1/02/19.

2) Circles :

i) In cartesian system, an eq$^n$ of a circle centred at origin (0,0) with radius a is $x^2+y^2=a^2$

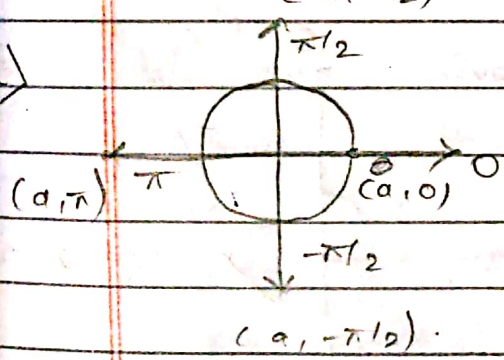
Now take $x = r\cos\theta$ & $y = r\sin\theta$.

$\therefore \quad r^2\cos^2\theta + r^2\sin^2\theta = a^2$

$$r^2 = a^2 \qquad \therefore \boxed{r = a}$$

which is polar form of a circle $x^2+y^2=a^2$.

ii) In cartesian system, an eq$^n$ of a circle centred at (a,0) with radius a is $(x-a)^2+y^2=a^2$




$\Rightarrow x^2 - 2ax + \cancel{a^2} + y^2 = \cancel{a^2}$

$\Rightarrow \cancel{r^2\cos^2\theta} = x^2+y^2 = 2ax.$

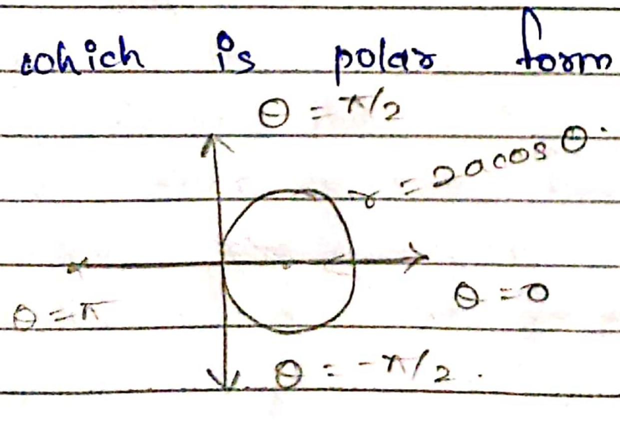
Now take, polar co-ordinate system

$r^2 = 2ax.$

$r\cancel{r} = 2a\cancel{r}\cos\theta$

$$\boxed{r = 2a\cos\theta.}$$

which is polar form of a circle $(x-a)^2+(y-0)^2=a^2$



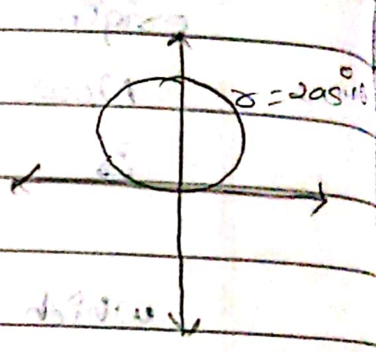
iv) In cartesian system, an eq$^n$ of a circle centred at $(0, a)$ with radius $a$ is $x^2 + (y-a)^2 = a^2$

