PARTIAL DIFFERENTIAL EQUATIONS

Definition

An equation which involves several independent variables (usually denoted *x, y, z, t,* .....), a dependent function *u* of these variables, and the partial derivatives of the dependent function *u* with respect to the independent variables such as

$$F(x, y, z, t, \ldots\ldots, u_x, u_y, u_z, u_t, \ldots\ldots, u_{xx}, u_{yy}, \ldots\ldots, u_{xy}, \ldots\ldots)=0$$

is called a partial differential equation.

Partial differential equations are used to formulate,and thus aid the solution of, problems involving functions of several variables; such as the propagation of sound or heat, electrostatics, electrodynamics, fluid flow, and elasticity.

Examples

i. $u_t=k(u_{xx}+u_{yy}+u_{zz})$ [linear three-dimensional heat equation]

ii. $u_{xx}+u_{yy}+u_{zz}=0$ [Laplace equation in three dimensions]

iii. $u_{tt}=c^2(u_{xx}+u_{yy}+u_{zz})$ [linear three-dimensional wave equation]

Order of PDE

- The order of a partial differential equation is the order of the highest derivative occurring in the equation.
- All the above examples are second order partial differential equations.
- $u_t = uu_{xxx} + \sin x$ *is an example for third order* partial differential equation.

Degree of PDE

- The power of the highest order derivative in a differential equation is called the degree of the partial differential equation.

- A partial differential equation is said to be **linear** if the unknown function u(.,.) and all its partial derivatives appear in an algebraically linear form, 'that is, of the first degree.
- The equation is called linear if the unknown function only appears in a linear form.

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = d(x, y)$$

Almost linear partial differential equations

$$P(x, y)u_x + Q(x, y)u_y = R(x, y, u)$$

- linear & nonlinear:
  - *linear* differential equation: all terms linear in unknown and its derivatives
  - e.g.
    - $x''+ax'+bx+c=0$ – linear
    - $x'=t^2x$ – linear
    - $x''=1/x$ – nonlinear

Homogeneous PDE

- A partial differential equation is called **homogeneous** if function on the right hand side of a partial differential equation is zero.
- The partial differential equation is called **non-homogeneous** if $f \neq 0$, that is function on the right hand side of a partial differential equation is not zero.

Formation of Partial Differential equations

Partial Differential Equation can be formed either by elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables .

SOLVED PROBLEMS

1.Eliminate two arbitrary constants a and b from $(x-a)^2+(y-b)^2+z^2=R^2$ here R is known constant

.

(OR) Find the differential equation of all spheres of fixed radius having their centers in x y- plane.

solution

$$(x-a)^2+(y-b)^2+z^2=R^2 \ldots\ldots(1)$$

Differentiating both sides with respect to x and y

$$2z\frac{\partial z}{\partial x}=-2(x-a)$$

$$2z\frac{\partial z}{\partial y}=-2(y-b)$$

$$\frac{\partial z}{\partial x}=p,\frac{\partial z}{\partial y}=q$$

$$\therefore x-a=-pz, y-b=-qz$$

By substituting all these values in (1)

$$p^2 z^2 + q^2 z^2 + z^2 = R^2$$

$$\Rightarrow z^2 = \frac{R^2}{p^2 + q^2 + 1}$$

or

$$z^2 = \frac{R^2}{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

2. Find the partial Differential Equation by eliminating arbitrary functions from $z = f(x^2 - y^2)$

SOLUTION

$$z = f(x^2 - y^2)..........(1)$$

$$d.w.r.to.xandy$$

$$\frac{\partial z}{\partial x} = f^{'}(x^2 - y^2) \times 2x......(2)$$

$$\frac{\partial z}{\partial y} = f^{'}(x^2 - y^2) \times -2y......(3)$$

By $\dfrac{(2)}{(3)}$

$$\frac{\left(\dfrac{\partial z}{\partial x}\right)}{\left(\dfrac{\partial z}{\partial y}\right)} = \frac{-x}{y}$$

$$\frac{p}{q} = \frac{-x}{y} \Rightarrow py + qx = 0$$

3.Find Partial Differential Equation by eliminating two arbitrary functions from

$$z = yf(x) + xg(y)$$

SOLUTION

$$z = yf(x) + xg(y)......(1)$$

Differentiating both sides with respect to x and y

$$\frac{\partial z}{\partial x} = yf'(x) + g(y)........(2)$$

$$\frac{\partial z}{\partial y} = f(x) + xg'(y)........(3)$$

Again d . w .r. to x and yin equation (2)and(3)

$$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$$

$$x \times (2) + y \times (3) \ldots\ldots to \ldots get$$

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=$$

$$xg(y)+yf(x)+xy(f'(x)+g'(y))$$

$$=z+xy\left(f'+g'\right)$$

$$\Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z+xy\left(\frac{\partial^2 z}{\partial x\partial y}\right)$$

Different Integrals of Partial Differential Equation

1. Complete Integral (solution)

Let $F(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) = F(x, y, z, p, q) = 0 ......(1)$

be the Partial Differential Equation.

