Chapter 1

Function

Definition 1.0.1. *Let A and B be two non-empty sets. A function f from A to B is a set of ordered pairs*

$$f \subseteq A \times B$$

with property that for each element x in A there is a unique element y in B such that $(x, y) \in f$. The statement "f is a function from A to B" is usually represented symbolically by

$$f : A \to B.$$

If a function from A to B. Then A is called **domain** of f denoted by *dom* f, its members are the first co-ordinates of the ordered pairs belonging to f and the set B is called **co-domain**. If $(x, y) \in f$, it is customary to write $y = f(x)$, y is called the **image** of x; and x is a **pre-image** of y. The set consisting of all the images of the elements of A under the function f is called the **range** of f. It is denoted by $f(A)$.

$$\text{The range of } f = \{f(x) : \text{ for all } x \in A\}.$$

For example

1. $f(x) = x^2$ for $x \in \mathbb{R}$ represents a function from $\mathbb{R}$ to $\mathbb{R}$.
2. $f = \{(1, 1), (2, 3), (3, 5), (3, 7), (4, 7), (5, 12)\}$, then f is not a function because $(3, 5)$ and $(3, 7)$ have same first component.

1.0.1 Classification of Functions

Function can be classified mainly into two groups.

(1) **Algebraic function:** A function which consist of a finite number of terms of involving powers and roots of the independent variable x and the four fundamental operations of addition, subtraction, multiplication and division is called algebraic function. Three particular cases of algebraic functions are:

(i) **Polynomial function:** A function of the form $a_0x^n + a_1x^{n-1} + ... + a_n$, where n is positive integer and $a_0, a_1, a_2, ..., a_n$ are real constants and $a_0 \neq 0$ is called a polynomial of x in degree n e.g. $f(x) = x^2 + x + 5$ is a polynomial of degree 2.

(ii) **Rational function:** A function of the form $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials in $x, g(x) \neq 0$ is called a rational function, e.g. $F(x) = \frac{x^3-x+7}{x+4}$.

(iii) **Irrational function:** A function involving radicals are called irrational functions. $f(x) = \sqrt[x]{x} + x + 5$ is an irrational function.

(2) **Transcendental function:** A function which is not algebraic is called Transcendental function.

(i) **Trigonometric function:** The six functions $\sin\ x, \cos\ x, \tan\ x, \cot\ x$, $cosec\ x$ and $\sec\ x$, where x is angle measured in radians are called trigonometric function.

(ii) **Inverse trigonometric function:** The six functions $\sin^{-1}\ x, \cos^{-1}\ x, \tan^{-1}\ x, \cot^{-1}\ x, cosec^{-1}\ x$ and $\sec^{-1}\ x$ are called trigonometric function.

(iii) **Exponential function:** A function $f(x) = a^x (a > 0)$ satisfying the law $a^1 = a$ and $a^x \cdot a^y = a^{x+y}$ is called the exponential function.

(iv) **Logarithmic function:** The inverse of the exponential function is called the logarithmic function. So, if $y = a^x (a > 0, a = 1, x \in \mathbb{R}, y > 0)$ then $x = \log_a(y)$ is called Logarithmic function.

1.0.2 Types of Functions

The functions can be different types.

(1) **One-to-One Function:** A function f from A to B is one-to-one or injective, if for all elements x_1, x_2 in A such that $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
e.g. Let $A = \{1, 2, 3\}, B = \{a, b, c, d\}$ and let $f(1) = a, f(2) = c$ and $f(3) = d$. Then f is injective since the different elements 1,2,3 in A are assign to the different elements a, c, d respectively in B.

(2) **Many-One Function:** A function f from A to B is said to be many-one if and only if two or more elements of A have same image in B. e.g. Let $f(x) = x^2$, x is any real number and $f : \mathbb{R} \to \mathbb{R}$. Then f is many-one function.

(3) **Into Function:** A function f from A to B is said to be Into if and only if there exist at least one element in B which is not image of any element in A.
e.g. Let $A = \{1, 2, 3\}, B = \{a, b, c, d\}$ and let $f(1) = a, f(2) = c$ and $f(3) = d$. Then f is into since b is not an image of any element in A.

(4) **Onto Function:** A function f from A to B is said to be onto(surjective) if and only if every element of B is the image of some element in A.
e.g. Let $f(x) = x$, x is any real number and $f : \mathbb{R} \to \mathbb{R}$. Then f is onto function.

(5) **Bijective Function:** A function f from A to B is said to be bijective if and only if f is both one-to-one and onto.

Problems

1.1 Limit of a Function

Definition 1.1.1. *The function approaches $l(l \in \mathbb{R})$ as x approaches a from either sides if given $\epsilon > 0$ there exist $\delta > 0$ such that*

$$|f(x) - l| < \epsilon \text{ for } 0 < |x - a| < \delta.$$

We write, in this case

$$\lim_{x \to a} f(x) = l.$$

1.1.1 Precise Definitions of One-sided Limits

Definition 1.1.2. *The function f approaches l as x approaches a from left side if given $\epsilon > 0$ there exist $\delta > 0$ such that*

$$|f(x) - l| < \epsilon \text{ for } a - \delta < |x - a| < \delta.$$

We write, in this case

$$\lim_{x \to a-} f(x) = l.$$

Definition 1.1.3. *The function f approaches l as x approaches a from right side if given $\epsilon > 0$ there exist $\delta > 0$ such that*

$$|f(x) - l| < \epsilon \text{ for } a < |x - a| < a + \delta.$$

We write, in this case

$$\lim_{x \to a+} f(x) = l.$$

1.2 Continuity of Function

Definition 1.2.1. *A function $f(x)$ is said to be **continuous** at $x = a$ if the following conditions are satisfied*

(i) $f(a)$ *is defined*

(ii) $\lim_{x \to a+} f(x) = f(a)$ *(Right-continuous)*

(iii) $\lim_{x \to a-} f(x) = f(a)$ *(Left-continuous)*

(iv) $\lim_{x \to a} f(x) = f(a)$

*If f does not satisfies one or more of the above conditions, then f is said to be **discontinuous** at $x = a$, and a is called a point of discontinuity.*

Chapter 2

Derivative of a function

Definition 2.0.1. *Let a function $f(x)$ be defined on a closed interval [a,b], then the function $f(x)$ is said to be differentiable at x if*

$$\lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

exists finitely . The limit is called the derivative of $f(x)$ with respect to x and its denoted by $f'(x)$ or $\frac{df}{dx}$

2.0.1 Derivative of Some Standard Functions

Formula 2.0.1. *If k is constant, then* $\frac{d}{dx}(k) = 0$

Formula 2.0.2. $\frac{d}{dx}(x^n) = nx^{n-1}$

Formula 2.0.3. $\frac{d}{dx}(a^{mx}) = ma^{mx} \log_e a$

Formula 2.0.4. $\frac{d}{dx}(e^{ax}) = ae^{ax}$

Formula 2.0.5. $\frac{d}{dx}(\log(ax+b)) = \frac{a}{ax+b}$

Formula 2.0.6. $\frac{d}{dx}(\sin(bx+c)) = b\cos(bx+c)$

Formula 2.0.7. $\frac{d}{dx}(\cos(bx+c)) = -b\sin(bx+c)$

Formula 2.0.8. $\frac{d}{dx}(\tan x) = \sec^2 x$

Formula 2.0.9. $\frac{d}{dx}(\cot x) = -cosec^2 x$

Formula 2.0.10. $\frac{d}{dx}(\sec x) = \sec x \tan x$

Formula 2.0.11. $\frac{d}{dx}(cosecx) = -cosec\, x\ \cot x$

2.0.2 Linearity Properties, Product rule, Quotient rule

Linearity Properties: *Let f and g be two differentiable functions of x and α be any scaler. Then*

(1) $\frac{d}{dx}(f+g) = \frac{df}{dx} + \frac{dg}{dx}$

(2) $\frac{d}{dx}(\alpha f) = \alpha \frac{df}{dx}$

Product Rule of Differentiation: *Let f and g be two differentiable functions of x. Then*

$$\frac{d}{dx}(fg) = f\frac{dg}{dx} + g\frac{df}{dx}$$

Quotient Rule for Differentiation: *Let f and g be two differentiable functions of x. Then*

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

2.0.3 Derivative of Composite Function and Implicit Function

Let $y = f(t)$ and $t = g(x)$, where f and g are both differentiable functions. Then $y = f(g(x)) = f \circ g(x)$ is a function of x. Hence

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

This is known as the ***chain rule****.*

Definition 2.0.2. *When an equation is written in the form $y = f(x)$ it is said to be an* ***explicit function*** *of x.*

A simple rule of differentiating an implicit function is summarized as:

$$\frac{d}{dx}[f(y)] = \frac{d}{dy}[f(y)] \times \frac{dy}{dx}$$

2.1 Higher order Derivatives

*If $y = f(x)$ is a differential function, then its **first order derivative** w.r.t x is denoted by $\frac{dy}{dx}$ or $\frac{d}{dx}[f(x)]$ or y' or Dy*

*The derivative of $\frac{dy}{dx}$ w.r.t x is called the **second order derivative** of y w.r.t x and is denoted by $\frac{d^2y}{dx^2}$ or $\frac{d^2}{dx^2}[f(x)]$ or y'' or D^2y*

*In general, the n^{th} **order derivative** of y w.r.t x is denoted by $\frac{d^ny}{dx^n}$ or $\frac{d^n}{dx^n}[f(x)]$ or y_n or D^ny*

2.1.1 Some standard results on n^{th} derivative

Formula 2.1.1. *If $y = e^{ax+b}$, then $y_n = a^ne^{ax+b}$.*

Formula 2.1.2. *If $y = a^{bx}$, then $y_n = b^na^{bx}(loga)^n$.*

Formula 2.1.3. *If $y = \sin(ax + b)$,then $y_n = a^n \sin(\frac{n\pi}{2} + ax + b)$.*

Formula 2.1.4. *If $y = \cos(ax + b)$, then $y_n = a^n \cos(\frac{n\pi}{2} + ax + b)$.*

Formula 2.1.5. *If $y = e^{ax}\cos(bx+c)$, then $y_n = r^ne^{ax}\cos(bx+c+n\theta)$, wherer $= \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$*

Formula 2.1.6. *If $y = e^{ax}\sin(bx+c)$, then $y_n = r^ne^{ax}\sin(bx+c+n\theta)$, wherer $= \sqrt{a^2 + b^2}$ and $\theta = \tan^{-1}\frac{b}{a}$*

Formula 2.1.7. *If $y = (ax+b)^m$, then $y_n = a^nm(m-1)(m-2)...(m-n+1)(ax+b)^{m-n}$, if $m >= n$*

Note 2.1.1. *If m is a positive integer and $m = n$, then $y_n = n!a^n$*

Note 2.1.2. *If m is a positive integer and $m < n$, then $y_n = 0$*

Note 2.1.3. *If $m = -1$ i.e. if $y = \frac{1}{(ax+b)}$, then $y_n = \frac{(-1)^nn!a^n}{(ax+b)^{n+1}}$.*

Formula 2.1.8. *If* $y = \log(ax+b)$, *then* $y_n = \dfrac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$

Problems

Example 2.1.1. *Find the* n^{th} *derivatives of the following function*

$(i) cos^4 x$ C.W $(ii) sinx sin2x sin3x$ H.W $(iii) sin^2 x sin2x$ H.W $(iv) e^{2x} sin3x cos2x$ C.W $(v) e^{3x} cos^2 x$ H.W

$(vi) \frac{x^3}{(x+1)(x+2)}$ H.W $(vii) \frac{2x-1}{(x-2)(x-3)}$ H.W $(viii) \frac{3x+1}{(x+1)^2(x-2)}$ C.W $(ix) \frac{x^2+4x+1}{x^3+2x^2-x-2}$ H.W Hint $(x) log(4x+3)e^{5x+7}$ C.W

$(xi) e^{2x+4} + 6^{2x+4}$ C.W $(xii) \frac{x}{(x+1)^4}$ H.W

Example 2.1.2. *find the* n^{th} *derivatives of the function* $\frac{x^4}{(x-1)(x-2)}$ → C.W

Solution: $\frac{x^4}{(x-1)(x-2)} = \frac{x^4}{x^2-3x+2} = x^2 + 3x + 7 + \frac{15x-14}{(x-1)(x-2)}$

now, let $\frac{15x-14}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$

$\therefore 15x - 14 = A(x-2) + B(x-1)$

taking x=1, we get 1=-A i.e. A=-1

taking x=2, we get 16=B i.e. B=16.

$\therefore y = x^2 + 3x + 7 + \frac{-1}{(x-1)} + \frac{16}{(x-2)}$

$\therefore y_n = 0 - \frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{16(-1)^n n!}{(x-2)^{n+1}} = (-1)^n n! [\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}]$

Example 2.1.3. *If* $y = \frac{ax+b}{cx+d}$, *then find* y_n Hint H.W

Example 2.1.4. *Find* n^{th} *derivative of function* $y = \frac{1}{(x-1)(x-2)}$ H.W

Example 2.1.5. *If* $y = sinax + cosax$ *prove that* $y_n = a^n[1+(-1)^n sin2ax]^{\frac{1}{2}}$

Hint H.W

sin(x+y) = sinx cosy + cosx siny
sin(x-y) = sinx cosy - cosx siny
cos(x+y) = cosx cosy - sinx siny
cos(x-y) = cosx cosy + sinx siny

2SC = S+S
2CS = S-S
2CC = C+C
-2SS = C-C

Example 2.1.6. *Find* y_n *if* $\frac{x^m - 1}{x-1}$ C.W where ~~n~~ m is fixed.

Example 2.1.7. *Find* y_n *if* $e^{x\cos\theta}\cos(x\sin\theta)$ C.W θ is fixed.

2.2 Leibnitz's Theorem

Statement: *"If f and g are two functions of x possessing derivatives of nth order, then*

$$(fg)_n = \binom{n}{0} f_n g + \binom{n}{1} f_{n-1}g_1 + \binom{n}{2} f_{n-2}g_2 + \cdots + \binom{n}{r} f_{n-r}g_r + \cdots + \binom{n}{n} f g_n, \tag{2.2.0.1}$$

where f_n *and* g_n *are derivative of order n of f and g respectively".*

2.2.1 Some Useful Values

1. $\binom{n}{0} = 1$
2. $\binom{n}{1} = n$
3. $\binom{n}{2} = \frac{n(n-1)}{2\cdot 1}$
4. $\binom{n}{3} = \frac{n(n-1)(n-2)}{3\cdot 2\cdot 1}$
5. $\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-(r-1))}{r(r-1)\cdots 1}$
6. $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
7. $\binom{n}{r} = \binom{n}{n-r}$
8. $\binom{n}{n} = 1$

Problems

Example 2.2.1. *Find* n^{th} *derivative of function* $y = x^2 5^{3x}$. C.W

Solution: *Choose* $f(x) = x^2$ *and* $g(x) = 5^{3x}$.

Then $f_n(x) = 3^n(\log 5)^n 5^{3x}$ *and* $g_1 = 2x, g_2 = 2$.

Then from equation (2.2.0.1), we have

$$(y)_n = 3^n(\log 5)^n 5^{3x} x^2 + n3^{n-1}(\log 5)^{n-1} 5^{3x} 2x + n(n-1)3^{n-2}(\log 5)^{n-2} 5^{3x}.$$

Example 2.2.2. *Find n^{th} derivative of function $y = x^2 \sin(3x + 5)$* H.W

Example 2.2.3. *Find n^{th} derivative of function $y = x^3 \log(7x + 5)$* H.W

Example 2.2.4. *Find n^{th} derivative of function $y = x^2 e^x \cos(4x - 3)$* C.W

Example 2.2.5. *If $y = a\cos(\log x) + b\sin(\log x)$ then show that $x^2y_2 + xy_1 + y = 0$ and $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$* C.W

Solution: *Here* $y = a\cos(\log x) + b\sin(\log x)$

$$\therefore y_1 = -\frac{a\sin(\log x)}{x} + \frac{b\cos(\log x)}{x}$$

$$\therefore xy_1 = -a\sin(\log x) + b\cos(\log x)$$

Differentiating again w.r.t.x, we get

$$xy_2 + y_1 = -\frac{a\cos(\log x)}{x} - \frac{b\sin(\log x)}{x}$$

$$\therefore xy_2 + y_1 = \tfrac{-1}{x}\left[a\cos(\log x) + b\sin(\log x)\right]$$

$$\therefore xy_2 + y_1 = \tfrac{-y}{x}$$

$$\therefore x^2y_2 + xy_1 + y = 0$$

Differentiating n times using Leibnitz's theorem , we get

$$x^2y_{n+2} + n(2x)y_{n+1} + \tfrac{n(n-1)}{2}(2)y_n + xy_{n+1} + n(1)y_n + y_n = 0$$

$\therefore x^2y_2 + xy_1 + y = 0$ *and* $x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$

Example 2.2.6. *If* $y = (x^2-1)^n$ *then prove that* $(x^2-1)y_{n+2} + 2xy_{n+1} - n(n+1)y_n = 0$ H.W

Example 2.2.7. *If* $y = tan^{-1}\left(\frac{a-x}{a+x}\right)$ *then prove that* $(x^2+a^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$ C.W $n>1$

Example 2.2.8. *If* $y = e^{msin^{-1}x}$ *then prove that* $(1-x^2)y_{n+2} - x(2n+1)y_{n+1} - (n^2+m^2)y_n = 0$ C.W

Example 2.2.9. *If* $logy = tan^{-1}x$ *then prove that* $(1+x^2)y_{n+2} + ((2n-2)x-1)y_{n+1} + n(n+1)y_n = 0$ H.W

Example 2.2.10. *If* $y = (sin^{-1}x)^2$ *then show that* $(1-x^2)y_{n+2} - x(2n+1)y_{n+1} - n^2y_n = 0$ C.W

Example 2.2.11. *If* $y = (tan^{-1}x)^2$ *then show that* $(1+x^2)y_2 + 2x(x^2+1)y_1 = 2$ H.W

Example 2.2.12. *If* $y = e^x logx$ *then prove that* $xy_{n+2} + (n+1-x)y_{n+1} - (n+1)y_n = exp(x)$ $n \geqslant 1$ H.W

2.3 Expansion of Function

2.3.1 Power Series

An infinite series in the form $a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$ *where the* a*'s are constants is called a power series in* x .
Similarly ,An infinite series in the form $a_0 + a_1(x-a) + a_2(x-a)^2 + ... + a_n(x-a)^n + ...$ *is called a power series in (x-a).*

2.3.2 Taylor's Series

Statement: *If $f(x)$ possess derivative of all order at aand a is any constant, then f(x) can be expanded into a power series of x*

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + ... + \frac{(x-a)^n}{n!}f^n(a) + ... \quad (2.3.2.1)$$

Note 2.3.1. *Another form of Taylor's Series is*

$$f(x+a) = f(a) + \frac{(x)}{1!}f'(a) + \frac{(x)^2}{2!}f''(a) + ... + \frac{(x)^n}{n!}f^n(a) + ... \quad (2.3.2.2)$$

Example 2.3.1. *Find Taylor's series expansion of $f(x) = x^5 + 2x^4 - x^2 + x + 1$ around $a = -1$.* H.W

Solution: *Since*

$$f(x) = x^5 + 2x^4 - x^2 + x + 1 \Rightarrow f(-1) = 0,$$

$$f'(x) = 5x^4 + 8x^3 - 2x + 1 \Rightarrow f'(-1) = 0.$$

$$f''(x) = 20x^3 + 24x^2 - 2 \Rightarrow f''(-1) = 2.$$

$$f'''(x) = 60x^2 + 48x \Rightarrow f'''(-1) = 12.$$

$$f^{iv}(x) = 120x + 48 \Rightarrow f^{iv}(-1) = -72$$

$$f^{v}(x) = 120 \Rightarrow f^{iv}(-1) = 120$$

Using Taylor's series expansion (2.3.2.2), we get

$$\begin{aligned} x^5 + 2x^4 - x^2 + x + 1 &= f(-1) + \frac{(x+1)}{1!}f'(-1) + \frac{(x+1)^2}{2!}f''(-1) \\ &+ \frac{(x+1)^3}{3!}f'''(-1) + \frac{(x+1)^4}{4!}f^{iv}(-1) + \frac{(x+1)^5}{5!}f^{v}(-1) \\ &= (x+1)^2 + 2(x+1)^3 - 3(x+1)^4 + (x+1)^5 \end{aligned}$$

Problems

Example 2.3.2. *Find Taylor's series expansion of e^x, $sinx$. $cosx$, $sinhx$. $coshx$. $log(1+x)$, $tanx$* c.w at a=0 point i.e Maclaurin's series.

Example 2.3.3. *Find Taylor's series expansion for $log_e cosx$ about the point $\pi/3$. Hence find the approximate value od $log_e cos61$·* c.w

Example 2.3.4. *Expand $tan^{-1}x$ in powers of $(x-\frac{\pi}{4})$* c.w

Example 2.3.5. *Calculate the approximate value of $\sqrt[3]{28}$ correct up to* four ~~for~~ *decimal place by using an approximate Taylor's theorem*

Example 2.3.6. *Arrange $7+(x+2)+3(x+2)^3+(x+2)^4-(x+2)^5$ in powers of x* c.w

Example 2.3.7. *Expand $\frac{1}{\sqrt{x}}$ in series of power of (x-1) and hence show that $\sqrt{2}=1+\frac{1}{4}+\frac{3}{32}+\frac{5}{28}+...$* H.W

Example 2.3.8. *Expand $sin(\frac{\pi}{4}+\theta)$ in powers of θ and hence find the value of $sin46$* c.w

Example 2.3.9. *Expand $logsinx$ in the powers of (x-2)* H.W

2.3.3 Maclaurin's Series

If we put $a=0$ in (2.3.2.2), then we get

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+...+\frac{x^n}{n!}f^n(0)+.... \qquad (2.3.3.1)$$

It is known as Maclaurin's series.

Example 2.3.10. *Find Maclaurin's series of $f(x)=\cos x$.*
Solution: *Since*

$$f(x)=\cos x \Rightarrow f(0)=1.$$

$$f'(x)=-\sin x \Rightarrow f'(0)=0,$$

$$f''(x) = -\cos x \Rightarrow f''(0) = -1.$$

$$f'''(x) = \sin x \Rightarrow f'''(0) = 0.$$

$$f^{iv}(x) = \cos x \Rightarrow f^{iv}(0) = 1$$

Continuing in same way and using (2.3.3.1), we get

$$\begin{aligned} \cos x &= f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + \ldots.. \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots. \end{aligned}$$

Example 2.3.11. *Find Maclaurin's series expansion of* $e^x, \sin x, \cos x$, $\sinh x, \cosh x, \log(1+x), \tan x, \sec x$ repeat

Example 2.3.12. *Using Maclaurin's series, expand* $f(x) = \dfrac{e^x}{e^x+1}$ *as far as term in* x^3 C-W

Example 2.3.13. *Find Maclaurin's series expansion of* $f(x) = \sin x/2$ C-W

Example 2.3.14. *Find Maclaurin's series expansion of*

$$f(x) = \log(1 + \sin x)$$ H-W

Example 2.3.15. *Using Maclaurin's series, expand* C-W

$$f(x) = \sqrt{(1 + \sin x)}$$

2.4 L'hospital's rule TUT

Note 2.4.1. *The indeterminate from* $\frac{0}{0}$
If $f(a) = 0$ *and* $g(a) = 0$, *then* $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$

Note 2.4.2. *The indeterminate from* $\frac{\infty}{\infty}$
If $\lim\limits_{x\to\infty} f(x) = \infty$ *and* $\lim\limits_{x\to\infty} g(x) = \infty$, *this form reduced to* $\frac{0}{0}$

form if we write $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}=\lim\limits_{x\to a}\dfrac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$

Note 2.4.3. *The indeterminate from* $0\times\infty$

If $\lim\limits_{x\to\infty} f(x)=0$ *and* $\lim\limits_{x\to\infty} g(x)=\infty$, *then* $\lim\limits_{x\to a}[f(x)\cdot g(x)]$ *takes the form* $0\times\infty$.

In this we can write $f(x)\cdot g(x)=\dfrac{f(x)}{\frac{1}{g(x)}}$ *in this case the new form is* $\frac{0}{0}$*type and L'Hospital's rule is applicable.*

Note 2.4.4. *The indeterminate from* $\infty-\infty$

$\lim\limits_{x\to\infty} f(x)=\infty$ *and* $\lim\limits_{x\to\infty} g(x)=\infty$,*then* $\lim\limits_{x\to a}[f(x)-g(x)]$ *takes the form* $\infty-\infty$ *In this case, we can write*

$$f(x)-g(x)=\frac{\frac{1}{g(x)}-\frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}$$

in this case the new form is $\frac{0}{0}$*type and L'Hospital's rule is applicable.*

Note 2.4.5. *The indeterminate from* $0^0, \infty^0, 1^\infty$

consider $\lim\limits_{x\to a}[f(x)]^{g(x)}$

when

(i) $\lim f(x)=0$ *and* $\lim g(x)=0$,

(ii)$\lim f(x)=\infty$ *and* $\lim g(x)=0$,

(iii)$\lim f(x)=1$ *and* $\lim g(x)=\infty$,

For the above three cases, we write

$$L=\lim_{x\to a}[f(x)]^{g(x)}$$

$\therefore\ \log L=\lim\limits_{x\to a} g(x)\log f(x)$ *in each of the three cases, the function on the right hand side has the form* $0\times\infty$ *and its limit will be* e^L

Evaluate the following:

$(i) \lim\limits_{x \to 0} \dfrac{2\sin x - sin2x}{x^3}$ $(ii) \lim\limits_{x \to 0} \dfrac{e^x + e^{-x} - x^2 - 2}{\sin^2 x - x^2}$ $(iii) \lim\limits_{x \to a} \dfrac{\log(x-a)}{\log(e^x - e^a)}$

$(iv) \lim\limits_{x \to 0} \log_{\tan x} \tan 2x$ $(v) \lim\limits_{x \to 1}[\frac{x}{x-1} - \frac{1}{\log x}]$ $(vi) \lim\limits_{x \to 0}(\frac{1}{x})^{1-\cos x}$

$(vii) \lim\limits_{x \to 0}(\sec^2 x)^{\cot^2 x}$ $(viii) \lim\limits_{x \to \frac{\pi}{2}}(\cos x)^{\cos x}$ $(ix) \lim\limits_{x \to a} \log(2 - x/a) cot(x-a)$

$(x) \lim\limits_{x \to 0} \dfrac{1 - \cos x + \sin x + \log(1-x)}{x \tan^2 x \cancel{(1+x)}}$ $(xi) \lim\limits_{x \to 0}(\sin x + 1)^{\frac{1}{x}}$

* Limit of a functions of two variables.

Let $f: E \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function where E is an open set containing point (x_0, y_0). (It may happen that function f is not defined at point (x_0, y_0)). Let $l \in \mathbb{R}$. If given $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that $|f(x,y) - l| < \varepsilon$, whenever $0 < |(x,y) - (x_0, y_0)| < \delta$, then l is called the limit of function f at (x_0, y_0), and it is denoted by $\lim_{(x,y) \to (x_0, y_0)} f(x,y) = l$.

e.g.: (1) $\lim_{(x,y) \to (2,3)} 3xy = 18$

~~Here~~ Let $\varepsilon > 0$ be any arbitrary positive number.

Now we want to find $\delta > 0$ such that

$|3x \cdot y - 18| < \varepsilon$ whenever $|(x,y) - (2,3)| < \delta$.

Now $|(x,y) - (2,3)| < \delta$

$\Rightarrow \sqrt{(x-2)^2 + (y-3)^2} < \delta$

$\Rightarrow |x-2| < \delta$ & $|y-3| < \delta$

$\Rightarrow 2-\delta < x < 2+\delta$ & $3-\delta < y < 3+\delta$

This gives $-(15\delta + 3\delta^2) < 3xy - 18 < 15\delta + 3\delta^2$

But we want $|3xy - 18| < \varepsilon$ so take

$15\delta + 3\delta^2 = \varepsilon$

$\Rightarrow 15\delta + 3\delta^2 - \varepsilon = 0$

$\Rightarrow \delta = \dfrac{-15 \pm \sqrt{225 + 12\varepsilon}}{6} = -\dfrac{5}{2} + \sqrt{\varepsilon/3 + \dfrac{25}{4}}$
$(\because \delta > 0)$

Thus if we take $\delta = -\frac{5}{2} + \sqrt{\varepsilon/3 + 25/4}$

then $|3xy - 18| < \varepsilon$ whenever $|(x,y) - (2,3)| < \delta$.

Therefore $\lim_{(x,y)\to(2,3)} f(x,y) = 3xy = 18$.

<2> Let $f: \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ defined as

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}, \quad (x,y) \neq (0,0).$$

Then show that limit of f at $(0,0)$ doesn't exist.

<u>Thm:</u> Let a function $y = \phi(x)$ is continuous at point $(a, \phi(a)) = (a, b)$ and $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and is equal to $l \in \mathbb{R}$, then $\lim_{x\to a} f(x, \phi(x))$ exists and equal to l.

<u>Ex:</u> (i) $\lim_{(x,y)\to(0,0)} \tan^{-1}(y/x) = ?, \quad (x,y) \neq (0,0)$

(ii) $\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2 + y^2}, \quad (x,y) \neq (0,0)$

* <u>Continuity:</u>

Let $f: E \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a function and $(a,b) \in E$. be any fixed point. The function f is said to be continuous at point (a,b) if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Ex! Discuss the continuity of the following functions.

(i) $f(x, y) = \tan^{-1}(y/x)$, $x \neq 0$

$= 0$, $x = 0$.

