Thursday

1-10-18

INFINITE SERIES

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* Sequence :-

If for each $n \in \mathbb{N}$, a number a_n is assigned, then the numbers $a_1, a_2, a_3, \ldots, a_n, \ldots$ is said to form a sequence, that means sequence is a function from set of natural numbers to the set $S = \{a_1, a_2, a_3, \ldots, a_n, \ldots\}$. Thus, if $f : \mathbb{N} \to S$ is a sequence then we can write $f(n) = a_n$, where a_n is called n^{th} term of a sequence.

It is denoted by $\{a_n\}_{n=1}^{\infty}$ (or) $\{a_n\}$.

For e.g

(1) $1, 2, 3, \ldots, n, \ldots$ is a sequence whose n^{th} term is 'n' & it is denoted by $\{n\}_{n=1}^{\infty}$.

(2) $1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots$ is a sequence whose n^{th} term is '$\frac{1}{n}$' & it is denoted by $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$

(3) $1, -1, 1, -1, \ldots$ is a sequence whose n^{th} term is $(-1)^{n+1}$ & it is denoted by $\{(-1)^{n+1}\}_{n=1}^{\infty}$

* Series :-

The sum of terms of sequence is called a series.

If the no. of terms of a series is limited or finite then, the series is called a finite series & if no. of terms are infinite, then the series is called infinite series.

It is denoted by $a_1 + a_2 + \dots + a_n + \dots = \sum_{n=1}^{\infty} a_n$

(or)

$\sum a_n$

Note:

→ If we take sum of first n terms of the infinite series '$\sum_{n=1}^{\infty} a_n$', then it is called partial sum of a given series. & it is denoted by 'S_n', that means $S_n = a_1 + a_2 + \dots + a_n$.

* Definition:-

1) Convergent series:

→ The ~~iff~~ infinite series '$\sum_{n=1}^{\infty} a_n$' is said to be a ~~con~~ convergent series if for some finite unique number ~~S~~ s, $\lim_{n\to\infty} S_n = s$

Eg. Let us consider a series $\frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \dots$

$$= \sum_{n=1}^{\infty} \frac{1}{n.(n+1)}$$

Here, the partial sum of given series is,

$$S_n = \frac{1}{1\times2} + \frac{1}{2\times3} + \frac{1}{3\times4} + \dots + \frac{1}{n(n+1)}$$

$$\therefore S_n = \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \dots + \left(\cancel{\frac{1}{n-1}} - \cancel{\frac{1}{n}}\right) + \left(\cancel{\frac{1}{n}} - \frac{1}{n+1}\right)$$

$$S_n = 1 - \frac{1}{n+1}$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[1 - \frac{1}{n+1}\right]$$

$$= 1 - 0$$

$\lim_{n\to\infty} S_n = 1$, which is finite number.

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Hence, given series is a convergent series.

2) Divergent series.

→ The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be ~~dive~~ divergent if $\lim_{n\to\infty} S_n$ is either $+\infty$ or $-\infty$.

~~For.~~ eg.

Let us consider a series $1+2+3+\dots = \sum_{n=1}^{\infty} n$

Here, the partial sum of given series is

$S_n = 1+2+3+\dots+n \Rightarrow \frac{n(n+1)}{2}$.

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2}$

$\longrightarrow \infty$

Hence, given series is a ~~dive~~ divergent series.

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3) Oscillating Series.

→ The infinite series '$\sum_{n=1}^{\infty} a_n$' is said to be oscillating series (or) oscillatory series If the limit of S_n is not unique as $n \to \infty$.

~~N/B~~ The series '$\sum a_n$' is finitely oscillating if the ~~kind~~ limit of S_n is different finite value as $n \to \infty$, otherwise '$\sum a_n$' is infinitely oscillating.

(Eg.) Let us consider an infinite series $1 - 1 + 1 - 1 + \dots$

$$= \sum_{n=1}^{\infty} (-1)^{n+1}$$

→ Here, we can observe that if n is even number, then ~~we can that~~

$$\lim_{n \to \infty} S_n = 0 \;\&\; \lim_{n \to \infty} S_{n+1} = 1,$$

∴ Here, the limit of partial sum is not unique, i.e. here the limit is 0 ~~and~~ as well as 1.

∴ The given series is oscillating series. In fact, it is finitely oscillating series.

Note:

→ The convergence of a series '$\sum a_n$' remains unaffected by addition or removal of finite number of terms of a series and by multiplication of each term with a finite number of a given series.