So $x^2 + y^2 - 2ay + a^2 = a^2$

$x^2 + y^2 = 2ay$

$r^{\cancel{2}} = 2a\cancel{r}\sin\theta$

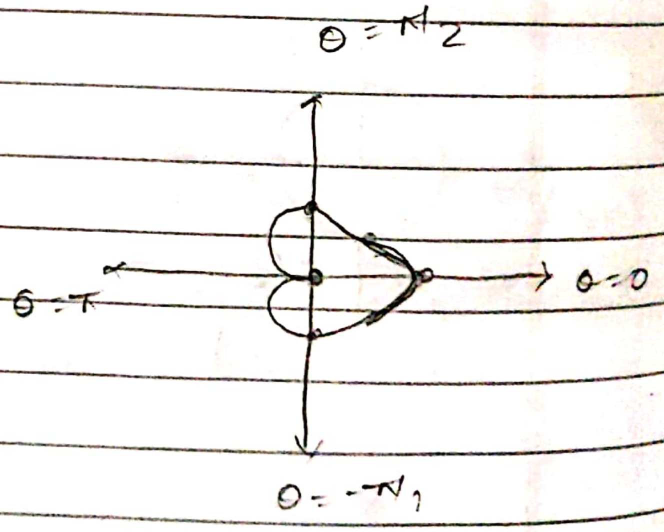
$$\boxed{r = 2a\sin\theta}$$



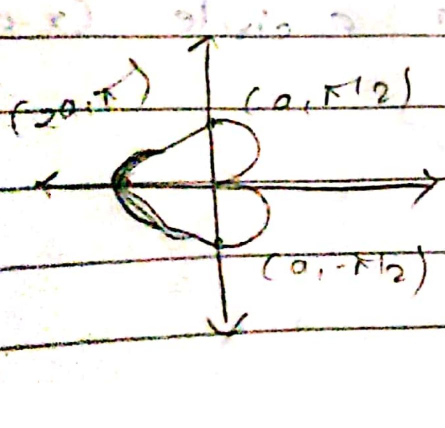

3) Cardioids :

① $r = a(1 + \cos\theta)$

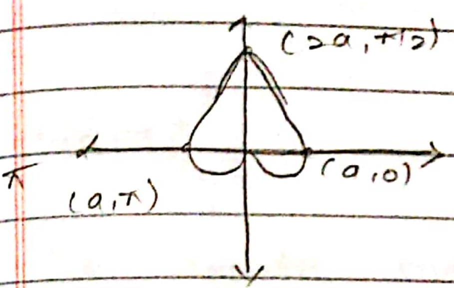
| $\theta$ | 0 | $\pi/3$ | $\pi/2$ | $2\pi/3$ | $\pi$ |
|---|---|---|---|---|---|
| $r$ | $2a$ | $3a/2$ | $a$ | $a/2$ | 0 |



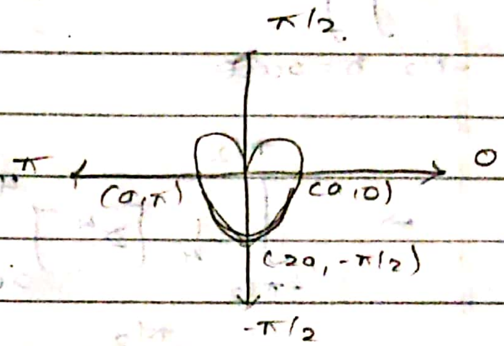

② $r = a(1 - \cos\theta)$



3) $r = a(1+\sin\theta)$




4) $r = a(1-\sin\theta)$



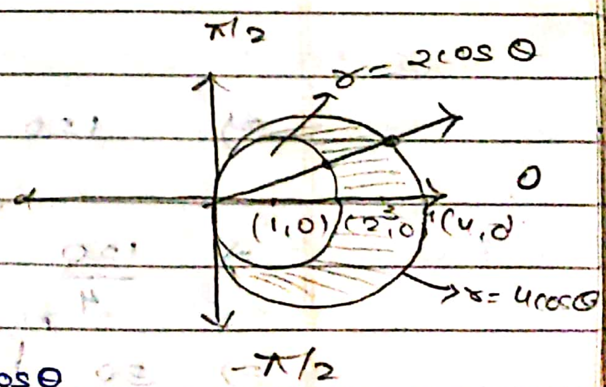

Q.1) Evaluate : $\iint_R r^3 \, dr \, d\theta$, where R is the region bounded between the curves $r = 2\cos\theta$ & $r = 4\cos\theta$.

Sol^n: We are given a region R which is bounded between the circles $r = 2\cos\theta$ & $r = 4\cos\theta$.

Here $\because r = 2\cos\theta$ represents a circle centred at C(1,0) with radius 1.

$r = 4\cos\theta$ represents a circle centred at C(2,0) with radius 2.

$\therefore$ We have




To decide the limits we take a ray from origin within the region.

$\therefore$ limits of $r$ are $r = 2\cos\theta$ to $r = 4\cos\theta$.

& limits of $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$ as region of integration is in 4th & 1st quadrant.

$$\int_{-a}^{a} f = 2\int_{0}^{a} f \rightarrow \text{even}$$
$$= 0 \rightarrow \text{odd}$$

