12/08/19.

3. Partial Differential Equation.

Formation of PDE :

Q.1) $(x-a)^2 + (y-b)^2 + z^2 = R^2$.

partially derivative w.r.t x.

Solⁿ: $2(x-a) + 0 + 2z \cdot \frac{\partial z}{\partial x} = 0$.

$$x - a = -z \frac{\partial z}{\partial x}.$$

partially derivative w.r.t. y.

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$y - b = -z \cdot \frac{\partial z}{\partial y}$$

let $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$.

So $x - a = -pz$

$y - b = -qz$.

Substituting values in eqⁿ.

$$p^2z^2 + q^2z^2 + z^2 = R^2$$

$$\boxed{z^2 = \frac{R^2}{p^2+q^2+z^2}}.$$

Q.2) Construct P.D.E by eliminating arbitary constant for the given relation.

$z = (x^2+a)(y^2+b).$

Solⁿ: derivative w.r.t. x.

$$\frac{\partial z}{\partial x} = (x^2+a)(0) + (y^2+b)(2x).$$

Similarly

$$\frac{\partial z}{\partial y} = (x^2+a)\cdot(2y) + (y^2+b)(0):$$

let $\frac{\partial z}{\partial x} = p$ $\quad \frac{\partial z}{\partial y} = q.$

$$(y^2+b)(2x) = p$$
$$(x^2+a)(2y) = q.$$

So $z = \frac{q}{2y} \cdot \frac{p}{2x}$

$$z = \frac{1}{4}\left(\frac{qp}{xy}\right).$$

$$\therefore \boxed{4xyz = pq.}$$

Q.3) Find the PDE by eliminating arbitrary funⁿ:

$$z = f(x^2-y^2)$$

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Sol^n:
$$\frac{\partial z}{\partial x} = f'(x^2-y^2)\cdot 2x \quad \longrightarrow (1)$$

$$\frac{\partial z}{\partial y} = -f'(x^2-y^2)\, 2y \quad \longrightarrow (2)$$

divide eq^n (1) by (2)

$$\frac{\left(\frac{\partial z}{\partial x}\right)}{\left(\frac{\partial z}{\partial y}\right)} = -\frac{x}{y}$$

$\therefore$ let $\frac{\partial z}{\partial x} = p \qquad \frac{\partial z}{\partial y} = q$

$$\therefore \quad \frac{p}{q} = -\frac{x}{y}$$

$$\therefore \quad \boxed{py + qx = 0}$$

Q.4) Find PDE

$$z = f y(x) + g x(y)$$

Sol^n:
$$\frac{\partial z}{\partial x} = y f'(x) + g(y)$$

$$\frac{\partial z}{\partial y} = f(x) + x g'(y)$$

let $\frac{\partial z}{\partial x} = p \qquad \frac{\partial z}{\partial y} = q$

$p = y f'(x) + g(y) \quad q = f(x) + x g'(y).$

Now again derivating w.r.t x & y.

$$\frac{\partial^2 z}{\partial x^2} = y f''(x) + \cancel{g(y)}$$

$$\frac{\partial^2 z}{\partial y^2} = \cancel{f(x)} + x g''(y).$$

$$\frac{\partial^2 z}{\partial x \partial y} = \cancel{g''(y)}.\ f'(x) + g'(y)$$

multiplying $\frac{\partial z}{\partial x}$ by x & $\frac{\partial z}{\partial y}$ by y,

$$px + qy = x\left[y f'(x) + g(y)\right] + y\left[f(x) + x g'(y)\right]$$

$$= \left[g x(y) + y f(x)\right] + xy\left(\frac{\partial^2 z}{\partial x \partial y}\right)$$

$$\boxed{px + qy = z + xy\left(\frac{\partial^2 z}{\partial x \partial y}\right)}$$

Q.1) $2z = (ax + y)^2 + b$

Q.2) $z = ax + a^2y^2 + b$

Q.3) $x^2 + y^2 + (z - c)^2 = a^2$.

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① $2z = (ax+y)^2 + b$

Sol^n:

$\cancel{2}\frac{\partial z}{\partial x} = \cancel{2}(ax+y)\cdot(a)$

$\cancel{2}\frac{\partial z}{\partial y} = \cancel{2}(ax+y)$

$p = a(ax+y)$

$q = ax+y$ $\qquad \therefore \; a = \frac{q-y}{x}$.

