IMP
1 $\int$ = area & length
2 $\int$ = volume in 3D
3 $\int$ = volume in 4D.

Page
Date

01.19

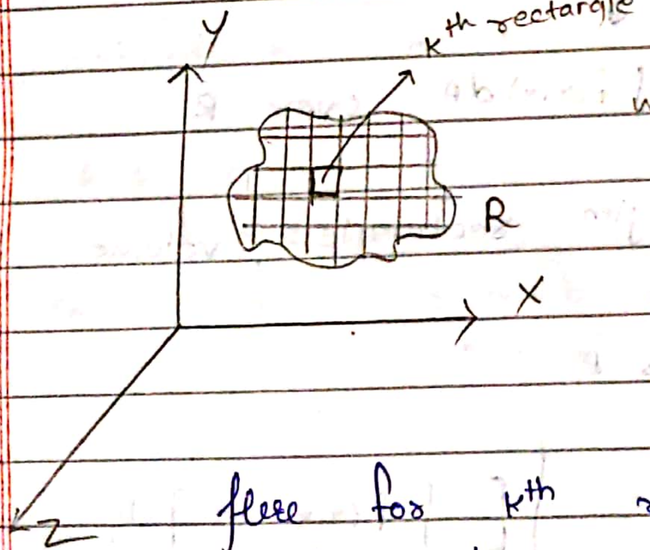
5. IMPROPER AND MULTIPLE INTEGRALS.

Multiple Integrals :

↳ Integration is used to find area and length of a curve ~~or~~ ~~functi~~ within some defined region. But when we want to find some integrals of more complicated regions and volumes of some solid regions ~~then~~ in higher dimension then the concept of multiple integral is used.

Double Integrals :

Let $z = f(x,y)$ be a continuous function of two variables x and y.
We want to find volume of a curve or function within some arbitary region R.



we have taken any arbitary region R then in order to find volume we had taken one k^{th} rectangle & we will find volume of that

Here for k^{th} rectangle the area is given by

$$dA_k = \delta x_k \, \delta y_k \quad \underline{\underline{or}} \quad \delta y_k \, x_k.$$

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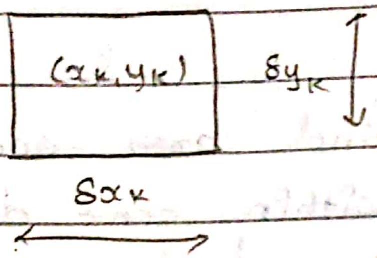
$\therefore$ Volume of k^{th} rectangle is given by,

k^{th} rectangle



$$\text{Volume} = l \times b \times h = A \times h.$$

$$\cancel{dA_k} = \cancel{\delta x_k}\ \cancel{\delta y_k}. \qquad V_k = \underbrace{f(x_k, y_k)}_{h} \cdot \underbrace{dA_k}_{A}.$$

So, the approximate volume is

$$V = \sum_{k=1}^{n} f(x_k, y_k)\, dA_k.$$

$\therefore$ The exact volume of f over the region R is given by

$$V = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k)\, dA_k.$$

$$\downarrow$$

This is nothing but $\iint f(x,y)\, dA$ over R

→ Similary for $(i\,j)^{th}$ rectangle, volume is

double integral over R

$$\iint_R f(x,y)\, dA = \int \left[\int f(x,y)\, dx \right] dy$$
$$= \int \left[\int f(x,y)\, dy \right] dx.$$

→ The integral inside the bracket is known as inner integral, which can be always evaluated first and the integral outside the bracket is known as outer integral.

→ Here dxdy or dydx is known as order of integration

→ There are two types of regions for the evaluation of double integrals.

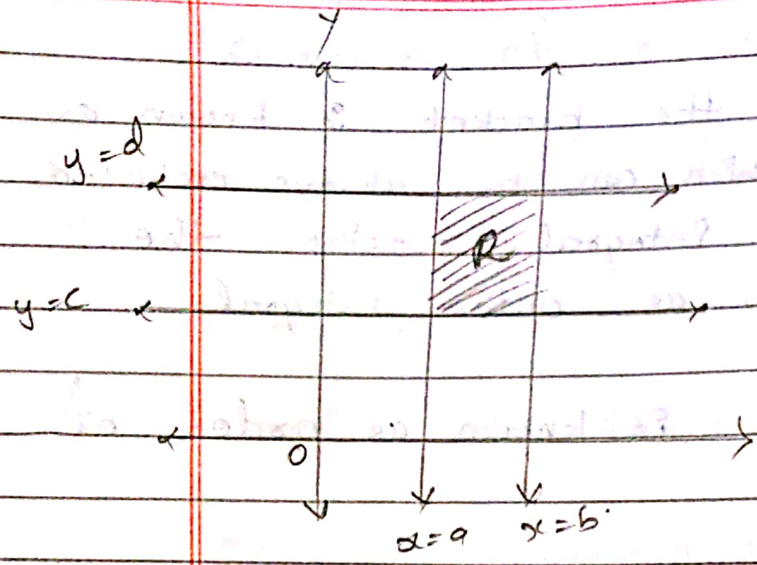
1. Rectangular region
2. Non-rectangular region.

04/01/19

1) Rectangular Region : Let $f(x,y)$ be a continuous funⁿ defined on a rectangular region R such that

$$R : a \le x \le b$$
$$c \le y \le d$$

Then $x = a$; $x = b$ $(a < b)$
$y = c$; $y = d$ $(c < d)$



Here we can observe that for the rectangular region all limits of x & y must be constants.

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Fubini's Theorem :

Let $f(x,y)$ be a continuous function defined on a rectangular region

$R : \; a \le x \le b$

$\quad c \le y \le d$.

then $$\iint_R f(x,y)\, dA = \int_{y=c}^{d} \int_{x=a}^{b} f(x,y)\, dx\, dy$$

$$= \int_{x=a}^{b} \int_{y=c}^{d} f(x,y)\, dy\, dx .$$

by fixing limit we can change

Q1) Verify the Fubini's theorem for

$$\int_1^2 \int_2^3 7xy^2 \, dx\,dy.$$

Soln: We are given $\int_1^2 \int_2^3 7xy^2 \, dx\,dy$.

Here we can observe that the limits are $2 \le x \le 3$ & $1 \le y \le 2$.