→ Geometric progression, (G.P)

$$1 + r + r^2 + \dots + r^{n-1} = \frac{1-r^n}{1-r}, \text{ where}$$

r is the ratio of each term.

eg. Examine the nature of a series:

$$1 + \frac{3}{4} + \frac{9}{16} + \frac{27}{64} + \dots$$

→ We are given a series i.e $1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \dots = \sum_{n=0}^{\infty} (3/4)^n$

which is in G.P with ratio $r = 3/4$.

Here, the partial sum of given series is

$$S_n = 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots + \left(\frac{3}{4}\right)^{n-1}$$

$$S_n = \frac{1-(3/4)^n}{1-(3/4)}$$

$$\Rightarrow 4\left[\frac{1-(3/4)^n}{1}\right]$$

Now, $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left[4\left(1-\left(\frac{3}{4}\right)^n\right)\right]$

$$= \lim_{n\to\infty} 4 - \lim_{n\to\infty} 4\cdot\left(\frac{3}{4}\right)^n$$

$= 4 - 0$

$= 4$, which is finite.

$\therefore$ The given series is convergent series.

* **Some Useful limits:-**

1. $\lim_{n\to\infty} x^n = 0$, if $|x| < 1$.

2. $\lim_{n\to\infty} x^n = \infty$, if $|x| > 1$.

3. $\lim_{n\to\infty} \cancel{\log} \frac{\ln(n)}{n} = 0$.

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4. $\lim_{n\to\infty} n^{1/n} = 1.$

5. $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e^1 . = e$

6. $\lim_{n\to\infty} \frac{1}{n^n} = 0.$

7. $\lim_{n\to\infty} \left(\frac{n!}{n}\right)^{1/n} = \frac{1}{e}.$

* Positive term series.

→ An infinite series is said to be positive term series if all the terms of the series after some particular terms of a series are positive

For. e.g.

$$-8-1-3+1+2+3+9+4+\ldots$$

* Necessary Condition for a series Σa_n to be convergent :-

→ If the series Σa_n is convergent then, $\lim_{n\to\infty} a_n = 0.$

But, the converse of the above statement is not true, i.e, if $\lim_{n\to\infty} a_n = 0$, then '$\sum a_n$' may or may not converge.

For e.g

→ Consider $\sum_{n=1}^{\infty} 1/n$.

Here, $a_n = \frac{1}{n}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n}$$
$$= 0.$$

but the given series $\sum_{n=1}^{\infty} 1/n$ is not convergent i.e divergent series.

[*] Different types of test for Convergent of an infinite series:

1) (i) Zero test,

if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ is always divergent.

Eg.

Test the convergence of following series.

1. $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$

→ We are given series $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$, Here n^{th} term is $a_n = \frac{n^2-1}{n^2+1}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-1}{n^2+1}$$

$$= \lim_{n\to\infty} \frac{1-1/n^2}{1+1/n^2}$$

$$\Rightarrow 1 \neq 0$$

∴ By zero test, the given series is divergent

2. $\sum_{n=1}^{\infty} \frac{3n}{n+1}$

→ We are given series $\sum_{n=1}^{\infty} \frac{3n}{n+1}$, Here n^{th} term is $a_n = \frac{3n}{n+1}$, then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{n+1}$$

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$$= \lim_{n\to\infty} \frac{3}{1+1/n}$$

$$\Rightarrow 3 \neq 0$$

$\therefore$ By zero test, the given series is divergent.

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2) (II) Geometric Series Test

→ The series ~~a r~~ $a + ar + ar^2 + \dots + ar^{n-1} + \dots$ is

(i) Convergent if $|r| < 1$. & its series sum is $\frac{a}{1-r}$.
↳ $-1 < r < 1$

(ii) Divergent if $r \geq 1$

(iii) Oscillating if ~~$r = 1$~~ $r \leq -1$.

If $r = -1$, then finitely oscillating series. & if $r < -1$, then infinitely oscillating series.