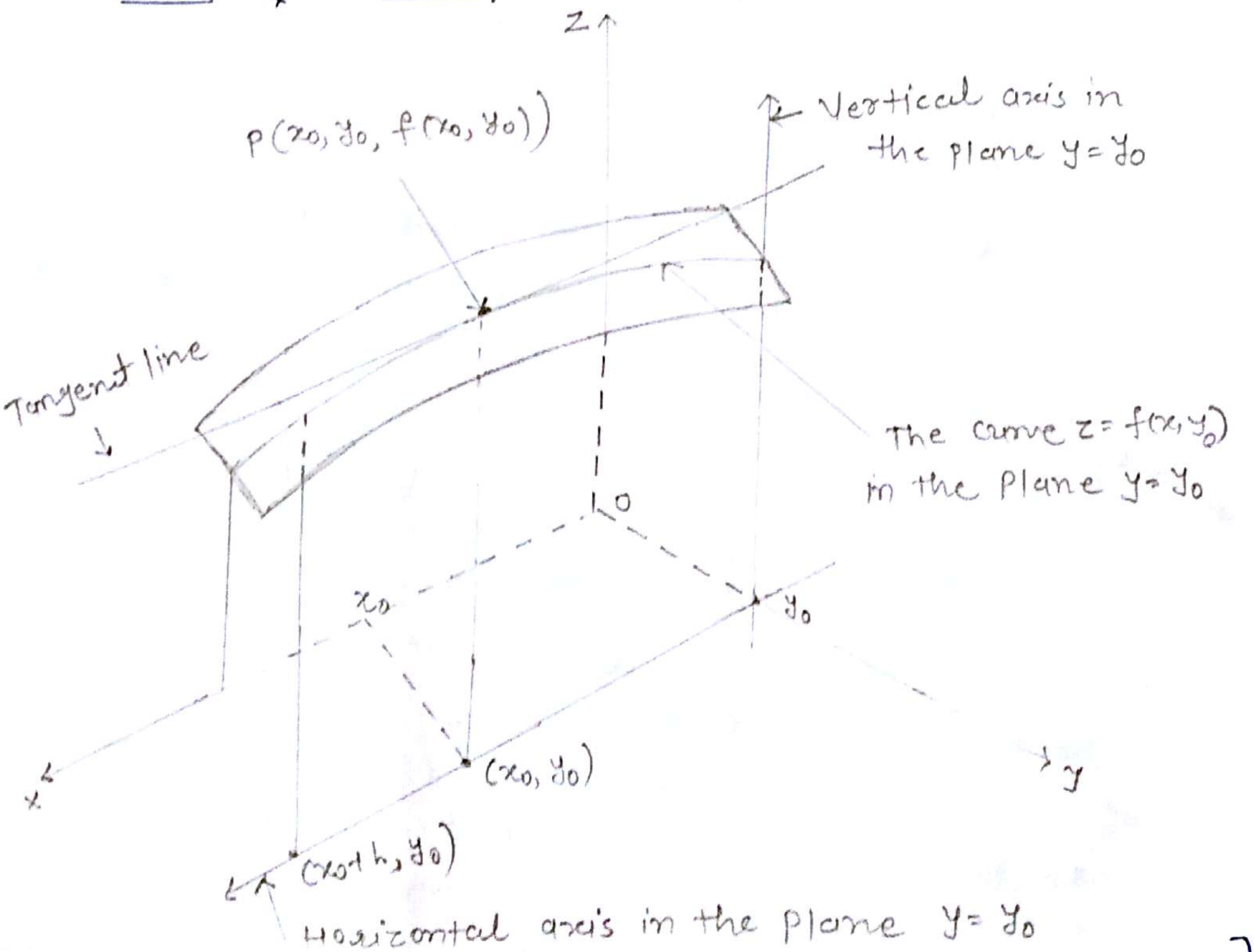
(Discuss the continuity at point $(0,1)$).

(ii) $f(x, y) = \dfrac{x^2 - y^2}{x + y}$, $(x, y) \neq (0,0)$

$= 0$ if $(x, y) = (0,0)$.

(Discuss the continuity at point $(0,0)$.).

* Partial derivative of a Function of Two variables



[The intersection of the Plane $y = y_0$ with the surface $z = f(x, y)$]

Let (x_0, y_0) be a point in the domain of function f. Then the graph of the function is the surface $z = f(x, y)$. The vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x_0, y_0)$ (see the above figure). The curve is the graph of the function $z = f(x, y_0)$ in the plane $y = y_0$. The horizontal coordinate in this plane is x; the vertical coordinate is z; and the variable y is constant with the value $y = y_0$.

We define the partial derivative of f with respect to x at the point (x_0, y_0) as the ordinary derivative of the curve $z = f(x, y_0)$ with respect to x at the point $x = x_0$. To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ rather than d.

⇒ Partial Derivative with respect to x:

The partial derivative of $z = f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\left.\frac{dz = f(x, y_0)}{dx}\right|_{x = x_0} = \left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

Provided the limit exists.

The slope of the curve $z = f(x, y_0)$ at the point $P(x_0, y_0, f(x_0, y_0))$ in the plane $y = y_0$ is the value of the partial derivative of f with respect to x at point (x_0, y_0). The partial derivative $\partial f / \partial x$ at (x_0, y_0) gives the rate of change of f with respect to x when y is fixed with value y_0.

Notation: → $\frac{\partial f}{\partial x}(x_0, y_0)$ or $f_x(x_0, y_0)$

"Partial derivative of f with respect to x at point (x_0, y_0)."

→ f_x, $\frac{\partial f}{\partial x}$ or $\frac{\partial z}{\partial x}$ means partial derivative of f (or z) with respect to x. These symbols generally used when you denote partial derivative as a function.

* Partial derivative with respect to y:

The partial derivative of $z = f(x,y)$ with respect to y at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \left.\frac{dF}{dx}\right|_{y=y_0} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h},$$

provided the limit exists. where $F = f(x_0, y)$

→ The following examples show, the values of these partial derivatives are usually different at a given point (x_0, y_0).

EX: Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$. where $f(x,y) = x^2 + 3xy + y - 1$.

Ans: For finding $\frac{\partial f}{\partial x}$, we treat y as a constant.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 + 3xy + y - 1) = 2x + 3y$$

$$\therefore \left.\frac{\partial f}{\partial x}\right|_{(4,-5)} = 2\cdot(4) + 3(-5) = -7$$

Similarly $\left.\frac{\partial f}{\partial y}\right|_{(4,-5)} = 13.$

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Ex: The plane $x = 1$ intersects the Paraboloid $z = x^2+y^2$ in a Parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$.

Sol: The slope is the value of the partial derivative $\frac{\partial z}{\partial y}$ at $(1, 2)$.

$$\therefore \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(x^2+y^2) = 2y$$

$$\therefore \left.\frac{\partial z}{\partial y}\right|_{(1,2)} = 2.(2) = 4$$

* **Partial derivatives and Continuity:**

A function f can have partial derivatives with respect to both x and y at a point without the function being continuous there. The following example shows that the partial derivative of the given function exist at point (x_0, y_0) but f is discontinuous.

e.g.:
$$f(x,y) = \begin{cases} 0 & \text{if } xy \neq 0 \\ 1 & \text{if } xy = 0 \end{cases}$$

Here $$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(x+h, 0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{0}{h} = 0$$

& $$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(0, y+h) - f(0,0)}{h} = 0.$$

First we approach $(0,0)$ along the line $y=x$

then $$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} 0 = 0$$

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and if we approach $(0,0)$ through the x-axis then $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} 1 = 1.$

Therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ doesnot exists.

Hence f is not continuous at point $(0,0)$.

Ex: Find the partial derivative of the following functions at point $(0,0)$.

(1) $f(x,y) = \dfrac{x(x^2-y^2)}{x^2+y^2}$, $(x,y)\neq(0,0)$

$= 0$ if $(x,y) = (0,0)$.

Sol^n: Here $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h}$

$= \lim_{h\to 0} \dfrac{h^3/h^2}{h} = \lim_{h\to 0} 1 = 1$

& $f_y(0,0) = \lim_{h\to 0} \dfrac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} 0 = 0.$

<2> $f(x,y) = x^2\tan^{-1}(y/x) + y^2\tan^{-1}(x/y)$, $(x,y)\neq(0,0)$

$= 0$ $(x,y)=(0,0)$.

* The following examples shows that the given functions are continuous at given point but at that point partial derivatives of that function at given point may or may not exists.

Ex: show that the function

$$f(x,y) = \begin{cases} (x+y)\sin\left(\frac{1}{x+y}\right), & x+y \neq 0 \\ 0 & \text{if } x+y=0 \end{cases}$$

is continuous at $(0,0)$ but its partial derivatives f_x and f_y do not exist at $(0,0)$.

Sol^n: Let $\varepsilon > 0$. Then we want to find $\delta > 0$ such that $|f(x,y) - f(0,0)| < \varepsilon$.

Now $|f(x,y) - f(0,0)| = \left|(x+y)\cdot\sin\left(\frac{1}{x+y}\right)\right|$

$$\leq |x+y| \leq |x| + |y|$$

$$\leq 2\sqrt{x^2+y^2}.$$

Now if we choose $\delta < \varepsilon/2$, then

$|f(x,y) - f(0,0)| < \varepsilon$ whenever $0 < \sqrt{x^2+y^2} < \delta$.

Therefore $\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0)$.

Hence, the given function is continuous at point $(0,0)$.

Now $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$

$$= \lim_{h\to 0} \frac{(0+h)\sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h\to 0} \sin(1/h)$$

which is not exists.

Now $f_y(0,0) = \lim_{h\to 0} \sin(1/h)$ which does not exist.

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Ex! show that the function

$$f(x,y) = \frac{x^2+y^2}{|x|+|y|}, \quad (x,y) \neq (0,0)$$

$$= 0 \quad \text{if } (x,y) = (0,0).$$

is continuous at $(0,0)$ but its partial derivative f_x and f_y do not exist at $(0,0)$.

Ex! show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+2y^2}, & (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

is not continuous at $(0,0)$ but its partial derivatives f_x and f_y exist at $(0,0)$.

Thm! (sufficient condition for continuity)

A sufficient condition for a function f to be continuous at a point (x_0, y_0) is that one of its first order partial derivatives exists and is bounded in the neighbourhood of (x_0, y_0) and that the other exists at (x_0, y_0).

* **Partial Derivative of higher order:**

Let $E \subset \mathbb{R}^2$ be an open subset (non empty), and let $f: E \subset \mathbb{R}^2 \to \mathbb{R}$ be real valued function. Then its partial derivatives f_x and f_y are again a functions of x and y. Therefore we can take its partial derivative with respect to x and y.

The partial derivative of second order are

(i) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$

(ii) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

(iii) $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{yx}$

(iv) $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{xy}$

~~(#)~~ These partial derivative are defined in the following way.

Now $f_{xy}(x, y) = (f_x(x,y))_y$

$$= \lim_{k \to 0} \frac{f_x(x, y+k) - f_x(x, y)}{k}$$

But $f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$

$$\therefore f_{xy}(x, y) = \lim_{k \to 0} \frac{\lim_{h \to 0} \frac{f(x+h, y+k) - f(x, y+k)}{h} - \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}}{k}$$

$$= \lim_{k \to 0} \lim_{h \to 0} \frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk}$$

Similarly,

$$f_{yx}(x, y) = \lim_{h \to 0} \lim_{k \to 0} \frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk}$$

$$f_{xx}(x, y) = (f_x(x, y))_x = \lim_{h \to 0} \frac{f_x(x+h, y) - f_x(x, y)}{h}$$

& $f_{yy}(x, y) = (f_y(x, y))_x = \lim_{k \to 0} \frac{f_y(x, y+k) - f_y(x, y)}{k}$

Ex: Find $f_{xy}(0,0)$, $f_{yx}(0,0)$, $f_{xx}(0,0)$, $f_{yy}(0,0)$ for the following functions.

(1) $f(x,y) = x^2 \tan^{-1} y/x - y^2 \tan^{-1} x/y$, $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$

Sol^n: Here $f_x(x,y) = \begin{cases} 2x \tan^{-1} y/x - y & \text{if } (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$

similarly $f_y(x,y) = x - 2y \tan^{-1} x/y$ if $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$

Now $f_{xy}(0,0) = \lim\limits_{k \to 0} \dfrac{f_x(0,k) - f_x(0,0)}{k} = \lim\limits_{k \to 0} \dfrac{-k-0}{k} = -1.$

$f_{yx}(0,0) = \lim\limits_{h \to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{h-0}{h} = 1$

$f_{xx}(0,0) = \lim\limits_{h \to 0} \dfrac{f_x(h,0) - f_x(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0-0}{h} = 0.$

& $f_{yy}(0,0) = \lim\limits_{k \to 0} \dfrac{f_y(0,k) - f_y(0,0)}{k} = \lim\limits_{k \to 0} \dfrac{0-0}{k} = 0.$

Ex: Find $f_{xy}(0,0)$, $f_{yx}(0,0)$, $f_{xx}(0,0)$ & $f_{yy}(0,0)$ for the following functions:

$f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$, $(x,y) \neq (0,0)$

$= 0$ if $(x,y) = (0,0)$.

Ans: $f_{xx}(0,0) = f_{yy}(0,0) = 0$, $f_{xy}(0,0) = -1$, $f_{yx}(0,0) = 1$

But for any $(x,y) \neq (0,0)$ we have

$$f_{xy}(x,y) = f_{yx}(x,y).$$

Therefore this example shows that, $f_{xy} \neq f_{yx}$ in general.

* **Differentiation of a function of two variables:**

Let S be a non-empty open subset of $\mathbb{R}^2$ and $f: S \to \mathbb{R}$ be a real function and $(x+h, y+k)$ is a point in nbhd of (x,y). The function f is said to be differentiable at point $(x,y) \in S$, if

$$f(x+h, y+k) = f(x,y) + Ah + Bk + \varepsilon\rho \quad \text{where}$$

(i) A & B are independent of h & k.

(ii) $\rho = \sqrt{h^2+k^2}$ is such that as $\rho \to 0 \Rightarrow \varepsilon \to 0$

i.e $$\lim_{\rho \to 0} \frac{f(x+y, y+k) - f(x,y) - (Ah+Bk)}{\rho} = 0.$$

Ex: Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x-y}, & (x,y) \neq (1,-1) \\ = 0 & (x,y) = (1,-1) \end{cases}$$

is continuous and differentiable at $(1,-1)$.

Soln: We have

$$\lim_{(x,y)\to(1,-1)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(1,-1)} (x+y) = 0 = f(1,-1).$$

Therefore, the function is continuous at $(1,-1)$.

The Partial derivative are given by

$$f_x(1,-1) = \lim_{h\to 0} \frac{f(1+h,-1) - f(1,-1)}{h} = \lim_{h\to 0} \frac{1}{h}\left[\frac{(1+h)^2-1}{(1+h)+1} - 0\right]$$

$$= \lim_{h\to 0} \frac{2+h}{2+h} = 1.$$

similarly,

& $f_y(1,-1) = 1.$

Therefore, the partial derivatives exist at $(1,-1)$.

Now, we have

$$f_x(x,y) = \frac{(x-y)\,2x - (x^2-y^2)}{(x-y)^2} = \frac{(x-y)^2}{(x-y)^2} = 1, \quad (x,y) \neq (1,-1)$$

& $f_x(1,-1) = 1.$

since $\lim_{(x,y)\to(1,-1)} f_x(x,y) = \lim_{(x,y)\to(1,-1)} 1 = 1 = f_x(1,-1),$

The partial derivative f_x is continuous at $(1,-1)$.

similarly, the Partial derivative f_y is also continuous at $(1,-1)$. Therefore f is differentiable at point $(1,-1)$.

Thm: If function $z = f(x,y)$, defined on an open set $E \subset \mathbb{R}^2$ and differentiable at point $(x,y) \in E$, then its Partial derivatives f_x and f_y exist at Point (x,y).

Remark: In the definition of differentiablity, The constants A & B are

$A = f_x$ & $B = f_y$

Note! The converse of the above mentioned theorem is not true, i.e. if the Partial derivatives $f_x(x,y)$ and $f_y(x,y)$ of a function f exist, it is not necessary that the function f is differentiable at point (x,y).

Ex: show that the function

$$f(x,y) = \begin{cases} \dfrac{x^3+2y^3}{x^2+y^2} & , (x,y) \neq (0,0) \\ 0 & , (x,y) = (0,0) \end{cases}$$

Possesses partial derivatives $f_x(0,0)$ & $f_y(0,0)$, But f is not differentiable at $(0,0)$.

Soln:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h-1}{h} = 1$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{2k-0}{k} = 2.$$

Therefore, the partial derivatives $f_x(0,0)$ & $f_y(0,0)$ exists.

Now Let $\rho = \sqrt{h^2+k^2}$. Then

$$\lim_{\rho\to 0} \frac{f(h,k) - f(0,0) - (h+2k)}{\rho}$$

$$= \lim_{\rho\to 0} \frac{\dfrac{h^3+2k^3}{h^2+k^2} - h - 2k}{\rho}$$

$$= \lim_{\rho\to 0} \frac{\cancel{h^3} + \cancel{2k^3} - \cancel{h^3} - \cancel{2k^3} - 2h^2k - k^2h}{\rho}$$

$$= \lim_{\rho\to 0} - \frac{hk(2h+k)}{\rho}.$$

let $h = r\cos\theta$, $y = r\sin\theta$ then if $\rho\to 0 \Rightarrow r\to 0$ for arbitrary θ.

$$\therefore \lim_{\rho\to 0} - \frac{hk(2h+k)}{\rho} = -\lim_{r\to 0}\left[\cos\theta\cdot\sin\theta\,(2\cos\theta + \sin\theta)\right]$$

$= -[\cos\theta . \sin\theta \; (\sin\theta + 2\cos\theta)]$

The limit depends on θ and does not tend to zero for arbitrary θ. Hence, the given function is not differentiable.

Ex: Discuss the differentiablity of f at point $(0,0)$.

$$f(x, y) = \frac{x^2 y^2}{(x^2+y^2)^2}, \quad (x, y) \neq (0,0)$$

$$= 0, \quad (x, y) = (0,0).$$

Thm: If $z = f(x, y)$ is a real valued function, defined on non-empty open set $E \subset \mathbb{R}^2$ and if $f_x(x, y)$, $f_y(x, y)$ exist and continuous at point $(x, y) \in E$, then

(i) The function f is differentiable at point (x, y)

(ii) For $(x, y) \in E$, $\delta z = \frac{\partial z}{\partial x}\delta x + \frac{\partial z}{\partial y}\delta y + \varepsilon \delta$, where

$\varepsilon \to 0$ as $\delta = \sqrt{(\delta x)^2 + (\delta y)^2} \to 0$.

where $\delta z = f(x+\delta x, y+\delta y) - f(x, y)$.

δx & δy are increment in x and y respectively

Thm: If a function $z = f(x, y)$, defined on an open set $E \subset \mathbb{R}^2$, is differentiable at point $(a,b) \in E$, then the function f is continuous at $(a,b) \in E$.

Note: The converse of the above theorem is not true.

consider
$$f(x, y) = \frac{x^3 + 2y^3}{x^2+y^2}, \quad (x, y) \neq (0,0)$$

$$= 0 \quad (x, y) = (0,0)$$

Then f is continuous at $(0,0)$. But not diff. at $(0,0)$.

* **Total differential:**

If $z = f(x,y)$ is a real function defined on non-empty open set $E \subset \mathbb{R}^2$, which is diff. at point $(x,y) \in E$ then

$$\delta z = f_x \,\delta x + f_y\, \delta y + \varepsilon \rho .$$

where $\varepsilon \to 0$ as $\rho \to 0$.

Here principal part $f_x\,\delta x + f_y\,\delta y$ of δz is called total derivative of z and it is denoted by δz. (or dz)

Thus $\delta z = f_x\,\delta x + f_y\,\delta y$. which can also be written as

$$dz = f_x\,dx + f_y\,dy .$$

* **Thm: Young's theorem:**

Let f be a real function defined on non empty open set $E \subset \mathbb{R}^2$. If f_x, f_y exist in some nbhd of (x,y) and are both differentiable at point (x,y) with respect x and y then $f_{xy} = f_{yx}$ at point (x,y).

Note: converse of the above theorem is not true.

For that consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Then $f_{xy}(0,0) = f_{yx}(0,0)$. But The functions f_x & f_y are not differentiable at point $(0,0)$.

* **Thm: Schwartz's theorem:**

Let $f: E \subset \mathbb{R}^2 \to \mathbb{R}$ be a function such that its partial derivatives f_x, f_y, f_{xy} & f_{yx} exist on an open set S containing point (x,y). If both the partial derivatives f_{xy} and f_{yx} are continous at point (x, y) then we have $f_{xy}(x,y) = f_{yx}(x,y)$

Note: converse of the above thm is not true. The following example serves as a counter example.

$$f(x, y) = \frac{x^2y^2}{x^2+y^2}, \quad (x,y) \neq (0,0)$$
$$= 0 \quad (x,y) = (0,0).$$

Examples:

(1) Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ = 0 & (x,y) = (0,0) \end{cases}$$

has partial derivatives $f_x(0,0)$, $f_y(0,0)$ but the partial derivatives are not continuous at $(0,0)$.

(2) For the function

$$f(x,y) = \begin{cases} \dfrac{y(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ = 0 & (x,y) = (0,0) \end{cases}$$

compute $f_x(0,0)$, $f_y(x,0)$, $f_x(0,0)$ and $f_y(0,0)$, if they exist.

(3) Find the first order partial derivatives for the following functions at the specified point.

(i) $f(x,y) = \ln(x/y)$ at $(2,3)$.

(ii) $f(x,y) = \cot^{-1}(x+y)$ at $(1,2)$.

(iii) $f(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}$ at $(6,7)$.

(iv) $f(x,y) = \ln\left(\dfrac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right)$ at $(3,4)$

(v) $f(x,y,z) = (x^2+y^2+z^2)^{-1/2}$ at $(2,1,2)$.

(vi) $f(x,y,z) = \ln\left(x+\sqrt{y^2+z^2}\right)$ at $(2,3,4)$

(vii) $f(x,y,z) = (xy)^{\sin z}$ at $(3,5,\pi/2)$

*. **Derivatives of composite functions:**

Let $z = f(x,y)$ be a function of two independent variables x and y. suppose that x & y are themselves functions of some independent variable say t, i.e. $x = \phi(t)$, $y = \psi(t)$. Then $z = f[\phi(t), \psi(t)]$ is a composite function of the independent variable t. If partial derivatives f_x, f_y are continuous functions of x & y and $x = \phi(t)$ and $y = \psi(t)$ are differentiable functions of t, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$

Now, let x & y be functions of two independent variables u & v, say $x = \phi(u,v)$, $y = \psi(u,v)$. Then $z = f[\phi(u,v), \psi(u,v)]$ is a composite function of two independent variables u & v. If the functions f, ϕ and ψ have continuous partial derivatives with respect to their arguments then

$$\frac{\partial z}{\partial u} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u}.$$

& $$\frac{\partial z}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial v}.$$

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If function z is of n variable ie $z = f(x_1, x_2, \ldots, x_n)$ and the partial derivatives of f with respect to all its variables are continuous and $x_1, x_2, \ldots x_n$ are differentiable functions of some independent variable t, then

$$\frac{dz}{dt} = \frac{\partial f}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f}{\partial x_2}\cdot\frac{dx_2}{dt} + \cdots + \frac{\partial f}{\partial x_n}\cdot\frac{dx_n}{dt}.$$

Examples:

1. Find $\frac{df}{dt}$ at $t=0$.

(i) $f(x,y) = x\cos y + e^x \sin y$,
where $x = t^2+1$, $y = t^3+t$

sol: when $t=0$ we get $x=1$, $y=0$, using the chain Rule we get

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

$$= (\cos y + e^x \sin y)(2t) + (-x\sin y + e^x\cos y)\cdot(3t^2+1)$$

$$\therefore \left.\frac{df}{dt}\right|_{t=0} = e.$$

(ii) $f(x,y,z) = x^3 + xz^2 + y^3 + xyz$ where $x = e^t$, $y = \cos t$, $z = t^3$.

Ex:

(1) If $z = f(x,y)$, $x = e^{2u} + e^{-2v}$, $y = e^{-2u} + e^{2v}$, then show that $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]$.

(2) If $z = f(ax+by)$, then $b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0$ where a and b are constant.

(3) If $z = \log[(x^2-y^2)/(x^2+y^2)]$ then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.

(4) If $u = f(x-y, y-z, z-x)$ then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

(5) If $z = y + f(u)$, $u = x/y$ then $u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$.

(6) If $\omega = f(u,v)$, $u = \sqrt{x^2+y^2}$, $v = \cot^{-1}(y/x)$ then

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \frac{1}{x^2+y^2}\left[(x^2+y^2)\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right].$$

(7) If $z = f(x,y)$, $x = u\cos\alpha - v\sin\alpha$,
$y = u\sin\alpha + v\cos\alpha$ where α is constant. Then

$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

(8) If $z = \ln(u^2+v)$, $u = e^{x+y^2}$,
$v = x+y^2$, then $2y\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$.

* Find $\frac{d\omega}{dt}$ in the following problem:

(i) $\omega = x^2+y^2+z^2$ where $x = \cos t$, $y = \ln(t+1)$
$z = e^t$. at $t = 0$.

(ii) $\omega = z\ln y + y\ln z + xyz$, at $t = 0$.
where $x = \sin t$, $y = t^2+1$, $z = \cos^{-1}t$,

(iii) $\omega = e^x\sin(y+2z)$, $x = t$, $y = 1/t$, $z = t^2$
at $t = 1$.

* Derivatives of functions defined implicitly:

Some surfaces in $\mathbb{R}^3$ are described by cartesian equations of the form $F(x, y, z) = 0$. An equations like this is said to provide an implicit representation of the surface. For example the equation $x^2 + y^2 + z^2 - 1 = 0$ represents the surface of unit sphere with ~~radii~~ center at the origin. Sometimes it is possible to solve the equation $F(x, y, z) = 0$ for one of the variables in terms of the other two, say for z in terms of x & y. This leads to one or more equations of the form $z = f(x, y)$.

In the general case one can not obtain an explicit formula for z in terms of x & y. For example $y^2 + xz + z^2 - e^z - 4 = 0$, in this equation one can not easily write z in terms of x & y. In this situation one can deduce various properties of the partial derivatives $\partial f/\partial x$ & $\partial f/\partial y$ without an explicit knowledge of $z = f(x, y)$.

Let there is a function $z = f(x, y)$ such that $F(x, y, f(x, y)) = 0$. for all (x, y) in some open set S, although we may not have explicit formulas for calculating $f(x, y)$. Then

$$\frac{\partial f}{\partial x} = -\frac{\partial F/\partial x}{\partial F/\partial z} \quad \& \quad \frac{\partial f}{\partial y} = -\frac{\partial F/\partial y}{\partial F/\partial z}$$

Ex! Assume that the equation $y^2 + xz + z^2 - e^z - c = 0$ defines z as a function of x & y say $z = f(x, y)$. Find a value of the constant c such that $f(0, e) = 2$ and compute the partial derivatives $\partial f/\partial x$ & $\partial f/\partial y$ at point $(x, y) = (0, e)$.

Sol: when $x=0$, $y=e$ and $z=2$ then from the equation we get $c=4$.

Let $F(x,y,2) = y^2 + xz + z^2 - e^z - 4$.

$\therefore \quad \frac{\partial f}{\partial x} = \frac{z}{x+2z-e^z}$ & $\frac{\partial f}{\partial y} = -\frac{2y}{x+2z-e^z}$

when $x=0$, $y=e$ and $z=2$ we have

$\frac{\partial f}{\partial x} = \frac{2}{e^2-4}$, $\frac{\partial f}{\partial y} = \frac{2e}{e^2-4}$.

Ex: Find $\partial z/\partial x$ if the equation $yz - \ln z = x+y$ defines z as a function of the two independent variables x & y and the partial derivatives exist.

Ex: Find $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ at a given point.

(i) $z^3 - xy + yz + y^3 - 2 = 0$ at $(1,1,1)$

(ii) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$ at $(2,3,6)$

(iii) $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0$ at (π, π, π).

(iv) $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$ at $(1, \ln 2, \ln 3)$

Ex: Find the value of $\frac{dy}{dx}$ at given point. (where y is a diff function of x).

(i) $y^3 + y^2 - 5y - x^2 + 4 = 0$ at Point $(1,1)$

(ii) $xy + y^2 - 3x - 3 = 0$ at $(-1,1)$

(iii) $xe^y + \sin xy + y - \ln 2 = 0$ at $(0, \ln 2)$.

* Homogeneous Functions :

A function f of two variable is said to be homogeneous of degree n in x & y if it can be written in any one of the following forms

(i) $f(\lambda x, \lambda y) = \lambda^n f(x,y)$

(ii) $f(x,y) = x^n g(y/x)$

(iii) $f(x,y) = y^n g(x/y)$.

where λ is a scalar.

* Some examples of homogeneous functions.

f	degree of homogeneity.
$x^2 + xy$	2
$\tan^{-1}(y/x)$	0
$\frac{1}{x+y}$	-1
$\sqrt{x}/\sqrt{x^2+y^2+z^2}$	$-1/2$.

* The function $f(x,y) = (x^2+y)/(x+y^2)$ is not homogeneous.

Thm: Euler's theorem :

If $f(x,y)$ is a homogeneous function of degree n in x & y and has continuous first and second order partial derivatives, then

(i) $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

(ii) $$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f.$$

Ex: If $u(x,y) = \cos^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, $0<x, y<1$, then

Prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\frac{1}{2}\cot u$.

sol: The given function can be written as

$\cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$. Here $\cos u$ function is a homogeneous function of degree $\frac{1}{2}$.