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$$\therefore \int_{\theta=-\pi/2}^{\pi/2}\int_{r=2\cos\theta}^{4\cos\theta} r^3\, dr\, d\theta$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \frac{1}{4}\left[r^4\right]_{2\cos\theta}^{4\cos\theta} d\theta$$

$$\Rightarrow \frac{1}{4}\int_{-\pi/2}^{\pi/2} (4)^4\cos^4\theta - (2)^4\cos^4\theta \; d\theta$$

$$\Rightarrow \frac{(4^4-2^4)}{4}\int_{-\pi/2}^{\pi/2}\cos^4\theta \; d\theta.$$

$$\Rightarrow \frac{4^4-2^4}{4}\, 2\int_{0}^{\pi/2} \cos^4\theta \; d\theta \quad \left[\because \cos^4\theta \text{ is even}\right]$$

$$\Rightarrow (64-4)\, 2 \int_{0}^{\pi/2} (\cos^2\theta)^2\, d\theta.$$

$$\Rightarrow 120\int_{0}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta.$$

$$\Rightarrow \frac{120}{4}\int_{0}^{\pi/2} 1 + 2\cos(2\theta) + \cos^2(2\theta)\; d\theta.$$

$$\Rightarrow 30\int_{0}^{\pi/2}\left[1 + 2\cos(2\theta) + \frac{1+\cos(4\theta)}{2}\right] d\theta.$$

$$\Rightarrow 30\left[\theta + \frac{\cancel{2}\sin(2\theta)}{\cancel{2}} + \frac{\theta}{2} + \frac{\sin 4\theta}{8}\right]_{0}^{\pi/2}$$

$$\Rightarrow 30\left[\frac{\pi}{2} + 0 + \frac{\pi}{4} + 0\right] \Rightarrow 30\left[\frac{3\pi}{4}\right]$$

$$\Rightarrow 30 \times \frac{3\pi}{4} \rightarrow \frac{90\pi}{4} \Rightarrow \boxed{\frac{45\pi}{2}}$$

NOTE:

To decide the limits of $r$ and $\theta$. we will take a ray within the region.

limits of $r$ is nearest & farthest distⁿ from the origin. & limits of $\theta$ is evaluated quadrant wise in the anti-clockwise direction.

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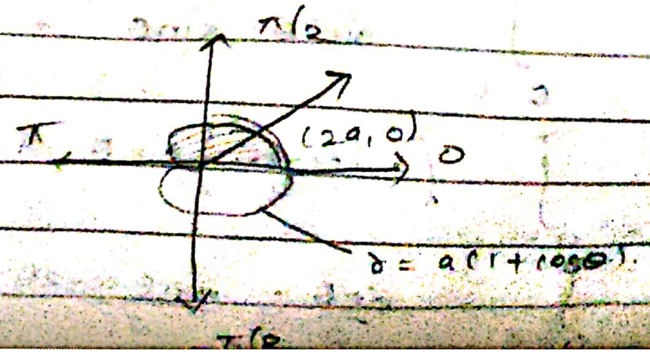
Q.2) Find $\iint_R r\sin\theta \, dr d\theta$. where R is the region bounded in the cardioid $r = a(1+\cos\theta)$ above the initial line.

Solⁿ: We are given region R bounded in the cardioid $r = a(1+\cos\theta)$ above the initial line i.e $\theta = 0$.

So We have



limits of $r$ is $r=0$ to $r = a(1+\cos\theta)$

$\theta$ is $\theta=0$ to $\theta = \pi$.

So $$\int_{\theta=0}^{\pi}\int_{r=0}^{r=a(1+\cos\theta)} r\sin\theta \, dr\, d\theta$$

$$\Rightarrow \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$$

$$\Rightarrow \int_{\theta=0}^{\pi} \frac{\sin\theta}{2}\left[a^2(1+\cos\theta)^2 - 0\right] d\theta$$

$$\Rightarrow \int_{\theta=0}^{\pi} \frac{\sin\theta}{2}\left[a^2(1+2\cos\theta+\cos^2\theta)\right) d\theta.$$

$$\Rightarrow a^2 \int_{\theta=0}^{\pi} \frac{\sin\theta}{2}(\sin^2\theta + 2\cos\theta)\, d\theta.$$

$$\Rightarrow \frac{a^2}{2}\int_{\theta=0}^{\pi} \sin^3\theta + 2\cos\theta\sin\theta \, d\theta.$$

$$\Rightarrow \frac{a^2}{2}\int_{\theta=0}^{\pi} \sin^3\theta + \sin 2\theta \, d\theta.$$

$$\Rightarrow \frac{a^2}{2}\int_{\theta=0}^{\pi} \sin^2\theta\cdot\sin\theta + \sin 2\theta\, d\theta.$$

$$\Rightarrow \frac{a^2}{2}\int_{\theta=0}^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)\sin\theta + \sin 2\theta \, d\theta$$