Eg. Test the convergence of following infinite series:

(1) $1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \dots$

H.W. 1) $\sum_{n=0}^{\infty} \frac{3^{2n}}{2^{3n}}$ {Hint : $r = 9/8 = 1.1 > 1$ i.e. Divergent

(3) $\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n}}{2^{3n}}$

→ We are given $\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n}}{2^{3n}}$, it can be written as $\sum_{n=0}^{\infty} (-1)^n \left(\frac{3^2}{2^3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \left(\frac{9}{8}\right)^n$

$= 1 - \frac{9}{8} + \left(\frac{9}{8}\right)^2 + \ldots$

which is geometric series with ratio (r) $= \frac{-9/8}{1} = \frac{-9}{8} = -1.1 < 1$

∴ By Geometric series test, given series is infinitely oscillating series.

(3)(T) P-Series Test

→ The series of the form : $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \ldots$

then this type of series is called p-series (or) Hyperharmonic series.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ is :

(i) Convergent : if $p > 1$

(ii) Divergent if $p \leq 1$

H.W. 1) $\sum_{n=1}^{\infty} \frac{1}{n^{0.5}}$ 2) $\sum_{n=1}^{\infty} 4 n^{-7}$

Eg. Test the convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n}$ (Harmonic series)

→ We are given $\sum_{n=1}^{\infty} \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots$ which is p-series with power $p=1$.

∴ By p-series test, given series is divergent.

(4)(T) Comparision Test

→ Statement :

(1) Limit form

Let Σa_n & Σb_n be two positive term series such that $\lim_{n\to\infty} \frac{a_n}{b_n} = l$, which is finite & non-zero, Then both the series Σa_n & Σb_n converges or diverges together i.e if Σb_n is convergent, then Σa_n is also convergent & if Σb_n is divergent, then Σa_n is also divergent.

(2) Let Σa_n & Σb_n be two positive term series.
(i) if Σb_n is convergent & $0 \le a_n \le b_n$ for all n, then Σa_n is also convergent. & $\Sigma a_n \le \Sigma b_n$

(ii) If Σb_n is divergent & $0 \le b_n \le a_n$ for all n, then Σa_n is also divergent & $\Sigma a_n \ge \Sigma b_n$.

Note:

→ for the comparision test another Σb_n can be obtained by considering ~~b_n~~.

$b_n = \frac{\text{Highest power term of } n \text{ in numerator of } a_n}{\text{Highest power term of } n \text{ in denominator of } a_n.}$

E.g. Test the converge of following series:

(1) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

⟹ We are given $a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} = \frac{1}{(n^3+1)^{1/2}}$

Let us consider $b_n = \frac{n^0}{n^{3/2}} = \frac{1}{n^{3/2}}$

Now, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{(n^3+1)^{1/2}} \times n^{3/2}$

$= \lim_{n\to\infty} \frac{\cancel{n^{3/2}}}{\cancel{n^{3/2}}\left(1+\frac{1}{n^3}\right)^{1/2}}$

$$= \frac{1}{(1+0)^{1/2}}$$

$= 1$, which is finite & non-zero.

∴ By comparision test, both series Σa_n & Σb_n converges or diverges together.

But, here $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$

which is p-series with power $p = 3/2 = 1.5 > 1$

∴ ~~Series~~ ~~is~~ By p-series test, Σb_n is convergent.

Hence, Σa_n is also convergent.

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Eg:(2) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \ldots$

→ We are given $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \ldots = \sum_{n=1}^{\infty} \frac{n+1}{n^p}$

Here, $a_n = \frac{n+1}{n^p}$, let $b_n = \frac{n}{n^p} = \frac{1}{n^{p-1}}$

Now, $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+1}{n^{p}} \times \frac{n^{p-1}}{1}$

$= \lim_{n\to\infty} \frac{n+1}{n}$

$= \lim_{n\to\infty} \frac{n(1+1/n)}{n}$

$= 1$, which is finite & non-zero.

$\therefore$ By comparision test, both the series Σa_n & Σb_n converges or diverges together.

But here, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^{p-1}}$ is p-series with power $p-1$.

$\therefore$ By p-series test, if $p-1>1$ i.e $p>2$, then Σb_n is convergent & if $p-1\leq 1$ i.e $p\leq 2$, then Σb_n is divergent, hence for $p>2$ Σa_n is convergent & for $p\leq 2$ Σa_n is divergent.

(3) $\sum_{n=1}^{\infty} \frac{2n^5+3}{n^3-5}$

$\rightarrow$ We are given $\sum_{n=1}^{\infty} \frac{2n^5+3}{n^3-5}$

H.W. $\dfrac{n^p}{\sqrt{n+1}+\sqrt{n}}$

So, Here, $a_n = \dfrac{2n^5+3}{n^3-5}$ Let $b_n = \dfrac{n^5}{n^3} = n^2$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^5+3}{n^3-5} \times \frac{1}{n^2}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^5}(2+3/n^5)}{n^3(1-5/n^3).n^2}$$

$$= \frac{2+0}{1-0}$$

$= 2$, which is finite & non-zero.