Therefore by Euler's thm we have

$$x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u$$

$$\therefore \; -x\sin u \cdot \frac{\partial u}{\partial x} - y(\sin u)\cdot\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$\therefore \; x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

Ex: If $u(x,y) = x^2\tan^{-1}(y/x) - y^2\tan^{-1}(x/y)$

$x>0$, $y>0$, then evaluate

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}.$$

sol: We have $u(\partial x, \partial y) = \partial^2 u(x,y)$. Therefore $u(x,y)$ is a homogeneous function of degree 2.

Therefore by Euler's thm we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u.$$

Remark: Euler's thm on homogeneous functions also holds for n variables. Thus if f is a function of n variable $x_1, x_2, \ldots x_n$ and of degree m. Then

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \cdots + x_n\frac{\partial f}{\partial x_n} = mf.$$

Examples.

(1) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x+y}\right)$ then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$$

<2> If $u = \sin^{-1}\frac{x^3+y^3}{x+y}$ then show that

$$x.\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u.$$

<3> If $u = x f(y/x) + g(y/x)$, then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

<4> If $\tan u = \frac{x^2y^2}{x+y}$ then show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{3}{2}\sin 2u$$

<5> $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 4u - \sin 2u.$$

<6> If $u = e^{x^2+y^2}$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u.$

<7> If $u = \operatorname{cosec}^{-1}\left[\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right]^{1/2}$ then show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{144}(13+\tan^2 u).$$

Thm: If z is a homogeneous function of x & y and $z = f(u)$, then

(i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{nf}{f'}$ (where $f' \neq 0$)

(ii) $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(g' - 1)$

where $g = \frac{nf}{f'}$ and "'" denotes the differentiation with respect to u, n is the degree of homogeneity of z.

EX: If $u = \sin^{-1} \frac{x+y}{\sqrt{x}+\sqrt{y}}$, then prove that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \tan u.$$

SOL: Here $z = \sin u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$. which is a homogeneous function of x & y of degree $\frac{1}{2}$.

Therefore we have

$$\therefore \quad x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} \cdot \frac{\sin u}{\frac{d}{du}(\sin u)} = \frac{1}{2} \tan u.$$

Examples:

(1) if $z = \sin^{-1} \frac{\sqrt{x^2+y^2}}{xy}$, then prove that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = \tan^3 u.$$

<2> If $z = x^n f_1(y/x) + y^{-n} f_2(x/y)$ then prove that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial x} = n^2 z.$$

<3> If $u = \operatorname{cosec}^{-1}\left(\frac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}\right)$, then prove that

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = \frac{1}{36} \tan u \left(7 + \tan^2 u\right).$$

● : solutions of Examples which are on Page NO: 13.

(2) Let $u = \sin^{-1}\left(\frac{x^3 + y^3}{x+y}\right)$

Then $\sin u = \frac{x^3+y^3}{x+y}.$

∴ By Euler's modified method we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\sin u}{\cos u} = 2\tan u.$$

(4) Let $\tan u = \frac{x^2 y^2}{x+y}$

Then degree of homogeneity of $z = \tan u$ is 3. Therefore by Euler's modified method we have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\frac{\tan u}{\sec^2 u}$$

$$= \frac{3}{2}\sin 2u$$

(5) Let $\tan u = \frac{x^3+y^3}{x-y}$. Then $z = \tan u$ is homogeneous function of degree ~~3~~ 2.

Therefore By Euler's modified method we have

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(g'-1)$$

when $g = \frac{2\tan u}{\sec^2 u} = 2\sin u \cdot \cos u$
$= \sin 2u$

$\therefore g' = 2\cos 2u$

$$\therefore x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u(2\cos 2u - 1)$$
$$= \sin 4u - \sin 2u$$

(6) Let $u = e^{x^2+y^2}$.

$\Rightarrow \log_e u = x^2+y^2$

Then $z = \log_e u$ is homogeneous function of degree 2.

Therefore By Euler's modified method

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cdot(\log u)\, u$$

(7) Let $u = \operatorname{cosec}^{-1}\left[\frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}\right]^{1/2}$. Then

$$\operatorname{cosec}^2 u = \frac{x^{1/2}+y^{1/2}}{x^{1/3}+y^{1/3}}$$

which is a homogeneous function of degree $1/6$

Therefore by Euler's method (modified)

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = g(g'-1)$$

where $g = \frac{nf}{f'} = \frac{1}{6} \csc^2 u \cdot \frac{1}{-2\csc^2 u \cdot \cot u}$

$$= -\frac{1}{12} \tan u$$

$$\therefore g' = -\frac{1}{12} \sec^2 u$$

$$\therefore x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{-1}{12} \tan u \left(\frac{-1}{12} \sec^2 u - 1\right)$$

$$= \frac{\tan u \cdot \sec^2 u}{144} + \frac{1}{12} \tan u$$

$$= \frac{\tan u}{144} [\sec^2 u - 1 + 13]$$

$$= \frac{\tan u}{144} [13 + \tan^2 u]$$

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<2> Let $z = x^n f_1(y/x) + y^{-n} f_2(y/x)$

Take $u_1 = x^n f_1(y/x)$ & $u_2(x,y) = y^{-n} f_2(y/x)$.

Then u_1 is homogeneous function of degree n and u_2 is homogeneous function of degree $-n$.

Therefore By Euler's theorem we have

$$x \frac{\partial u_1}{\partial x} + y \frac{\partial u_1}{\partial y} = n u_1 \quad \text{--- (1)}$$

$$x^2 \frac{\partial^2 u_1}{\partial x^2} + 2xy \frac{\partial^2 u_1}{\partial x \partial y} + y^2 \frac{\partial^2 u_1}{\partial y} = n(n-1) u_2 \quad \text{--- (2)}$$

and $x \frac{\partial u_2}{\partial x} + y \frac{\partial u_2}{\partial y} = n u_2$ ——— (3)

$$x^2 \frac{\partial^2 u_2}{\partial x^2} + 2xy \frac{\partial^2 u_2}{\partial x \partial y} + y^2 \frac{\partial^2 u_2}{\partial y^2} = -n(-n-1)u_2 \quad \text{---} (4)$$

Therefore

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} + x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial x}$$

$$= nu_1 + n(n-1)u_1 - n(-n-1)u_2 - nu_2$$

($\because$ From equations ①, ② ③ and (4) and $z = u_1 + u_2$)

$$= \cancel{nu_1} + n^2 u_1 - \cancel{nu_1}$$
$$+ n^2 u_2 + \cancel{nu_2} - \cancel{nu_2}$$

$$= n^2(u_1 + u_2) = n^2 z.$$

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<3> Let $\operatorname{cosec} u = \dfrac{x^{1/2} + y^{1/2}}{x^{1/3} + y^{1/3}}$ which is homogeneous function of degree $1/6$.

Therefore by Euler's modified method we have

$$x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = g(g'-1)$$

where $g = \frac{1}{6} \operatorname{cosec} u \cdot \dfrac{1}{-\operatorname{cosec} u \cdot \cot u} = \dfrac{-1}{6} \tan u$

$$\therefore g' = -\frac{\sec^2 u}{6}$$

$$\therefore x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = -\frac{\tan u}{6}\left(\frac{\sec^2 u}{6} - 1\right)$$

$$= \frac{\tan u \cdot \sec^2 u}{36} + \frac{\tan u}{6}$$

$$= \frac{\tan u}{36}\left[\sec^2 u + 7 - 1\right] = \frac{\tan u}{36}\left[7 + \tan^2 u\right]$$

* Taylor's Theorem:

Let a function f defined in some domain $E \subseteq \mathbb{R}^2$ have continuous partial derivatives upto $(n+1)^{th}$ order in some neighborhood of a point $P(x_0, y_0)$ in E. Then for some point (x_0+h, y_0+k) in this neighborhood, we have

$$f(x_0+h, y_0+k) = f(x_0, y_0) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x_0, y_0)$$

$$+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x_0, y_0) + \cdots +\cdots$$

$$+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x_0, y_0) + R_n$$

where R_n is the remainder term given by

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x_0+\theta h, y_0+\theta k), \quad 0<\theta<1.$$

th

END

Department of Mathematical Sciences

MA141 Engineering Mathematics I

Unit : 3 Applications of Partial derivative

Taylor is of f is $f(x)$ is given by

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \dots$$

This can be written in the alternative form (by replacing $x - x_0$ by h):

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \dots$$

This expansion can be generalised to functions of two or more variables. Indeed, for functions of two variables we find:

$$f(x_0 + h,\ y_0 + k) \simeq f(x_0, y_0) + hf_x(x_0, y_0) + kf_y(x_0, y_0)$$

where, assuming h and k to be small, we have ignored higher-order terms involving powers of h and k. We define δf to be the change in $f(x, y)$ resulting from small changes to x_0 and y_0. Thus:

$$\delta f = f(x_0 + h,\ y_0 + k) - f(x_0, y_0)$$

and so $\delta f \simeq hf_x(x_0, y_0) + kf_y(x_0, y_0)$. Using the notation δx and δy instead of h and k for small increments in x and y respectively we may write

$$\delta f \simeq \delta x.f_x(x_0, y_0) + \delta y.f_y(x_0, y_0)$$

Finally, using the more common notation for partial derivatives, we write

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y.$$

Informally, the term δf is referred to as the absolute error in $f(x, y)$ resulting from errors $\delta x, \delta y$ in the variables x and y respectively. Other *measures* of error are used. For example the relative error in a variable f is defined as $\frac{\delta f}{f}$ and the percentage relative error is $\frac{\delta f}{f} \times 100$.

We denote δ by d.

Key formulas

- Error in f is df
- Approximate is $f + df$
- Relative error in f is $\dfrac{df}{f}$
- Percentage relative error in f is $100\dfrac{df}{f}$

(1) If the sides and angle of a plane triangle vary in such a way that its circum radius remains constant then Prove that $\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=0.$

Ans. We know that from circum radius R is a radius of a circle which passes through three vertices of triangle. We have following formula for circum radius R with If the sides a, b, c and angles A, B, C of a triangle.

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R$$

Variation in a, b, c is da, db dc respectively .. Variation in A, B, C is dA, dB, dC respectively.

From above formula we have,

$$\left.\begin{array}{l}\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R \\ a=2R\sin A\Rightarrow da=2R\cos A dA\Rightarrow dA=\frac{da}{2R\cos A} \\ b=2R\sin B\Rightarrow db=2R\cos A dB\Rightarrow dB=\frac{db}{2R\cos B} \\ c=2R\sin C\Rightarrow dc=2R\cos A dC\Rightarrow dC=\frac{dc}{2R\cos C}\end{array}\right\}$$

We know that sum of angles of triangle is 180'.

$$\left.\begin{array}{l}A+B+C=\pi \\ dA+dB+dC=0 \\ \frac{da}{2R\cos A}+\frac{db}{2R\cos B}+\frac{dc}{2R\cos C}=0 \\ \frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=0\end{array}\right\}$$

(2) Find the percentage error in calculating the area of a rectangle when an error of 3% made in measuring each of its sides.

(3) The power dissipated in a resistor is given by $P=\frac{E^2}{R}$. Using calculus find the approximate percentage change in P when E is increased by 3 percentage and R is decreased by 2 percentage.(Ans. 2%)

(4) The deflection at the centre of a rod of length l and diameter d supported at its ends and loaded at the centre with a weight w varies as wl^3d^{-4}. What is the percentage increase in the deflection corresponding to the percentage increase in the w, l and d of 3,2 and 1 respectively? (Ans. 5% increase)

(5) The period T of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$. Find the maximum percentage error in T due to possible errors up to 1 percentage in l and 2.5 percentage in g.(Ans. 1.75%)

(6) Find the possible percentage error in computing the parallel resistance r of three resistances r_1, r_2 and r_3 if if r_1, r_2, r_3 are each in error by +1.2 percentage. (Ans. 1.2%)

(7) Find the percentage error in the area of ellipse when an error of +0.05 percentage made in measuring semi-major and semi-minor axes.(Ans. 0.1)

(8) If the ideal gas law is used to find P when T and V are given but there is an error of 0.3 percentage in measuring T and an error of 0.8 percentage in measuring V. Find the greatest percentage error in P.(Ans.1.1%)

(9) The diameter and altitude of a can in the shape of a right circular cylinder are measured as 40 cm and 64 cm respectively. The possible error in each measurement is ±5 percentage. Find approximately the maximum possible error in the computed value for the volume and the lateral surface. (Ans.Volume 15% and surface area 10%)

(10) The legs of a right angle triangle are 6 cm and 8 cm long. Find approximately the greatest error in length of hypotenuse, if an error of 0.1 cm is made in measuring. (Ans.1.4%)

(11) The dimension of box are measured to be 10 cm,12cm and 15 cm. Find the approximate error in the volume of box is V and an error of +0.02 cm is made in each measurement.(Ans 9)

(12) A company has contracted when the resistors having resistance R_1 and R_2 are connected in parallel resistance R of the combination is given by $R = \frac{R_1R_2}{R_1 + R_2}$, where $R_1 = 2\Omega$ and $R_2 = 6\Omega$. However when finally measured the change of 0.013Ω in R_1 and -0.023Ω in R_2 was found. Find approximately the greatest error in R.(Ans 0.005875ohm)

(13) As a result of the radius of a cone changes from 30 cm to 30.1 cm and its height changes from 60 cm to 59.5 cm. Find the approximate change in volume of a cube.

(14) If the fiber glass sheet costs Rs 45 per square feet. Find approximate the greatest cost of fiber glass sheet 3.012 feet and 5.982 feet long.

Ex. Find approximate value of $(4.1)^2 + (2.9)^2$ by applying concept of partial derivative.

Ans. $u = x^2 + y^2,\ \ x = 4,\ dx = 0.1,\ y = 3,\ dy = -0.1$

Approximate value is $= x^2 + y^2 + u_x dx + u_y dy$

$= 16 + 9 + 8(0.1) + (6)(-0.1) = 25.2$

EX. Find approximation:

(1) $\sqrt[5]{(3.8)^2+2(2.1)^3}$

(2) $\sqrt[4]{(5.1)^2(2.9)+2(2.9)}$

(3) $\sqrt{(299)^2+(399)^2}$

(4) $\sqrt{(0.98)^2+(2.01)^2+(1.94)^2}$

(5) $\log(3\sqrt{1.03}+4\sqrt{0.98}-1)$

(6) $\sin 58^\circ \cos 46^\circ$

(7) $\sin 29^\circ \cos 58^\circ$

(8) $\cos 62^\circ \sin 45^\circ$.

Tangent plane and Normal line:

Let f(x,y,z)=0 be any surface. Then equation of Tangent plane is

$$(x-x_1)\left(\frac{\partial f}{\partial x}\right)_p+(y-y_1)\left(\frac{\partial f}{\partial y}\right)_p+(z-z_1)\left(\frac{\partial f}{\partial z}\right)_p=0$$

and equation of Normal line is

$$\frac{(x-x_1)}{\left(\frac{\partial f}{\partial x}\right)_p}=\frac{(y-y_1)}{\left(\frac{\partial f}{\partial y}\right)_p}=\frac{(z-z_1)}{\left(\frac{\partial f}{\partial z}\right)_p}$$

EX: Find tangent plane and normal line on surface of $x^2+y^2+z^2=4$ at point $(1,1,1)$.

Answer. Here we have following formula for tangent plane and normal line of surface

f(x, y,z)=0 at point p(x0, y0,z0).

$$\left.\begin{array}{l}(x-x_0)(f_x)_p+(y-y_0)(f_y)_p+(z-z_0)(f_z)_p=0\\[2ex] \dfrac{(x-x_0)}{(f_x)_p}=\dfrac{(y-y_0)}{(f_y)_p}=\dfrac{(z-z_0)}{(f_z)_p} \quad or \quad \dfrac{(f_x)_p}{(x-x_0)}=\dfrac{(f_y)_p}{(y-y_0)}=\dfrac{(f_z)_p}{(z-z_0)}\end{array}\right\}$$

Here

$$f=x^2+y^2+z^2-4 \;\&\; p(1,\ 1,1)=p(x_0,\ y_0,\ z_0)$$
$$f_x=2x\Rightarrow(f_x)_p=2$$
$$f_y=2y\Rightarrow(f_y)_p=2$$
$$f_z=2z\Rightarrow(f_z)_p=2$$

Now put this values in above formula

$$\left.\begin{array}{l}(x-1)(2)+(y-1)(2)+(z-1)(2)=0 \Rightarrow 2x+2y+2z-6=0 \Rightarrow x+y+z=3 \\ \dfrac{(x-1)}{2}=\dfrac{(y-1)}{2}=\dfrac{(z-1)}{2} \quad or \quad \dfrac{2}{(x-1)}=\dfrac{2}{(y-1)}=\dfrac{2}{(z-1)}\end{array}\right\}$$

Ex: Find the equation of tangent plane and normal line to the surface

$2x^2+y^2+2z=3$ at the point (2,1,-3).

Ex: Find the equation of tangent plane and normal line to the surface $z=\sqrt{1-x^2-y^2}$ at point $(\frac{2}{3},\frac{2}{3},\frac{1}{3})$.

Ex: Find the equation of tangent plane and normal line to the surface $z=\tan^{-1}\left(\frac{y}{x}\right)$ at point (1,1).

Ex: Find the equation of tangent plane and normal line to the surface $3x^2-4y^2+z^2-8zx+9y+80=0$ at point (3,4,5).

Ex: Find the equation of tangent plane and normal line to the surface $xyz=a^2$ at point (α,β,γ).

<u>Maxima and Minima of Functions of two Variables:</u>

Let $z=f(x,y)$ be any surface. Put $f_x=0$ *and* $f_y=0$. Solve this two expression and find value of x and y. Obtain value of $r=f_{xx},\ t=f_{yy}$ *and* $s=f_{xy}$ and put value of s and y and observe following consition.

I. If $rt-s^2>0$, then function has maxima or minima. If $r>0$ then minima and if $r<0$ then maxima.

II. If $rt-s^2<0$, then function has neither maxima nor minima and such point is saddle or critical point.

III. If $rt-s^2=0$, then this method fails.

EX. Find Maxima and Minima for the following functions:

(1) $f(x,y)=x^3+y^3-3x-12y+20$

Answer.

$$\left.\begin{array}{l} f(x,\ y) = x^3 + y^3 - 3x - 12y + 20 \\ f_x = 3x^2 - 3 \Rightarrow f_x = 0 \Rightarrow f_x = 3x^2 - 3 = 0 \Rightarrow x^2 = 1 \Rightarrow x = \underline{+}1 \\ f_y = 3y^2 - 12 \Rightarrow f_y = 0 \Rightarrow f_y = 3x^2 - 12 = 0 \Rightarrow y^2 = 4 \Rightarrow y = \underline{+}2 \end{array}\right\}$$

Therefore stationary points/points of inflection /critical points are (1, 2), (-1, -2), (-1, 2) and (1, -2).

$$\left.\begin{array}{l} f(x,\ y) = x^3 + y^3 - 3x - 12y + 20 \\ f_x = 3x^2 - 3 \Rightarrow r = f_{xx} = 6x \ \&\ s = f_{xy} = 0 \\ f_y = 3y^2 - 12 \Rightarrow t = f_{yy} = 6y \end{array}\right\}$$

Point	*r = 6x*	*s = 0*	*t = 6y*	$tr - s^2$	
(1, 2)	+6	0	12	72>0	Min point
(-1, -2)	-6	0	-12	72>0	Max point
(-1, 2)	-6	0	12	-72<0	Saddle
(1, -2).	6	0	-12	-72<0	Saddle

Maximum point is (-1 ,-2) and maximum value is of function f is 38 and Minimum point is (1, 2) and minimum value is of function f is 2.

(2) $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

(3) $f(x,y) = 2x^4 + y^2 - x^2 - 2y$

(4) $f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$

Lagrange's Method of Undetermined multipliers:

Ex: If the sum of three positive numbers is unity, then find the maximum value of their product.

Ans. Suppose that three positive numbers are $x,\ y\ \&\ z$.

Here we want to find maximum of $f = xyz\ over\ x + y + z = 1$. Because sum of three positive number is 1.

We will apply Lagrange's method of multipliers.

$$\left.\begin{array}{l} f_x + \lambda g_x = 0 \Rightarrow yz + \lambda = 0 \Rightarrow \lambda = -yz \\ f_y + \lambda g_y = 0 \Rightarrow xz + \lambda = 0 \Rightarrow \lambda = -xz \\ f_z + \lambda g_z = 0 \Rightarrow xy + \lambda = 0 \Rightarrow \lambda = -xy \end{array}\right\}$$

On solving above three equations, we get

$$-yz = -xz = -xy \Rightarrow x = y = z$$
$$x + y + z = 1 \Rightarrow 3x = 1$$
$$x = y = z = \frac{1}{3} \Rightarrow f = xyz = \frac{1}{27}$$

Thus maximum value is $\frac{1}{27}$.

Ex: In a triangle find maximum value of $\sin A \sin B \sin C$, where A, B and C are three angles of triangle.

Ex: Find the extreme values of $x - 2y + 5z$ on the sphere $x^2 + y^2 + z^2 = 30$.

Ex: Maximize $x^l y^m z^n$ subject to $x + y + z = a$.

Ex: If $\frac{3}{x} + \frac{4}{y} + \frac{5}{z} = 6$, then find the values of x, y, z such that $x + y + z$ is minimum.

Ex: Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Jacobians

Let $u = f(x, y)$ and $v = g(x, y)$ be two functions. Then Jacobian of u and v with respect to x and y is defined as

$$J\left(\frac{u, v}{x, y}\right) = \frac{\partial(u,\ v)}{\partial(x,\ y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

Similarly for three functions in three variables is given by

$$J\left(\frac{u,\ v, w}{x,\ y, z}\right) = \frac{\partial(u,\ v, w)}{\partial(x,\ y, z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Ex: If $u = e^x \cos y$ and $v = e^x \sin y$, then find $\frac{\partial(u, v)}{\partial(x, y)}$.

Answer. . Jacobian is given by $J\left(\frac{u,\ v}{x,\ y}\right)=\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}=\begin{vmatrix} e^x\cos y & -e^x\sin y \\ e^x\sin y & e^x\cos y \end{vmatrix}=e^{2x}$

Ex: If $x=r\cos\theta$ and $y=r\sin\theta$, then find $\frac{\partial(r,\theta)}{\partial(x,y)}$ and $\frac{\partial(x,y)}{\partial(r,\theta)}$.

Ex: If $u=xyz$, $v=xy+yz+zx$ and $w=x+y+z$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

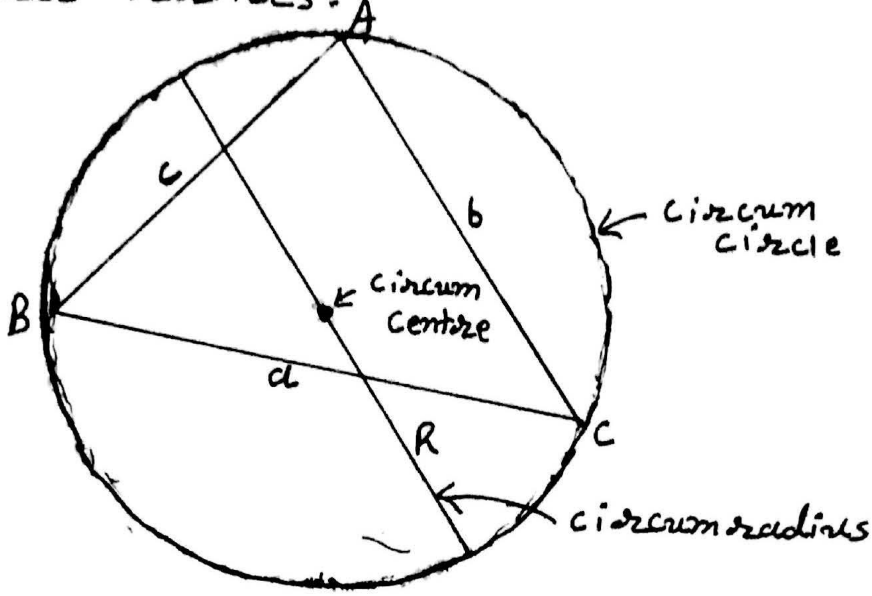
Ex: If $u=\frac{x}{y-z}, v=\frac{y}{z-x}, w=\frac{z}{x-y}$ then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Q: (1)

If the sides and angles of a plane triangle vary in such a way that its circum radius remains constant Prove, that $\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=0$

$\Rightarrow$ We know that circumradius is a radius of the unique circle which passes through each of the tringle's three vertices.

Only for understanding



For ΔABC we know that

$$\boxed{\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2R} \quad \text{—} (1)$$

where, R is circum-radius, which will remain constant

From (1)

$$a = 2R\sin A$$
$$\Rightarrow da = 2R\cos A\, dA$$
$$\Rightarrow \frac{da}{\cos A} = 2R\, dA \quad \text{—} (a)$$

Similarly

$$b = 2R\sin B$$
$$\Rightarrow db = 2R\cos B\, dB$$
$$\Rightarrow \frac{db}{\cos B} = 2R\, dB \quad \text{—} (b)$$

&
$$c = 2R\sin C$$
$$\Rightarrow dc = 2R\cos C\, dC$$
$$\Rightarrow \frac{dc}{\cos C} = 2R\, dC \quad \text{—} (c)$$

adding (a), (b) & (c)

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 2R\,(dA + dB + dC) \quad \text{—— (2)}$$

For any ΔABC we know that

$$A + B + C = 180^\circ$$

$$\Rightarrow \boxed{dA + dB + dC = 0} \quad \text{—— (3)}$$

From (2) & (3)

$$\boxed{\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0}$$

Q: (3)

The Power dissipated in a resistor is a loss of power due to heat transfor. In Physics, it is given by

$$P = \frac{V^2}{R}$$

only for Understanding.

where P is power dissipated
V is voltage
& R is resistance of resistor

Here, we have given

$$P = \frac{E^2}{R} \quad \text{---} \ (1)$$

& $\frac{dE}{E} \times 100 = 3$, $\frac{dR}{R} \times 100 = -2$

Now, applying log on both sides of eqn (1)

$$\Rightarrow \log P = \log\left(\frac{E^2}{R}\right).$$

$$\Rightarrow \log P = 2\log E - \log R$$

$$\Rightarrow \frac{dP}{P} = 2\frac{dE}{E} - \frac{dR}{R}$$

multiply 100 on both sides of above equation

$$\therefore \frac{dP}{P} \times 100 = 2\frac{dE}{E} \times 100 - \frac{dR}{R} \times 100$$

$$= 2 \times 3 - (-2)$$

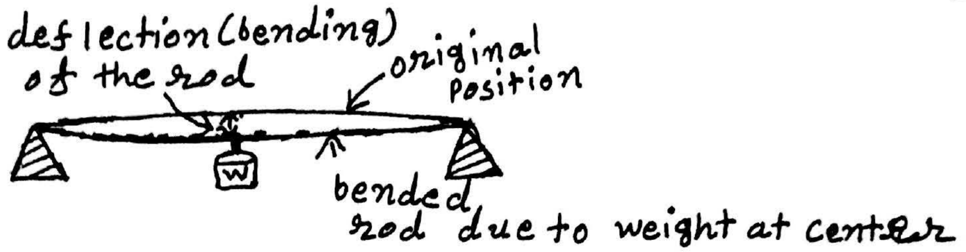
$$= 6 + 2$$

$$= 8$$

Hence, there is approximately <u>8 percentage increase in P</u> when E is increased by 3 percentage and R is decreased by 2 percentage.

Q: (4)

don't draw in Exam



Here, we have given that deflection or bending of a rod (D) at the center which can be measured in angle or distance is dependent on the length of rod l, diameter d and weight w loaded at the center.

$$\therefore D \propto w l^3 d^{-4}$$

$$\Rightarrow D = k w l^3 d^{-4} \quad \text{——} \quad (1)$$

where k is constant

Applying log on both sides of eqn (1).

$$\therefore \log D = \log k w l^3 d^{-4}$$

$$\Rightarrow \log D = \log k + \log w + 3\log l - 4\log d$$

$$\Rightarrow \frac{\delta D}{D} = 0 + \frac{\delta w}{w} + 3\frac{\delta l}{l} - 4\frac{\delta d}{d}$$

multiplying 100 on both sides of above eqn

$$\therefore \frac{\delta D}{D} \times 100 = \frac{\delta w}{w} \times 100 + 3\frac{\delta l}{l} \times 100 - 4\frac{\delta d}{d} \times 100 \quad (2)$$

Here, we also have provided that the percentage increase in w, l and d are 3, 2 and 1 respectively.

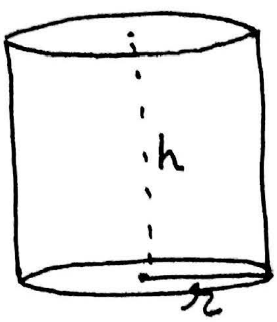
So, $\frac{\delta w}{w} \times 100 = 3$, $\frac{\delta w}{l} \times 100 = 2$ & $\frac{\delta d}{d} \times 100 = 1$

From eq^n (2),

$$\frac{\delta D}{D} \times 100 = 3 + (3 \times 2) - (4 \times 1)$$
$$= 3 + 6 - 4$$
$$= 5$$

Hence, There is 5 percentage increase in the deflection corresponding to the percentage increase in w, l and d of 3, 2 and 1 respectively.