Here we want to prove that

$$\int_{y=1}^{2} \int_{x=2}^{3} 7xy^2 \, dx\,dy = \int_{x=2}^{3} \int_{y=1}^{2} 7xy^2 \, dy\,dx.$$

$$L.H.S = \int_{y=1}^{2} \int_{x=2}^{3} 7xy^2 \, dx\,dy$$

$$= \int_{y=1}^{2} 7y^2 \left(\frac{x^2}{2}\right)_2^3 dy$$

$$= \int_{y=1}^{2} 7y^2 \cdot \frac{1}{2} \left[x^2\right]_2^3 dy$$

$$= \frac{7}{2} \int_{y=1}^{2} y^2 \left[9-4\right] dy$$

$$= \frac{35}{2} \int_{y=1}^{2} y^2 \, dy$$

$$= \frac{35}{2} \left[\frac{y^3}{3}\right]_1^2$$

$$= \frac{35}{2} \cdot \frac{1}{3} (8-1)$$

$$= \frac{35 \times 7}{6} = \boxed{\frac{245}{6}}$$

Now, R.H.S $= \int_{x=2}^{3} \int_{y=1}^{2} 7xy^2 \, dy \, dx$

$$= \int_2^3 7x \left[\frac{y^3}{3}\right]_1^2 dx$$

$$= \frac{7}{3} \int_2^3 x \left[y^3\right]_1^2 dx$$

$$= \frac{7}{3} \int_2^3 x [8-1] \, dx$$

$$= \frac{49}{3} \int_2^3 x \, dx$$

$$= \frac{49}{3} \left[\frac{x^2}{2}\right]_2^3$$

$$= \frac{49}{3 \times 2} \left[x^2\right]_2^3$$

$$= \frac{49}{6} [9-4]$$

$$= \frac{49 \times 5}{6} = \boxed{\frac{245}{6}}$$

$\therefore$ L.H.S = R.H.S.

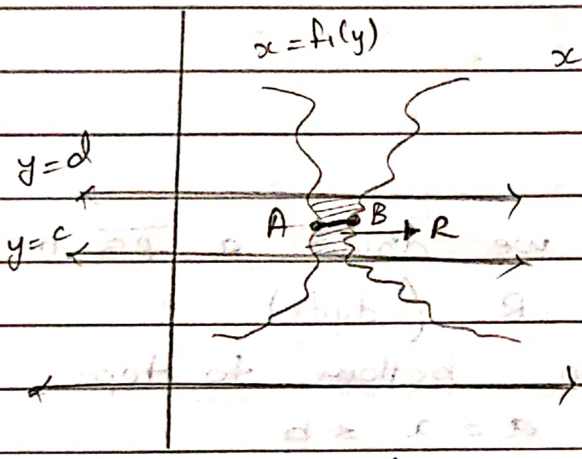
Thus Fubini's theorem is verified. Proved.

② Non-Rectangular Region :

→ When limit of x as a function :

let $f(x,y)$ be a continuous function defined on a region R bounded by $x = f_1(y)$: $x = f_2(y)$ and $y = c$; $y = d$ $(c < d)$

∴ we have



For $dx\,dy$ //

A = L.L.

B = U.L.

$\overline{AB}$ ∥ X-axis.

For $dy\,dx$ vice versa.

$dx\,dy$

→ Left to right.

For limits of x, we take $\overline{AB}$ ∥ x-axis within the region $(dx\,dy)$.

∴ limits are $f_1(y) \le x \le f_2(y)$

& $c \le y \le d$

$dy\,dx$ ∥ y axis

↑ top bottom

Follow the direction from left to right.

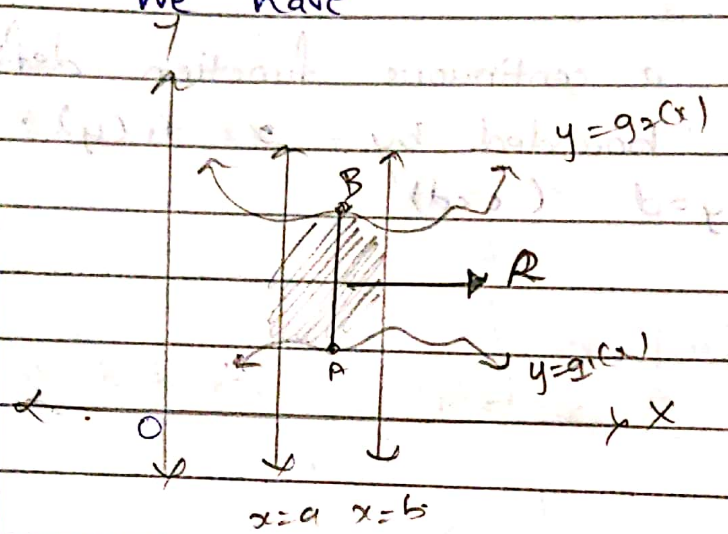
$$\iint_R f(x,y)\,dA = \int_{y=c}^{d} \int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy$$

→ When limit of y as a function :

let $f(x,y)$ be a continuous function defined on a region R bounded by $y = g_1(x)$; $y = g_2(x)$

$x = a$ & $x = b$ $(a < b)$

$\therefore$ We have



For limits of y we draw a $\overline{AB} \parallel$ Y-axis within the region R (dydx)

Follow the direction bottom to top.

$\therefore$ limits are $a \le x \le b$

& $g_1(x) \le y \le g_2(x)$

i.e. $$\iint_R f(x,y)\, dA = \int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx.$$

10/01/19

Q.1) Evaluate : $$\int_0^2 \int_0^y e^{x+y} \cdot dA.$$

Sol^n: We are given $\int_0^2 \int_0^y e^{x+y}\, dA. = \int_{y=0}^{2} \int_{x=0}^{y} e^x \cdot e^y\, dx\, dy$

Page
Date

$$\Rightarrow \int_{y=0}^{2} e^{y} \left[e^{x}\right]_{0}^{y} dy$$

$$\Rightarrow \int_{y=0}^{2} e^{y} \left[e^{y}-1\right] dy$$

$$\Rightarrow \int_{y=0}^{2} \left(e^{2y} - e^{y}\right) dy$$

$$\Rightarrow \left[\frac{e^{2y}}{2}\right]_{0}^{2} - \left[e^{y}\right]_{0}^{2}$$

$$\Rightarrow \frac{e^{4}}{2} - \frac{1}{2} - e^{2} + 1$$

$$\Rightarrow \boxed{\frac{e^{4}}{2} - e^{2} + \frac{1}{2}}$$

Ans.

Q.2) $\int_{0}^{1}\int_{0}^{x^2} xy^2 \, dA.$

Sol^n: We are given $\int_{0}^{1}\int_{0}^{x^2} xy^2 \, dA = \int_{x=0}^{x=1}\int_{y=0}^{y=x^2} xy^2 \, dy\, dx$

$$\Rightarrow \int_{x=0}^{1} x \cdot \left[\frac{y^3}{3}\right]_{0}^{x^2} dx$$

$$\Rightarrow \int_{x=0}^{1} x \cdot \left[\frac{x^6}{3}\right] dx \cdot \Rightarrow \frac{1}{3}\int_{x=0}^{1} x^7 \, dx.$$

$$\Rightarrow \frac{1}{3}\left[\frac{x^8}{8}\right]_0^1$$

$$\Rightarrow \frac{1}{3}\left[\frac{1}{8}-0\right] \Rightarrow \boxed{\frac{1}{24}}$$

Ans.