$$\Rightarrow \frac{a^2}{2}\int_{\theta=0}^{\pi} \frac{\sin\theta}{2} - \frac{\sin\theta\cos 2\theta}{2} + \sin 2\theta\, d\theta$$

$$\Rightarrow \frac{a^2}{2}\int_{\theta=0}^{\pi} \cancel{\frac{\sin\theta}{2} + \sin 2\theta \, d\theta} - \frac{a^2}{2}\int_{\theta=0}^{\pi} \cancel{\frac{\sin\theta \cos 2\theta}{2}} \, d\theta$$

let $\cancel{\cos 2\theta} = t$

$\cancel{-2\sin\theta \, d\theta} = dt.$

$$\Rightarrow \frac{a^2}{2}\cancel{\left[-\frac{\cos\theta}{2}\right]_0^{\pi}} + \cancel{\left[-\frac{\cos 2\theta}{2}\right]_0^{\pi}}$$

| $\theta$ | $0$ | $\pi$ |
|---|---|---|
| $t$ | $1$ | $-1.$ |

$$- \frac{a^2}{2}\int_{+1} t \, dt.$$

$$\Rightarrow \frac{a^2}{2}\cancel{\left[\frac{1}{2}+1\right]} - \cancel{\left[\frac{-1}{2}+1\right]} \cancel{- \frac{a^2}{2}\left[\frac{t^2}{2}\right]_{-1}^{1}}$$

$$\rightarrow \frac{a^2}{2}\int_{\theta=0}^{\pi} \sin\theta \, d\theta + \int_{\theta=0}^{\pi} \sin 2\theta \, d\theta - \int_{\theta=0}^{\frac{1}{2}} \cos^2\theta . \sin\theta \, d\theta$$

$(\cos\theta)^2 . \sin\theta \, d\theta.$

$$\Rightarrow \frac{a^2}{2}\left[ -\cos\theta - \frac{\cos 2\theta}{2} - \frac{\cos^3\theta}{3} \right]_0^{\pi}$$

$$\Rightarrow -\frac{a^2}{2}\left[ \cos\theta + \frac{\cos 2\theta}{2} + \frac{\cos^3\theta}{3} \right]_0^{\pi}$$

$$\Rightarrow -\frac{a^2}{2}\left[ -1 + \frac{1}{2} - \frac{1}{3} \right] - \left[ 1 + \frac{1}{2} + \frac{1}{3} \right].$$

$$\Rightarrow -\frac{a^2}{2}\left[ \frac{-1}{2} - \frac{1}{3} \right] - \left[ \frac{3}{2} + \frac{1}{3} \right]$$

$$\Rightarrow -\frac{a^2}{2}\left[ \frac{-5}{6} \right] - \left[ \frac{11}{6} \right]$$

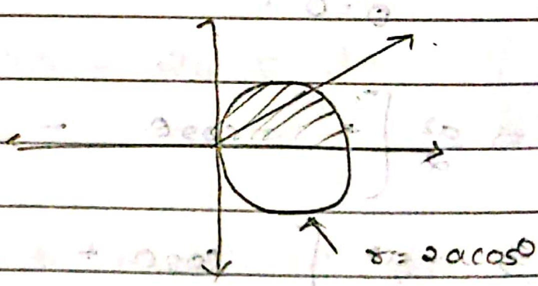
$$\Rightarrow \frac{a^2}{2}\left[ \frac{5}{6} + \frac{11}{6} \right] \Rightarrow \frac{16 a^2}{12} \Rightarrow \boxed{\frac{4a^2}{3}.}$$

(Q/8). H.W.

$\iint_R r\sin\theta \, dr d\theta$. where R is region bounded in the cardioid $r = a(1+\cos\theta)$. above the initial line & in 1st quadrant.

Q03) Find $\iint_R r^2 \sin\theta \, dr d\theta$. where R is region bounded by the semi-circle $r = 2a\cos\theta$ above the initial line.

Solⁿ: We are given a region which is bounded by the semi-circle $r = 2a\cos\theta$ above the initial line.