$\therefore$ By comparision test, both the series Σa_n & Σb_n converges or diverges together.

$$\therefore \sum_{n=1}^{\infty} = \sum_{n=1}^{\infty} n^2$$

Now, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} n^2$

$= \infty \neq 0$

$\therefore$ By zero-test, Σb_n is ~~a~~ divergent. Hence Σa_n is also divergent.

(5)(T) Ratio Test : (~~De~~ D' Alembert's Ratio test)

→ Statement :

Let $\sum_{n=1}^{\infty} a_n$ be a positive term series, such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$.

Then,
(i) $l < 1$, then series is convergent
(ii) $l > 1$, then " " divergent.
(iii) $l = 1$, then test failes.

i.e this test ~~chett~~ can only not gives the converge of series

(6)(T) Root Test : [Cauchy's Root test]

→ Statement :

Let $\sum_{n=1}^{\infty} a_n$ be a positive term series such that $\lim_{n\to\infty} (a_n)^{1/n} = l$. Then,

(i) $l < 1$, the series is convergent
(ii) $l > 1$, the " " divergent
(iii) $l = 1$, then test failes.
i.e. this test can only not gives the converge of series.

NOTE:-

→ Most propably, if the series contain factorial term then ratio test may be useful & if the series contains n^{th} power then root test may be useful.

Eg. Test the converge of following series:

(1) $\frac{1!}{1} + \frac{2!}{2^2} + \frac{3!}{3^3} + \dots$

→ We are given $\frac{1!}{1} + \frac{2!}{2^2} + \frac{3!}{3^3} + \dots$ i.e $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Here, $a_n = \frac{n!}{n^n}$, ~~Then~~

Then $a_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{(n+1)}} \times \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}\cancel{(n+1)}}{(n+1)^n \cancel{(1+n)}} \times \frac{n^n}{\cancel{n!}}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{n^n}}{\cancel{n^n}(1+1/n)^n}$$

$= \frac{1}{e} < 1$

~~ratio~~

$\therefore$ By ~~ratio~~ test given series is converg-ent.

(2) $\sum_{n=1}^{\infty} n!\, e^{-n}$

$\rightarrow$ We are given $\sum_{n=1}^{\infty} n!\, e^{-n}$

Here, $a_n = n!\, e^{-n}$

Then $a_{n+1} = (n+1)!\, e^{-(n+1)}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!\, e^{-(n+1)}}{n!\, e^{-n}}$$

$$= \lim_{n\to\infty} \frac{\cancel{n!}\,(n+1)\, \cancel{e^{-n}}\, e^{-1}}{\cancel{n!}\, \cancel{e^{-n}}}$$

$$= \cancel{e^{-1}}\ \infty$$

$$= \cancel{1/e}\ \cancel{<1}\ \infty > 1$$

$\therefore$ By ratio test, given series is divergent.

(3) $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \frac{1.2.3.4}{3.5.7.9} + \ldots$

$\rightarrow$ We are given $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \ldots$

$$= \cancel{\lim} \sum_{n=1}^{\infty} \frac{1.2.3\ldots n}{3.5.7\ldots(2n+1)}$$

Here $a_n = \dfrac{1\cdot2\cdot3\ldots n}{3\cdot5\cdot7\ldots(2n+1)}$

Then, $a_{n+1} = \dfrac{1\cdot2\cdot3\ldots n(n+1)}{3\cdot5\cdot7\ldots(2n+1)(2n+3)}$

Now, $\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = \cancel{\lim}$

$$= \lim_{n\to\infty} \frac{\cancel{1\cdot2\cdot3\ldots n}(n+1)}{\cancel{3\cdot5\cdot7\ldots(2n+1)}(2n+3)} \times \frac{\cancel{3\cdot5\cdot7\ldots(2n+1)}}{\cancel{1\cdot2\cdot3\ldots n}}$$

$$= \lim_{n\to\infty} \frac{n+1}{2n+3}$$

$$= \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}(2+1/n)}$$

$$= \frac{1}{2} < 1$$

$\therefore$ By ratio test, given series is convergent

(4) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

→ We are given $\cancel{(\log n)^n}$ $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$

Here $a_n = \frac{1}{(\log n)^n}$

Now, $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left[\frac{1}{(\log n)^n}\right]^{1/n}$

$= \lim_{n\to\infty} \frac{1}{\log n}$

$= 0 < 1$

$\therefore$ By root test, given series is convergent series.