The complete integral of equation (1) is given by $\phi(x, y, z, a, b) = 0 ....(2)$

OR A Solution which contains a number of arbitrary constants equal to the independent variables, is called a complete integral

2. Particular solution

A solution obtained by giving particular values to the arbitrary constants in a complete integral is called particular solution .

3.Singular solution

The eliminant of a , b between

$$\phi(x, y, z, a, b) = 0$$

$$\frac{\partial \phi}{\partial a} = 0, \frac{\partial \phi}{\partial b} = 0$$

when it exists , is called singular solution

4. General solution

In equation (2) assume an arbitrary relation of the form $b = f(a)$. Then (2) becomes

$$\phi(x, y, z, a, f(a)) = 0 \text{..........}(3)$$

Differentiating (2) with respect to a,

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} f'(a) = 0 \text{..........}(4)$$

The eliminant of (3) and (4) if exists, is called general solution

Standard types of Non linear PDE of the first order

TYPE-I (Equations involving Only p and q)

The Partial Differential equation of the form

$$f(p,q)=0$$

has solution $z=ax+by+c$

with $f(a,b)=0$

TYPE-II (***Clairaut's*** form)

The Partial Differential Equation of the form

$z=px+qy+f(p,q)$ is called ***Clairaut's*** form of *pde* , it's solution is given by

$$z=ax+by+f(a,b)$$

SOLVED PROBLEMS

1.Solve the *PDE* $p^2 - q = 1$

Solution

Complete solution is given by

$$z = ax + by + c$$

with $$a^2 - b = 1$$

$$\Rightarrow b = a^2 - 1$$

$$z = ax + (a^2 - 1)y + c$$

2.Solve the *PDE* $pq + p + q = 0$

Solution

The complete solution is given by

$$z = ax + by + c$$

with

$$ab + a + b = 0$$

$$a = \frac{-b}{b+1}$$

$$\therefore z = \frac{-b}{b+1}x + by + c.......(1)$$

Where b,c are arbitrary constants

3.Solve the *PDE* $z = px + qy + \sqrt{1 + p^2 + q^2}$

Solution

The PDE $z = px + qy + \sqrt{1 + p^2 + q^2}$

is in Clairaut's form

complete solution of (1) is

$$z = ax + by + \sqrt{1 + a^2 + b^2} \,.......(2)$$

Where a,b are arbitrary constants

TYPE-III (Equations Not involving Independent Variables)

If the *PDE* is given by $f(z, p, q) = 0$

Since z is a function of x and y,

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$= pdx + qdy$$

Let us assume that $q = ap$

The given *PDE* $f(z, p, q) = 0$

becomes $f(z, p, ap) = 0$

Solving it for p i.e. $p = \phi(z, a)$

$$\therefore dz = \phi(z, a)dx + a\phi(z, a)dy$$

$$\frac{dz}{\phi(z, a)} = dx + ady$$

integrating $\int \frac{dz}{\phi(z, a)} = x + ay + b$

Which is a complete integral

4. Solve the pde $zpq = p + q$

Solution

Assume $q = ap$

Substituting in given equation

$$zpap = p + ap$$

$$p = \frac{1+a}{az}, q = \frac{1+a}{z}$$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\Rightarrow dz = \frac{1+a}{az} dx + \frac{1+a}{z} dy$$

$$zadz = (1+a)(dx + ady)$$

Integrating on both sides

$$\frac{a}{2}z^2 = (1+a)(x+ay) + b$$

Where a,b are arbitrary constants

TYPE-IV (Separable Equations)