$a = \frac{p}{q}$.

$\therefore \; \boxed{p = aq}$.

$\therefore \; p = \left(\frac{q-y}{x}\right)q$.

$\therefore \; px = q^2 - qy$

$\therefore \; \boxed{px - q^2 + qy = 0}$ //.

Q.2) $z = ax + a^2y^2 + b$.

Sol^n:

$\frac{\partial z}{\partial x} = a + 0 + 0$

$\frac{\partial z}{\partial y} = 0 + 2a^2y + 0$.

$p = a$.

So $\boxed{q = 2p^2y}$

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Q.3) $x^2 + y^2 + (z-c)^2 = a^2$.

Sol^n: $\frac{\partial z}{\partial x} = 2x + 2(z-c) \cdot \frac{\partial z}{\partial x} = 0$

$\frac{\partial z}{\partial y} = 2y + 2(z-c) \frac{\partial z}{\partial y} = 0$

$$\frac{p}{q} = \frac{x + (z-c)p}{y + (z-c)q}$$

$$py + pq\cancel{(z-c)} = xq + \cancel{(z-c)pq}$$

$\therefore$ $\boxed{py - qx = 0}$.

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Q.1). $z = xy + f(x^2+y^2)$

Sol^n: $\cancel{\frac{\partial z}{\partial x}} = y \cancel{+} \cancel{f'}(x^2+y^2) \cdot (2x)$.

let us suppose $x^2 + y^2 = u$.

So $z = xy + f(u)$. $\longrightarrow$ (1)

differentiate w.r.t. x

$$\frac{\partial z}{\partial x} = y + f'(u) \cdot \frac{\partial u}{\partial x}.$$

$$p = y + f'(u) \cdot (2x). \longrightarrow (2)$$

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differentiate w.r.t. y

$$\frac{\partial z}{\partial y} = x + f'(u)\frac{\partial u}{\partial y}$$

$$q = x + f'(u)\, 2y \longrightarrow (3)$$

Comparing eq^n (2) & (3)

$$f'(u) = \frac{p-y}{2x} \qquad f'(u) = \frac{q-x}{2y}$$

$$\frac{p-y}{\cancel{2}x} = \frac{q-x}{\cancel{2}y}$$

$$py - y^2 = qx - x^2.$$

$$\therefore \boxed{py - qx - y^2 + x^2 = 0.}$$

Q.2) $z = f(x + at) + g(x - at).$

$\cancel{\frac{\partial z}{\partial x} = f'(x+a}$

let us suppose $\quad x + at = u$

$z = f(u) + g(v).$ $\qquad x - at = v$

So $\frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x} + g'(v)\frac{\partial v}{\partial x}.$

$$p = f'(u) + g'(v). \longrightarrow (1)$$

$$\frac{\partial z}{\partial t} = f'(u)\cdot\frac{\partial u}{\partial t} + g'(v)\cdot\frac{\partial v}{\partial t}.$$

$$\frac{\partial z}{\partial t} = f'(u)\cdot a + g'(v)\cdot -a. \longrightarrow (2)$$

multiplying eq^n (1) with a & add with eq^n (2)

$$a\frac{\partial z}{\partial x} = af'(u) + a\,g'(v).$$

$$\frac{\partial z}{\partial t} = af'(u) - a\,g'(v).$$

$$a\frac{\partial z}{\partial x} - \frac{\partial z}{\partial t} = 2a\,g'(v)$$

$$a\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} = 2a\,f'(u)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(u) + g''(v) \longrightarrow (3)$$

$$\frac{\partial^2 z}{\partial t^2} = a^2 f''(u) + a^2 g''(v). \longrightarrow (4)$$

differentiate eq^n (1) w.r.t t.

$$\frac{\partial^2 z}{\partial x\,\partial t} = f''(u)\cdot\frac{\partial u}{\partial t} + g''(v)\,\frac{\partial v}{\partial t}. \longrightarrow (5)$$

From eq^n (4)

$$\frac{\partial^2 z}{\partial t^2} = a^2\left[f''(u) + g''(v)\right]$$

$$\boxed{\frac{\partial^2 z}{\partial t^2} = a^2\cdot\frac{\partial^2 z}{\partial x^2}.}$$

$f(x^2+y^2,\ z-xy)$

$z = y^2 + 2g(1/x + \log y)$.