(5) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n \cdot x^n, \; x > 0$

$\rightarrow$ We are given $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n \cdot x^n, \; x > 0$

Here $a_n = \left(\frac{n+1}{n+2}\right)^n \cdot x^n$

Now, $\lim_{n\to\infty} \cancel{\frac{n+1}{n+2}} (a_n)^{1/n} = \lim_{n\to\infty} \left[\left(\frac{n+1}{n+2}\right)^n \cdot x^n\right]^{1/n}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right) \cdot x \Rightarrow x \lim_{n\to\infty} \frac{\cancel{n}(1+1/n)}{\cancel{n}(1+2/n)}$

$= x.$

$\therefore$ By root test, if $x < 1$, then given series is convergent & if $x > 1$, then given series is divergent.

If $x = 1$, then root test ~~failes~~ failes.

For $x = 1$, given series became

$$\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n (1)^n = \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n$$

Here, in this case, $a_n = \left(\frac{n+1}{n+2}\right)^n$

Now, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^n$

$$= \lim_{n\to\infty} \frac{\cancel{n^n}(1+1/n)^n}{\cancel{n^n}(1+2/n)^n}$$

$$= \frac{e}{e^2}$$

$$= \frac{1}{e} \neq 0$$

$\therefore$ By zero test, given series is divergent for $x = 1$.

* Cauchy's Integral test :-

Let $f(x)$ be a non-Negative Decresing & Integrable function on $[1, \infty]$.

Let $I_n = \int_1^n f(x)\,dx$

→ If $\lim_{n\to\infty} I_n$ exist & finite then

series $\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + f(3) + \dots$

is convergent

otherwise

Divergent

Note The series sum is always lies between

$I = \lim_{n\to\infty} I_n$ and $I + f(1)$

$(I_n, I_n + f(1))$

lies between

Test the convergent of following series

(1) $\sum_{n=1}^{\infty} \frac{1}{n(\log n)^2}$

sol^n.. we are given $\sum_{n=2}^{\alpha} \frac{1}{n(\log n)^2}$

$$a_n = \frac{1}{n(\log n)^2} \quad wh\ n \geq 2$$

let $f(x) = \frac{1}{x(\log x)^2}$

$$= (\log x)^{-2} \cdot \frac{1}{x} \qquad x \geq 2$$

$$= \cancel{\frac{(\log x)^{-2+1}}{(-1)}}$$

$$= \cancel{-\frac{1}{\log x}}$$

Here we can observed that $f(x)$ is non negative Decresing & integrable On $[2, \alpha]$ now $I_n = \int_2^n (\log x)^{-2} \cdot \frac{1}{x}$

$$= \left[\frac{(\log x)^{-1}}{(-1)}\right]_2^n$$

$$= -\left[\frac{1}{(\log x)}\right]_2^n$$

$$= -\left[\frac{1}{(\log n)} - \frac{1}{(\log(2))}\right]$$

$$= -\left[\frac{\log(2)}{\log(n)}\right]$$

$$\therefore \lim_{n\to\infty} I_n = \lim_{n\to\infty} -\left[\frac{1}{\log(n)} - \frac{1}{\log(2)}\right]$$

$\log(\infty) = \infty$

$$= -\left[\frac{1}{\log(\infty)} - \frac{1}{\log 2}\right]$$

$\frac{1}{\infty} = 0$

$$= -\left[-\frac{1}{\log(2)}\right]$$

$$= \frac{1}{\log(2)}$$ which is finite

$\therefore$ By ~~Chauchy's~~ Cauchy's Integral Test given Series is convergent

(3) $\sum_{n=1}^{\infty} n . e^{-n^2}$

Sol^n:- $\sum_{n=1}^{\infty} n . e^{-n^2}$

Here $a_n = n . e^{-n^2}$, $n \geq 1$

Let, $f(x) = x . e^{-x^2}$, $x \geq 1$

Here we can observed that $f(x)$ is non-negative, decresing & Integral on $[1, \alpha)$ now,