Q: (9)



Here, we have given a can in the shape of a right circular cylinder, whose diameter is 40 cm & altitude is 64 cm

$\therefore$ $d = 2r = 40$ cm & $h = 64$ cm

But there is ± 5 percentage error in each measurement

So, $\frac{\delta d}{d} \times 100 = \pm 5$ & $\frac{\delta h}{h} \times 100 = \pm 5$

Now, we know that volume of this can is given by $V = \pi r^2 h$

$$V = \frac{\pi}{4} d^2 h \quad \text{(1)} \quad (\because d = 2r)$$

and lateral surface is given by

$$S = 2\pi r h$$

$$S = \pi d h \quad \text{—} (2)$$

Now applying log on both sides of eqn (1) we have,

$$\therefore \log v = \log(\pi/4\, d^2 h)$$

$$\Rightarrow \log v = \log \pi/4 + 2\log d + \log h$$

$$\Rightarrow \frac{\delta v}{v} = 0 + 2\frac{\delta d}{d} + \frac{\delta h}{h}$$

$$\Rightarrow \frac{\delta v}{v} \times 100 = 2\frac{\delta d}{d} \times 100 + \frac{\delta h}{h} \times 100$$

$$\Rightarrow \frac{\delta v}{v} \times 100 = 2[\pm 5] + [\pm 5]$$

$$= \pm 5 \text{ or } \pm 15$$

Now applying log on both sides of eqn (2) we have

$$\therefore \log s = \log(\pi d h) = \log \pi + \log d + \log h$$

$$\Rightarrow \frac{\delta s}{s} = 0 + \frac{\delta d}{d} + \frac{\delta h}{h}$$

$$\Rightarrow \frac{\delta s}{s} \times 100 = \frac{\delta d}{d} \times 100 + \frac{\delta h}{h} \times 100$$

$$\Rightarrow \frac{\delta s}{s} \times 100 = \pm 5 \pm 5$$

$$= 0 \text{ or } \pm 10$$

So, maximum error in volume is 15% and in lateral surface is 10% due to ±5 percentage error in measurement of d & h.

Now computed value of $v = \pi/4\, d^2 h$

$$= \pi/4 \times (40)^2 \times 64$$

$$= 25600\,\pi \text{ cm}^3$$

So maximum error in computed value of v is

$$dv = \frac{25600 \times 15}{100} \times \pi = 3840\pi \text{ cm}^3$$

and computed value of lateral surface $S = \pi dh$
$= \pi \times 40 \times 64$
$= 2560\pi \ cm^2$

So, maximum error in computed value of latera surface S is $ds = 2560\pi \times \frac{10}{100}$
$= 256\pi \ cm^2$

Q: (5) we know that the period T of a simple pendulam is given by

$$T = 2\pi\sqrt{\frac{l}{g}} \quad \text{—— (i)}$$

Also, we have provided there are possible errors upto 1 percentage in l and 2.5 percentage in g.

$$\therefore \frac{\delta l}{l} \times 100 = \pm 1 \quad \& \quad \frac{\delta g}{g} \times 100 = \pm 2.5$$

Now, applying log on both sides of eq (i)

$$\therefore \log T = \log\left(2\pi\sqrt{\frac{l}{g}}\right)$$

$$\Rightarrow \log T = \log 2\pi + \frac{1}{2}\left[\log l - \log g\right]$$

$$\Rightarrow \frac{\delta T}{T} = 0 + \frac{1}{2}\left[\frac{\delta l}{l} - \frac{\delta g}{g}\right]$$

$$\Rightarrow \frac{\delta T}{T} \times 100 = \frac{1}{2}\left[\frac{\delta l}{l} \times 100 - \frac{\delta g}{g} \times 100\right]$$

$$\Rightarrow \frac{\delta T}{T} \times 100 = \frac{1}{2}\left[\pm 1 - (\pm 2.5)\right]$$

$$\Rightarrow \boxed{\frac{\delta T}{T} \times 100 = \pm 0.75 \text{ or } \pm 1.75}$$

Hence, there is maximum error upto 1.75 percentage in T due to possible errors upto 1 percentage in l and 2.5 percentage in g.

Q: (6)

Here, we have given

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \quad \text{— (1)}$$

& there is error by 1.2 percentange in measuring each resistance.

i.e $\quad \frac{dr_1}{r_1} \times 100 = \frac{dr_2}{r_2} \times 100 = \frac{dr_3}{r_3} \times 100 = 1.2$

Now from (1)

$$-\frac{1}{r^2} dr = -\frac{1}{r_1^2} dr_1 - \frac{1}{r_2^2} dr_2 - \frac{1}{r_3^2} dr_3$$

$$\Rightarrow -\frac{1}{r} \frac{dr}{r} = -\left[\frac{1}{r_1} \frac{dr_1}{r_1} + \frac{1}{r_2} \frac{dr_2}{r_2} + \frac{1}{r_3} \frac{dr_3}{r_3}\right]$$

$$\Rightarrow \frac{1}{r} \frac{dr}{r} \times 100 = \frac{1}{r_1} \frac{dr_1}{r_1} \times 100 + \frac{1}{r_2} \frac{dr_2}{r_2} \times 100 + \frac{1}{r_3} \frac{dr_3}{r_3} \times 100$$

$$\Rightarrow \frac{1}{r} \frac{dr}{r} \times 100 = \frac{1}{r_1} \times 1.2 + \frac{1}{r_2} \times 1.2 + \frac{1}{r_3} \times 1.2$$

$$\Rightarrow \frac{1}{r} \frac{dr}{r} \times 100 = 1.2 \times \left[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right]$$

$$\Rightarrow \frac{1}{r} \frac{dr}{r} \times 100 = 1.2 \frac{1}{r} \quad (\because \text{from eq}^n \text{ (1)})$$

$$\Rightarrow \boxed{\frac{dr}{r} \times 100 = 1.2}$$

Hence, there is 1.2 percentage possible error in computing the paralled resistance r due to 1.2 percentage errer in each resistances r_1, r_2 & r_3.

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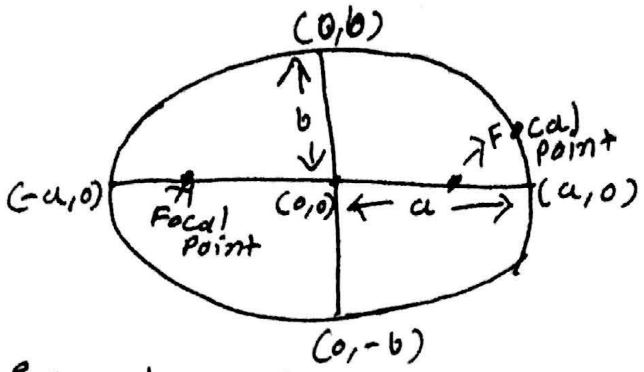
Q: (7)



As, show in above figure if we consider an ellipse, whose semi-major and semi-minor axis are a & b respectively. Then area of such ellipse is given by

$$A = \pi a b \quad \text{— (1)}$$

Here an error of 0.05 percentage made in measuring semi-major and semi-minor axis.

So, $\frac{da}{a} \times 100 = 0.05$ & $\frac{db}{b} \times 100 = 0.05$

Now, applying log on both sides of eqn (1)

$$\therefore \log A = \log(\pi a b)$$

$$\Rightarrow \log A = \log \pi + \log a + \log b$$

$$\Rightarrow \frac{dA}{A} = 0 + \frac{da}{a} + \frac{db}{b}$$

$$\Rightarrow \frac{dA}{A} \times 100 = \frac{da}{a} \times 100 + \frac{db}{b} \times 100$$

$$\Rightarrow \frac{dA}{A} \times 100 = 0.05 + 0.05$$

$$\Rightarrow \boxed{\frac{dA}{A} \times 100 = 0.1}$$

Hence, There is 0.1 percentage error in the area of ellipse due to 0.05 percentage error in measurment of semi-major and semi-minor axis.

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Q: (8) The ideal gas law can be represented as

$$PV = nRT$$

$$\Rightarrow P = \frac{nRT}{V} \quad — (1)$$

We have provided that there is an error 0.3 percentage in measuring T and an error ~~mea~~ of 0.8 percentage in measuring V.

So, $\frac{dT}{T} \times 100 = \pm 0.3$ & $\frac{dV}{V} \times 100 = \pm 0.8$

Now, applying log on both sides of eqn (1)

$$\log P = \log\left(\frac{nRT}{V}\right)$$

$$\Rightarrow \log P = \log n + \log R + \log T - \log V$$

$$\Rightarrow \frac{dP}{P} = \frac{dT}{T} - \frac{dV}{V}$$

$$\Rightarrow \frac{dP}{P} \times 100 = \frac{dT}{T} \times 100 - \frac{dV}{V} \times 100$$

$$\Rightarrow \frac{dP}{P} \times 100 = \pm 0.3 - \pm 0.8$$

$$\Rightarrow \boxed{\frac{dP}{P} \times 100 = \pm 0.5 \text{ or } \pm 1.1}$$

Hence, the greatest percentage error in P is 1.1 percentage due to 0.3 percentage error in T and 0.8 percentage error in V.

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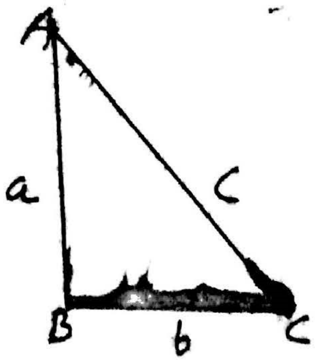
Q: (10)



For a right angle triangle ΔABC

$$a^2+b^2=c^2$$

$$\Rightarrow c=\sqrt{a^2+b^2} \quad \text{—} (1)$$

Here, we have given $a=6\,cm$ & $b=8\,cm$

So, $c=\sqrt{36+64}=\sqrt{100}=10\,cm$

Now from (1)

$$\therefore dc=\frac{1}{2\sqrt{a^2+b^2}}\cdot 2a\,da+\frac{1}{2\sqrt{a^2+b^2}}\cdot 2b\,db$$

$$\Rightarrow dc=\frac{a}{\sqrt{a^2+b^2}}da+\frac{b}{\sqrt{a^2+b^2}}db$$

dividing above eqn by $\sqrt{a^2+b^2}$ on both sides

$$\Rightarrow \frac{dc}{\sqrt{a^2+b^2}}=\frac{a}{a^2+b^2}da+\frac{b}{a^2+b^2}db$$

Here, we have given an error of 0.1 cm is made in measuring

$$\therefore da=db=\pm 0.1\,cm$$

$$\Rightarrow \frac{dc}{c}=\pm\frac{6}{36+64}\times 0.1\pm\frac{8}{36+64}\times 0.1$$

$$\Rightarrow \frac{dc}{c}=\pm\frac{6}{100}\times 0.1\pm\frac{8}{100}\times 0.1$$

$$\Rightarrow \frac{dc}{c}\times 100=\pm 0.6\pm 0.8$$

$$\Rightarrow \boxed{\frac{dc}{c}\times 100=\pm 1.4 \text{ or } \pm 0.2}$$

Hence, the greatest percentage error in length of hypotenuse is 1.4 percentage.

Q: (11)

Surface of rectangular parallelepiped of sides a, b and c is given by

$$S = 2ab + 2bc + 2ac \quad — (1)$$

Given that

$$da = db = dc = \delta$$

Now, from eqⁿ (1)

$$dS = \frac{\partial S}{\partial a} da + \frac{\partial S}{\partial b} db + \frac{\partial S}{\partial c} dc$$

$$\Rightarrow dS = (2b+2c) da + (2a+2c) db + (2b+2a) dc$$

$$\Rightarrow dS = (2b+2c)\delta + (2a+2c)\delta + (2b+2a)\delta$$

$$\Rightarrow dS = (4a+4b+4c)\delta$$

$$\Rightarrow \boxed{dS = 4\delta(a+b+c)}$$

The dimension of box are measured to be 10 cm, 12 cm and 15 cm.

let $a = 10\text{ cm}$, $b = 12\text{ cm}$ & $c = 15\text{ cm}$

Also, ~~an~~ error of 0.02 cm is made in each measurement

$$\therefore da = db = dc = 0.02\text{ cm}$$

Now volume of box is given by

$$V = a \cdot b \cdot c$$

$$\Rightarrow dV = bc\, da + ac\, db + ab\, dc$$

$$\Rightarrow dV = 12 \times 15 \times 0.02 + 10 \times 15 \times 0.02 + 10 \times 12 \times 0.02$$

$$\Rightarrow dV = [180 + 150 + 120] \times 0.02$$

$$\Rightarrow dV = 450 \times 0.02$$

$$\Rightarrow \boxed{dV = 9\text{ cm}^3}$$

Q: (12) ~~Part~~ Here $R = \frac{R_1 R_2}{R_1 + R_2}$ — (1)

where $R_1 = 2\Omega$ and $R_2 = 6\Omega$

The change of 0.013Ω in R_1 and of -0.023Ω in R_2 was found

So $dR_1 = 0.013\Omega$ & $dR_2 = -0.023\Omega$

Now from eq^n (1)

$$\therefore dR = \frac{\partial R}{\partial R_1} dR_1 + \frac{\partial R}{\partial R_2} dR_2$$

$$\Rightarrow dR = \frac{R_2(R_1+R_2) - R_1 R_2}{(R_1+R_2)^2} dR_1 + \frac{R_1(R_1+R_2) - R_1 R_2}{(R_1+R_2)^2} dR_2$$

$$\Rightarrow dR = \frac{R_2^2}{(R_1+R_2)^2} dR_1 + \frac{R_1^2}{(R_1+R_2)^2} dR_2$$

Substituting values of R_1, R_2, dR_1 & dR_2

$$\therefore dR = \frac{36}{64} \times 0.013 + \frac{4}{64}(-0.023)$$

$$\Rightarrow dR = 0.0073125 - 0.0014375$$

$$\Rightarrow \boxed{dR = 0.005875\Omega}$$

Q:(13) We know that volume of a cone is given by

$$V = \frac{\pi}{3} r^2 h \quad — (1)$$

Here radius of a cone changes from 30 cm to 30.1 cm and height changes from 60 cm to 59.5 cm

So, $r = 30$ cm & $dr = 0.1$ cm
& $h = 60$ cm & $dh = -0.5$ cm

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Now from eqn (1)

$$dv = 2r\frac{\pi}{3}h\,dr + \frac{\pi r^2}{3}\,dh$$

Now, substituting values of r, h, dr & dh

$$\therefore\ dv = 2 \times 30 \times \frac{\pi}{3} \times 60 \times 0.1 + \frac{\pi(30)^2}{3} \times (-0.5)$$

$$\Rightarrow dv = 120\pi - 150\pi$$

$$\Rightarrow \boxed{dv = -30\pi\ cm^3}$$

So, volume of a cone changes by $30\pi\ cm^3$.

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JQP

MA141
Unit-3
Matrix Algebra - I

10 Hrs. 17%

$1\frac{1}{2}$ Lec
- Definition of Matrix
- Types of matrices and their properties
- Determinant and their properties

2. Lec
- Rank and nullity of a matrix
- Determination of rank

1 Lec - Gauss Jordan method for computing inverse.

2 Lec - Triangularization of matrices by Gauss-Elimination process

3 Lec - Solution of system of linear equations.

☆ Matrices :-

By an $m \times n$ matrix we mean a meaningful arrangement of mn objects in m rows and n columns in the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

The objects $a_{11}, a_{12}, \ldots, a_{mn}$ are called the elements of the matrix. Each element of the matrix can be a real or a complex number or a function of one or more variables or any other objects. The element a_{ij} is the common element of i^{th} row and j^{th} column.

→ Matrix is represented by $A = [a_{ij}]_{m \times n}$ or $(a_{ij})_{m \times n}$

Types of Matrices.

1) **Vectors :-** A vector is a matrix with only one row or column. Its entries are called the components of the vector.

- A matrix of order $1 \times n$, that is, it has one row and n columns is called a row vector or a row matrix of order n and is written as

$$[a_{11}\ a_{12} \dots a_{1n}] \text{ or } [a_1\ a_2 \dots a_n]$$

- A matrix of order $m \times 1$, that is, it has one column and m rows is called a column vector or a column matrix of order m and is written as

$$\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{m1} \end{bmatrix} \text{ or } \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

2) **Square matrices :-** A matrix A of order $m \times n$ in which $m = n$, that is number of rows is equal to the number of columns is called a square matrix of order n.

→ The elements a_{ii} are called elements of the principal diagonal or the main diagonal.

→ The elements a_{ij}, when $i \neq j$ are called the off-diagonal elements.

3) **Null matrix :-** A matrix A of order $m \times n$ in which all the elements are zero is called a null matrix or a zero matrix and is denoted by O.

4) **Diagonal matrix :-** A square matrix A in which all the off - diagonal elements a_{ij}, $i \neq j$ are zero is called a diagonal matrix. A diagonal matrix is denoted by D and written as
$D = diag[a_{11}\ a_{12} \ldots a_{nn}]$.

→ If all the elements of a diagonal matrix of order n are equal, that is $a_{ii} = \alpha$ for all i, then the matrix is called a scalar matrix of order n.

→ If all the elements of a diagonal matrix of order n are 1, then the matrix is called an identity matrix of order n and denoted by I_n.

5) **Equality of two matrices :-**

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$ are said to be equal, when

i) they are of the same order, that is $m = p$, $n = q$ and

ii) Corresponding elements are equal, that is $a_{ij} = b_{ij}$ for all i, j.

6) **Submatrix :-**

A matrix obtained by omitting some rows and/or columns from a given matrix A is called a submatrix of A. The given matrix A, itself can be taken as the submatrix of A.

☆ Matrix Algebra :-

1) Multiplication of a matrix by a scalar :-
Let k be a scalar (real or complex) and $A = [a_{ij}]_{m\times n}$ be a given matrix. Then

$$B_{m\times n} = [k\, a_{ij}]_{m\times n}$$
$$= k\,[a_{ij}] = k A_{m\times n}.$$

is called scalar multiplication of a matrix A by scalar k.

2) Addition of two matrices :-
Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be matrices of same order then sum of these two matrices is denoted by A+B and defined as

$$A+B := [a_{ij} + b_{ij}]_{m\times n}.$$

→ Matrices of different orders cannot be added.

3) Multiplication of two matrices :-
Let $A_{m\times n}$ & $B_{p\times q}$ be two matrices then the product $C = A\cdot B$ can be defined only if $n = p$, that is no of columns of left hand matrix is equal to the no of rows of right hand matrix which are in multiplication.

→ Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{jk}]_{n\times q}$ then $C = A\cdot B = [c_{ik}]_{m\times q}$

Where $c_{ik} = a_{i1} b_{1k} + a_{i2} b_{2k} + \dots + a_{in} b_{nk}$

$$= \sum_{j=1}^{n} a_{ij} b_{jk}.$$

☆ ~~Minors:~~

☆ **Determinants :-**

With every square matrix A of order n, we associate a determinant of order n which is denoted by $\det(A)$ or $|A|$. The determinant has a value and this value is real if matrix is real and may be complex, if the matrix is complex. A determinant of order n is defined as

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

Where M_{ij} are the minors of a_{ij} of order $(n-1) \times (n-1)$.

☆ **Minors :-**

Let A be any square matrix, then minor M_{ij} of order $(n-1)$ is the determinant of submatrix obtained from A by removing ith row and jth column.

→ In general minors are determinant of any ~~sabm~~ square submatrix of given matrix A obtained by eliminating certain no of rows and certain no of columns.

☆ Singular and Non-singular matrix :-
The matrix A is said to be singular if $|A| = 0$ and non-singular if $|A| \neq 0$.

☆ Properties of determinants.
1) If there is zero row or zero column in matrix then the value of determinant is zero
2) $|A| = |A^T|$
3) $|A+B| \neq |A| + |B|$
4) $|A \cdot B| = |A| \cdot |B|$

→ It is possible that for two matrice A and B, the product AB is defined but BA may not be defined.

→ If both the product AB and BA are defined, then both the matrices AB and BA are square matrices.

For example

$A_{3\times2}$ & $B_{2\times3} \Rightarrow AB_{3\times3}$ & $BA_{2\times2}$.

→ $AB \neq BA$ in general

→ $AB = 0$ does not always imply that either $A = 0$ or $B = 0$

→ If $AB = AC$ that does not always imply $B = C$.

☆ Properties of the matrix addition and scalar multiplication :-

Let A, B, C are matrices which are conformable for addition and α, β be scalars. Then

1) $A+B = B+A$ (commutative)
2) $(A+B)+C = A+(B+C)$ (Associative)
3) $A+0 = A$
4) $A+(-A) = 0$, Here 0 represents null matrix
5) $\alpha(A+B) = \alpha A + \alpha B$
6) $(\alpha+\beta)A = \alpha A + \beta A$
7) $\alpha(\beta A) = \alpha\beta A$.
8) $1A = A$ and $0A = 0$

Here 1 & 0 are scalars on left hand side and on right hand side 0 is a null matrix.

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☆ Transpose of a matrix :-

The matrix obtained by interchanging the corresponding rows and columns of a given matrix A is called the transpose matrix of A and is ~~defined~~ as denoted by A^T or A'.

If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

then

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

☆ Properties of matrix multiplication :-

1) If A, B, C are matrices of order $m \times n$, $n \times p$ and $p \times q$ respectively, then

$$(AB)C = A(BC) \quad \text{(associative law)}$$

is a matrix of order $m \times q$.

2) If A is a matrix of order $m \times n$ and B, C are matrices of order $n \times p$, then

$$A(B+C) = AB + AC \quad \text{(left distributive law)}$$

3) If A, B are matrices of order $m \times n$ and C is a matrix of order $n \times p$, then

$$(A+B)C = AC + BC \quad \text{(right distributive law)}$$

4) If A is a matrix of order $m \times n$ and B is a matrix of order $n \times p$, then

$$\alpha(AB) = A(\alpha B) = (\alpha A)B \quad \text{for any scalar } \alpha.$$

☆ Rank of a matrix by minors :-

The rank of a matrix A is denoted by $r(A)$ or $\rho(A)$ is the order of the largest non-zero minor of A.

∴ The rank of a matrix is the largest value of r such that at least one $r \times r$ submatrix of A has determinant non-zero and all submatrix of A having order larger than r have determinant zero.

→ For any matrix of order $m \times n$, the rank is always less than or equal to $\min(m, n)$
i.e. $\rho(A_{m\times n}) \leq \min(m, n)$.

→ For square matrix $A_{n\times n}$

$$\rho(A) = n \text{ iff } |A| \neq 0.$$

if $|A| = 0$ then $\rho(A) < n$.

→ $\rho(A) = 0$ iff $A = O$ (i.e null matrix).

☆ Method of obtaining rank by minor.

1st way Let A be $m \times n$ matrix.

Step-1 :- Find determinants of highest order square submatrices of A. These submatrices will have order $= \min(m, n)$.
If any of them have non-zero determinant then $\rho(A) = \min(m, n)$.

Step-2 If all minors of order $= \min(m, n)$ have determinant value 0, then $\rho(A) < \min(m, n)$. Now find out determinants of square submatrices of A of order $\min(m,n) - 1$, and check whether any of them have non-zero value or not. If any of them have non-zero value then $\rho(A) = \min(m,n) - 1$, otherwise continue the same procedure by reducing order of square submatrices until we get a submatrix with non-zero determinant.

Note:- Any matrix which is not null matrix has rank ≥ 1.

2nd way

Instead of starting from higest order submatrix we could also start with least order submatrix i.e of order 1×1.

Step-1 Check whether given matrix is null matrix or not. If it is not null matrix then $\rho(A) \geq 1$.

Step-2 Check whether ~~any~~ square submatrix of order 2×2 has non-zero determinant or not. If any of them has non-zero determinant then $\rho(A) \geq 2$.

Step-3 Continue this process till we get an order r for which all square submatrix of order $r \times r$ have determinant zero and atleast ~~then $\rho(A) =$~~ one square submatrix of order $r-1$ has determinant non-zero. Then $\rho(A) = r-1$.

Ex:- Find rank of a matrix $A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$

Solⁿ => Here given matrix is of the order 2×3

$\therefore \rho(A) \leq \min(2,3) = 2$

$\therefore \rho(A) = 1$ or 2 as A is not null matrix.

Now let's consider submatrices of A of order 2×2, which are as follows.

$A_1 = \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$, $A_2 = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$, $A_3 = \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix}$

$\therefore |A_1| = \begin{vmatrix} 2 & 3 \\ 3 & 5 \end{vmatrix} = 10 - 9 = 1 \neq 0$

$\therefore |A_2| = \begin{vmatrix} 3 & 4 \\ 5 & 7 \end{vmatrix} = 21 - 20 = 1 \neq 0$

$\therefore |A_3| = \begin{vmatrix} 2 & 4 \\ 3 & 7 \end{vmatrix} = 14 - 12 = 2 \neq 0$

Here, minor of order 2 has non-zero determinant.

$\therefore \rho(A) = 2$

Note:- Here, it is sufficient to check only one submatrix of order 2×2 has determinant value non-zero.

A Find the rank of following matrices by minors.

1) $\begin{bmatrix} 1 & 3 & -4 \\ -1 & -3 & 4 \\ 2 & 6 & -8 \end{bmatrix}$ Ans:- rank = 1

2) $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -1 \\ 1 & -1 & 2 \\ 5 & 4 & -5 \end{bmatrix}$ Ans:- rank = 2

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3) $\begin{bmatrix} 1 & 1 & 1 \\ p & q & r \\ p^3 & q^3 & r^3 \end{bmatrix}$ H.W

Ans :- $|A| = (p-q)(q-r)(r-p)(p+q+r)$

$\therefore$ rank$(A) = 3$ if $p \neq q \neq r$ & $p+q+r \neq 0$

$= 2$ if $p \neq q \neq r$ & $p+q+r = 0$

$= 2$ if either $p = q$ or $q = r$ or $p = r$.

$= 1$ if $p = q = r$

4) $\begin{bmatrix} 2 & 1 & 5 & -1 \\ -1 & 2 & 5 & 3 \\ 3 & 2 & 9 & -1 \end{bmatrix}$ → Ans :- 2

H.W

5) $\begin{bmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{bmatrix}$ C.W

Ans :- rank $= 3$ if $a \neq 1, a \neq -2$

& rank $= 1$ if $a = 1$

$r(A) = 2$, if $a = -2$

6) $\begin{bmatrix} a & b & c \\ b & a & c \end{bmatrix}$ H.W

Ans :- rank $= 2$ if $a \neq b$

rank $= 1$ if $a = b$

★ Find all values of μ for which rank of the matrix

$\mu = 1, 2, 3$

Hint C.W

$$A = \begin{bmatrix} \mu & -1 & 0 & 0 \\ 0 & \mu & -1 & 0 \\ 0 & 0 & \mu & -1 \\ -6 & 11 & -6 & 1 \end{bmatrix}$$

If $a = -2$

$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix}$

is equal to 3.

(7) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 11 \end{bmatrix}$ CW

$r(A) = 2$

H.W.

(8) $\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix}$ CW

$r(A) = 2$

☆ Triangular Matrix :-

1) Upper triangular matrix :-

A square matrix is called upper triangular matrix if all elements below the principal diagonal are zero.

2) Lower triangular matrix :-

A square matrix is called lower triangular matrix if all elements above the principal diagonal are zero.

☆ Elementary Row Operations :-

Information

1) Interchange of two rows :-

If we interchange i^{th} row of given matrix with j^{th} row then we denote this operation by $R_i \leftrightarrow R_j$

2) Multiplication of a row by a nonzero constant :-

If we multiply ~~i^{th}~~ all elements of i^{th} row of given matrix by a nonzero constant k then we will denote it by $R_i \rightarrow kR_i$.

3) Addition of a constant multiple of one row to another row :-

If we multiply j^{th} row of given matrix by a constant k and then add it to i^{th} row element wise then we denote this operation by $R_i \rightarrow R_i + kR_j$.

☆ Effects of elementary row operations :-

1) If any two rows are interchanged, then the value of the determinant is multiplied by (-1).

2) If a row is multiplied by a constant k then the value of the determinant is multiplied by k.

3) The value of a determinant is unchanged if we add constant multiple of one row to another row.

☆ Row Equivalent Matrices :-

Matrices obtained by performing elementary row operation on a given matrix are called row equivalent matrices.

→ Row-equivalent matrices have the same rank.
→ Every matrix is row-equivalent to itself.

☆ Row-Echelon Form of a Matrix

A matrix is in row-echelon form if the following are satisfied.

i) If the i^{th} row contains all zeros, it is true for all subsequent rows.

ii) If a column contains a non-zero entry of any row, then every subsequent entry in this column is zero, that is if the i^{th} and $(i+1)^{th}$ rows are both non-zero rows, then the initial non-zero entry of the $(i+1)^{th}$ row appear in a right hand column

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than that of the i^{th} row.

(iii) Rows containing all zeros occur only after all non-zero rows.

Note :- If A is a square matrix, then the row-echelon form is an upper trianglular matrix.

→ In similar way one can also discuss column-echelon form and it will be lower triangular matrix for a square matrix.

★ **Rank of Matrix by Row-Echelon Form :-**

The number of non-zero rows in the row echelon form of a matrix A gives the rank of the matrix A.

Ex:-1 Reduce the following matrices to row-echelon form and find it's rank.

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$$ CW

Sol^n:-

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} \boxed{1} & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 14 & 12 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 + 2R_1 \end{matrix} \left(\begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + 2R_1 \end{matrix} \right)$$

$$\sim \begin{bmatrix} \boxed{1} & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 + 2R_2 \; (R_3 \to R_3 + 2R_2)$$

$$\sim \begin{bmatrix} \boxed{1} & 3 & 5 \\ 0 & \boxed{1} & 6/7 \\ 0 & 0 & 0 \end{bmatrix} \quad -\tfrac{1}{7}R_2 \quad (R_2 \to -\tfrac{1}{7}R_2)$$

This is the row echelon form of A.
Since the number of non-zero rows in the row echelon form is 2, we get $r(A) = 2$.