Q.3). $z = f\left(\frac{y}{x}\right)$

Sol^n: let us take $\frac{y}{x} = u$.

So $z = f(u) \longrightarrow (1)$

$$\frac{\partial z}{\partial x} = f'(u) \cdot \frac{\partial u}{\partial x}$$

$$p = f'(u) \cdot \left(-\frac{y}{x^2}\right). \longrightarrow (2)$$

$$\frac{\partial z}{\partial y} = f'(u) \cdot \frac{\partial u}{\partial y}$$

$$q = f'(u) \cdot \left(\frac{1}{x}\right) \longrightarrow (3)$$

From eq^n (2) & (3)

$$-\frac{p x^{\cancel{2}}}{y} = q \cancel{x}.$$

$$\therefore \boxed{px + qy = 0.}$$

Q. 4). $z = x f(y) + y g(x)$

Sol^n: $\frac{\partial z}{\partial x} =$

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Q.5). $f(x^2+y^2,\ z-xy) = 0.$

Sol^n: let $x^2+y^2 = u$

$z-xy = v.$

$f(u,v) = 0$

$df = 0.$

Now differentiate w.r.t. x.

$\cancel{\frac{\partial f}{\partial x}} = \cancel{f'(u)\cdot\frac{\partial u}{\partial x}} \cancel{+ f'(v)}$ $\left[df = \frac{\partial f}{\partial u}\cdot du + \frac{\partial f}{\partial v}\cdot dv.\right]$

$$df = \frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x} = 0.$$

$$= \frac{\partial f}{\partial u}[2x] + \frac{\partial f}{\partial v}\cdot\left[\frac{\partial z}{\partial x} - y\right] = 0$$

$$\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y} = 0$$

$$= \frac{\partial f}{\partial u}\cdot[2y] + \frac{\partial f}{\partial v}\left[\frac{\partial z}{\partial y} - x\right] = 0.$$

$$2x\frac{\partial f}{\partial u} + (p-y)\frac{\partial f}{\partial v} = 0 \longrightarrow (2)$$

$$2y\frac{\partial f}{\partial u} + (q-x)\frac{\partial f}{\partial v} = 0 \longrightarrow (3)$$

Now, taking ratio.

$$\frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = \frac{\cancel{f}\,(p-y)}{\cancel{p}x} \times \frac{\cancel{p}y}{\cancel{f}\,(q-x)}$$

$$\therefore \quad \frac{\frac{\partial f}{\partial u}}{\frac{\partial f}{\partial v}} = \frac{p-y}{q-x} \qquad \frac{y-p}{\cancel{p}x} = \frac{x-q}{\cancel{p}y}$$

$$\therefore \quad \cancel{(q-x)\frac{\partial f}{\partial u} = (p-y)\frac{\partial f}{\partial v}}.$$

$$y^2 - py = x^2 - qx.$$

$$\therefore \quad \boxed{x^2 - y^2 - qx + py = 0}$$

Q. 6). $z = y^2 + 2g\left(\frac{1}{x} + \log y\right)$

$\frac{1}{x} + \log y = u.$

Sol^n:

$$\frac{\partial z}{\partial x} = 2g'(u) \cdot \frac{\partial u}{\partial x}$$

$$p = 2g'(u) \cdot \left(\frac{-1}{x^2}\right).$$

$$\frac{\partial z}{\partial y} = 2y + 2g'(u) \cdot \frac{\partial u}{\partial y}$$

$$q = 2y + 2g'(u) \cdot \frac{1}{y}.$$

Now, $2g'(u) = -px^2$

$2g'(u) = (q-2y)y.$

$\cancel{-px^2 = 2y^2 - qy}$

$-px^2 = (q-2y)y$

$$\boxed{px^2 + qy = 2y^2.}$$

6/3/19

Non-linear PDE of 1st order :

TYPE - I (Eq.^{n} involving only p & q)

The PDE of the form $f(p,q) = 0$
has sol^n $z = ax + by + c$
with $f(a,b) = 0$

TYPE - II (Clairaut's form)