$$I_n = \int_1^n x . e^{-x^2}$$

let's take $x^2 = t$
$2x dx = dt$
$x dx = dt/2$

if $x = 1 \Rightarrow t = 1$
$x = n \Rightarrow t = n^2$ } Limit

$$I_t = \int_1^{n^2} e^{-t} \cdot dt/2$$

$$= \frac{1}{2} \int_1^{n^2} e^{-t} . dt \Rightarrow \frac{1}{2} \left[\frac{e^{-t}}{(-1)} \right]_1^{n^2}$$

$$= -\frac{1}{2}\left[e^{-t}\right]_{1}^{n^2}$$

$$I_n = -\frac{1}{2}\left[e^{-n^2} - e^{-1}\right]$$

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} -\frac{1}{2}\left[e^{-n^2} - e^{-1}\right]$$

$$= -\frac{1}{2}\left[0 - \frac{1}{e}\right]$$

$$= \frac{1}{2e}$$ which is finite

$\therefore$ By ~~cauchy's~~ cauchy's Integral Test given Series is convergent.

H.W

(1) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Alternating series :-

A series is said to be an Alternating series if terms of series are alternatively possitive & Negative

The genral form of an alternating series is $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, whr $a_n > 0$

for e.g. $1-2+3-4+5-6+\ldots$

for the conversion of alternating series we have the following test

[8] Lebinitz's test

statement :- an alternating series

$$\sum_{n=1}^{\infty} (-1)^{n+1} \cdot a_n \quad \text{wh } a_n > 0$$

is conversion if it satishfied following condn convergent if

(i) each term is Numerically less than it's preceding term

i.e $a_1 > a_2 > a_3 > a_4 \ldots a_n > a_{n+1} \ldots$

e.g $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots$

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(ii) $\lim_{n \to \infty} a_n = 0$

Note: if in this case if $\lim_{n \to \infty} a_n \neq 0$ then series is oscilatting series

2 To prove $cond^n$ no. 1. It is sufficient to prove that

$$a_n > a_{n+1} \text{ for all } n$$

i.e $a_n - a_{n+1} > 0$ for all n

Example

test for convergent series

(1) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

we are given $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

$= 1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \ldots$ which is an alternating series thearfore

$a_n = \frac{1}{\sqrt{n}}$ Then $a_{n+1} = \frac{1}{\sqrt{n+1}}$

we know that

$$\sqrt{n} < \sqrt{n+1}$$

$$\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n+1}}$$

$$a_n > a_{n+1}$$

∴ each term is Numerically less than it's preceding term

Now,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}}$$

$$= \frac{1}{\sqrt{(\infty)+1}}$$

$$= 0$$

∴ By Lebinitz's test Given series is convergent series

(2) $$\frac{x}{1+x} - \frac{x^2}{(1+x)^2} + \frac{x^3}{(1+x)^3} - \frac{x^4}{(1+x)^4}$$

whr $0 < x < 1$

We are given

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{(1+x)^n}$$

which is an Alternating series

Then

$$a_n = \frac{x^n}{(1+x)^n} \quad \text{Then}$$

$$a_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$$

Here to prove cond^n (1) we try to prove that

$$a_n - a_{n+1} > 0 \quad \text{for all } n$$

Consider $a_n - a_{n+1}$

$$= \frac{x^n}{1+x^n} - \frac{x^{n+1}}{1+x^{n+1}}$$

$$= \frac{x^n + x^{2n+1} - x^{n+1} - x^{2n+1}}{(1+x^n)(1+x^{n+1})}$$

$$= \frac{x^n - x^{n+1}}{(1+x^n)(1+x^{n+1})}$$

$0 < x < 1$

$(1-x)$

$$= \frac{x^n(1-x)}{(1+x^n)(1+x^{n+1})} > 0$$

$$\lim_{n\to\infty} x^n = 0 , \ |x|<1$$

which is greter than 0 which is $0 < x < 1$

$$a_n - a_{n+1} > 0$$

$\therefore$ $a_n > a_{n+1}$, for all n

$\therefore$ Each term is Numerically less than it's preceding term

Now,

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{x^n}{1+x^n}$$

$$= \frac{\lim\limits_{x\to\infty} x^n}{\lim\limits_{x\to\infty} 1+x^n} = \frac{0}{1+0}$$

$$= 0$$

$\therefore$ By Lebinitz's test given series is convergent

H.W

(2) $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$ convergent

(3) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{n}{2n-1}$ oscilatting