0537 \\ 0.0428 \\ 0.0225 \\ 0.0200 \\ 0.0422 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} -0.262 \\ -0.344 \\ -0.782 \\ -0.280 \\ -0.223 \\ -0.118 \\ -0.104 \\ -0.220 \end{bmatrix}, \quad \Phi_3 = \begin{bmatrix} -0.0694 \\ 0.428 \\ -0.719 \\ 0.412 \\ 0.160 \\ 0.269 \\ 0.125 \\ -0.112 \end{bmatrix}, \quad \Phi_4 = \begin{bmatrix} 0.402 \\ 0.812 \\ -0.294 \\ -0.227 \\ -0.100 \\ -0.147 \\ -0.0921 \\ 0.0263 \end{bmatrix},$$

$$\Phi_5 = \begin{bmatrix} -0.294 \\ -0.829 \\ 0.326 \\ 0.262 \\ 0.0714 \\ 0.184 \\ 0.0972 \\ -0.0369 \end{bmatrix}, \quad \Phi_6 = \begin{bmatrix} -0.567 \\ -0.0060 \\ 0.820 \\ 0.0557 \\ 0.0472 \\ -0.00519 \\ 0.0559 \\ 0.0215 \end{bmatrix}, \quad \Phi_7 = \begin{bmatrix} -0.809 \\ -0.0156 - 0.0708\,i \\ -0.5329 + 0.0973\,i \\ 0.1984 + 0.0240\,i \\ -0.00566 + 0.0155\,i \\ 0.0559 - 0.00818\,i \\ 0.0561 - 0.0233\,i \\ 0.0188 + 0.00960\,i \end{bmatrix}, \quad \Phi_8 = \begin{bmatrix} -0.809 \\ -0.0156 + 0.0708\,i \\ -0.5329 - 0.0973\,i \\ 0.1984 - 0.0240\,i \\ -0.00566 - 0.0155\,i \\ 0.0559 + 0.00818\,i \\ 0.0561 + 0.0233\,i \\ 0.0188 - 0.00960\,i \end{bmatrix}.$$

The power iterated eigenvalue and eigenvector produced are:

$$k = 1.090 \tag{25}$$

$$\phi = \begin{bmatrix} 0.262 \\ 0.344 \\ 0.782 \\ 0.280 \\ 0.223 \\ 0.118 \\ 0.104 \\ 0.220 \end{bmatrix} \tag{26}$$

4.2 Results

A $k \approx 1.090$ indicates the reactor is supercritical ($k > 1$), meaning the neutron population would increase over generations. A hand derivation is impractical for G = 8, so we rely solely on the numerical results. The power iteration method is designed to converge to the largest eigenvalue, and after just 1 iteration, $k = 1.090021$ is the initial approximation (the value may still adjust with further iterations).

The eigenvalue computation yielded complex conjugates $k_7 = -4.694 \times 10^{-18} + 3.097 \times 10^{-18} i$ and $k_8 = -4.694 \times 10^{-18} - 3.097 \times 10^{-18} i$, with magnitudes on the order of 10^{-18}. These are likely numerical artifacts due to round-off errors in the matrix inversion or eigenvalue solver, given their negligible size compared to the dominant eigenvalue $k_2 = 1.090$. The corresponding eigenvectors would be complex, potentially indicating oscillatory modes, but their small magnitude suggests they are not physically significant. The power iteration method correctly identified the dominant real eigenvalue and eigenvector, which are sufficient for determining the reactor's criticality.