Q.3) Evaluate : $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$.

Sol^n: We are given $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$ which is not in a proper form.

$\therefore$ ~~So~~ Proper form of given integral is

$$\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dy\,dx.$$

$$\Rightarrow \int_0^1 \cancel{\left[x^2(y) + \frac{y^3}{3}\right]_x^{\sqrt{x}}}\,dx. \qquad \int_0^1 \left[x^2(y) + \frac{y^3}{3}\right]_x^{x^{1/2}} dx$$

$$\Rightarrow \int_{x=0}^1 \left[x^2(\sqrt{x}) + \frac{(\sqrt{x})^3}{3}\right] - \left[x^2(x) - \frac{x^3}{3}\right] dx$$

$$\Rightarrow \int_{x=0}^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - \left(x^3 + \frac{x^3}{3}\right)\right) dx.$$

Page
Date

$$\Rightarrow \int_{x=0}^{1} x^{5/2} + \frac{x^{3/2}}{3} - \frac{4x^3}{3} \, dx .$$

$$\Rightarrow \left[\frac{2x^{7/2}}{7} + \frac{2x^{5/2}}{5(3)} - \frac{4}{3} \cdot \frac{x^4}{4} \right]_0^1$$

$$\Rightarrow \frac{2}{7} + \frac{2}{15} - \frac{4}{3}$$

$$\Rightarrow \frac{30 + 14 - 35}{105} \Rightarrow \frac{9}{105} \Rightarrow \boxed{\frac{3}{35}}$$

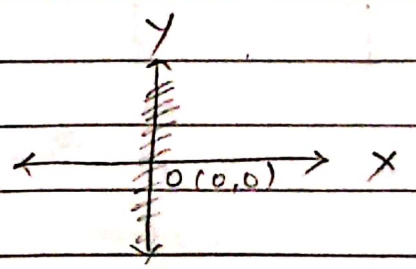
Ans.

Some useful curves :

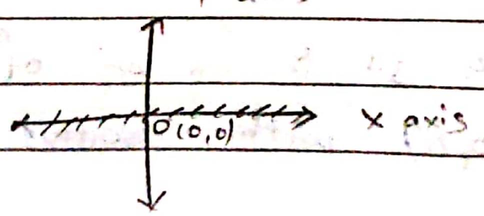
① Straight line :

→ An equation of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is given by $$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}$$

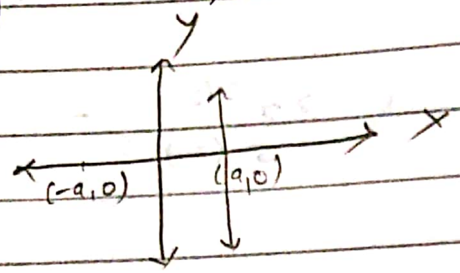
→ $x = 0$ (Y-axis)



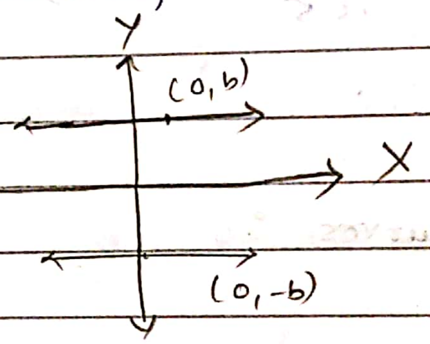
→ $y = 0$ (x-axis)



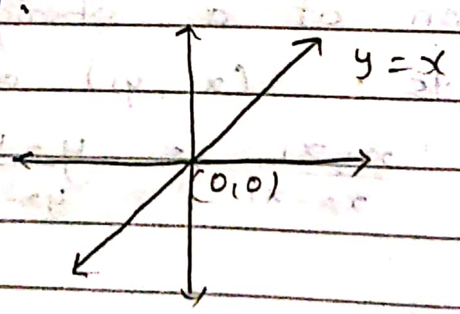
→ $x = a$: It is an eq^n of straight line passing through $(a, 0)$ & parallel to Y-axis.

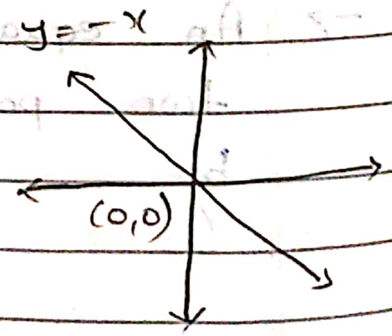


→ $y = b$: It is an eq^n of straight line passing through $(0, b)$ & parallel to X-axis.



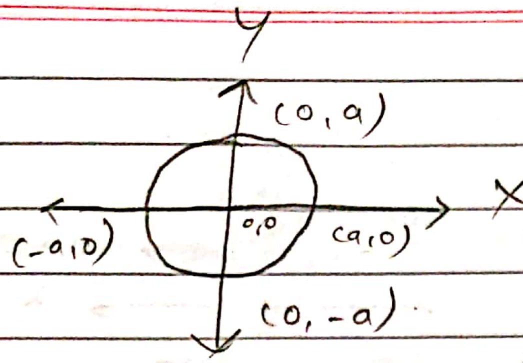
→ $Y = x$: It is an eq^n of straight line passing through $(0, 0)$.



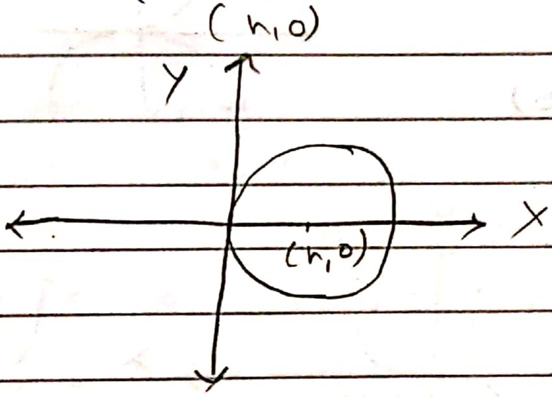


② circles

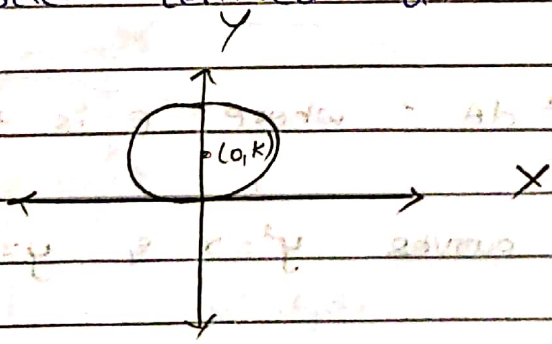
→ $x^2 + y^2 = a^2$: It is an eq^n of a circle centred at $(0, 0)$ with radius a.