☆ Reduce the following matrices to row-echelon form and find their ranks.

1) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{bmatrix}$ Ans :- 2

C.W

2) $\begin{bmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 4 & -1 & 5 \end{bmatrix}$ Ans :- 3

C.W

3) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ Ans :- 4

H.W

4) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 1 & -2 \\ 3 & 2 & 4 & 2 \end{bmatrix}$ Ans :- 2

H.W

5) $\begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$ Ans :- 2

C.W

6) $\begin{bmatrix} 1 & -2 \\ 0 & 0 \\ -3 & 6 \end{bmatrix}$ Ans :- 1.

H.W

Date :

☆ Reduced - Row Echelon Form :-

In row echelon form of A, if we make every entry above the first non-zero entry of row zero by row operations then it is called reduced - row echelon form of A.

→ If A is a non-singular square matrix then it's reduced - row echelon form is an identity matrix of same order as of A.

☆ Inverse of a Matrix :-

The inverse of an $n \times n$ matrix $A = [a_{ij}]_{n\times n}$ is denoted by A^{-1} and is an $n \times n$ matrix such that

$$AA^{-1} = A^{-1}A = I$$

where I is the unit matrix.

→ If A has an inverse, then A is called non-singular matrix. If A has no inverse, then A is called singular matrix.

→ If A has an inverse, the inverse is unique.

☆ Existence of The Inverse :-

The inverse A^{-1} of an $n \times n$ matrix A exists if and only if rank $A = n$ thus if and only if $\det A \neq 0$.

Hence A is non-singular if rank $A = n$, and is singular if rank $A < n$.

Determination of the Inverse by the Gauss - Jordan Method.

For any square matrix $A_{n\times n}$.

Step-I Check whether $|A| \neq 0$ or not.

(If $|A| = 0$ then inverse of A does not exist.)

Step-II If $|A| \neq 0$, then consider $I_{n\times n}$ and write the augmented matrix $[A \mid I]_{n\times 2n}$.

Step-III Use elementary row operations on the augmented matrix $[A \mid I]$ and convert into the augmented matrix $[I \mid B]$.

i.e $[A \mid I] \xrightarrow[\text{row - operations}]{\text{Elementary}} [I \mid B]$

Then B is the inverse of A and denoted by A^{-1}.

Ex-1 Using Gauss - Jordan method, find the inverse of the matrix

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$$

C.W

Soln: Here $|A| = -1(-4-3) - 1(12+1) + 2(9-1)$

$= 7 - 13 + 16 = 10 \neq 0$.

$\therefore A^{-1}$ exists.

Now, consider

$$[A \mid I] = \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 3 & -1 & 1 & 0 & 1 & 0 \\ -1 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \quad R_1 \to (-1) R_1.$$

Date :

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \qquad \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right] \qquad R_2 \to \tfrac{1}{2} R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array}\right] \qquad \begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 7/2 & 3/2 & 1/2 & 0 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right] \qquad R_3 \to \left(-\tfrac{1}{5}\right) R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/10 & 2/10 & 3/10 \\ 0 & 1 & 0 & -13/10 & -2/10 & 7/10 \\ 0 & 0 & 1 & 4/5 & 1/5 & -1/5 \end{array}\right] \qquad \begin{array}{l} R_1 \to R_1 - 3/2 R_3 \\ R_2 \to R_2 - 7/2 R_3 \end{array}$$

$$A^{-1} = \begin{bmatrix} -7/10 & 2/10 & 3/10 \\ -13/10 & -2/10 & 7/10 \\ 4/5 & 1/5 & -1/5 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{10} \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix}$$

☆ Use - Gauss - Jordan Method to find the inverse of following matrices if exist.

1) H.W $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ Ans :- $A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$

2) H.W $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$ Ans :- $\frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

3) C.W $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ Ans :- $\frac{1}{4}$ -16 ? $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

4) H.W $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ Ans :- $\frac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$

5) C.W $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Ans :- $A^{-1} = A$.

6) C.W $\begin{bmatrix} 1 & 2 & 5 \\ 0 & -1 & 2 \\ 2 & 4 & 10 \end{bmatrix}$ Ans :- A^{-1} does not exist

7) $\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{bmatrix}$ Ans :- $\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix}$

Date :

☆ Solution of Linear System of Equations :-

One can solve a system of n equations in n-unknowns, $AX = b$ (provided $|A| \neq 0$) by the inverse matrix method and the Cramer's rule.

Here, we are going to discuss a method for solving a general system of m equations in n unknown given by.

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

which can be written in matrix form

$$AX = B$$

Where.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times}$$

→ Here if a column vector $b = 0$ then the system is known as Homogeneous system of linear equations.

→ If a column vector $b \neq 0$ then the system is known as Non homogeneous system of linear equations.

→ The matrix

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right] \quad \text{---} \quad (*)$$

is called the augmented matrix and has m rows and $(n+1)$ columns. The augmented matrix describes completely the system of equations. The solution of the system of equations is an n-tuple $(x_1, x_2, \ldots, x_n)$ that satisfies all the equations.

There are three possibilities:

1) The system has a unique solution.
2) The system has no solution.
3) The system has infinite number of solutions.

→ The system of equation is said to be consistent, if it has atleast one solution and inconsistent, if it has no solution.

→ Using the concepts of ranks and row-echelon form, we now obtain the solution of the linear system of equations (if exist).

★ Gauss-Elimination Method for Non-Homogeneous Eq^{ns}:-

Step-1 Consider the augmented matrix of order $m \times (n+1)$ as $[A|B]$ represented by ⊛

$$[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

for given non-homogeneous system of m equations in n-unknowns.

Step-2 Reduce $[A|B]$ to the row echelon form by using elementary row operations.

Step-3 Find the rank($[A|B]$) and rank(A).

Date :

Step-4 Apply Consistency test as follow.

Case - I

If rank (A) = rank ([A|B]) = n = no of variables

then system has a unique solⁿ.

Case - II

If rank (A) = rank ([A|B]) < n = no of variables.

then system has infinitely many solutions.

Case - III

If rank (A) ≠ rank ([A|B]) then system has no solution.

Step-5 If the system is consistent (i.e case - I or case - II) then obtain solution by.

Case - I by using the back substitution method.

Case - II Let's assume rank (A) = rank ([A|B]) = r and $r < n$ then take (n - r) no. of variables arbitrary constants $k_1, k_2, \dots, k_r$ such that we have r number of eqns in r - unknowns. Now use back - substitution Method :-

Ex :- 1 Solve the system of equations (if possible) using Gauss - elimination method.

C.W.

i) $2x + y - z = 4$

$x - y + 2z = -2$

$-x + 2y - z = 2$

Solⁿ :- Given system can be written as

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix}$$

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Now $[A|B] = \left[\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 1/2 & -1/2 & 2 \\ 0 & -3/2 & 5/2 & -4 \\ 0 & 5/2 & -3/2 & 4 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{1}{2} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 + R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 1/2 & -1/2 & 2 \\ 0 & \boxed{1} & -5/3 & 8/3 \\ 0 & 0 & 8/3 & -8/3 \end{array}\right] \quad \begin{array}{l} R_2 \to (-2/3) R_2 \\ R_3 \to R_3 - 5/2 R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 1/2 & -1/2 & 2 \\ 0 & \boxed{1} & -5/3 & 8/3 \\ 0 & 0 & \boxed{1} & -1 \end{array}\right] \quad R_3 \to (3/8) R_3$$

Here rank([A|B]) = 3

and rank (A) = 3

$\therefore$ rank(A) = rank([A|B]) = 3 = no of variables

$\therefore$ Given system has unique solution.

Using the back substitution method, we obtain the solution as

$$z = -1,$$

$$y - \frac{5}{3} z = \frac{8}{3} \Rightarrow y = \frac{8}{3} - \frac{5}{3} \Rightarrow y = 1,$$

& $$x + \frac{1}{2} y - \frac{1}{2} z = 2 \Rightarrow x = 2 - \frac{1}{2} - \frac{1}{2} \Rightarrow x = 1,$$

Therefore, the system of equations has the unique solution

$$(x, y, z) = (1, 1, -1).$$

ii) $2x + z = 3$

C.W

$x - y + z = 1$

$4x - 2y + 3z = 3$

Solⁿ:- Given system can be written as

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}$$

Now

$$[A|B] = \left[\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 4 & -2 & 3 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1/2 & 3/2 \\ 0 & -1 & 1/2 & -1/2 \\ 0 & -2 & 1 & -3 \end{array}\right] \quad \begin{array}{l} R_1 \to \frac{1}{2} R_1 \\ R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 4R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1/2 & 3/2 \\ 0 & \boxed{1} & -1/2 & 1/2 \\ 0 & 0 & 0 & -2 \end{array}\right] \quad \begin{array}{l} R_2 \to (-1) R_2 \\ R_3 \to R_3 + 2R_2 \end{array}$$

$\therefore$ rank$([A|B]) = 3$ and rank$(A) = 2$

$\therefore$ rank$([A|B]) \neq$ rank(A).

$\therefore$ Given system is inconsistent.

Hence, the system of equations has no solution.

iii) $x - y + z = 1$

C.W

$2x + y - z = 2$

$5x - 2y + 2z = 5$.

Solⁿ:- Given system can be written as

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 5 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

Now

$$[A|B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & -1 & 2 \\ 5 & -2 & 2 & 5 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 5R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 1 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to \frac{1}{3} R_2 \\ R_3 \to R_3 - 3R_2 \end{array}$$

$\therefore$ rank$([A|B]) = 2$ & rank$(A) = 2$

$\therefore$ rank$([A|B]) =$ rank$(A) = 2 < 3 =$ no of unknown

$\therefore$ The system has ~~infinitely~~ infinite number of solution.

Here we need to assume $3-2 = 1$ variable as an arbitrary constant say $z = k_1$

Then by back substitution method.

$$y - z = 0$$

$$\therefore y = z \Rightarrow y = k_1$$

$$\& \quad x - y + z = 1$$

$$\therefore x - k_1 + k_1 = 1$$

$$\therefore \boxed{x = 1}$$

$\therefore$ Therefore the solution of the system is given by

$$(x, y, z) = (1, k_1, k_1)$$

where k_1 is an arbitrary constant.

☆ Solve the following system of equations (if possible) using Gauss elimination method.

1) $4x - 3y - 9z + 6w = 0$

$2x + 3y + 3z + 6w = 6$

C.W

$4x - 21y - 39z + 6w = -24$

Ans :- (x, y, z, w)
$= \left(1 + k_1 - 2k_2, \frac{4 - 5k_1 - 2k_2}{3}, \ldots\right.$

k_1, k_2 are arbitrary.

2) $x + 2y - 2z = 1$

$2x - 3y + z = 0$

C.W

$5x + y - 5z = 1$

Ans :- $(x, y, z) = (1, 1, 1)$.

$3x + 14y - 12z = 5$.

3) $x - 4y + 7z = 8$

$3x + 8y - 2z = 6$

H.W

Ans :- No solution

$7x - 8y + 26z = 3$

4) $x + 4y + 7z = 1$

Ans :- $(x, y, z) = (1 + k_1, -2k_1, k_1)$

$2x + 3y + 8z = 2$

k_1 is arbitrary

H.W

$x + 2y + 3z = 1$

☆ Gauss - Elimination Method for Homogeneous Eq^{ns}.

In Homogeneous system of eq^{ns} $b = 0$

$\therefore$ rank([A|B]) = rank(A).

Hence, Homogeneous system is always consistent.

Therefore, there are only two possibilities :

1) The system has a unique solution.

2) The system has infinite number of solution.

$\Rightarrow$ Gauss - Elimination method can be used as follow.

X

Step-1 Obtain row-echelon form of A and find rank(A).

Date :

Step-2

Case-I If rank (A) = n = no of unknowns then system has unique sol^n and $(x_1, x_2, \ldots, x_n) = (0, 0, \ldots, 0)$ is always sol^n of homogeneous system.

$\therefore$ The system has only sol^n $(0, 0, \ldots, 0)$ which is also known as trivial sol^n.

Case-II If rank (A) < n = no of unknowns, then system has infinitely many solutions.

Now, if rank (A) = r < n then take (n-r) no of varibles as arbitrary constants $k_1, k_2, \ldots, k_r$ such that we have r number of eq^{ns} in r-unknowns. Now use back-substitution method. These solutions are known as non-trivial solutions.

op lenty

X

☆ Solve following homogeneous systems by Gauss-Elimination method.

H.W 1) $3x + y + 2z = 0$, $x - 2y + 3z = 0$, $x + 5y - 4z = 0$

Ans :- $(x, y, z) = (-k_1, k_1, k_1)$ & k_1 is arbitrary.

H.W 2) $3x - 11y + 5z = 0$, $4x + y - 10z = 0$, $4x + 9y - 6z = 0$

Ans :- $(x, y, z) = (0, 0, 0)$.

C.W 3) $2x - y - 3z + w = 0$, $x + y + z + w = 0$, $2x - 7y - 13z - w = 0$

$-x + 5y + 9z + w = 0$

Ans :- $(x, y, z, w) = \left(\frac{2(k_1 - k_2)}{3}, \frac{-5k_1 - k_2}{3}, k_1, k_2 \right)$

where k_1, k_2 are arbitrary.

☆ Rank - Nullity Theorem :-

Consider the homogeneous system of equations

$$AX = 0$$

Where A is an $m \times n$ matrix. Then set of all solutions of this system is known as null space and it's dimension is called the nullity of A. Therefore we can write

$$\boxed{\text{rank}(A) + \text{nullity}(A) = n}$$

Ex:-1 Check that for what values of λ and μ the equations $x+y+z=6$, $x+2y+3z=10$, $x+2y+\lambda z=\mu$ have

C.W

i) a unique solution

ii) infinite number of solutions.

iii) no solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

Ex:-2 Find the value of k so that the equations $x+y+3z=0$, $4x+3y+Kz=0$, $2x+y+2z=0$ have a non-trivial solution. $K=8$

C.W

Ex:-3 Find the conditions on a, b and c such that the system $x+2y+3z=a$, $2x+5y+3z=b$, $x+8y=c$ to be consistent. Ans. for all values of a, b, c eq^n are consistent

H.W

(1)
$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

(1) unique sol^n :- $\lambda \neq 3$, μ any

(2) infinite no :- $\lambda = 3$, $\mu = 10$

(3) no sol^n :- $\lambda = 3$, $\mu \neq 10$

(2)
$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 4 & 3 & K & 0 \\ 2 & 1 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -1 & K-3 & 0 \\ 0 & -1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 3-K & 0 \\ 0 & -1 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 3-K & 0 \\ 0 & 0 & -1-K & 0 \end{array}\right]$$

$K \neq -1$

(3)
$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 2 & 5 & 3 & b \\ 1 & 8 & 0 & c \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & -3 & b-2a \\ 0 & 6 & -3 & c-a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & -3 & b-2a \\ 0 & 0 & 15 & c-a-b+2a \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 1 & -3 & b- \\ 0 & 0 & 1 & \frac{c+a-b}{15} \end{array}\right]$$

1

Complex number :-

A complex number z is an ordered pair (x,y) of real numbers x and y, written

$$z = (x, y) = x + iy \qquad \text{where } i = \sqrt{-1}$$

x is called real part

y is " imaginary part.

$$x = \operatorname{Re} z \;,\quad y = \operatorname{Im} z$$

* Note:- If $\operatorname{Re} z = 0$ then $z = iy$ is called purely imaginary numbers.

If $I(z) = 0$ then $z = x$ is called purely real number

i can be written as $(0,1) = 0 + i1$.

Def^n

If $z = x + iy$ is a complex number then the complex conjugate of z is defined as $x - iy$ and is denoted by $\bar{z}$.

$$\operatorname{Re}(z) = \frac{z + \bar{z}}{2} \qquad \operatorname{Im}(z) = \frac{z - \bar{z}}{2}.$$

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2$$

Algebra of complex numbers :-

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are two complex numbers, then

(1) equality :- $z_1 = z_2$ iff $x_1 = x_2$ and $y_1 = y_2$

i.e their real parts and imaginary part are equal.

* Addition :-

$$z_1 \pm z_2 = (x_1 \pm i y_1) + (x_2 \pm i y_2) = (x_1 + x_2) \pm i (y_1 + y_2)$$

* Multiplication :-

$$z_1 \cdot z_2 = (x_1 + i y_1)(x_2 + i y_2)$$
$$= (x_1 \cdot x_2 - y_1 y_2) + i (x_1 y_2 + y_1 x_2)$$

* Division :-

Assuming $z_2 \neq 0$, $\frac{z_1}{z_2} = \frac{x_1 + i y_1}{x_2 + i y_2}$

$$= \frac{x_1 + i y_1}{x_2 + i y_2} \times \frac{x_2 - i y_2}{x_2 - i y_2}$$

$$= \frac{(x_1 x_2 + y_1 y_2) + i (x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$$

provided $x_2^2 + y_2^2 \neq 0$.

Results :-

(i) $\overline{(z_1 \pm z_2)} = \bar{z}_1 \pm \bar{z}_2$

(ii) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

(iii) $\overline{(z_1 \cdot z_2)} = \bar{z}_1 \cdot \bar{z}_2$

(iv) $z = \bar{z}$ iff z is real.

Geometrical representation of complex numbers:-

The complex number $z = x + iy$ can be expressed as the ordered pair (x, y) of the real numbers. this can be represented by a point P in the xy-plane called a complex plane or (we denote this complex plane by $\mathbb{C}$) Argand plane, where x is called real axis and y-axis is called an imaginary axis.

$|z|$ = distance between $(0,0)$ & point (x,y) and it is defined as $|z| = \sqrt{x^2 + y^2}$.

* Polar form of a non zero complex numbers.

Let $z = x+iy$ be a non zero complex number and let r and θ be polar coordinate of order pair (x, y). Since $x = r\cos\theta$, $y = r\sin\theta$ the complex number z can be written as

$$z = x+iy = r\cos\theta + i\, r\sin\theta$$

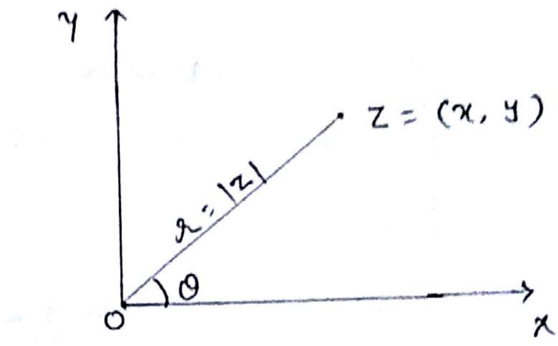
$$= r\,(\cos\theta + i\sin\theta) \qquad \text{—— (1)}$$

where $r = \sqrt{x^2+y^2}$ i.e. $r = |z|$.

Define $\cos\theta + i\sin\theta = e^{i\theta}$ where θ is to be measured in radians.

Then equation (1) will be

$$z = r\,e^{i\theta}$$



(Complex plane)

Thus representation of ~~β~~ z is called exponential Polar form of z.

→ If $z = 0$, then the coordinate θ is undefined so it is understood that $z \neq 0$ whenever polar coordinates are used.

* argument and Principal argument of z.

We know that $z = r e^{i\theta}$ for a non zero complex number, where $r = |z|$ and θ is a real number represents the angle, measured in radians, that z makes with the positive real axis.

That means θ has an infinite number of possible values including negative, that differ by integral multiples of 2π.

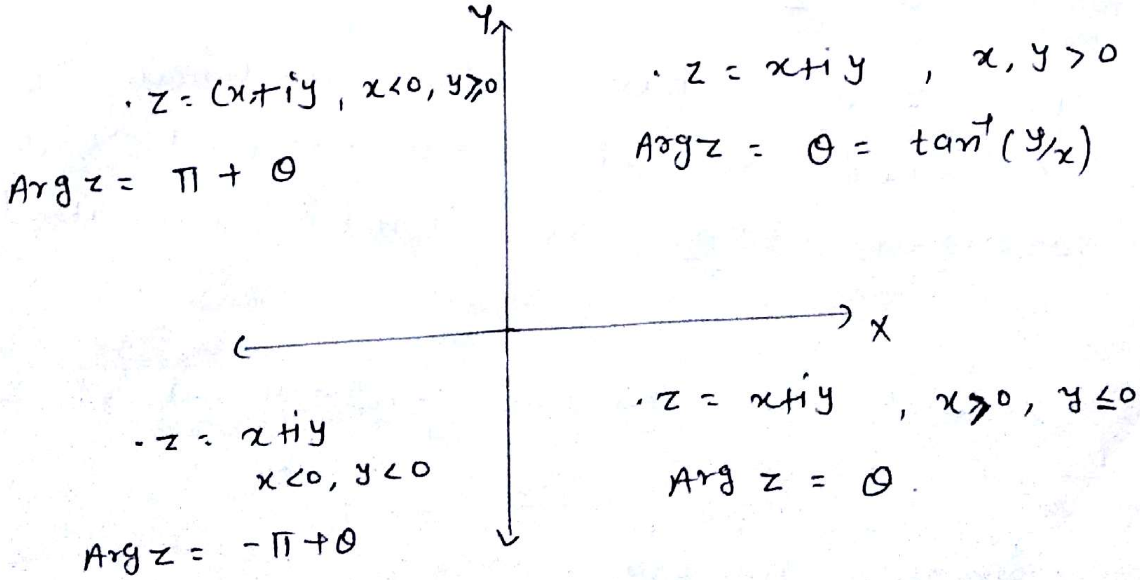
These values of θ can be obtained from the equation $\tan\theta = y/x$, where the quadrant containg the point corresponding to z must be specified.

Each value of θ is called argment of z, and the set of all such values is denoted by $\arg z$.

The value θ which is in $(-\pi, \pi]$ is called principal value of $\arg z$, and it is denoted by $\operatorname{Arg} z$.

Therefore $\arg z = \operatorname{Arg} z + 2n\pi, \quad n \in \mathbb{Z}$.

The following diagram helps to find the principlal argument of a complex number $z \neq 0$.



where $\theta = \tan^{-1}(y/x) \in [-\pi/2, \pi/2]$

* Properties of argument.

(i) $\arg(z) = \arg(\bar{z})$

(ii) $\arg(z_1 \cdot z_2) = \arg(z_1) + \arg(z_2)$

(iii) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

(iv) ~~arg~~ $\arg(z^n) = n \arg(z)$.

where z, z_1, z_2 are non zero complex numbers.

Note: (i) $|z_1 + z_2| \leq |z_1| + |z_2|$ $\quad (z_1, z_2 \in \mathbb{C})$

(ii) $|z_1 - z_2| \geq \big| |z_1| - |z_2| \big|$

Ex: Express the following numbers in the polar form. and finds its argument & principal argument

(a) $\sqrt{3} + i$ (b) $-\sqrt{3} + i$ (c) $-\sqrt{3} - i$ H.W (d) $\sqrt{3} - i$ H.W.

Ans: (a) $\sqrt{3} + i$

Here $z = \sqrt{3} + i$ $\therefore$ $x = \sqrt{3}$ & $y = 1$

Therefore $r = \sqrt{x^2 + y^2} = 2$.

Now observe that the given complex number is in first quadrent so

$\text{Arg}(\sqrt{3} + i) = \tan^{-1}(1/\sqrt{3}) = \pi/6$.

& $\arg(\sqrt{3} + i) = \pi/6 + 2n\pi$, $n \in \mathbb{Z}$.

so Polar form of $\sqrt{3} + i$ is $2(\cos \pi/6 + i \sin \pi/6)$

& Exponential form of $\sqrt{3} + i$ is $2e^{i\pi/6}$

(b) Here $z = -\sqrt{3}+i$ so $x = -\sqrt{3}$ & $y = 1$

$\therefore r = 2$. Also observe that $\sqrt{3}+i$ is in 2^{nd} quadrant so $Arg(-\sqrt{3}+i) = \pi + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$$= \pi - \pi/6$$

$$= 5\pi/6.$$

Therefore

$$\arg(-\sqrt{3}+i) = \left\{ \frac{5\pi}{6} + 2n\pi : n \in \mathbb{Z} \right\}$$

so Polar form of $\sqrt{3}+i$ is $2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

and exponential form of $-\sqrt{3}+i$ is $2e^{i\frac{5\pi}{6}}$.

H.W

Ex: Find the moduli and arguments of the following complex numbers. Also find principal argument.

(i) $\frac{1-i}{1+i}$ (ii) $\frac{3-i}{2+i} + \frac{3+i}{2-i}$

(iii) $1+i$ iv) $3+3\sqrt{3}i$ v) $\frac{1+2i}{1-(1-i)^2}$

vi) $\frac{(1+i)^2}{1-i}$ vii) $\sqrt{\frac{1+i}{1-i}}$ viii) $(4+2i)(-3+\sqrt{2}i)$

* DE Moivre's Formula:

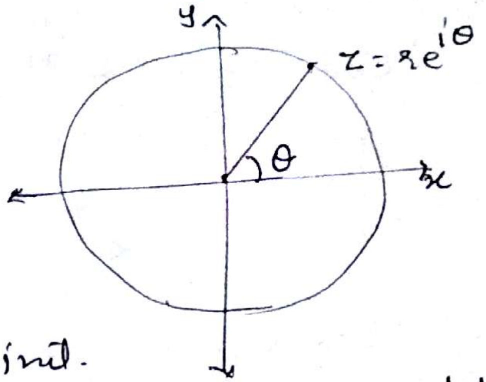
Let $n \in \mathbb{Z}$ then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

i.e. $(e^{i\theta})^n = e^{in\theta}$.

* consider any complex number $z = re^{i\theta}$ which is lying on a circle centered at the origin with radius r. (see Following fig.)

As θ increased z moves around the circle in the counterclockwise direction.



So in perticular when θ is increased by 2π we will reach at the original point. Also the same is true when θ is decreased by 2π. So one can say that two complex numbers $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ are equal if and only if

$$r_1 = r_2 \quad \text{and} \quad \theta_1 = \theta_2 + 2k\pi, \quad k \in \mathbb{Z}.$$

This observation help us to find the n^{th} roots of complex number.

* **Roots of complex numbers:**

We want to find n^{th} roots of complex number $z_0 = r_0 e^{i\theta_0}$ (where $n \in \mathbb{N}$).

Assume that $z = re^{i\theta}$ is the n^{th} root of complex number z_0.

Therefore $$z^n = r_0 e^{i\theta_0}$$

i.e. $$r^n e^{in\theta} = r_0 e^{i\theta_0}$$

This gives $r^n = r_0$ and $n\theta = \theta_0 + 2k\pi$, $k \in \mathbb{Z}$.

$\therefore$ $r = r_0^{1/n}$ and $\theta = \dfrac{\theta_0}{n} + \dfrac{2k\pi}{n}$, $k \in \mathbb{Z}$.

So the complex numbers

$$z = r_0^{1/n} e^{i\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)}, \quad k \in \mathbb{Z} \qquad \text{—— (1)}$$

are the n^{th} roots of z_0.

Let $\omega = e^{\frac{i2\pi}{n}}$ then $\omega^n = 1$. so equation (1) can be written as

$$z = r_0^{1/n} e^{\frac{i\theta_0}{n}} . \omega^k \quad , \quad k \in \mathbb{Z}.$$

Therefore n distinct n^{th} roots of complex number z_0 are

$k=0$ then $z = r_0^{1/n} e^{\frac{i\theta_0}{n}} = c_0$

$k=1$ then $z = r_0^{1/n} e^{\frac{i\theta_0}{n}} \omega = c_0\omega$

$\vdots$

$k=n-1$ then $z = r_0^{1/n} e^{\frac{i\theta_0}{n}} . \omega^{n-1} = c_0\omega^{n-1}$

where $\omega = e^{\frac{i2\pi}{n}}$ which is n^{th} root of unity.

Note: Continued products means product of all the roots.

Ex! Find all values of the following.

(i) $(-1)^{1/5}$ (ii) $(1-i)^{2/3}$ (iii) $(1+i)^{1/8}$. H.W

Ans: (i) Let $z_0 = (-1)$.

then $z_0 = e^{i\pi}$. so Here $\theta_0 = \pi$ & $r_0 = 1$

Therefore all values of $(-1)^{1/5}$ are

$$e^{i\frac{\pi}{5}}, e^{i\pi/5}.e^{i\frac{2\pi}{5}}, e^{i\pi/5}.e^{i\frac{4\pi}{5}}, e^{i\pi/5}.e^{i\frac{6\pi}{5}}, e^{i\pi/5}.e^{i\frac{8\pi}{5}}.$$

i.e $e^{i\pi/5}, \quad e^{i3\pi/5}, \quad e^{i\pi}, \quad e^{i\frac{7\pi}{5}}, \quad e^{i9\pi/5}$

(ii) $(1-i)^{2/3} = (-2i)^{1/3}$

so take $z_0 = -2i = 2e^{i\pi/2}$

Therefore $r_0 = 2$ & $\theta_0 = \pi/2$ & $n = 3$

Now $c_0 = r_0^{1/3} e^{\frac{i\theta_0}{3}}$ and $\omega_0 = e^{\frac{i2\pi}{3}}$

$= 2^{1/3} \cdot e^{i\pi/6}$

so all values of $(-2i)^{1/3}$ are

$2^{1/3} e^{i\pi/6}$, $2^{1/3} e^{\frac{i5\pi}{6}}$, $2^{1/3} e^{i9\pi/6}$.

