Linearization of Blackjack Expected Values

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1 Linearization

The goal in card-counting methods for any strategy decision in blackjack is to linearize the problem such that expected value can be found by a linear combination of cards that have been removed from the deck. This analysis is basically the same as the analysis of Peter Griffin [1], except that he linearized as linear combination of cards remaining in the deck. Because card counters always count cards as they are removed, the removal linearization seems more direct.

Imagine an arbitrary deck of N cards (such as 52 × number of decks or a subset of such a deck) in which k cards have been removed. There are $\binom{N}{k}$ possible ways to remove k cards. Let Y_i be expected value for a given strategy decision (*e.g.*, hit or strand, or how much to bet) for the i^{th} member of the $\binom{N}{k}$ subsets. Our goal is to approximate Y_i by a linear combination of removed cards or:

$$Y_i \approx \sum_j X_{ij}\rho_j^{(k)}$$

where $X_{ij} = 1$ if card j has been removed or $X_{ij} = 0$ if has not been removed, and $\rho_j^{(k)}$ is the weight to assign card j (from 1 to N) when removing exactly k cards. The goal is to determine the weights $\rho_j^{(k)}$.

2 Least Squares Analysis

The $\rho_j^{(k)}$ weighting factors can be found by the method of least squares. The goal is to choose $\rho_j^{(k)}$ such that the metric summed over all subsets:

$$\Phi = \sum_i \left(\sum_j X_{ij}\rho_j^{(k)} - Y_i \right)^2$$

is minimized. By standard least squares methods, the equation for $\rho_j^{(k)}$ is

$$\sum_i \sum_j X_{ij} X_{im} \rho_j^{(k)} = \sum_i X_{im} Y_i$$

Splitting out the diagonal terms (and changing index in second term for clarity), the N equations in the N unknowns are

$$\left(\sum_i X_{ij}^2 \right) \rho_j^{(k)} + \sum_{m \neq j} \left(\sum_i X_{ij} X_{im} \right) \rho_m^{(k)} = \sum_i X_{im} Y_i$$

for j from 1 to N.

Because X_{ij} is 1 or 0, $X_{ij}^2 = X_{ij}$. From definition of X_{ij},

$$\sum_i X_{ij}^2 = \sum_i X_{ij} = \binom{N-1}{k-1}$$

In other words, this sum counts the number of subsets i that removes card j. Similarly,

$$\sum_i X_{ij} X_{im} = \binom{N-2}{k-2}$$

or counts the number subsets that remove both cards j and m. Next, the term $\sum_i X_{im} Y_i$ is the sum expected values for all subsets that remove card j. Let μ be the mean expectation with no cards removed and let E_j be the change in expected value after removing card j or $\mu + E_j$ is the expected value with only card j removed. This expected value is given by

$$\mu + E_j = \frac{\sum_i X_{im} Y_i}{\binom{N-1}{k-1}}$$

The equation thus simplifies to

$$\begin{aligned}
\binom{N-1}{k-1} \rho_j^{(k)} + \sum_{m \neq j} \binom{N-2}{k-2} \rho_m^{(k)} &= \binom{N-1}{k-1} (\mu + E_j) \\
\rho_j^{(k)} + \frac{k-1}{N-1} \sum_{m \neq j} \rho_m^{(k)} &= \mu + E_j \\
\mathsf{A} \boldsymbol{\rho}^{(k)} &= \boldsymbol{\mu}^*
\end{aligned}$$

Here A is a matrix with all diagonal element equal to 1 and all off-diagonal elements equal to $(k-1)/(N-1)$, $\boldsymbol{\rho}^{(k)}$ is vector of card weights, and $\boldsymbol{\mu}_j^* = \mu + E_j$ is expected value with only card j removed.

The weights are given by

$$\boldsymbol{\rho}^{(k)} = \mathsf{A}^{-1} \boldsymbol{\mu}^*$$

Like A, A^{-1} has constant value on all diagonal elements and a different constant value on all off diagonal elements:

$$\mathsf{A}^{-1} = \begin{cases} \frac{1+k(N-2)}{k(N-k)} & i = j \\ -\frac{k-1}{k(N-k)} & i \neq j \end{cases}$$

The solution becomes

$$\rho_j^{(k)} = \frac{1 + k(N-2)}{k(N-k)} (\mu + E_j) - \frac{k-1}{k(N-k)} \sum_{m \neq j} (\mu + E_m)$$

The sum of effects is zero leading to

$$E_j = -\sum_{m \neq j} E_m$$

Substituting gives

$$\begin{aligned}
\rho_j^{(k)} &= \frac{1 + k(N-2)}{k(N-k)} (\mu + E_j) - \frac{(k-1)(N-1)\mu}{k(N-k)} + \frac{k-1}{k(N-k)} E_j \\
\rho_j^{(k)} &= \frac{1 + k(N-2) - (k-1)(N-1)}{k(N-k)} \mu + \frac{N-1}{N-k} E_j \\
\rho_j^{(k)} &= \frac{\mu}{k} + \frac{N-1}{N-k} E_j
\end{aligned}$$

Finally, substituting into the initial expected value approximation gives

$$Y_i \approx \mu + \frac{N-1}{N-k} \sum_{j=1}^{k} E_j$$

References

[1] Peter A. Griffin. *The Theory of Blackjack*. Sixth Edition. Huntington Press, Las Vegas, NV, 1999.