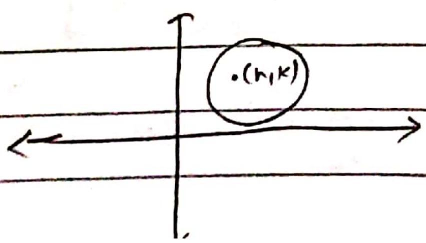
→ $(x-h)^2 + y^2 = a^2$: It represents an eq^n of circle centred at $(h, 0)$ with radius a.



→ $x^2 + (y-k)^2 = a^2$: It represent an eq^n of circle centred at $(0, k)$ with radius a.



→ $(x-h)^2 + (y-k)^2 = a^2$: It represents an eq^n of circle centred at (h, k) with radius a.



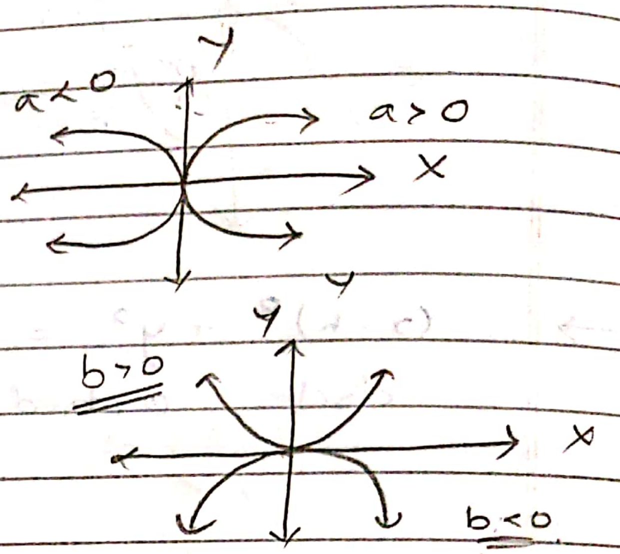
(3) Parabolas :

→ $y^2 = 4ax$ $(a>0)$

→ $y^2 = 4ax$ $(a<0)$

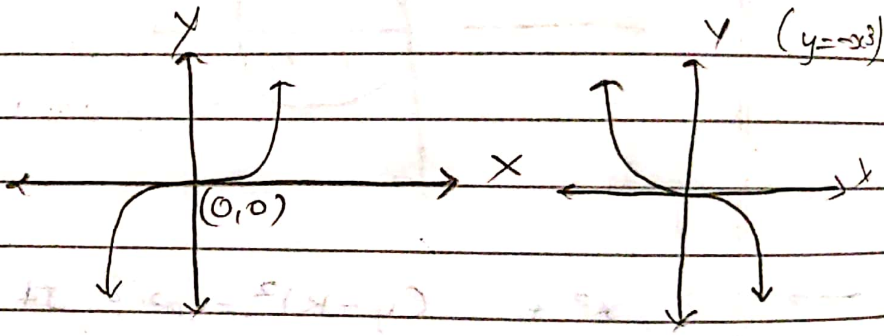
→ $x^2 = 4by$ $(y>0)$

→ $x^2 = 4by$ $(y<0)$



(4) $y = x^3$

$y = -x^3$

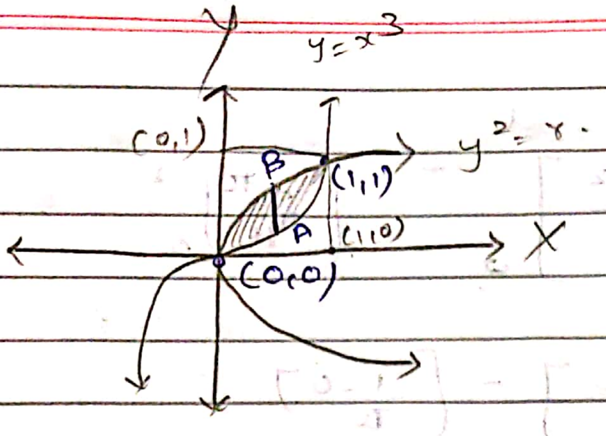


Q01) Evaluate $\iint_R y^2 \, dA$; where R is the region bounded by the curves $y^2 = x$ & $y = x^3$.

Solⁿ: We are given that R is bounded by the curves $y^2 = x$ & $y = x^3$.

∴ We have

Page
Date



$y^2 = x$ & $y = x^3$

$(x^3)^2 = x$

$x^6 = x$

$x^6 - x = 0$

$x(x^5 - 1) = 0$

$\therefore \boxed{x = 0 \quad y = 0}$ & $x^5 = 1$ $\boxed{x = 1}$ $\boxed{y = 1}$

(1) A $\Rightarrow$ $y = x^3$

$\Rightarrow$ B $\Rightarrow$ $y^2 = x$ $\therefore$ $y = \pm\sqrt{x}$

but 1^{st} quadrant so $y = \sqrt{x}$.

$\Rightarrow$ (2) $x = 0$ & $x = 1$. (check it by extending region & we get limits).

$$\iint_R y^2\,dA = \int_{x=0}^{1}\int_{y=x^3}^{x^{1/2}} y^2\,dy\,dx$$

Here we consider order of integration as $dy\,dx$.

$\therefore$ We take $\overline{AB} \parallel$ Y-axis within the region.

$\therefore$ for the limits we can observe that point A lies on $y = x^3$ & B lies on $y^2 = x$

$\therefore$ limits of y are (1).

Now for limits of x we observe that our region of integration is extended from $x = 0$ to $x = 1$ on X-axis.

$\therefore$ limits of x are (2)

$$\Rightarrow \int_{x=0}^{1}\int_{y=x^3}^{\sqrt{x}} y^2\,dy\,dx.$$

$$\Rightarrow \int_{x=0}^{1}\left[\frac{y^3}{3}\right]_{x^3}^{x^{1/2}} dx$$

$$\Rightarrow \frac{1}{3}\int_{x=0}^{1}\left(x^{3/2} - x^{9}\right)dx.$$

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$$\Rightarrow \frac{1}{3}\left[\frac{2x^{5/2}}{5}\right]_0^1 - \left[\frac{x^{10}}{10}\right]_0^1$$

$$\Rightarrow \frac{1}{3}\left[\frac{2}{5}-0\right] - \left[\frac{1}{10}-0\right]$$

$$\Rightarrow \frac{1}{3}\left[\frac{2}{5}-\frac{1}{10}\right]$$

$$\Rightarrow \frac{1}{3} \times \frac{3}{10} \Rightarrow \boxed{\frac{1}{10}.}$$

HW.