Hence all values of $(1-i)^{2/3}$ are

$2^{1/3} e^{i\pi/6}$, $2^{1/3} e^{i5\pi/6}$, $2^{1/3} e^{i9\pi/6}$.

Ex: Find continued Product of all the values of $\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]^{3/4}$

Ans: $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ can be written as

$\cos\pi/3 + i\sin\pi/3$.

Therefore

$$\left[\frac{1}{2} + i\sqrt{3}/2\right]^{3/4} = \left[\cos\pi/3 + i\sin\pi/3\right]^{3/4}$$

$$= \left[\cos\pi + i\sin\pi\right]^{1/4}$$ (using De moivre's)

$$= (-i)^{1/4}$$

Now we find 4th root of complex number $(-i)$.

so take $z_0 = (-i)$

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Therefore $z_0 = e^{i\pi}$, $r_0 = 1$, $\theta = \pi$.

and $n = 4$.

Now $c_0 = r_0^{1/4} e^{\frac{i\theta_0}{4}}$

$= e^{i\pi/4}$

and $\omega = e^{\frac{i2\pi}{4}} = e^{i\pi/2}$

Therefore all 4^{th} roots of (i) are

$e^{i\pi/4}$, $e^{i3\pi/4}$, $e^{i5\pi/4}$, $e^{i7\pi/4}$.

Hence these are all values of $\left[\frac{1}{2} + i\frac{\sqrt{3}}{2}\right]^{3/4}$.

Now continued product is the product of all values. Hence continued product is

$$e^{i\pi/4} \cdot e^{i3\pi/4} \cdot e^{i5\pi/4} \cdot e^{i7\pi/4} = e^{i\left[\pi/4 + \frac{3\pi}{4} + \frac{5\pi}{4} + \frac{7\pi}{4}\right]}$$

$$= e^{i16\pi/4}$$

$$= e^{i\pi 4}$$

$$= \cos 4\pi + i \sin 4\pi$$

$$= 1.$$

Ex: Solve the following equations:

H.W (i) $x^6 - i = 0$ (ii) $x^7 + x^4 + i(x^3+1) = 0$

(iii) $x^4 - x^3 + x^2 - x + 1 = 0$.

Ans: (ii) $x^7 + x^4 + i(x^3+1) = 0$ —— (1)

$\therefore x^4(x^3+1) + i(x^3+1) = 0$ $\therefore (x^3+1)(x^4+i) = 0$

$\therefore \quad (x^3+1) = 0 \quad$ and $\quad (x^4+i) = 0$

First we find all values of x such that

$x^3+1=0 \quad$. i.e $\quad x = (-1)^{1/3}$

So Take $\quad z_0 = (-1)$

Then $\quad z_0 = e^{i\pi} \quad$ so $r_0 = 1, \quad \theta_0 = \pi \quad n=3.$

Now $\quad c_0 = r_0^{1/n} e^{i\frac{\theta_0}{n}} = e^{i\pi/3}$

& $\quad \omega = e^{i2\pi/3}$

- So all values of $(-1)^{1/3}$ are

$e^{i\pi/3}, \quad e^{i\pi}, \quad e^{i\frac{\pi 5}{3}}$

Now $\quad x^4+i=0 \quad$ implies $\quad x^4 = (-i)^{1/4}$

Take $\quad z_0 = (-i) = e^{i\pi}, \quad r_0 = 1, \quad \theta = \pi, \quad n=4$

Therefore $\quad c_0 = r_0^{1/4} e^{i\pi/4} = e^{i\pi/4}$

and $\omega = e^{i\pi/2}$

- Therefore all values of $(-i)^{1/4}$ are

$e^{i\pi/4}, \quad e^{i3\pi/4}, \quad e^{i5\pi/4}, \quad e^{i7\pi/4}$

Hence roots of equation (1) are

$e^{i\pi/3}, \ e^{i\pi}, \ e^{i5\pi/3}, \ e^{i\pi/4}, \ e^{i3\pi/4}, \ e^{i5\pi/4}, \ e^{i7\pi/4}.$

(iii) $\quad x^4 - x^3 + x^2 - x + 1 = 0$

Ans. multiply both the side by $(x+1)$ Then

$x^5+1 = 0$

$\therefore \quad x = (-1)^{1/5}$

Take $z_0 = -1 = e^{i\pi}$, so $r_0 = 1$, $\theta_0 = \pi$, $n = 5$

Therefore $c_0 = r_0^{1/n} e^{\frac{i\theta_0}{n}} = e^{i\pi/5}$

and $\omega = e^{\frac{i2\pi}{5}}$

Therefore all values of $(-1)^{1/5}$ are

$e^{i\pi/5}$, $e^{i\pi/5} \cdot e^{i2\pi/5}$, $e^{i\pi/5} \cdot e^{i4\pi/5}$, $e^{i\pi/5} \cdot e^{i6\pi/5}$,

$e^{i\pi/5} \cdot e^{i8\pi/5}$.

i.e. $e^{i\pi/5}$, $e^{i3\pi/5}$, $e^{i\pi}$, $e^{i7\pi/5}$, $e^{i9\pi/5}$.

Now we know that $x^5 + 1 = 0$

i.e $(x+1)(x^4 - x^3 + x^2 - x + 1) = 0$

and there are only four roots of $x^4 - x^3 + x^2 - x + 1 = 0$. Also we know that the complex roots are occurs in the pair so the roots of the equation $x^4 - x^3 + x^2 - x + 1 = 0$ are $e^{i\pi/5}$, $e^{i3\pi/5}$, $e^{i7\pi/5}$, $e^{i9\pi/5}$.

<u>Ex:</u> If α, β, γ and δ are the roots of $x^4 + x^3 + x^2 + x + 1 = 0$, then find its value and show that $(1-\alpha)(1-\beta)(1-\gamma)(1-\delta) = 5$

H.W <u>Ex:</u> Using De moivre's theorem prove that
$2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$
where $x = 2\cos\theta$.

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Ex Express $\dfrac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4}$ in the form $x+iy$

$$\Rightarrow \frac{(\cos\theta + i\sin\theta)^8}{(\sin\theta + i\cos\theta)^4} = \frac{(\cos\theta + i\sin\theta)^8}{(\frac{1}{i}\sin\theta + \cos\theta)^4 (i)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{1 \times (\cos\theta - i\sin\theta)^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{(\cos(-\theta) + i\sin(-\theta))^4}$$

$$= \frac{(\cos\theta + i\sin\theta)^8}{(\cos\theta + i\sin\theta)^{-4}}$$

$$= (\cos\theta + i\sin\theta)^{12} = \cos 12\theta + i\sin 12\theta$$

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$$= \frac{(\cos\theta + i\sin\theta)^8}{([(\cos\theta + i\sin\theta)]^{-1})^4} = \frac{(\cos\theta + i\sin\theta)^8}{((\cos\theta + i\sin\theta))^{-4}} = (\cos\theta + i\sin\theta)^{+12}$$

H.W Prove that $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n$

$$= 2^{n+1} \cos^n \theta/2 \ \cos \frac{n\theta}{2}.$$

H.W Express $\dfrac{(\cos 2\theta + i\sin 2\theta)^{2/3} (\cos\theta - i\sin\theta)^2}{(\cos 3\theta - i\sin 3\theta)^2 (\cos 5\theta + i\sin 5\theta)^{1/3}}$

H.W Simplify $\dfrac{(\cos 5\theta - i\sin 5\theta)^{2/5} (\cos \frac{2}{5}\theta + i\sin \frac{2}{5}\theta)^5}{(\cos 3\theta + i\sin 3\theta)^{1/3} (\cos \frac{2}{3}\theta - i\sin \frac{2}{3}\theta)^{18}}$

H.W Prove that the general value of θ which satisfies the Eq^n.

$(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots (\cos n\theta + i\sin n\theta) = 1$ is

$\dfrac{4m\pi}{n(n+1)}$ where m is any integer.

Exponential function

→ Define the exponential function e^z by as

$$e^z = e^x \cdot e^{iy} \quad \text{where} \quad z = x+iy$$

where $e^{iy} = \cos y + i\sin y$
y is measured in radian

→ one can see that e^z reduces to the usual exponential function when $y=0$.

→ We often write $\exp(z) = e^z$.

e.g. Find number $z = x+iy$ such that
$e^z = 1+i$

Ans: $z = \frac{1}{2}\ln 2 + \left(2n + \frac{1}{4}\right)\pi i$

$(n = 0, \pm 1, \pm 2, \ldots)$

* **Logarithmatic function:**

→ How one can find ω such that
(1) — $e^\omega = z_0$ where z_0 is any fixed complex number.

→ To solve the above question let us take
$z_0 = r_0 e^{i\theta_0}$ where $(-\pi < \theta_0 \le \pi)$

and $\omega = u + iv$.

Then equation (1) takes the form

$$e^{u} \cdot e^{iv} = r_0 e^{i\theta_0}$$

i.e. $e^{u} = r_0$ and $v = \theta_0 + 2n\pi, n \in \mathbb{Z}$.

i.e $w = \ln r_0 + i(\theta_0 + 2n\pi), n \in \mathbb{Z}$.

Thus if we take

$$\log z = \ln r_0 + i(\theta_0 + 2n\pi), n \in \mathbb{Z}$$

then $e^{\log z} = z, \quad (z \neq 0)$

which ~~suggest~~ motivate to take

(2) $\longrightarrow$ $\log z = \ln r_0 + i(\theta_0 + 2n\pi), n \in \mathbb{Z}$ as a definition of the logarithmatic function of a complex number $z = r_0 e^{i\theta_0}$.

e.g. if $z = -1 - \sqrt{3}i$ then $\log z = ?$

* If we take $n = 0$ in eq (2) then this value of $\log z$ is called Principal value of $\log z$ and it is denoted by $\operatorname{Log} z$.

$\therefore \operatorname{Log} z = \ln r_0 + i\theta_0$.

i.e $\operatorname{Log} z$ is well defined and single valued.

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and $\log z = \text{Log } z + 2n\pi i$, $(n \in \mathbb{Z})$

Ex: (i) 1, (ii) (-1)

Ex: $\text{Log}(i^3) = \text{Log}(-i)$

$$= \ln 1 - i\,\pi/2 = -\pi/2\, i$$

& $3\,\text{Log}(i) = 3\left(\ln 1 + i\frac{\pi}{2}\right) = \frac{3\pi}{2} i$

$\therefore$ $\text{Log}(i^3) \neq 3\,\text{Log}(i)$

* Complex exponents:

when $z \neq 0$ and the exponent is any complex number, then the function z^c is defined as $z^c = e^{c \log z}$

where $\log z$ denotes the multiple-valued logarithmatic function.

Ex: Find i^{-2i}

Here $z = i$ and $c = -2i$

Therefore $\log z = \ln 1 + i(\pi/2 + 2n\pi)$, $n \in \mathbb{Z}$

$$= i(\pi/2 + 2n\pi), \quad n \in \mathbb{Z}$$

$\therefore$ $i^{-2i} = e^{-2i\left(i\,(\pi/2 + 2n\pi)\right)}$, $n \in \mathbb{Z}$

$$= e^{2(\pi/2 + 2n\pi)}, \quad n \in \mathbb{Z}.$$

$\therefore$ All values of i^{-2i} are real numbers.

Ex! 1) Find the general value of $\log(1+i)+\log(1-i)$

C.W 2) Prove that i^{i} is real and find the value of $\sin(\log i^{i})$.

C.W 3) Prove that $\mathrm{Log}(-ei) = 1-\frac{\pi}{2}i$

C.W 4) Prove that $\mathrm{Log}(1+i)^2 = 2\,\mathrm{Log}(1+i)$.

5) $\mathrm{Log}(1-i) = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$

6) $\mathrm{Log}(-1+i)^2 \neq 2\,\mathrm{Log}(-1+i)$

7) $\log(i^2) = 2\log i$ when

$\log z = \ln r + i\theta$

$\left(r>0,\ \frac{\pi}{4}<\theta<\frac{9\pi}{4}\right)$

8) $\log(i^2) \neq 2\log i$ when

$\log z = \ln r + i\theta$

$\left(r>0,\ \frac{3\pi}{4}<\theta<\frac{11\pi}{4}\right)$.

* Trigonometric functions:

Euler's formula tells us that

$e^{ix} = \cos x + i\sin x$ and $e^{-ix} = \cos x - i\sin x$

for every real number x.

This gives

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \quad \text{and} \quad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

These motivate to define a sine and cosine functions of complex variable z as follows:

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \quad \text{and} \quad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

These functions are also called circular function of complex numbers.

* $\sin(-z) = -\sin z$, $\quad \cos(-z) = \cos z$

$e^{iz} = \cos z + i \sin z$

$\sin(z_1 + z_2) = \sin z_1 \cdot \cos z_2 + \cos z_1 \cdot \sin z_2$

$\cos(z_1 + z_2) = \cos z_1 \cdot \cos z_2 - \sin z_1 \cdot \sin z_2$

$\sin 2z = 2\sin z \cdot \cos z$, $\quad \cos 2z = \cos^2 z - \sin^2 z$

$\sin(z + \pi/2) = \cos z$, $\quad \sin(z - \pi/2) = -\cos z$

$\sin^2 z + \cos^2 z = 1.$

$\sin(z + 2\pi) = \sin z$, $\quad \sin(z + \pi) = -\sin z$

$\cos(z + 2\pi) = \cos z$, $\quad \cos(z + \pi) = -\cos z$

For real number y

$$\sinh y = \frac{e^{y} - e^{-y}}{2} \quad \text{and} \quad \cosh y = \frac{e^{y} + e^{-y}}{2}$$

Other hyperbolic functions are defined as follows

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}, \qquad \operatorname{cosech} x = \frac{1}{\sinh x}$$

* $\cosh^2 x - \sinh^2 x = 1$

$\operatorname{sech}^2 x + \tanh^2 x = 1$

$\coth^2 x - \operatorname{cosech}^2 x = 1$

* The hyperbolic sine and hyperbolic cosine function of a complex number defined as

$$\sinh z = \frac{e^{z} - e^{-z}}{2}, \qquad \cosh z = \frac{e^{z} + e^{-z}}{2}$$

* $\sinh(-z) = -\sinh z, \qquad \cosh(-z) = \cosh z$

$\cosh^2 z - \sinh^2 z = 1$

$\sinh(z_1 + z_2) = \sinh z_1 \cdot \cosh z_2 + \cosh z_1 \cdot \sinh z_2$

$\cosh(z_1 + z_2) = \cosh z_1 \cdot \cosh z_2 + \sinh z_1 \cdot \sinh z_2$.

* Let $z = x + iy$.

$\sinh z = \sinh x \cdot \cos y + i \cosh x \cdot \sin y$

$\cosh z = \cosh x \cdot \cos y + i \sinh x \cdot \sin y$

$|\sinh z|^2 = \sinh^2 x + \sin^2 y$

$|\cosh z|^2 = \sinh^2 x + \cos^2 y$.

* $\sin(iy) = i\sinh y$ and $\cos(iy) = \cosh y$

$$\sin z = \sin x \cosh y + i \cos x \cdot \sinh y$$

$$\cos z = \cos x \cdot \cosh y - i \sin x \cdot \sinh y.$$

$-i\sinh(iz) = \sin z$, $\cosh(iz) = \cos z$

$-i\sin(iz) = \sinh z$, $\cos(iz) = \cosh z$

* Relation between circular and Hyperbolic functions.

(i) $\sin(iz) = i\sinh z$ and $\sinh z = -i\sin iz$

(ii) $\cos(iz) = \cosh z$

(iii) $\tan(iz) = i\tanh z.$

Ans: i) $\sin(iz) = \dfrac{e^{i^2 z} - e^{-i^2 z}}{2i}$

$$= \frac{e^{-z} - e^{z}}{2i} = -i\left(\frac{e^{-z} - e^{z}}{2}\right) = \left(\frac{e^{z} - e^{-z}}{2}\right) i$$

$$= i\sinh z.$$

* Separate real and Imaginary parts.:

(i) $\sin(x \pm iy)$ (ii) $\cos(x \pm iy)$ (iii) $\tan(x \pm iy)$

(iv) $\sinh(x \pm iy)$ (v) $\cosh(x \pm iy)$ (vi) $\tanh(x \pm iy)$

Ans: (vi) $\tanh(x+iy) = \dfrac{\sinh(x+iy)}{\cosh(x+iy)}$

$$= \frac{2\sinh(x+iy)\ \cosh(x-iy)}{2\cosh(x+iy)\cdot\cosh(x-iy)}$$

$$= \frac{\sinh 2x + \sinh(2yi)}{\cosh 2x + \cosh(2yi)}$$

$$= \frac{\sinh 2x + i\sin(2y)}{\cosh 2x + \cos 2y} \qquad \left(\because \cosh z = \cos(iz),\ \sinh z = -i\sin(iz)\right)$$

$$= \frac{\sinh 2x}{\cosh 2x + \cos 2y} + i\frac{\sin(2y)}{\cosh 2x + \cos 2y}.$$

$$\tanh(x-iy) = \frac{2\sinh(x-iy)}{2\cosh(x-iy)} \times \frac{\cosh(x+iy)}{\cosh(x+iy)}$$

$$= \frac{\sinh(x-\cancel{iy}+x+\cancel{iy}) + \sinh(\cancel{x}+iy-\cancel{x}-iy)}{\cosh(x-\cancel{iy}+x+\cancel{iy}) + \cosh(\cancel{x}-iy+\cancel{x}+iy)}$$

$$= \frac{\sinh(2x) + \sinh(-2iy)}{\cosh(2x) + \cosh(-2iy)}$$

$$= \frac{\sinh(2x) - \sinh(2iy)}{\cosh(2x) + \cosh(2iy)} \qquad \left(\because \cosh(-z) = \cosh z,\ \sinh(-z) = -\sinh(z)\right)$$

$$= \frac{\sinh(2x) - i\sin(2y)}{\cosh(2x) + \cos(2y)}$$

$$= \frac{\sinh(2x)}{\cosh(2x)+\cos(2y)} - i\frac{\sin(2y)}{\cosh(2x)+\cos(2y)}$$

Ex: Prove that $(1+\cos\theta + i\sin\theta)^n + (1+\cos\theta - i\sin\theta)^n = 2^{n+1}\cos^n\theta/2 \cdot \cos\frac{n\theta}{2}$.

Ex: Prove that the general value of θ which satisfies the equation $(\cos\theta + i\sin\theta)\cdot(\cos 2\theta + i\sin 2\theta)\ldots(\cos n\theta + i\sin n\theta) = 1$ is $\frac{4m\pi}{n(n+1)}$, where $m \in \mathbb{Z}$.

SOLUTION OF CUBIC EQUATIONS BY "CARDON'S METHOD"

Method :- consider the cubic equation $ax^3 + bx^2 + cx + d = 0$ — (1)

since $a \neq 0$, Dividing by a, we get an equation as

$$x^3 + lx^2 + mx + n = 0.$$

To remove the x^2 term, put $y = x - \left(-\frac{l}{3}\right)$ or

$x = y - \frac{l}{3}$. so that the resulting equation is

of the form $$y^3 + py + q = 0 \quad \text{— (2)}$$

To solve (2), put $y = u + v$

$$y^3 = u^3 + v^3 + 3uv(u + v)$$

$$= u^3 + v^3 + 3uvy$$

$$\therefore y^3 - 3uvy - (u^3 + v^3) = 0. \quad \text{— (3)}$$

Compairing (2) & (3), we get..

$uv = -p/3$, $u^3 + v^3 = -q$ or $u^3 + v^3 = -q$

and $u^3 v^3 = -\frac{p^3}{27}$.

$\therefore$ u^3, v^3 are the roots of the equation

$$t^2 + qt - \frac{p^3}{27} = 0.$$

because, for $ax^2 + bx + c = 0$

$$\alpha_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad \alpha_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$$\alpha_1 + \alpha_2 = \frac{-b}{a} \quad \& \quad \alpha_1 \cdot \alpha_2 = \frac{c}{a}$$

which gives $u^3 = \frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p^3}{27}}\right) = \lambda^3$ (say)

$v^3 = \frac{1}{2}\left(-q - \sqrt{q^2 + \frac{4p^3}{27}}\right)$

The three values of u are λ, $\lambda\omega$, $\lambda\omega^2$, where ω is one of the imaginary cube roots of unity.

From $uv = \frac{-p}{3}$, we have $v = \frac{-p}{3u}$.

$\therefore$ When $u = \lambda$, $\lambda\omega$ and $\lambda\omega^2$

$v = \frac{-p}{3\lambda}$, $-\frac{p\omega^2}{3\lambda}$ and $-\frac{p\omega}{3\lambda}$.

Hence the three roots of (2) are

$\lambda - \frac{p}{3\lambda}$, $\lambda\omega - \frac{p\omega^2}{3\lambda}$, $\lambda\omega^2 - \frac{p\omega}{3\lambda}$.

So, Now, corresponding values of x can be found from the relation $x = y - \frac{p}{3}$.

Ex:- Solve by Cardan's method $x^3 - 3x^2 + 12x + 16 = 0$

Sol: ~~Here~~ consider the eqn $x^3 - 3x^2 + 12x + 16 = 0$ —— (1)

To remove the second term in (1),

put $y = x - 1$ or $x = y + 1$

then we have

$(y+1)^3 - 3(y+1)^2 + 12(y+1) + 16 = 0$

$\Rightarrow y^3 + 9y + 26 = 0.$ —— (2)

To solve (2), put $y = u + v$ so that

$y^3 - 3uvy - (u^3 + v^3) = 0.$ —— (3)

comparing ② and ③, we get

$uv = -3$ and $u^3 + v^3 = -26$.

$\therefore$ u^3, v^3 are the roots of the eqn

$$t^2 + 26t - 27 = 0.$$

$$\Rightarrow (t+27)(t-1) = 0.$$

$$\Rightarrow t = -27 \text{ or } t = 1.$$

$$u^3 = 27 \text{ or } v^3 = 1.$$

i.e $\boxed{u = -3}$ & $\boxed{v = 1}$

$$\therefore y = u + v = -3 + 1 = -2.$$

$$\Rightarrow x = y + 1 = -1.$$

$\therefore$ -1 is one of the root of eqn ①

Now Dividing ① by $x+1$, we get.

$$x^2 - 4x + 16 = 0.$$

$$\therefore x = \frac{4 \pm \sqrt{(16 - 64)}}{2}$$

$$= 2 \pm i2\sqrt{3}.$$

$\therefore$ The required roots of the given equation are -1, $2 \pm i2\sqrt{3}$.

(2) $x^3 - 18x + 35 = 0$ —(1)

Sol. To solve (1) Let $x = u + v$.

$\Rightarrow x^3 - 3uv\,x - (u^3 + v^3) = 0$ —(2)

Compairing (1) & (2), we get.

$uv = 6$ & $u^3 + v^3 = -35$.

Now, u^3, v^3 are the roots of the equation

$t^2 + 35t + 216 = 0.$

$(t + 27)(t + 8) = 0$

$\Rightarrow t = -27$ or $t = -8$

$\therefore u^3 = -27$ & $v^3 = -8$.

$\Rightarrow u = -3, \quad v = -2.$

$\therefore x = u + v = -3 - 2 = -5.$

$\therefore -5$ is one of the root.

$\therefore x + 5$ is the factor of $x^3 - 18x + 35 = 0$.

Now, Divide $x^3 - 18x + 35$ by $x + 5$.

we get.

$$x^2 - 5x + 7 = 0$$

$$\begin{array}{r|l} & x^2 - 5x + 7 \\ \hline x+5 & x^3 - 18x + 35 \\ & x^3 + 5x^2 \\ \hline & -5x^2 - 18x + 35 \\ & -5x^2 - 25x \\ & +\quad\;\; + \\ \hline & 7x + 35 \end{array}$$

$$\therefore x = \frac{5 \pm \sqrt{25 - 28}}{2}$$

$$= \frac{5 \pm i\sqrt{3}}{2}$$

$\therefore$ The required roots are $-5, \; \dfrac{5 \pm i\sqrt{3}}{2}$.

(3). $x^3 - 15x - 126 = 0$ —— ①.

Sol. To solve ①, let $x = u+v$.

$\Rightarrow x^3 - 3uvx - (u^3+v^3) = 0$. —②

From ① & ②, we get.

$-3uv = -15 \Rightarrow uv = 5$ &

$u^3+v^3 = 126$.

Now, u^3 & v^3 are the roots of the equation.

$t^2 - 126t + 125 = 0$

$\Rightarrow (t-125)(t-1) = 0$

$\Rightarrow t = 125$ or $t = 1$.

$\therefore u^3 = 125, \; v^3 = 1$

$\Rightarrow u = 5 \; ; \; v = 1$

$\therefore x = u+v = 6$. is one of the root.

Now, Devide eqn ① by $x-6$, we get.

$x^2+6x+21$.

Hence $x^2+6x+21 = 0$.

$\therefore x = \dfrac{-6 \pm \sqrt{36-4(21)}}{2}$

$= -3 \pm i2\sqrt{3}$

$$
\begin{array}{r|l}
 & x^2+6x+21 \\
x-6 & x^3 - 15x - 126 \\
 & x^3 - 6x^2 \\
\hline
 & 6x^2 - 15x - 126 \\
 & 6x^2 - 36x \\
\hline
 & 21x - 126 \\
\end{array}
$$

$\therefore$ The roots are $6, \; -3+i2\sqrt{3}, \; -3-i2\sqrt{3}$.

(4) Solve the equation $x^3+x^2-16x+20=0$.

Sol: Instead of diminishing the roots of the given equation by $-\frac{1}{3}$, we first multiply its roots by 3. So that the equation becomes.

$$x^3+3x^2-144x+540=0 \quad -(1)$$

$$(\because (x-3\alpha_1)(x-3\alpha_2)(x-3\alpha_3)=0$$

$$\Rightarrow x^3-3(\alpha_1+\alpha_2+\alpha_3)x^2+9(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)x-27\alpha_1\alpha_2\alpha_3=0)$$

The given eq^n is. ~~comparing it with~~ $(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)=0$

$$\Rightarrow x^3-(\alpha_1+\alpha_2+\alpha_3)x^2+9(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)x-\alpha_1\alpha_2\alpha_3=0$$

To find $\alpha_1, \alpha_2, \alpha_3$ we would find $3\alpha_1$; $3\alpha_2$, $3\alpha_3$ which are the roots of the eq^n

$$x^3+3x^2-144x+540=0.$$

To remove the x^2 term, put $x=y-\frac{3}{3}=y-1$ in eq^n (1)

So that. $(y-1)^3+3(y-1)^2-144(y-1)+540=0.$

$$\Rightarrow y^3-147y+686=0. \quad -(2)$$

Now to solve eq^n (2), Let $y=u+v$.

then, $y^3-3uvy-(u^3+v^3)=0. \quad -(3)$

Compare (2) & (3), we get.

$3uv=147$ & $u^3+v^3=-686$

$\Rightarrow uv=49$, $u^3+v^3=-686$

$\Rightarrow (uv)^3=(343)^2$, $u^3+v^3=-686$

Now, u^3, v^3 are the roots of the quadratic eqn. $t^2 + 686t + (343)^2 = 0$.

$\Rightarrow (t+343)^2 = 0$

$\therefore t = -343$.

$\therefore u^3 = v^3 = -343$.

$\therefore u = v = -7$.

$\therefore y = u + v = -14$.

$\Rightarrow x = y - 1 = -15$. is one of the root of eqn ①

$\therefore (x+15)$ is the factor of ~~eq~~ $x^3 + 3x^2 - 144x + 540 = 0$

Now Devide ~~$x+15$~~ eqn ① by $x+15$, we get $x^2 - 12x + 36 = 0$.

$(x-6)^2 = 0$

$\Rightarrow x = 6, 6$

$\therefore$ The roots of eqn ① are $-15, 6, 6$.

$$
\begin{array}{r|l}
 & x^2 - 12x + 36 \\
x+15 & \cancel{x^3} + 3x^2 - 144x + 540 \\
 & \cancel{x^3} + 15x^2 \\
 & - \quad - \\
\hline
 & -12x^2 - 144x + 540 \\
 & \mp 12x^2 \mp 180x \\
\hline
 & 36x + 540 \\
 & 36x + 540 \\
\hline
 & 0
\end{array}
$$

$\therefore$ The ~~roots~~ roots of the given equation are $\frac{-15}{3}, \frac{6}{3}, \frac{6}{3}$, i.e. $-5, 2, 2$.

(6) Solve $9x^3 + 6x^2 - 1 = 0$.

Soln The given eqn is $9x^3 + 6x^2 - 1 = 0$ ——(1)

Here the term in x is missing, let us put $x = \frac{1}{y}$ in the given eqn. ($\because$ zero is never the root of eqn (1)).

Hence we get $9 + 6y - y^3 = 0$ (Dividing eqn (1) by x^3).

$\Rightarrow y^3 - 6y - 9 = 0$ ——(2)

To solve eqn (2) let $y = u + v$ so that we have $y^3 - 3uvy - (u^3 + v^3) = 0$ ——(3)

Comparing (2) & (3) we get,

$uv = 2$, $u^3 + v^3 = 9$

$\Rightarrow (uv)^3 = 8$, $u^3 + v^3 = 9$

Now u^3 & v^3 are the roots of the eqn

$t^2 - 9t + 8 = 0$

$\Rightarrow (t-8)(t-1) = 0$

$\Rightarrow t = 8$ or $t = 1$

$\therefore u^3 = 8$ and $v^3 = 1$

$\Rightarrow u = 2$ and $v = 1$

$\therefore \boxed{y = 3}$.