The *pde* of the form $f(x,p)=g(y,q)$ can be solved by assuming

$$f(x,p)=g(y,q)=a$$

$$f(x,p)=a \Rightarrow p=\phi(x,a)$$

$$g(y,q)=a \Rightarrow q=\Psi(y,a)$$

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

$$dz=\phi(x,a)dx+\Psi(y,a)dy$$

Integrate the above equation to get solution

5.Solve pde $pq = xy$

(or) $$\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = xy$$

Solution

$$\frac{p}{x} = \frac{y}{q}$$

Assume that

$$\frac{p}{x} = \frac{y}{q} = a$$

$$\therefore p = ax, q = \frac{y}{a}$$

$$dz = pdx + qdy = axdx + \frac{y}{a}dy$$

Integrating on both sides $z = a\frac{x^2}{2} + \frac{y^2}{2a} + b$

Where a,b are arbitrary constants

6.Solve the pde $(1-x)p+(2-y)q=3-z$

Solution

pde $(1-x)p+(2-y)q=3-z$

$$z = px + qy + (3 - p - 2q)$$

Complete solution of above pde is

$$z = ax + by + (3 - a - 2b)$$

7.Solve the pde $p^2+q^2=z$

Solution

Assume that $q = ap$

$$p^2 + q^2 = z$$

becomes $p^2 + a^2 p^2 = z$

$$p^2 = \frac{z}{1+a^2} \text{ so } p = \pm\sqrt{\frac{z}{1+a^2}}$$

$$\therefore dz = \phi(z,a)dx + a\phi(z,a)dy (\because dz = pdx + qdy)$$

$$dz = \pm\sqrt{\frac{z}{1+a^2}}dx + a(\pm\sqrt{\frac{z}{1+a^2}})dy$$

$$\mp\sqrt{1+a^2}\,\frac{dz}{\sqrt{z}} = dx + ady$$

Integrate to get the solution $\mp\sqrt{1+a^2}\,2\sqrt{z} = x + ay + b$

Where b is arbitrary constant

8. Solve the equation $p^2 + q^2 = x + y$

Solution

$$p^2 - x = y - q^2 = a$$

$$p = \sqrt{a + x}, q = \sqrt{y - a}$$

$$dz = pdx + qdy = \sqrt{a + x}dx + \sqrt{y - a}dy$$

integrating

$$z = \frac{2}{3}(a + x)^{\frac{3}{2}} + (y - a)^{\frac{3}{2}} + b$$

Lagrange's Linear Equation

Def: The linear partial differential equation of first order is called as Lagrange's linear Equation. This eq is of the form

$$P(x,y,z)\left(\frac{\partial z}{\partial x}\right)+Q(x,y,z)\left(\frac{\partial z}{\partial y}\right)=R(x,y,z)$$

$$or \qquad Pp+Qq=R$$

Where P,Q and R are functions x,y and z

The general solution of the partial differential equation $Pp+Qq=R$ is $F(u,v)=0$

Where F is arbitrary function of $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$

Here $u = c_1$ and $v = c_2$ are independent solutions

of the auxilary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Method of Solution

In order to solve the equation Pp+Qq=R

- Form the auxiliary equations
- Solve these auxiliary equations

  by the method of grouping

OR by the method of multipliers

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{ldx+mdy+ndz}{lP+mQ+nR}$$

OR both to get two independent solutions $u=c_1$ and $v=c_2$. Then F(u,v)=0 is the general solution of the equation Pp+Qq=R

Solved problems

1.Find the general solution of $x^2 p + y^2 q = (x+y)z$

Solution

auxiliary equations are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$ Integrating on both sides

$$u = \left(x^{-1} - y^{-1}\right) = c_1$$

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$\frac{d(x-y)}{(x-y)(x+y)} = \frac{dz}{(x+y)z}$$

$$\frac{d(x-y)}{(x-y)} = \frac{dz}{z}$$ Integrating on both sides

$$\log(x-y)=\log z+\log c_2$$

$$v=(x-y)z^{-1}=c_2$$

The general solution is given by $F(u,v)=0$

$$F(x^{-1}-y^{-1},(x-y)z^{-1})=0$$

2.solve $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

solution

Auxiliary equations are given by

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}$$

$$\frac{\frac{dx}{x^2}}{(y-z)} = \frac{\frac{dy}{y^2}}{(z-x)} = \frac{\frac{dz}{z^2}}{(x-y)}$$

$$\frac{\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2}}{(y-z)+(z-x)+(x-y)}$$

$$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

Integrating on both sides

$$u = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = a$$

$$\frac{x^{-1}dx}{x(y-z)} = \frac{y^{-1}dy}{y(z-x)} = \frac{z^{-1}dz}{z(x-y)}$$

$$\frac{x^{-1}dx + y^{-1}dy + z^{-1}dz}{x(y-z) + y(z-x) + z(x-y)}$$

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$ Integrating on both sides

$$v = xyz = b$$

The general solution is given by

$$F(x^{-1}+y^{-1}+z^{-1}, xyz) = 0$$