Evaluate $\iint_R y^2\,dA$ $dx\,dy$. where R is the region bounded by the curves $y = x^3$ & $y^2 = x$.

(Hint : Order of integration is $dx\,dy$ so we take $\overline{AB} \parallel$ x-axis only.)

∴ limits are $x = y^2$ to $x = y^{1/3}$

$y = 0$ to $y = 1$

11/01/19

Q.2) Evaluate : $\iint_R (x+y)\,dy\,dx.$, where R is the region bounded by $x = 2$, $y = x$, $y = x+2$ and y-axis.

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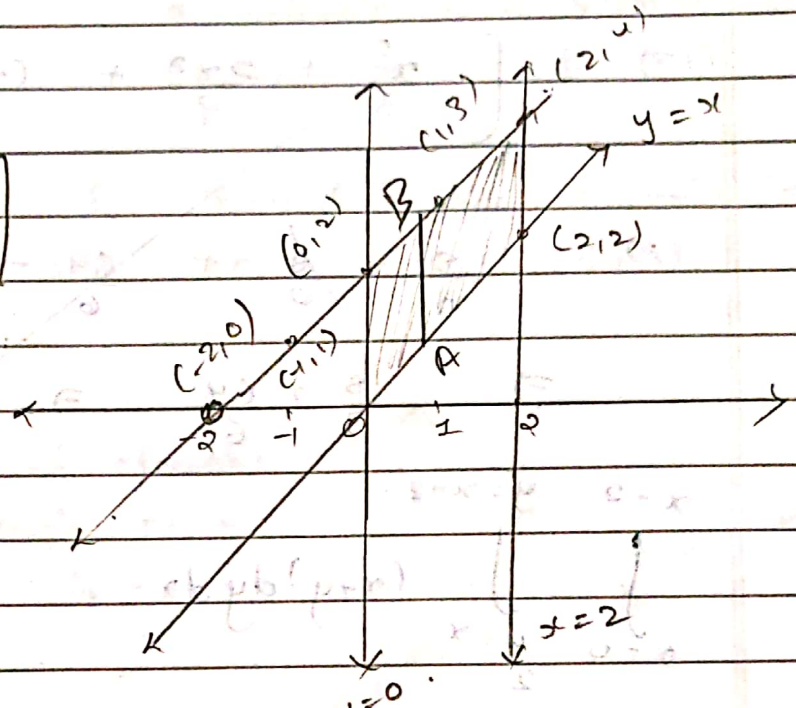
Page ____
Date ____

Solⁿ: We are given $\iint_R$ R is bounded by the curves $x=0$, $x=2$, $y=x$ & $y=x+2$.

∴ We have

$y = x+2$

x	-2	-1	0	1	2
y	0	1	2	3	4



Here order is $dydx$. So we take $\overline{AB} \parallel$ Y-axis

∴ limits are $y = x$ to $y = x+2$.

$x = 0$ to $x = 2$.

$$\Rightarrow \text{So} \iint_R (x+y)\,dydx = \int_{x=0}^{x=2} \int_{y=x}^{x+2} (x+y)\,dy\,dx$$

$$= \int_{x=0}^{2} \left[x(y) + \frac{y^2}{2} \right]_x^{x+2} dx$$

$$= \int_{x=0}^{2} \left(x(x+2) + \frac{(x+2)^2}{2} \right) - \left(x^2 + \frac{x^2}{2} \right) dx$$

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$$\Rightarrow \frac{8}{3} + \cancel{\frac{8}{2}} + \frac{64}{6} - \cancel{\frac{8}{2}}$$

$$\Rightarrow \frac{8}{3} + \frac{64}{6} \Rightarrow \frac{16+64}{6} \Rightarrow \frac{80}{6} \Rightarrow \frac{40}{3}$$

$$\int_{x=0}^{x=2} \int_{y=x}^{y=x+2} (x+y)\,dy\,dx.$$

$$\Rightarrow \int_{x=0}^{2} \left[xy + \frac{y^2}{2} \right]_{x}^{x+2} dx$$

$$\Rightarrow \int_{x=0}^{2} x(x+2) + \frac{(x+2)^2}{2} - \left[x^2 + \frac{x^2}{2} \right] dx$$

$$\Rightarrow \int_{x=0}^{2} x^2 + 2x + \frac{x^2+2x+4}{2} - \frac{3x^2}{2}\,dx$$

$$\Rightarrow \int_{x=0}^{2} \frac{\cancel{2x^2} + 4x + \cancel{x^2} + 2x + 4 - \cancel{3x^2}}{2}\,dx$$

$$\Rightarrow \int_{x=0}^{2} \frac{6x+4}{2}\,dx \Rightarrow \int_{x=0}^{2} 3x+2\,dx$$

$$\Rightarrow \frac{3}{2}\left[x^2\right]_0^2 + 2\left[x\right]_0^2$$

$$\Rightarrow \frac{3}{\cancel{2}}\left(\cancel{4}^{2}\right) + 2(2-0) \Rightarrow 6+4 = 10.$$

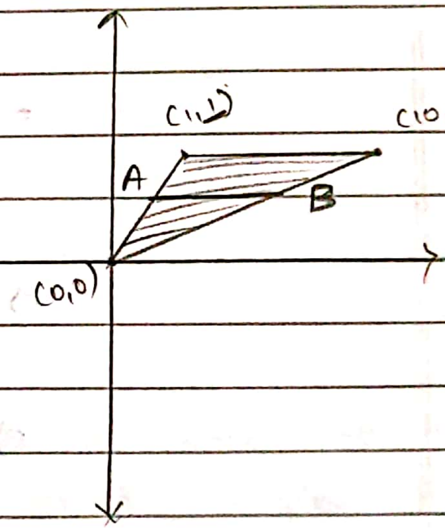
Page
Date

Q.3) Evaluate $\iint_R \left(\sqrt{xy-y^2}\right) dx\,dy$ where R is the region bounded by a triangle with vertices at (0,0), (10,1) & (1,1)

Sol^n: We are given R is bounded by a triangle with vertices at (0,0), (10,1) (1,1)

$\therefore$ We have

→ Here the eq^n of straight line passing through points (0,0) and (1,1) is



$$\frac{x-0}{1-0} = \frac{y-0}{1-0}$$

$$\therefore \quad x = y.$$

→ Also eq^n of straight line passing through points (0,0) and (10,1) is.

$$\frac{x-0}{10-0} = \frac{y-0}{1-0} \qquad \therefore \frac{x}{10} = y \qquad \therefore x = 10y.$$

→ And a straight line $y = 1$.