$\Rightarrow x = 1/y \Rightarrow x = \frac{1}{3}$ is one of the root of given eqn

Now Divide the given eqn by $3x - 1$, we get

$$3x^2 + 3x + 1 = 0$$

$$\therefore \quad x = \frac{-3 \pm \sqrt{9-12}}{2(3)}$$

$$\therefore \quad x = \frac{-3 \pm i\sqrt{3}}{6}$$

$$\begin{array}{r|l} & 3x^2 + 3x + 1 \\ \hline 3x - 1 & 9x^3 + 6x^2 - 1 \\ & 9x^3 - 3x^2 \\ \hline & 9x^2 - 1 \\ & 9x^2 - 3x \\ \hline & 3x - 1 \\ & 3x - 1 \\ \hline & 0 \end{array}$$

Hence the roots are $\frac{1}{3}$, $\frac{-3 + i\sqrt{3}}{6}$, $\frac{-3 - i\sqrt{3}}{6}$.

"Solution of biquadratic equations
By Ferrari method."

Consider the general quartic (biquadratic) equation

$$x^4 + bx^3 + cx^2 + dx + e = 0 \quad \text{--- (1)}$$

Now consider

$$\left(x^2 + \frac{b}{2}x\right)^2 = x^4 + bx^3 + \frac{b^2}{4}x^2$$

$$\therefore \quad x^4 + bx^3 = \left(x^2 + \frac{b}{2}x\right)^2 - \frac{b^2}{4}x^2 \quad \text{--- (2)}$$

Put this value in, equation (1) then we get

$$\left(x^2 + \frac{b}{2}x\right)^2 - \frac{b^2}{4}x^2 + cx^2 + dx + e = 0$$

$$\therefore \quad \left(x^2 + \frac{b}{2}x\right)^2 = \left(\frac{b^2}{4} - c\right)x^2 - dx - e \quad \text{--- (3)}$$

Now take any λ (Real number) and consider

$$\left(x^2 + \frac{b}{2}x + \lambda\right)^2 = \left(x^2 + \frac{b}{2}x\right)^2 + 2\left(x^2 + \frac{b}{2}x\right)\lambda + \lambda^2$$

$$= \left(\frac{b^2}{4} - c\right)x^2 - dx - e + 2\left(x^2 + \frac{b}{2}x\right)\lambda + \lambda^2$$

($\because$ by equation (3))

$$\therefore \quad \left(x^2 + \frac{b}{2}x + \lambda\right)^2 = \left(\frac{b^2}{4} - c + 2\lambda\right)x^2 + (b\lambda - d)x + \lambda^2 - e \quad \text{--- (4)}$$

Right hand side of the equation (4) is perfect square if and only if

$$(ba - d)^2 - 4\left(\frac{b^2}{4} - c + 2a\right)(a^2 - e) = 0 \quad -(5)$$

Equation (5) gives an equation which is qubic equation in a. By cardon's method one can solve that equation and find the value of a.

Now put this value in equation (4) then the R.H.S of equation will be a perfect square. (say R.H.S of (4) is $(x-\alpha)^2$)

Therefore from (4) we have

$$\left(x^2 + \frac{b}{2}x + a\right)^2 = (x-\alpha)^2$$

$$\Rightarrow \left(x^2 + \frac{b}{2}x + a - x + \alpha\right)\cdot\left(x^2 + \frac{b}{2}x + a + x - \alpha\right) = 0$$

From these equation one can obtain four roots of equation (1).

Ex: solve the following equations: (using cardon's or Ferrari method)

(1) $x^3 - 3x^2 + 3 = 0$

(2) $27x^3 + 54x^2 + 198x - 73 = 0$

(3) $x^3 - 7x^2 + 36 = 0$

(4) $6x^3 - 11x^2 - 3x + 2 = 0$

(5) $x^3 - 3x^2 + 1 = 0$

(6) $x^3 - 6x^2 + 5x + 8 = 0$

(7) $28x^3 - 9x^2 + 1 = 0$

(8) $x^3 + x^2 - 16x + 20 = 0$

(9) $x^3 - 3x^2 + 3 = 0$

(10) $x^3 - 27x + 54 = 0$

(11) $x^3 - 18x + 35 = 0$

(12) $x^3 - 15x = 126$

(13) $2x^3 + 5x^2 + x - 2 = 0$

(14) $9x^3 + 6x^2 - 1 = 0$

(15) $x^3 - 6x^2 + 6x - 5 = 0$

(16) $x^3 - 3x + 1 = 0$

(17) $27x^3 + 54x^2 + 198x - 73 = 0$

(18) $x^4 - 12x^3 + 41x^2 - 18x - 72 = 0$

(19) $x^4 - 2x^3 - 5x^2 + 10x - 3 = 0$

(20) $x^4 - 8x^2 - 24x + 7 = 0$

(21) $x^4 - 6x^3 - 3x^2 + 22x - 6 = 0$

(22) $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$

(23) $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$

(24) $x^4 - 10x^2 - 20x - 16 = 0$

(25) $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0$

(26) $x^4 + 12x - 5 = 0$

(27) $x^4 - 6x^3 + 3x^2 + 22x - 6 = 0$

(28) $x^4 - 8x^3 - 24x + 7 = 0$

(29) $x^4 - 10x^3 + 44x^2 - 104x + 96 = 0$

(30) $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$

(31) $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0$

(32) $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$

Infinite Series:

Introduction to sequences & series:

Sequences: If for each $n \in \mathbb{N}$, a number a_n is assigned, then the numbers $a_1, a_2, \cdots, a_n, \cdots$ is said to form a sequence
ie sequence is a function from the set $\mathbb{N}$ to the set $S = \{a_1, a_2, \cdots, a_n, \cdots\}$
Thus, $f: \mathbb{N} \to S$ is a sequence then we can write

$$f(n) = a_n,$$

where a_n is called n^{th} term of the sequence.
Symbolically, the sequence is denoted by $\{a_n\}_{n=1}^{\infty}$.

Illustration:

1. $1, 2, 3, 4, \cdots$ is a sequence whose n^{th} term is n & is denoted by $\{n\}$.
2. $1, 1/2, 1/3, 1/4, \cdots$ " " " " $1/n$ & " " "$\{1/n\}$
3. $1, -1, 1, -1, 1, -1, \cdots$ " " " " $(-1)^{n+1}$ " " "$\{(-1)^{n+1}\}$

Convergence of a sequence:

A sequence $\{a_n\}$ is said to ~~be~~ converge to a real number l if for $\varepsilon > 0$, $\exists\ N \in \mathbb{N} \ni$

$$|a_n - l| < \varepsilon, \ \forall\ n \geq N.$$

In this case we write $\boxed{\lim_{n\to\infty} a_n = l}$

If sequence is not convergent, it is said to be divergent.

Oscillating sequence: A sequence $\{a_n\}$ which is neither converges nor diverges is said to be an oscillating sequence.

eg. $\{1, -1, 1, -1, 1, -1 \cdots\}$ is an oscillating sequence
(It is not convergent as it has two subsequences having different limits.)

Examples: 1. Check the convergence of the following.

(1) $\{1/n\}_{n=1}^{\infty}$ (2) $\{\sqrt[n]{n}\}_{n=1}^{\infty}$

Solution: (1) $\{a_n\} = \{1/n\}_{n=1}^{\infty}$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/n = 0 < \infty$$

$\therefore \{1/n\}$ is said to be convergent & it converges to zero.

(2) $\{a_n\} = \{\sqrt[n]{n}\}_{n=1}^{\infty}$

$$\therefore \lim_{n\to\infty} n^{1/n} \quad (\infty^0)$$

$$= \lim_{n\to\infty} e^{1/n \ln n} = e^{\lim_{n\to\infty} \frac{\ln n}{n}} = e^0 = 1.$$

$$\boxed{\therefore \lim_{n\to\infty} n^{1/n} = 1}$$

Useful limits:

1. $\lim_{n\to\infty} x^n = 0$ if $|x|<1$
2. $\lim_{n\to\infty} x^n = \infty$ if $|x|>1$
3. $\lim_{n\to\infty} \frac{x^n}{n!} = 0, \forall x$
4. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$
5. $\lim_{n\to\infty} n^{1/n} = 1$
6. $\lim_{n\to\infty} (1+1/n)^n = e$
7. $\lim_{n\to\infty} 1/n^n = 0$.
8. $\lim_{n\to\infty} \left[\frac{n!}{n}\right]^{1/n} = \frac{1}{e}$
9. $\lim_{n\to\infty} \frac{a^{1/n}-1}{1/n} = \log a$.

* Series: The sum of sequence is called a series.

If the number of terms of a series is limited the series is called a finite series and if the number of terms of a series is infinite then such a series is called infinite series.

It is denoted by $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots + a_n + \dots$

* Convergence, Divergence and oscillation of a series:

let $S_n = a_1 + a_2 + \dots + a_n$ (partial sum of $\sum_1^{\infty} a_n$)

→ The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if $\lim_{n\to\infty} S_n < \infty$.

i.e for some number s, $\lim_{n\to\infty} S_n = s$.

→ The infinite series $\sum_{n=1}^{\infty} a_n$ is said to be divergent if $\lim_{n\to\infty} S_n$ is ∞ or $-\infty$

→ The series $\sum_1^{\infty} a_n$ is said to be oscillatary if the limit of S_n is not unique as n tends to infinity. The series $\sum a_n$ oscillates finitely if S_n tends to different finite values, otherwise $\sum a_n$ oscillates infinitely.

Note: $1 + r + r^2 + r^3 + \dots + r^{n-1} = \frac{1-r^n}{1-r}$ (G.P).

Examples:

Examine the nature of the following series

C.W. (1) $1 + 2 + 3 + \dots$

H.W. (2) $\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \frac{1}{9\cdot 13} + \dots$

C.W. (3) $4 - 9 + 5 + 4 - 9 + 5 + 4 - 9 + 5 + \dots$

C.W. (4) $1 + 3/4 + 9/16 + 27/64 + \dots$

H.W. (5) $\log 2 + \log 3/2 + \log 4/3 + \dots$

H.W. (6) $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$

(2) $\frac{1}{1\cdot 5}+\frac{1}{5\cdot 9}+\frac{1}{9\cdot 13}+\cdots$

(1) $1+2+3+\cdots$, ie. $\sum_{n=1}^{\infty} n$

Observe that $S_n = 1+2+\cdots+n = n(n+1)/2$

$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$

$\therefore$ Given series $\sum_{n=1}^{\infty} n$ diverges.

(4) $1+3/4+9/16+27/64+\cdots$

Given series is $\sum_{n=0}^{\infty}(3/4)^n = 1+3/4+(3/4)^2+(3/4)^3+\cdots$

$\therefore S_n = 1+3/4+(3/4)^2+\cdots+(3/4)^n$

$= \frac{1-(3/4)^{n+1}}{1-3/4} = 4\left[1-(3/4)^{n+1}\right]$

$\therefore \lim_{n\to\infty} S_n = 4$

$\therefore$ Given series is convergent.

* **Series of Positive Terms:**

An infinite series is said to be a positive term series if all the terms after some particular terms are positive.

e.g. $-\cancel{8}-1+2+4+6+\cdots$

Note: Series of positive terms either converges or diverges, it can never oscillate.

Necessary condition for the series $\sum a_n$ to be convergent:

If $\sum a_n$ is convergent then $\lim_{n\to\infty} a_n = 0$.

Converse need not be true.

e.g. $\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}}$

Here $a_n = \frac{1}{2\sqrt{n}}$ $\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{2\sqrt{n}} = 0$

But $S_n = \frac{1}{2\sqrt{1}}+\frac{1}{2\sqrt{2}}+\cdots+\frac{1}{2\sqrt{n}}$

For ~~K≥n~~, $k\le n$. $\sqrt{k}\le\sqrt{n} \Rightarrow \frac{1}{\sqrt{n}} \le \frac{1}{\sqrt{k}}$

$\therefore S_n \ge \frac{1}{2\sqrt{n}}+\frac{1}{2\sqrt{n}}+\cdots+\frac{1}{2\sqrt{n}}$

$= \frac{n}{2\sqrt{n}} = \frac{\sqrt{n}}{2}$

$\therefore \lim_{n\to\infty} S_n = \infty$ $\therefore \sum\frac{1}{2\sqrt{n}}$ diverges

* Geometric Series:

The series $1+r+r^2+r^3+\cdots$

(1) converges if $|r|<1$ & its sum is $\frac{1}{1-r}$.

(2) diverges if $r>1$

(3) oscillates if $r\leq -1$.

Ex. Test the convergence of the following:

C.W. (1) $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ C.W. (2) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$ (3) $\sum_{n=1}^{\infty} n \tan(1/n)$. C.W. (3) $\sum_{n=1}^{\infty} n\tan(1/n)$ (divergent)

H.W. (4) $\sum_{n=0}^{\infty} \frac{2^n-1}{3^n}$ Ans. (cnvg) H.W. (5) $\sum_{n=1}^{\infty} \frac{1}{2^n}$ (1, convergent) H.W. (6) $\sum_{n=1}^{\infty} 6\left(\frac{2}{3}\right)^n$ (book 18, convergent) (M 12, convergent)

Soln:- (1) $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots$,

which is geometric series with ratio $r=\frac{1}{2}<1$

$\therefore$ given series is convergent & sum is $\frac{1}{1-r} = \frac{1}{1-\frac{1}{2}} = 2$.

(2) $\sum_{n=1}^{\infty} \frac{n^2-1}{n^2+1}$.

Here $a_n = \frac{n^2-1}{n^2+1}$

Observe that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2-1}{n^2+1} = \lim_{n\to\infty} \frac{(1-1/n^2)}{(1+1/n^2)} = 1 \neq 0$.

$\therefore$ Given series is not convergent.

C.W.

Ex. A ball is dropped from a height of 20 m. Each time it strikes the ground it bounces vertically to a height that is 3/4 of the preceding height. Find the total distance the ball will travel if it is allowed to bounce indefinitely. (Ans. 80 m)

* Test of convergence:

Hyper Harmonic series OR p-series:

The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$

(i) converges if $p>1$

(2) diverges if $p\leq 1$.

Note: For $p=1$, $\sum_{n=1}^{\infty} \frac{1}{n}$ is known as harmonic series.

* Comparision Tests:

(I) Let $\sum a_n$ & $\sum b_n$ be two positive term series such that

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l \text{ (some finite number)}$$

Then, both the series converge or diverge together.

(i) if Σb_n is convergent then Σa_n is also convergent.

(ii) If Σb_n is divergent then Σa_n is also divergent.

(II) If Σa_n & Σb_n be non-negative series

(i) if Σb_n converges & $0 \le a_n \le b_n$, $\forall n \ge 1$ then Σa_n converges & $\Sigma a_n \le \Sigma b_n$.

(ii) if Σb_n diverges & $0 \le b_n \le a_n$, $\forall n \ge 1$ then Σa_n diverges & $\Sigma a_n \ge \Sigma b_n$.

Ex. Test the convergence of the series:

C.W. (1) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ H.W. (2) $\sum_{n=1}^{\infty} \frac{2n^2+3n}{5+n^5}$ H.W. (3) $\sum_{n=1}^{\infty} (\sqrt{n+1} - \sqrt{n})$

C.W. (4) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$ C.W. (5) $\sum_{n=1}^{\infty} \frac{n^p}{\sqrt{n+1}+\sqrt{n}}$ H.W. (6) $\sum_{n=1}^{\infty} \frac{4^n+5^n}{6^n}$

H.W. (7) $\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$, $x>0$ H.W. (8) $\sum_{n=1}^{\infty} \sqrt{n^4+1} - \sqrt{n^4-1}$.

(9) Ex. For which values of p does the series $\sum_{n=1}^{\infty} \frac{n+1}{n^p}$ is convergent.

Sol^n:- (1) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

$\forall n \in \mathbb{N}$, observe that $n^3+1 > n^3$

$\therefore \sqrt{n^3+1} > n^{3/2}$

$\therefore \frac{1}{\sqrt{n^3+1}} < \frac{1}{n^{3/2}}$, $\forall n \in \mathbb{N}$.

Take $b_n = 1/n^{3/2}$, then from p-series $\Sigma b_n = \Sigma 1/n^{3/2}$ is convergent

$\therefore$ By comparision test, $\Sigma a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ is also convergent.

(5) $\sum_{n=1}^{\infty} \frac{n^p}{\sqrt{n+1}+\sqrt{n}}$ i.e. $a_n = \frac{n^p}{\sqrt{n+1}+\sqrt{n}}$

Choose $b_n = \frac{1}{n^{1/2-p}}$ then

$$\frac{a_n}{b_n} = \frac{n^p}{\sqrt{n+1}+\sqrt{n}} \cdot \frac{1}{n^{-1/2+p}} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \left[\frac{1}{\sqrt{1+1/n}+1}\right]$$

$\therefore \lim_{n\to\infty} \frac{a_n}{b_n} = 1/2$ (finite)

$\therefore$ By comparision test, Σa_n & Σb_n converge or diverge together.

But the series $\Sigma b_n = \Sigma \frac{1}{n^{1/2-p}}$ is convergent when $1/2 - p > 1 \Rightarrow p < -1/2$.

& divergent when $p \ge -1/2$

$\therefore$ Given series is convergent if $p < -1/2$ & divergent if $p \ge -1/2$.

Test the convergence of the series

(iv) $\frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \cdots$

Here $a_n = \frac{n+1}{n^p}$, $n = 1, 2, \ldots$

Consider $b_n = \frac{1}{n^{p-1}}$ then

$$\frac{a_n}{b_n} = \frac{n+1}{n^p} \, n^{p-1} = \frac{n+1}{n}$$

$\therefore \lim_{n\to\infty} a_n/b_n = \lim_{n\to\infty} \frac{n+1}{n} = 1$ (finite)

$\therefore$ From comparision test, $\sum a_n$ & $\sum b_n$ both converge or diverge together.

But $\sum b_n = \sum \frac{1}{n^{p-1}}$ is convergent when $p-1>1$ i.e $p>2$

and divergent when $p-1 \leq 1$ i.e. $p \leq 2$.

$\therefore$ Given series is convergent if $p>2$ & divergent if $p \leq 2$.

*** D'Alembert's Ratio Test or Ratio Test:**

Let $\sum_{n=1}^{\infty} a_n$ be a series of positive terms. Then find $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$

(1) if $L<1$, $\sum a_n$ converges.

(2) if $L>1$, $\sum a_n$ diverges

(3) if $L=1$ test fails.

Ex. Test the following series for convergence and divergence:

C.W. (1). $\frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \cdots$

H.W. (2) $\frac{1}{10} + \frac{2!}{10^2} + \frac{3!}{10^3} + \cdots$

C.W. (3) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

H.W. (4) $\frac{1}{1\cdot 3\cdot 5} + \frac{1}{2\cdot 4\cdot 6} + \frac{1}{3\cdot 5\cdot 7} + \frac{1}{4\cdot 6\cdot 8} + \cdots$

H.W. (5) $\sum_{n=1}^{\infty} \frac{n^3+2}{2^n+2}$

C.W. (6) $\sum_{n=1}^{\infty} \frac{n}{e^{-n}}$

Sol^n:- (1) $\frac{1}{3} + \frac{1\cdot 2}{3\cdot 5} + \frac{1\cdot 2\cdot 3}{3\cdot 5\cdot 7} + \cdots = \sum a_n$ then

$$a_n = \frac{1\cdot 2\cdot 3 \cdots n}{3\cdot 5\cdot 7 \cdots (2n+1)} \qquad \therefore a_{n+1} = \frac{1\cdot 2\cdot 3 \cdots (n)(n+1)}{3\cdot 5\cdot 7 \cdots (2n+1)(2n+3)}$$

$$\therefore \frac{a_{n+1}}{a_n} = \frac{1\cdot 2\cdot 3 \cdots n(n+1)}{3\cdot 5\cdot 7 \cdots (2n+1)(2n+3)} \cdot \frac{3\cdot 5\cdot 7 \cdots (2n+1)}{1\cdot 2\cdot 3 \cdots n}$$

$$= \frac{n+1}{2n+3}$$

$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2} < 1 \Rightarrow$ Given series is convergent.

$\sum_{n=1}^{\infty} \frac{n!}{n^n}$, Here $a_n = \frac{n!}{n^n}$

$\therefore \frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)}{n+1} \cdot \frac{n^n}{(n+1)^n}$

$= \left[\left(\frac{n+1}{n}\right)^n\right]^{-1} = \left(\left[1+\frac{1}{n}\right]^n\right)^{-1}$.

$\therefore \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)^n\right)^{-1} = \frac{1}{e} < 1$

$\therefore$ Given series is convergent.

* **Cauchy's Root test OR Radical Test:**

Let $\sum_{n=1}^{\infty} a_n$ be the series of positive terms. Find $\lim_{n\to\infty} (a_n)^{1/n} = L$

(i) if $L<1$, $\sum a_n$ converges

(ii) if $L>1$, $\sum a_n$ ~~conv~~ diverges

(iii) if $L=1$ test fails.

Ex. Test the convergence of the following:

C.W. 1. $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$ (2) H.W. $\sum_{n=1}^{\infty} e^{-\sqrt{n}} x^n$, $x>0$ C.W. (3) $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n x^n$. $x>0$.

Solⁿ:- (1) $\sum_{n=2}^{\infty} \left(\frac{1}{\log n}\right)^n$ i.e. $u_n = \frac{1}{(\log n)^n}$

$\therefore \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(\left(\frac{1}{\log n}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = 0 < 1$

$\therefore$ From Cauchy's root test, given series is convergent.

(3). $\sum_{n=1}^{\infty} \left(\frac{n+1}{n+2}\right)^n x^n$. $x>0$.

Here $a_n = \left(\frac{n+1}{n+2}\right)^n x^n$ Then

$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \left(\frac{n+1}{n+2}\right) x = \lim_{n\to\infty} \left[\frac{1+1/n}{1+2/n}\right] x$

$\therefore \lim_{n\to\infty} a_n^{1/n} = x$

$\therefore$ The series is convergent if $x<1$ & divergent for $x>1$. For $x=1$, test fails

In this case,

$u_n = \left(\frac{n+1}{n+2}\right)^n = \left(\frac{1/n+1}{2/n+1}\right)^n \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(1+1/n)^n}{(1+2/n)^n} = \lim_{n\to\infty} \frac{e}{e^2} = \frac{1}{e} \neq 0$

$\therefore$ Given series is not convergent.

Cauchy Integral Test:

Let $f(x)$, $x \in [1, \infty)$ be a non-negative monotonic decreasing integrable function. Let $I_n = \int_1^n f(x)dx$. Then the series

$$\sum_{n=1}^{\infty} f(n) = f(1) + f(2) + f(3) + \cdots$$

is convergent if $\lim_{n\to\infty} I_n$ exists (i.e. finite), otherwise it is divergent.

Furthermore, the value of the sum of the series lies between $I = \lim_{n\to\infty} I_n$ and $I + f(1)$.

Ex. Test the convergence of the following:

C.W.

(1) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$.

Here $a_n = \frac{1}{n(\log n)^2}$, $n \geq 2$

Let $f(x) = \frac{1}{x(\log x)^2}$, $x \geq 2$

For $x \in [2, \infty)$, $f(x)$ is non negative decreasing function.

$$\therefore I_n = \int_2^n \frac{1}{x(\log x)^2}dx = \left\{ \frac{(\log x)^{-1}}{-1} \right|_2^n$$

$$= -\frac{1}{\log n} + \frac{1}{\log 2}$$

$$\therefore \lim_{n\to\infty} I_n = \frac{1}{\log 2} < \infty.$$

$\therefore$ From Cauchy integral test, given series is convergent.

C.W. (2). $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

C.W. (3) $\sum_{n=1}^{\infty} n e^{-n^2}$.

H.W. (4) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

Ans. convergent

H.W. (5) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2-1}}$

Ans. convergent.

Sol^n:- (3) $\sum_{n=1}^{\infty} n e^{-n^2}$, Here $a_n = n e^{-n^2}$, $n \geq 1$

Let $f(x) = x e^{-x^2}$, $x \geq 1$.

For $x \in [1, \infty)$, $f(x)$ is non-negative monotonic decreasing function.

For $I_n = \int_1^n x e^{-x^2} dx$

$x^2 = t \Rightarrow 2x\,dx = dt$, $t: 1 \to n^2$

$$I_n = \frac{1}{2}\int_1^{n^2} e^{-t}\,dt$$

$$\frac{1}{2}\left(\frac{e^{-t}}{-1}\right|_1^{n^2} = \frac{1}{2}\left[-e^{-n^2} + e^{-1}\right]$$

$$\therefore \lim_{n\to\infty} I_n = \frac{1}{2}(0 + e^{-1}) = \frac{e^{-1}}{2} = \frac{1}{2e} < \infty$$

$\therefore$ From Cauchy integral test, given series is convergent.

* Alternating Series:

In the previous notes, we have discussed different tests of convergence of series of positive terms. But there are series having positive as well as negative terms. We shall discuss the convergence of such series.

→ A series is said to be an alternating series whose terms are alternatively positive & negative.

The general form of this series is $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ where $a_n > 0$.

e.g. $1-2+3-4+5-\cdots$, $1,-1,1,-1,1,-1,\cdots$ is an alternating series.

Leibnitz's Test:

An alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$, where $a_n > 0$ is convergent if

(i) each term is numerically less than its preceeding term (i.e $\forall n$, $a_{n+1} < a_n$)

(ii) $\lim_{n\to\infty} a_n = 0$.

Note: If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} (-1)^n a_n$ is an oscillating series.

Examples:

C.W. 1. Show that the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ is convergent.

Solⁿ:- Given series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{\sqrt{n}}$

$\therefore a_n = \frac{1}{\sqrt{n}}$

Clearly $\sqrt{n+1} > \sqrt{n}, \forall n \Rightarrow \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \Rightarrow a_{n+1} < a_n, \forall n$

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$.

$\therefore$ By Leibnitz's test, the series is convergent.

C.W. 2. $\frac{x}{1+x} - \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} - \frac{x^4}{1+x^4} + \cdots$ $(0<x<1)$, Test the convergence.

Solⁿ:- Given series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{1+x^n}$. This is an alternating series

$a_n = \frac{x^n}{1+x^n} \Rightarrow a_{n+1} = \frac{x^{n+1}}{1+x^{n+1}}$

$$\therefore a_{n+1} - a_n = \frac{x^{n+1}}{1+x^{n+1}} - \frac{x^n}{1+x^n} = \frac{x^{n+1} + \cancel{x^{2n+1}} - x^n - \cancel{x^{2n+1}}}{(1+x^n)(1+x^{n+1})} = \frac{x^n(x-1)}{(1+x^n)(1+x^{n+1})}$$

$< 0 \quad (\because 0<x<1)$

$\Rightarrow a_{n+1} < a_n, \forall n.$

Also $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{x^n}{1+x^n} = 0.$

$\therefore$ By Leibnitz's test, the series is convergent.

H.W.

(3). $1-2x+3x^2-4x^3+\cdots$, $0<x<1$ (Test the convergence)

H.W.

(4) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\log(n+1)}$ (Test the convergence)

C.W.

(5) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, n}{2n-1}$ is oscillatory.

Sol: Here $a_n = \frac{n}{2n-1}$

$a_{n+1} - a_n = \frac{n+1}{2n+2-1} - \frac{n}{2n-1} = \frac{n+1}{2n+1} - \frac{n}{2n-1} = \frac{\cancel{2n^2}+2n-n-1-\cancel{2n^2}-n}{(2n+1)(2n-1)}$

$= \frac{-1}{4n^2-1} < 0.$

$\Rightarrow a_{n+1} < a_n,\ \forall n.$

Also $\lim_{n\to\infty} \frac{n}{2n-1} = \frac{1}{2} \neq 0.$

$\therefore$ Given series is oscillating.

(6). Show that the series $\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1}-\sqrt{n})$ is convergent.

* Absolute Convergence:

→ A series $\sum a_n$ is said to be absolutely convergent if the series $\sum |a_n|$ is convergent.

→ A series is said to be conditionally convergent if it is convergent but not converge absolutely.

Note: Every absolutely convergent series is convergent. (i.e if $\sum_{n=1}^{\infty} |a_n|$ converges, $\sum_{n=1}^{\infty} a_n$ also converges.)

C.W.

Eg. Ex 1. Consider the series $1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}+\cdots$

Given series is $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2} = \sum_{n=1}^{\infty} a_n$

$\therefore \sum |a_n| = \sum_{n=1}^{\infty} \left|\frac{1}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is convergent from p-series test.

C.W.

2. Find the value of x for which the series $\sum_{n=0}^{\infty} n!\, x^n$ is convergent

Sol: Here $a_n = n!\, x^n \Rightarrow a_{n+1} = (n+1)!\, x^{n+1}$

$\therefore \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)!\, x^{n+1}}{n!\, x^n}\right| = |(n+1)||x|$

$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{R} = \lim_{n\to\infty} (n+1)|x| = \infty \Rightarrow \boxed{R=0}$

Power series:

The general form of a power series is given by

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots,$$

which is in powers of $(x-a)$, where $c_1, c_2, c_3, \dots$ are constants.

Put $a=0$ then

$$\sum c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

is called a power series in powers of x.

The power series is convergent or divergent for some values of x. The power series in $x-a$ is always convergent in an interval with center at $x=a$. Such an interval is called an interval of convergence. The half length of this interval is known as the radius of convergence.

Ex. Find the interval of convergence for which the following series are convergent.