The PDE of form
$z = px + qy + f(p,q)$ is called clairaut's form of pde, it's sol^n is given by
$z = ax + by + f(a,b).$

Q.1) $p^2 - q = 1$

Sol^n: Complete sol^n is given by

$$z = ax + by + c$$

with $a^2 - b = 1$

Find any one value either a/b. Substitute.

$$\therefore b = a^2 - 1.$$

$$\boxed{z = ax + (a^2 - 1)y + c.}$$

Q.2) $pq + p + q = 0.$

Sol^n:

$$z = ax + by + c$$

with

$$ab + a + b = 0$$

$$ab + a = -b$$

$$a(b+1) = -b$$

$$a = \left(\frac{-b}{b+1}\right) \quad \Big/ \quad b = \frac{-a}{a+1}$$

$$\boxed{z = \left(\frac{-b}{b+1}\right)x + by + c.}$$

Q.3) $z = px + qy + \sqrt{1+p^2+q^2}$

Sol^n: $z = ax + by + \sqrt{1+a^2+b^2}$

Q.4) $pq = p+q$

Sol^n: $z = ax + by + c$

$ab = a+b$

$\cancel{b = \frac{a+b}{a}}$

$\cancel{b = 1+} b$

$ab - a = b$

$a(b-1) = b$

$a = \frac{b}{b-1}$

$$\boxed{z = \left(\frac{b}{b-1}\right)x + by + c.}$$

Q.5). $pq = 1$

Sol^n: $z = ax + by + c.$

$ab = 1$

$b = \frac{1}{a}$

$\therefore$ $\boxed{z = ax + \frac{y}{a} + c.}$

Q.6). $\sqrt{p} + \sqrt{q} = 1$

Sol^n: $z = ax + by + c.$

$\cancel{2ab +} a^2 \cancel{+ b^2 = 1}.$

$\cancel{b^2 = 1} - a^2$

$\sqrt{a} + \sqrt{b} = 1$

$\sqrt{a} = 1 - \sqrt{b}$

$b^2 = 1 - a^2 - a_2$

$b = \sqrt{1-a^2}.$

$a^2 = \cancel{1} 1 - 2\sqrt{b} + b.$

So, $z = ax + \sqrt{1-a^2}\, y + c.$

Q.7). $p^2+q^2 = npq.$ $a=1,\ b=-nb\ \ c=b^2.$

Sol^n: $z = ax+by+c.$

$a^2+b^2 = nab.$ $\Delta = b^2-4ac$

$= n^2b^2 - 4b^2$

$a^2 - nab + b^2 = 0$ $= (n^2-4)b^2.$

$\cancel{a(a-nb) = b^2}$ $a = \dfrac{nb \pm b\sqrt{n^2-4}}{2}.$

So $\boxed{z = \left(\dfrac{nb \pm b\sqrt{n^2-4}}{2}\right)x + by + c.}$

Q.8). $z = px+qy - 2\sqrt{pq}.$

Sol^n: $z = ax+by - 2\sqrt{ab}.$

Q.9). $(p+q)(z-px-yq) = 1.$

$\Rightarrow z - px - qy = 1/p+q.$

Sol^n: $\cancel{pz - p^2x - pqy + qz - pqx - q^2y = 1.}$

$z = ax+by+\dfrac{1}{a+b}$

$\Rightarrow z = px+qy+\dfrac{1}{p+q}$

$\Rightarrow \cancel{az - a^2x - aby + bz - abx - b^2y = 1.}$

$\cancel{z = \dfrac{1+a^2x+aby+abx+b^2y}{(a+b)}.}$

Q.10). $4xyz = pq + 2px^2y + 2qxy^2.$

Sol^n: $z = \dfrac{pq + 2px^2y + 2qxy^2}{4xy}$

Not clairaut

$z = \dfrac{pq}{4xy} + \dfrac{1}{2}px + \dfrac{1}{2}qy.$

$$\cancel{z = \frac{ab}{4xy} + \frac{ax}{2} + \frac{by}{2}}$$

Q.11). $(1-x)p + (2-y)q = 3-z.$

Sol^n: $\cancel{(1-x)p + (2-y)q}$

$z = 3 - (1-x)p + (2-y)q$

$z = 3 - (1-x)a + (2-y)b.$

$$\boxed{z = 3 - a + ax + 2b - by}$$