A to B $\quad x = y$ to $x = 10y$

$y = 0$ to $y = 1$.

→ We take $\overline{AB}$ ∥ to X-axis. So limits are

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So $\iint_R (\sqrt{xy-y^2})\, dx\, dy = \int_{y=0}^{1}\int_{x=y}^{10y} (\sqrt{xy-y^2}\, dx\, dy$

$$\Rightarrow \int_{y=0}^{1}\int_{x=y}^{10y} (xy-y^2)^{1/2}\, dx\, dy$$

$$= \int_{y=0}^{1} y^{1/2} (x-y)^{1/2}\, dx\, dy$$

$$= \int_{y=0}^{1} \left[y^{1/2}\, \frac{2(x-y)^{3/2}}{3} \right]_y^{10y} dy$$

$$\Rightarrow \int_{y=0}^{1} y^{1/2}\, \frac{2(9y)^{3/2}}{3} - y^{1/2}(0)$$

$$\Rightarrow \frac{2\times 3^3}{3} \int_{y=0}^{1} y^{1/2}\, y^{3/2}\, dy$$

$$\Rightarrow 2\times 9 \int_{y=0}^{1} y^2\, dy$$

$$\Rightarrow 18 \left[\frac{y^3}{3}\right]_0$$

$$\Rightarrow 18\left[\frac{1}{3}\right] \Rightarrow 6$$

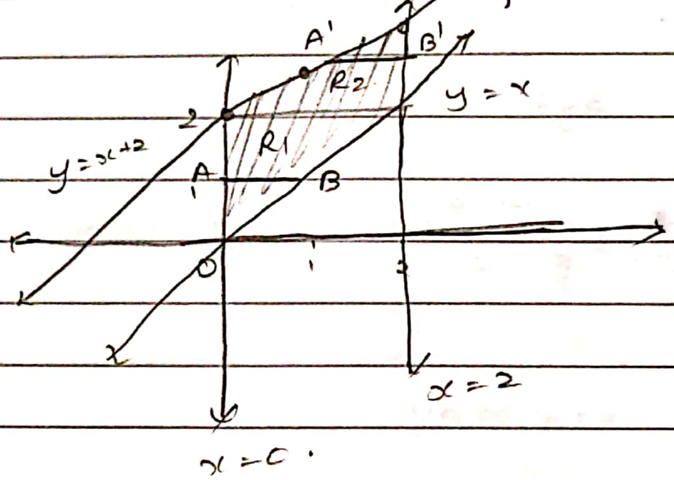
$$\Rightarrow 6$$

Extra

Q.2) by another method.

$$\iint_R (x+y)\,dx\,dy \qquad Y\text{-axis},\ x=2,\ y=x,\ y=x+2.$$

Soln: We are given R, which is bounded by the curves $x=0$, $x=2$, $y=x$ & $y=x+2$.



0	1	2
2	3	4

Here order is $dx\,dy$. So we take $\overline{AB} \parallel$ X-axis.
Here we can observe that region R is divided into R_1 & R_2 by taking order of integration as $dx\,dy$.

For region R_1, $\overline{AB} \parallel$ X-axis

So $A = x = 0$

$B = x = y$ limits of y: $y=0$ to $y=2$.

$R_1 = \cancel{\int_{y=0}^{y=2}\int_{x=0}^{x=y} (x+y)\,dx\,dy}$ For R_2, $\overline{AB} \parallel$ X-axis

$x=y-2$ $A = \cancel{x+2}$ to $B = x=2$

$y=2$ to $y=4$

Page
Date

$$R_1 = \int_{y=0}^{2}\int_{x=0}^{y} (x+y)\,dx\,dy$$

$$= \int_{y=0}^{2} \left[\frac{x^2}{2} + y(x)\right]_0^y dy$$

$$= \int_{y=0}^{2} \left(\frac{y^2}{2} + y^2\right) dy$$

$$= \int_{y=0}^{2} \frac{3y^2}{2}$$

$$= \frac{3}{2}\left[\frac{y^3}{3}\right]_0^2 \Rightarrow \frac{8}{2} \Rightarrow 4$$

$$R_2 = \int_{y=2}^{4}\int_{x=y-2}^{x=2} (x+y)\,dx\,dy$$

$$= \int_{y=2}^{4} \left[\frac{x^2}{2} + y(x)\right]_{y-2}^{2}$$

$$= (2+2y) - \left(\frac{(y-2)^2}{2} + y(y-2)\right)$$

$$= 2 + 2y - \left(\frac{y^2 - 4y + 4}{2} + y^2 - 2y\right)$$

$$= 4 + 4y - (y^2 - 4y + 4 + 2y^2 - 4y)$$

$$= \cancel{4} + 4y - 3y^2 + 12y - \cancel{4}$$

$$= \left[-3y^2 + 12y\right]_2^4$$

$$= 64\left[-\frac{\cancel{8}y^3}{\cancel{3}} + \frac{\cancel{12}^{6}y^2}{\cancel{2}}\right]_2^4$$

$$= \left[-y^3 + 6y^2\right]_2^4$$

$$= (-64 + 96) - (-8 + 24)$$

$$= 32 - 16$$

$$= 16/_2 = 8$$

So $R = R_1 + R_2$

$$= 4 + 8 = 12$$

Date
19.

Change of order of integration :

↳ Let us consider an integral $\int_{y=c}^{d} \int_{x=f_1(y)}^{f_2(y)} f(x,y)\,dx\,dy \rightarrow (1)$

↳ Here we want to change the order of integration. that means instead of dxdy we want to write dydx

↳ For that we must have to convert limits of x provided in eqⁿ (1) into constant form and limits of y provided in (1) into variable form using concept of line segment.

↳ After that, eqⁿ (1) can be written as:

$$\int_{x=a}^{b} \int_{y=g_1(x)}^{g_2(x)} F(x,y)\,dy\,dx \longrightarrow (2)$$

↳ Hence this whole process of redefining the limits and taking the order of integration dydx instead of dxdy is known as change of order of integration.