C.W. (1) $x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \frac{x^4}{4^2} + \dots$ H.W. (2) $\sum_{n=0}^{\infty} \frac{x^n}{n+2}$

Soln:- Here,

$$a_n = (-1)^{n+1} \frac{x^n}{n^2}$$

$$\therefore a_{n+1} = (-1)^n \frac{x^{n+1}}{(n+1)^2}$$

$$\therefore \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^n x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(-1)^{n+1} x^n}\right| = \left|\frac{x\, n^2}{(n+1)^2}\right|$$

$$\therefore \frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x| \left|\frac{n^2}{(n+1)^2}\right| = |x|.$$

$\therefore$ From De Alembert ratio test, given series is convergent if $|x|<1$.

The series is divergent if $|x|>1$

For $x=1$, the series becomes

$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$ which is convergent alternating series

& for $x=-1$, the series

$-1 - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \dots$ is also convergent (From p-series test.

(2)

* Convergence of some standard power series:

(1) $e^x = \sum_{n=0}^{\infty} x^n/n!$ C.W (2) $\log(1+x)$ C.W (3) $\sin x$ C.W (4) $\cos x$ H.W (5) $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n-1} x^{2n+1}$ H.W

Here $a_n = x^n/n!$

$a_{n+1} = \frac{x^{n+1}}{(n+1)!}$ $\therefore \frac{a_{n+1}}{a_n} = \frac{x^{n+1}}{(n+1)!} \frac{n!}{x^n} = \frac{x}{n+1}$

$$\frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{x}{n+1} = 0$$

$\Rightarrow R = \infty$

$\therefore$ By De' Alembert's ratio test the series is convergent for all x.

(2). $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$,

Here $a_n = (-1)^n \frac{x^{2n+1}}{(2n+1)!}$

$\therefore a_{n+1} = (-1)^{n+1} \frac{x^{2n+3}}{(2n+3)!}$

$$\therefore \frac{1}{R} = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right| = 0.$$

$\Rightarrow R = \infty$.

$\therefore$ By De' Alembert's ratio test. the series is convergent for all x.

Chandubhai S Patel Institute of Technology
CHARUSAT

Assignment

Engineering Mathematics - I (Section-I)

Matrix Theory

Q A. Define Rank of a Matrix and find the rank of following matrices by using minors:

$$(1)\begin{pmatrix}1&2&3\\1&4&2\\2&6&5\end{pmatrix}\ (2)\begin{pmatrix}1&2&3\\2&4&7\\3&6&10\end{pmatrix}\ (3)\begin{pmatrix}4&6&2\\2&3&1\\-2&-3&-1\end{pmatrix}\ (4)\begin{pmatrix}1&2&3\\2&3&1\\-2&-3&-1\end{pmatrix}$$

$$(5)\begin{pmatrix}1&2&1\\-1&0&2\\2&1&-3\end{pmatrix}\ (6)\begin{pmatrix}1&2&3\\2&3&1\\-2&-3&-1\end{pmatrix}\ (7)\begin{pmatrix}5&6&7&8\\6&7&8&9\\11&12&13&14\\16&17&18&19\end{pmatrix}\ (8)\begin{pmatrix}0&1&2&-2\\4&0&2&6\\2&1&3&1\\1&3&4&1\end{pmatrix}$$

Q. B Check whether the following matrices are in row echelon form or not, if not convert in row echelon form and hence determine the rank and nullity:

$$(1)\begin{pmatrix}1&2&3\\0&0&1\\0&0&12\end{pmatrix}\ (2)\begin{pmatrix}0&2&1\\5&2&3\\1&1&0\end{pmatrix}\ (3)\begin{pmatrix}1&2&3\\1&4&2\\2&6&5\end{pmatrix}\ (4)\begin{pmatrix}5&6&3\\8&4&2\\0&6&9\end{pmatrix}\ (5)\begin{pmatrix}0&1&-3&-1\\1&0&1&1\\3&1&0&2\\1&1&-2&0\end{pmatrix}$$

$$(6)\begin{pmatrix}2&3&-1&-1\\1&-1&-2&-4\\3&1&3&-2\\6&3&0&-7\end{pmatrix}\ (7)\begin{pmatrix}8&1&3&6\\0&3&2&2\\-8&-1&-3&4\end{pmatrix}\ (8)\begin{pmatrix}2&1&-3&-6\\3&-3&1&2\\1&1&1&2\end{pmatrix}$$

Q. C Find the inverse of the following matrices using Gauss-Jordan method if, it exists:

$$(1)\begin{pmatrix}2&1&-1\\0&2&1\\5&2&-3\end{pmatrix}\ (2)\begin{pmatrix}8&4&3\\2&1&1\\1&2&1\end{pmatrix}\ (3)\begin{pmatrix}-1&2&2\\2&-1&2\\2&2&-1\end{pmatrix}\ (4)\begin{pmatrix}0&1&2\\1&2&3\\3&1&1\end{pmatrix}\ (5)\begin{pmatrix}1&2&-1\\3&-2&4\\1&3&2\end{pmatrix}$$

$$(6)\begin{pmatrix}1&1&1\\1&2&3\\1&4&9\end{pmatrix}$$

Q. D Test the consistency of the following systems and if it consistent find the solution by Gauss Elimination method:

(i) $x+2y+z=5,\ 3x-y+z=6,\ x+4y+z=7$

(ii) $x+y+z=4,\ 2x+5y-2z=3,\ x+7y-7z=5$

(iii) $x+y-z=0,\ 2x-y+z=3,\ 4x+2y-2z=2$

(iv) $9x+4y+3z=-1,\ 5x+y+2z=1,\ 7x+3y+4z=1$

(v) $3x+2y+4z=7,\ 2x+y+z=4,\ x+3y+5z=2$

(vi) $x+y+z=8,\ x-y+2z=6,\ 9x+5y-7z=44$

Q. E Find the condition on α for which the system of equations

$$3x-y+4z=3,\ x+2y-3z=-2,\ 6x+5y+\alpha z=-3$$

has a unique solution. Find the solution for $\alpha = -5$.

Q. F Find the value of k such that the system of equations

$$2x + 3y - 2z = 0,\ 3x - y + 3z = 0,\ 7x + ky - z = 0$$

has non-trivial solutions. Find the solutions.

Q. G Find the value of λ and μ so that the equations

$$2x + y + 2z = 5,\ x + 2y + 3z = 6,\ x + 2y + \lambda z = \mu$$

have (a) no solution (b) unique solution (c) infinite number of solution.

Q. H For which value of k, the equations

$$2x + 2y + 5z = 1,\ 3x + 2y + 5z = 2k,\ 3x + 4y + 10z = k^2$$

have solution? Find the solution in each case.

Q. I Show that the equations

$$x + 3z = kx,\ -5x + 4z = ky,\ -3x + 2y - 5z = kz$$

has non-trivial solution then $k^2 - 12k + 14 = 0$.

Infinite Series

Q. J Discuss the convergence of the following series and if it is convergent find the sum of the series:

Chandubhai S Patel Institute of Technology
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Assignment

MA 141 ENGINEERING MATHEMATICS I (SECTION I)

Higher Order Derivatives

Q 1. Find the n^{th} derivative of $\dfrac{x}{(x-1)(2x+3)}$.

Q 2. If $y=(2-3x)^{10}$, find y_9.

Q 3. Find the n^{th} derivative of $\dfrac{x+2}{x+1}+\log\dfrac{x+2}{x+1}$.

Q 4. Find the n^{th} derivative of $\dfrac{1}{x^2-6x+8}$.

Q 5. Find the n^{th} derivative of (i) $e^x\cos^3 x$ (ii) $e^x\sin^2 x$ (iii) $\cos^2 x$ (iv) $\sin^3 x$ (v) $\cos x\cos 2x\cos 3x$.

Q 6. Find the n^{th} derivative of (i) $\dfrac{3}{(x-1)(x+2)}$ (ii) $\dfrac{1}{x^2+3x+2}$ (iii) $\dfrac{1-x}{1+x}$ (iv) $\dfrac{3x}{(2x-1)(x+1)}$ (v) $\ln(2x^2+3x+1)$ (vi) $\dfrac{x}{x+4}$.

Q 7. If $y=(sin^{-1}x)^2$, show that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$.

Q 8. If $y^{\frac{1}{m}}+y^{-\frac{1}{m}}=2x$, prove that $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0$.

Q 9. If $y=tan^{-1}x$, prove that $(1+x^2)y_{n+1}+2nxy_n+n(n-1)y_{n-1}=0$.

Q 10. If $y=a\cos(\log x)+b\sin(\log x)$, show that $x^2y_2+xy_1+y=0$ and $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$.

Q 11. If $y=sin^{-1}x$, prove that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$.

Q 12. If $\cos^{-1}\left(\dfrac{y}{b}\right)=\log\left(\dfrac{x}{n}\right)^n$, prove that $x^2y_{n+2}+(2n+1)xy_{n+1}+2n^2y_n=0$.

Q 13. If $y=e^{m\cos^{-1}x}$, prove that (i)$(1-x^2)y_2-xy_1=m^2y$ and (ii) $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+m^2)y_n=0$.

Q 14. If $y=\sin\log(x^2+2x+1)$, prove that (i) $(x+1)^2y_2+(x+1)y_1+4y=0$ and (ii) $(x+1)^2y_{n+2}+(2n+1)(x+1)y_{n+1}+(n^2+4)y_n=0$.

Q 15. Using Maclaurin's series, expand the following functions $\sinh x, \tan x, e^{-x}, \ln(1+x), \frac{1}{1-x}$.

Q 16. Expand $\log_e x$ in powers of $(x-1)$ and hence evaluate $\log_e 1.1$ correct to 4 decimal places.

Q 17. Expand $f(x)=x^3+4x^2+8x+1$ in powers of $(x-1)$.

Q 18. Expand $f(x) = x^5 + 2x^4 - x^2 + x + 1$ in powers of $(x+1)$.

Q 19. Expand $f(x) = x^4 - 5x^3 + 5x^2 + x + 2$ in powers of $(x-2)$.

Q 20. Expand $f(x) = 7x^3 + 5x^2 + 3x + 1$ in powers of $(x-1)$.

Q 21. Expand $f(x) = 2x^3 + 7x^2 + 5x + 1$ in powers of $(x-2)$.

Q 22. Expand $f(x) = 3x^3 + 9x^2 + 5$ in powers of $(x-1)$.

Q 23. Evaluate the following:

(1) $\lim\limits_{x\to 1} \dfrac{1+\log x - x}{1-2x+x^2}$ (2) $\lim\limits_{x\to 0} \dfrac{xe^x - \log(1+x)}{x^2}$ (3) $\lim\limits_{x\to 0} \dfrac{x\cos x - \log(1+x)}{x^2}$

(4) $\lim\limits_{x\to 0} \dfrac{\log\sin x}{\cot x}$ (5) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\tan x}{\tan 3x}$ (6) $\lim\limits_{x\to 0} \dfrac{\log\tan x}{\log x}$ (7) $\lim\limits_{x\to 0} \dfrac{\log\tan 2x}{\log\tan x}$

(8) $\lim\limits_{x\to 1} \log(1-x)\cot\frac{\pi x}{2}$ (9) $\lim\limits_{x\to a} \dfrac{\log(x-a)}{\log(e^x - e^a)}$ (10) $\lim\limits_{x\to 0}(\cos x)^{\frac{1}{x^2}}$

(11) $\lim\limits_{x\to 0} x^x$ (12) $\lim\limits_{x\to 0}\left(\dfrac{2x+1}{x+1}\right)^{x^{-1}}$ (13) $\lim\limits_{x\to 1} \dfrac{x^x - x}{x-1-\log x}$ (14) $\lim\limits_{x\to 0} \dfrac{\log x}{\cot x}$

Partial Derivatives

Q 24. If $z(x+y) = x^2 + y^2$, show that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.

Q 25. If $u = sin^{-1}\left(\dfrac{x}{y}\right) + tan^{-1}\left(\dfrac{y}{x}\right)$, then find the value of $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y}$.

Q 26. If $u = log(x^2 + y^2) + tan^{-1}\left(\dfrac{y}{x}\right)$, show that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Q 27. If $x^x y^y z^z = c$, show that at $x = y = z$, $\dfrac{\partial^2 z}{\partial x \partial y} = -(xlogex)^{-1}$.

Q 28. If $f(x,y) = \dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{logx - logy}{x^2+y^2}$, show that $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + 2f(x,y) = 0$.

Q 29. If $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

Q 30. If $z = \sec^{-1}\dfrac{x^3+y^3}{x+y}$, show that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2\cot z$.

Q 31. If $u = \cos^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\cot u = 0$.

Q 32. If $u = \tan^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{4}\sin 2u$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = \dfrac{1}{16}(\sin 4u - 4\sin 2u)$.

Q 33. If $u = \cot^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \dfrac{1}{4}\sin 2u = 0$.

Q 34. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, $x \neq y$, show that

(i) $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$

(ii) $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x\partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = (1 - 4\sin^2 u)\sin 2u$.

Q 35. If $u = \tan\left[\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt[4]{x}+\sqrt[4]{y}}\right]$, then prove that $xu_x + yu_y = \dfrac{1+u^2}{4}\tan^{-1} u$.

Q 36. If $u = \sec^{-1}\left(\dfrac{x^3+y^3}{2x-y}\right)$, show that

(i) $xu_x + yu_y = 2\cot u$

(ii) $x^2u_x + 2xyu_{xy} + y^2u_{yy} = -2\cot u(2\operatorname{cosec}^2 u + 1)$.

Q 37. If $u = \cot^{-1}\dfrac{x^3+y^3}{x^2+y^2}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\sin 2u$.

Q 38. If $u = \sin^{-1}\dfrac{x^2+y^2}{x+y}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$.

Q 39. If $u = \sin^{-1}\dfrac{x+y}{\sqrt{x}+\sqrt{y}}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{1}{2}\tan u$.

Q 40. If $u = \sin^{-1}\dfrac{x^2y^2}{x^2+y^2}$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2\tan u$.

Q 41. Find approximation:
(i) $\sqrt[4]{(5.1)^2(2.9)+2(2.9)}$ [**Ans: 3**] (ii) $\sqrt[5]{(3.8)^2+2(2.1)^3}$ [**Ans: 2.01**] (iii) $\sqrt[3]{(4.1)^2+3(3.8)^2}$ [**Ans: 3.916667**] (iv) $\sqrt{(299)^2+(399)^2}$ [**Ans: 498.6**] (v) $\sin 58^\circ\cos 46^\circ$ [**Ans:** $\dfrac{\sqrt{3}}{2\sqrt{2}} - \dfrac{\pi}{180}\left(\dfrac{2+\sqrt{3}}{2\sqrt{2}}\right)$] (vi) $\sin 29^\circ\cos 58^\circ$ [**Ans:** $\dfrac{1}{4}+\dfrac{\sqrt{3}\pi}{120}$] (vii) $\sin 31^\circ\cos 58^\circ$ [**Ans:**

$\frac{1}{4} + \frac{3\sqrt{3}\pi}{720}$](viii) $\sin 44° \cos 62°$ [**Ans:** $\frac{1}{2\sqrt{2}} - \frac{\pi}{360\sqrt{2}}(1+2\sqrt{3})$]
(ix) $\log(3\sqrt{1.03} + 4\sqrt{0.98} - 1)$ [**Ans: 0.7790**] (x) $[(3.82)^2 + 2(2.1)^3]^{\frac{1}{5}}$
(xi) $\sqrt{(0.98)^2 + (2.01)^2 + (1.94)^2}$.

Q 42. If the sides and angles of a plane triangle vary in such a way that its circum radius remains constant, prove that $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos c} = 0$.

Q 43. In a plane triangle, if the sides a, b be constant, prove that the variations of its angles are given by the relations $\frac{dA}{\sqrt{a^2 - b^2 sin^2 A}} = \frac{dB}{\sqrt{b^2 - a^2 sin^2 B}} = -\frac{dC}{c}$, the letters having their usual significance.

Q 44. The power dissipated in a resistor is given by $P = \frac{E^2}{R}$. Using calculus, find the approximate percentage change in P when E is increased by 3 percentage and R is decreased by 2 percentage.

Q 45. The deflection at the center of a rod of length l and diameter d supported at its ends and loaded at the center with a weight w varies as wl^3d^{-4}. What is the percentage increase in the deflection corresponding to the percentage increase in w, l and d of 3,2 and 1 respectively?

Q 46. The diameter and altitude of a can in the shape of a right circular cylinder are measured as 40 cm and 64 cm respectively. The possible error in each measurement is ± 5 percentage. Find approximately the maximum possible error in the computed value for the volume and the lateral surface. Find the corresponding percentage error.

Q 47. The period T of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$. Find the maximum error in T due to possible errors upto 1 percentage in l and 2.5 percentage in g.

Q 48. Find the possible percentage error in computing the parallel resistance r of three resistances r_1, r_2, r_3 from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}$ if r_1, r_2, r_3 are each in error by 1.2 percentage.

Q 49. Find the percentage error in the area of ellipse, when an error of 0.05 percentage made in measuring semi-major and semi-minor axis. [**Ans: 0.1 percentage**]

Q 50. Find the percentage error in the area of an ellipse when an error of +1 percent made in measuring the major and minor axes.

Q 51. If the ideal gas law is used to find P when T and V are given but there is an error 0.3 percentage in measuring T and an error of 0.8 percentage in measuring V. Find approximate the greatest percentage error in P. [**Ans: 0.5 percentage decreases**]

Q 52. The legs of a right angle triangle are 6 cm and 8 cm long. Find approximately the greatest percentage error in length of hypotenuse, if an error of 0.1 cm is made in measuring. [**Ans: 1.4 percentage**].

Q 53. Find approximate error in its surface of rectangular parallelepiped of sides a, b and c in error δ is made in the measurement of each side. [**Ans:** $4\delta(a+b+c)$]

Q 54. The dimension of box are measured to be 10 cm, 12 cm and 15 cm. Find the approximate error in the volume of box is V and an error of 0.02 cm is made in each measurement. [**Ans:** $9cm^3$]

Q 55. A company has contracted when the resistors having resistance R_1 and R_2 are connected in parallel resistance R of the combination is given by $R = \frac{R_1R_2}{R_1+R_2}$, $R_1 = 2\Omega$ and $R_2 = 6\Omega$. However when finally measured the change of 0.013Ω in R_1 and of -0.023Ω in R_2 was found. Find approximately the greatest error in R. [**Ans: 0.005875** Ω]

Q 56. As a result of deformation of the radius of a cone changes from 30 cm to 30.1 cm and its height changes from 60 cm to 59.5 cm. Find the approximate change in Volume of a cube. [**Ans:** $30\pi cm^3$].

Q 57. What is approximate error in volume of rectangular parellopied $10'' \times 20'' \times 30''$ if an error of $-0.2''$ is made in measures of each side. Find also approximate volume. [**Ans:** $-220''$**cubic inch, volume is 5780 cubic inch**]

Q 58. A company has contracted to manufacture 10,00,000 closed wooden crates having dimensions 3 ft, 4 ft and 5 ft. The cost of wood to be used is Rs. 50 per square feet. If the machines that are used to cut the pieces of wood have a possible error of 0.05 feet in each dimension. Find approximate the greatest possible error in the estimate of cost of wood. [**Ans: 12,00,00,000 Rs.**].

Q 59. If the fibre glass sheet costs Rs. 45 per square feet. Find approximate the greatest cost of fibre glass sheet 3.012 feet wide and 5.982 feet long. [**Ans: 810.81 Rs.**].

Q 60. Find total differential of following.

(1) $w = 4x^3 - xy^2 + 3y - 7$ [**Ans:** $dw = (12x^2 - y^2)dx + (3 - 2xy)dy$].
(2) $w = y\tan x^2 - 2xy$ [**Ans:** $dw = (2xy\sec x^2 - 2y)dx + (\tan x^2 - 2x)dy$].
(3) $w = x\cos y - y\sin x$ [**Ans:** $dw = (\cos y - y\cos x)dx + (-x\sin y - \sin x)dy$].
(4) $w = xe^{2y} + e^{-y}$ [**Ans:** $dw = (e^{2y})dx + (2xe^{2y} - e^{-y})dy$].
(5) $w = \ln(x^2 + y^2 + z^2)$ [**Ans:** $dw = \left(\frac{2x}{x^2+y^2+z^2}\right)dx + \left(\frac{2y}{x^2+y^2+z^2}\right)dy + \left(\frac{2z}{x^2+y^2+z^2}\right)dz$].
(6) $w = \dfrac{xyz}{x+y+z}$ [**Ans:** $dw = \frac{yz(y+z)}{(x+y+z)^2}dx + \frac{xz(x+z)}{(x+y+z)^2}dy + \frac{xy(x+y)}{(x+y+z)^2}dz$].
(7) $w = x\tan^{-1}z - \frac{y^2}{z}$ [**Ans:** $dw = \tan^{-1}zdx - \frac{2y}{z}dy + (\frac{x}{1+z^2} + \frac{y^2}{z^2})dz$].

(8) $w = e^{yz} - \cos(xz)$[**Ans:** $dw = z\sin xz dx + ze^{yz}dy + (ye^{yz} + x\sin xz)dz$].

Q 61. Find $\dfrac{du}{dt}$ for following:

(1) $u = x^2 + 2xy + y^2, x = t\cos t, y = t\sin t$.
[**Ans:** $\frac{du}{dt} = (2x+2y)[-t\sin t + \cos t] + (2x+2y)[t\cos t + \sin t]$].

(2) $u = \ln(xy + y^2), x = e^t, y = e^{-t}$ [**Ans:** $\frac{du}{dt} = \frac{e^t}{xy+y^2}(y - x - 2y)$].

(3) $u = \tan^{-1}\left(\dfrac{y}{x}\right), x = \ln t, y = e^t$ [**Ans:** $\frac{du}{dt} = (\frac{-y}{x^2+y^2})(\frac{1}{t}) + (\frac{x}{x^2+y^2})e^t$].

(4) $u = xy + yz + zx, x = t\cos t, y = t\sin t$[**Ans:** $\frac{du}{dt} = (y+z)[-t\sin t + \cos t] + (x+z)[t\cos t + \sin t]$].

Q 62. If $u = \sqrt{x^2 + y^2 + z^2}$, where $x = \sin t, y = \cos t, z = \tan t$, find $\dfrac{du}{dt}$ at $t = \dfrac{\pi}{4}$.
[**Ans:** $\sqrt{2}$]

Q 63. If $z = x^2 - y^2$, where $x = e^t\cos t, y = e^t\sin t$, find $\dfrac{dz}{dt}$.
$[Ans : 2e^t[(x-y)\cos t - (x+y)\sin t]$

Q 64. If $u = 2x^2 - yz + xz^2$, where $x = 2\sin t, y = t^2 - t + 1, z = 3e^{-t}$, find $\dfrac{du}{dt}$ at $t = 0$.
[**Ans: 24**]

Q 65. If $z = e^{xy^2}$, where $x = t\cos t, y = t\sin t, t \in \mathbb{R}$, find $\dfrac{dz}{dt}$ at $t = \dfrac{\pi}{2}$.
[**Ans:** $-\dfrac{\pi^3}{8}$]

Q 66. Given $u = \ln\sqrt{x^2 + y^2}, x = re^s, y = re^{-s}$, find $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial s}$
[**Ans:** $\frac{\partial u}{\partial r} = \frac{xe^s + ye^{-s}}{x^2+y^2}, \frac{\partial u}{\partial s} = \frac{x^2-y^2}{x^2+y^2}$].

Q 67. Given $u = xy + xz + yz, x = r, y = r\cos t, z = r\sin t$, find $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial t}$.

Q 68. If $u = x - y^2, x = 2r - 3s + 4, y = -r + 3s - 6$, find $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial s}$.
[**Ans:** $\frac{\partial u}{\partial r} = 2 - 2r + 6s - 12, \frac{\partial u}{\partial s} = 6r - 18s + 33$]

Q 69. If z is a homogenous function of x and y and $x = e^u + e^{-v}, y = e^{-u} + e^v$, then show that $\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v} = x\dfrac{\partial z}{\partial x} - y\dfrac{\partial z}{\partial y}$.

Q 70. If $z = f(x,y), x = e^u\cos v, y = e^u\sin v$, show that $y\dfrac{\partial f}{\partial u} + x\dfrac{\partial f}{\partial v} = e^{2u}\dfrac{\partial f}{\partial u}$.

Q 71. Find the equation of the tangent plane and normal line to the surface
$x^2 + 2y^2 + 3z^2 = 12$ at $(1, 2, -1)$.

Q 72. Find the equation of the tangent plane and normal line to the surface $2x^2 + y^2 + 2z = 3$ at $(2, 1, -3)$.

Q 73. Find the equations of tangent plane and normal line to the surface $z = \sqrt{1 - x^2 - y^2}$ at $(\frac{2}{3}, \frac{2}{3}, \frac{1}{3})$ [**Ans: Tangent plane** $2x + 2y + z - 3 = 0$, **Normal line** $\frac{x - \frac{2}{3}}{2} = \frac{y - \frac{2}{3}}{2} = \frac{z - \frac{2}{3}}{1}$].

Q 74. Find the equations of tangent plane and normal line to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = 4$ at $(4, 1, 1)$. [**Ans: Tangent plane** $x + 2y + 2z - 8 = 0$, **Normal line** $2(x - 4) = y - 1 = (z - 1)$].

Q 75. Find the equations of tangent plane and normal line to the surface $xy + yz + zx = 1$ at $(3, -1, 2)$. [**Ans: Tangent plane** $x + 5y + 2z - 2 = 0$, **Normal line** $\frac{x-3}{1} = \frac{y+1}{5} = \frac{z-2}{2}$].

Q 76. Find the equations of tangent plane and normal line to the surface $xyz = a^2$ at (α, β, γ). [**Ans: Tangent plane** $x\beta\gamma + y\alpha\gamma + z\alpha\beta - 3\alpha\beta\gamma = 0$, **Normal line** $\frac{x-\alpha}{\beta\gamma} = \frac{y-\beta}{\alpha\gamma} = \frac{z-\gamma}{\alpha\beta}$].

Q 77. Find the equations of tangent plane and normal line to the surface $x^3 + 2xy^3 - 7x^2 + 3y + 1 = 0$ at $(1, 1)$.[**Ans: Tangent plane** $y - x = 0$, **Normal line** $\frac{(x-1)}{-1} = \frac{(y-1)}{1}$].

Q 78. Find the equations of tangent plane and normal line to the surface $z = \tan^{-1} \frac{y}{x}$ at $(1, 1)$.[**Ans: Tangent plane** $x - y + 2z - \frac{\pi}{2} = 0$, **Normal line** $2\frac{x-1}{1} = 2\frac{y-1}{-1} = \frac{z - \frac{\pi}{4}}{1}$].

Q 79. Find the equations of tangent plane and normal line to the surface $z = e^{-x} \sin y$ at $(0, \frac{\pi}{2})$. [**Ans: Tangent plane** $x + z - 1 = 0$, **Normal line** $\frac{x-0}{1} = \frac{y - \frac{\pi}{2}}{0} = \frac{z-1}{1}$].

Q 80. Test the relative maxima and minima for the following functions:

(i) $f(x, y) = 2x^4 + y^2 - x^2 - 2y$

(ii) $f(x, y) = x^3 + y^3 - 3x - 12y + 20$

(iii) $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$

(iv) $f(x, y) = 3x^2 + x^3 + 4xy + y^2$

Q 82. In a triangle find the maximum value of $\cos A \cos B \cos C$, where A, B and C are 3 angles. [**Ans:** $\dfrac{1}{8}$]

Q 83. Maximise $x^l y^m z^n$ subject to $x + y + z = a$.

Q 84. Maximise $x^2 y^3 z^4$ subject to $2x + 3y + 4z = 9$.

Q 85. Find the maximum value of xyz subject to $\dfrac{x^2}{25} + \dfrac{y^2}{16} + \dfrac{z^2}{9} = 1$. [**Ans:** $\frac{20}{\sqrt{3}}$]

Q 86. The sum of three positive numbers is unity. Find the maximum value of their product. [**Ans:** $\frac{1}{27}$]

Q 87. In a triangle find the maximum value of $\sin A \sin B \sin C$, where A, B and C are 3 angles. [**Ans:** $\dfrac{3\sqrt{3}}{8}$]

Q 88. If $(\sin z)y = (\cos z)x$, find $\dfrac{\partial z}{\partial y}$ and $\dfrac{\partial z}{\partial x}$ at a point $z = \pi, x = \frac{1}{3}, y = \frac{1}{6}$. [**Ans:** $\frac{\partial z}{\partial x} = 6$ **and** $\frac{\partial z}{\partial y} = 0$]

Q 89. If $z(z^2 + 3x) + 3y = 0$, then prove that $z_{xx} + z_{yy} = \dfrac{2z(x-1)}{(z^2+x)^3}$.

Q 90. If $z^3 - zx - y = 0$, then prove that $\dfrac{\partial^2 z}{\partial x \partial y} = -\dfrac{3z^2 + x}{(3z^2 - x)^3}$.

Q 91. If $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1} \frac{y}{x}$, evaluate $\dfrac{\partial(r,\theta)}{\partial(x,y)}$.

Q 92. If $x = r\cos\theta$, $y = r\sin\theta$, evaluate $\dfrac{\partial(r,\theta)}{\partial(x,y)}$ and $\dfrac{\partial(x,y)}{\partial(r,\theta)}$.

Q 93. If $u = x^2$, $v = y^2$, evaluate $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Q 94. If $u = \frac{y-x}{1+xy}$, $v = \tan^{-1} y - \tan^{-1} x$, evaluate $\dfrac{\partial(u,v)}{\partial(x,y)}$.

Q 95. If $u = xyz$, $v = xy + yz + zx$ and $w = x + y + z$ evaluate $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

Q 96. If $u = x(1-y)$, $v = xy$, evaluate $\dfrac{\partial(u,v)}{\partial(x,y)}$.