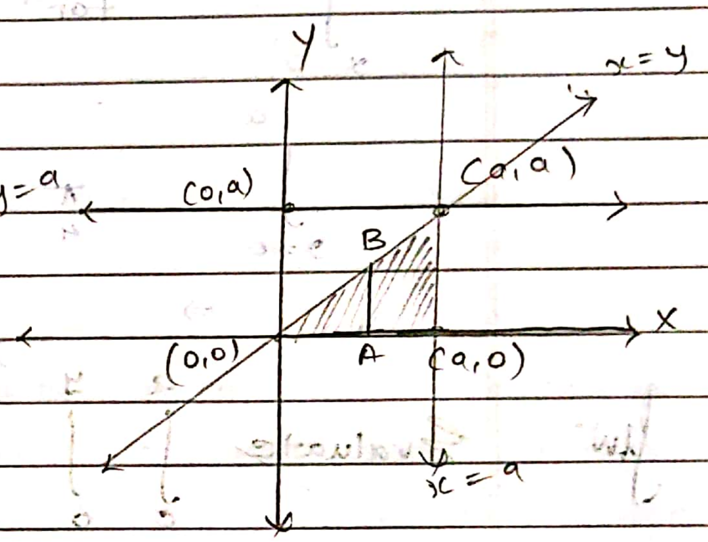
Q1) Change the order of integration in $\int_{0}^{a} \int_{y}^{a} \frac{x}{x^2+y^2}\,dx\,dy$ & hence evaluate it.

Sol^n: We are given $y = \int_{y=0}^{a} \int_{x=y}^{a} \frac{x}{x^2+y^2}\, dx\, dy$

& order of integration is $dx\,dy$.
That means here the region of integration is bounded by the curves $y=0, y=a, x=y, x=a$
& o-of-int as $dx\,dy$.

So We have,

y=0 & y=a shows the portion & then using rest of data we can find intersection portion & then in that we have to change the order.



Here we want to change order of integration. That means instead of $dx\,dy$, we want to write $dy\,dx$. So we take $\overline{AB} \parallel$ Y-axis within region.

$\therefore$ limits are: $y=0$ to $y=x$ & $x=0$ to $x=a$

Now, after changing the order of integration we have

$$\int_{x=0}^{a} \int_{y=0}^{x} \frac{x}{x^2+y^2}\, dy\, dx.$$

$$\int_{x=0}^{a}\int_{y=0}^{x} x\left(\frac{1}{x^2+y^2}\right) dy dx$$

$$\int_{x=0}^{a}\left[\cancel{x}\cdot\frac{1}{\cancel{x}}\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$\int_{x=0}^{a} \tan^{-1}(1) - \tan^{-1}(0)\, dx$$

$$\int_{x=0}^{a} \frac{\pi}{4}\, dx$$

$$\Rightarrow \frac{\pi}{4}[x]_0^a \Rightarrow \boxed{\frac{a\pi}{4}}$$

H.W. Evaluate $\int_0^a\int_0^y \frac{x}{x^2+y^2}\, dA$ using C.O.I.

Q.2) Find $\int_{x=0}^{1}\int_{y=e^x}^{e} \frac{1}{\log y}\, dy dx$ using C.O.I.

Sol^n: We are given $\int_{x=0}^{1}\int_{y=e^x}^{e} \frac{1}{\log y}\, dy dx$

Here the region is bounded by curves

i.e. $x=0$, $x=1$, $y=e^x$, $y=e \simeq 2.71$

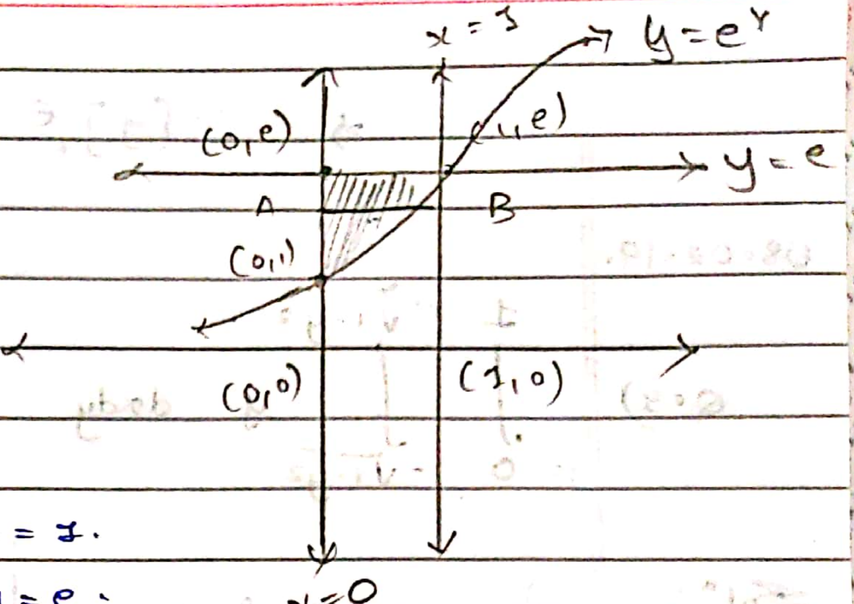
& order of integration as $dydx$.

$\therefore$ We have

$y = e^x$

x	-1	0	1	2
y	0.36	1	e	e^2

2.71 7.2



A : $x = 0$ to $x = 1$.

$y = 0$ to $y = e$.

$$\int\int \cancel{\frac{1}{\log y}} \cancel{dx}\, dy$$

Here we want to change order of integration i.e. instead of $dydx$ we have to write $dxdy$. So we take $\overline{AB} \parallel$ X-axis within region. Here we can observe that A lies on $x=0$

B on $y = e^x$

$\therefore \log y = x$

$x = 0$ to $x = \log y$

& limits of y are $y = 1$ to $y = e$.

$$\int_{y=1}^{e}\int_{x=0}^{\log y} \frac{1}{\log y}\, dx\, dy$$

$$\Rightarrow \int_{y=1}^{e} \frac{1}{\log y} [x]_0^{\log y}\, dy$$

$$\Rightarrow \int_{y=1}^{e} \left(\frac{\log y}{\log y} - 0\right) dy \Rightarrow \int_{y=1}^{e} 1\, dy.$$

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$\Rightarrow [y]_1^e \Rightarrow \boxed{e-1}$

08.02.19.

Q.3) $\displaystyle\int_{y=0}^{1}\int_{-\sqrt{1-y^2}}^{x=\sqrt{1-y^2}} y\, dx dy$ & hence find it.

Solⁿ: We are given $\displaystyle\int_{y=0}^{1}\int_{x=-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y\, dx dy$

i.e., here the region is bounded by the curves $y=0$, $y=1$, $x=-\sqrt{1-y^2}$; $x=\sqrt{1-y^2}$ & order is $dxdy$.

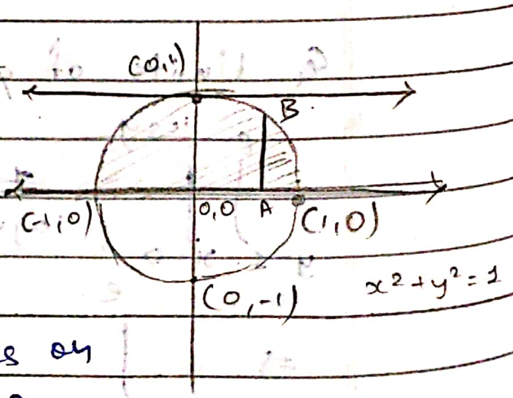
Here given that $x = \pm\sqrt{1-y^2}$

$x^2 = \pm\, 1-y^2$

$\boxed{x^2+y^2=1}$, which is eqⁿ of circle

$\therefore$ We have.

Here we want to change o. of int so we want to write $dydx$ instead of $dx\, dy$. So we take $\overline{AB}$ $\parallel$ Y-axis.



Here we can observe that A lies on $y=0$ & B lies on $x^2+y^2=1$.

limits of y are $y=0$ to $y=\sqrt{1-x^2}$

limits of x are $x=-1$ to $x=1$

Page
Date

$$\int_{x=-1}^{1}\int_{y=0}^{\sqrt{1-x^2}} y\,dy\,dx.$$

$$\Rightarrow \int_{x=-1}^{1} \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx.$$

$$\Rightarrow \frac{1}{2}\int_{x=-1}^{1} \left((1-x^2)^{1/2}\right)^2 - 0.\ dx$$

$$\Rightarrow \frac{1}{2} \left[x - \frac{x^3}{3}\right]_{-1}^{1}.$$

$$\Rightarrow \frac{1}{2} \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right)$$

$$\Rightarrow \frac{1}{2}. \left(\frac{2}{3} + \frac{2}{3}\right) \Rightarrow \frac{1}{2}\left(\frac{4}{3}\right) \Rightarrow \frac{2}{3}.$$

H.W. $\int_0^1\int_0^{\sqrt{1-x^2}} x\,dy\,dx$ & hence evaluate it.

Double Integrals in Polar Co-ordinate System:

In polar co-ordinate system, the function is dependent on (r,θ). Here we can observe that limits of θ must be in the constant form only and limits of r may or may not be in the constant form (i.e. either variable or constant form)

Hence for the polar system, the order of integration is $dr\,d\theta$.

Q.1) Evaluate : $\int_0^{\pi}\int_0^{a\sin\theta} \cancel{\sin\theta}\; r\,dr\,d\theta$.

Sol^n: We are given that $\int_0^{\pi}\int_0^{a\sin\theta} r\,dr\,d\theta$.

$$= \int_{\theta=0}^{\pi} \left[\frac{r^2}{2}\right]_{r=0}^{a\sin\theta} d\theta.$$

$$\Rightarrow \frac{1}{2}\int_0^{\pi}\left[a^2\sin^2\theta\right] - 0\, d\theta.$$

$$\Rightarrow \frac{a^2}{2}\int_0^{\pi} \sin^2\theta\, d\theta.$$

$$\Rightarrow \frac{a^2}{2}\int_0^{\pi}\left[\frac{1-\cos 2\theta}{2}\right]d\theta$$

$$\Rightarrow \frac{a^2}{2}\left[\frac{1}{2}\int_0^{\pi} d\theta - \frac{1}{2}\int_0^{\pi}\cos 2\theta\, d\theta\right]$$

$$\Rightarrow \frac{a^2}{2}\left[\frac{1}{2}\left[\theta\right]_0^{\pi} - \frac{1}{2(2)}\left[\sin 2\theta\right]_0^{\pi}\right]$$

$$\Rightarrow \frac{a^2}{2}\left[\frac{1}{2}(\pi) - 0\right]$$

$$\Rightarrow \boxed{\frac{\pi a^2}{4}}.$$

Q.2) $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta \, dr\, d\theta$

Sol^n: We are given $\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} r\sin\theta\, dr\, d\theta$.

$\Rightarrow \int_{\theta=0}^{\pi/2} \sin\theta \cdot \frac{1}{2}\left[r^2\right]_0^{a\cos\theta} d\theta$

$\Rightarrow \frac{1}{2}\int_{\theta=0}^{\pi/2} \sin\theta \cdot \left[a^2\cos^2\theta\right] d\theta$.

$\Rightarrow \frac{a^2}{2}\int_{\theta=0}^{\pi/2} \sin\theta \cdot \cos^2\theta \, d\theta$.

Now let ~~$\sin\theta = t$~~ $\cos\theta = t$

~~$\cos\theta d\theta$~~ $-\sin\theta d\theta = dt$

$\Rightarrow \frac{a^2}{2}\int_0^1 t^2\, dt$

θ	0	$\pi/2$
t	1	0

$\Rightarrow \frac{a^2}{2} \cdot \left[\frac{t^3}{3}\right]_0^1$

$\Rightarrow \frac{a^2}{2}\left[\frac{1}{3}\right] \Rightarrow \boxed{\frac{a^2}{6}}$

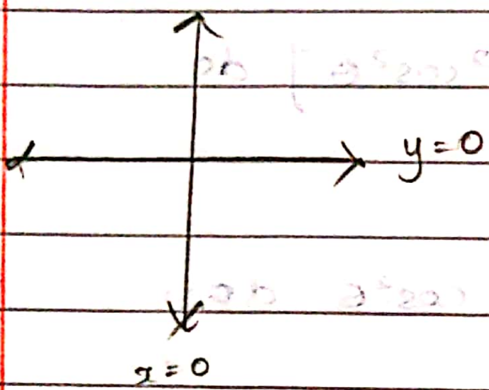
Some useful curves :

⇒ For the polar system, we take
$x = r\cos\theta$ & $y = r\sin\theta$.

1) Straight line :

Polar form of

$$x = 0 \qquad\qquad y = 0$$
$$r\cos\theta = 0 \qquad\qquad r\sin\theta = 0$$
$$\cos\theta = 0 \qquad\qquad \sin\theta = 0$$
$$\boxed{\theta = \frac{\pi}{2}} \qquad\qquad \boxed{\theta = \pi}$$



⇒ Polar form of straight lines $x = a$ & $y = b$ are as follows :

$$x = a$$
$$r\cos\theta = a$$
$$r = \frac{a}{\cos\theta} = \boxed{a\sec\theta}$$

$$y = a$$
$$r\sin\theta = a$$
$$r = \frac{a}{\sin\theta} = \boxed{a\,\mathrm{cosec}\,\theta}$$

⇒ $x = y$:

$$\boxed{\theta = \pi/4.}$$