$r = 0$ to $r = 2a\cos\theta$

$\theta = 0$ to $\theta = \pi/2$

$$\int_0^{\pi/2} \int_0^{2a\cos\theta} r^2 \sin\theta \, dr d\theta$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sin\theta}{3} \left[r^3\right]_0^{2a\cos\theta} d\theta$$

$$\Rightarrow \int_0^{\pi/2} \frac{\sin\theta}{3} \, 8a^3 \cos^3\theta \, d\theta.$$

$$\Rightarrow \frac{8a^3}{3} \int_0^{\pi/2} \sin\theta \cos^3\theta \, d\theta.$$

$\Rightarrow \dfrac{8a^3}{3}$

$\cos\theta = t$

$-\sin\theta\, d\theta = dt.$

| $\theta$ | $0$ | $\pi/2$ |
|---|---|---|
| $t$ | $1$ | $0$ |

$$\Rightarrow \frac{8a^3}{3}\int_0^1 t^3\, dt$$

$$\Rightarrow \frac{8a^3}{3}\left[\frac{t^4}{4}\right]_0^1 \Rightarrow \frac{\cancel{8}^{2}a^3}{\cancel{4}\cdot 3} \Rightarrow \boxed{\frac{2a^3}{3}}$$

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## Double Integrations by change of co-ordinate system:

Suppose we are given a problem in cartesian co-ordinate system i.e. $(x, y)$. Now we want to convert this system into new $(u, v)$ system.

For that we require

$x = f_1(u, v)$, $y = f_2(u, v)$ and $|J|$, where

$J$ is Jacobian of $xy$ with respect to $u$ & $v$.

i.e. $J = \dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix}$

With the help of these, we will change the co-ordinates of integration as follows:

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$$\iint_R g(x, y)\, dx\,dy = \iint_{R^*} g\left(f_1(u, v), f_2(u, v)\right) |J|\, du\,dv.$$

# Change of cartesian system into polar system:

→ To convert cartesian system into polar system we will take $x = r\cos\theta$, $y = r\sin\theta$ & $J = \frac{\partial(x,y)}{\partial(r,\theta)}$

$$= \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & r - \sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = 1(r)$$

$$= r.$$

$$\therefore \quad |J| = r.$$

$\therefore$ The order of integration is $dA = |J|\, dr\, d\theta$.

$$= r\, dr\, d\theta.$$

→ With the help of this, we have

$$\iint_R g(x,y)\, dx\, dy = \iint_{R^*} g(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

Here $R$ is the region in cartesian system and $R^*$ is the region in polar system.

Q.1) Evaluate $\iint_R (x^2+y^2)\, dA$, where R is the annular region between the curves $x^2+y^2=1$ & $x^2+y^2=5$ by changing into polar co-ordinates.

Sol^n: We are given $f(x,y) = x^2+y^2$ and the region is bounded between two circles $x^2+y^2=1$ & $x^2+y^2=5$

→ Here we want to convert given problem into polar form. For that we take

$x = r\cos\theta \quad y = r\sin\theta$, & $dA = r\,dr\,d\theta$.

→ Here $f(r,\theta) = r^2\cos^2\theta + r^2\sin^2\theta = r^2$.

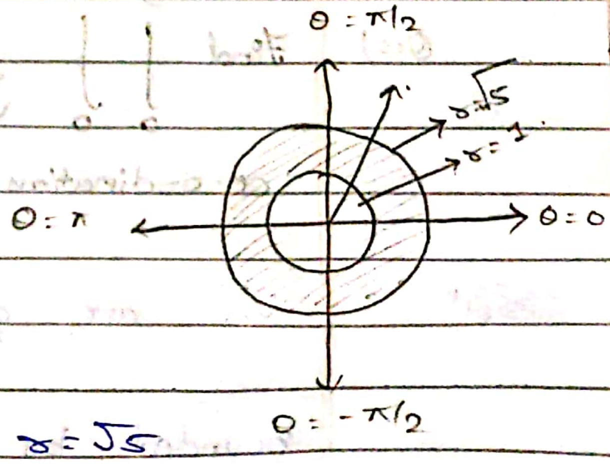
Also here $x^2+y^2=1$, the polar form is

$r^2 = 1 \quad \therefore r = 1$.

for $x^2+y^2=5$, the polar form is

$r^2 = 5 \quad \therefore r = \sqrt{5}$

∴ We have




→ Now, to decide the limits we take a ray from origin within region.

∴ limits are $r=1$ to $r=\sqrt{5}$

$\theta = 0$ to $\theta = 2\pi$.

$$\int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{5}} (r^2)\, r\,dr\,d\theta$$

$$\Rightarrow \int_{\theta=0}^{2\pi}\int_{r=1}^{\sqrt{5}} r^3\, dr\,d\theta$$

$$\Rightarrow \frac{1}{4}\int_{\theta=0}^{2\pi} \left[r^4\right]_1^{\sqrt{5}} d\theta$$

$$\Rightarrow \frac{1}{4}\int_{\theta=0}^{2\pi} (5)^{4/2} - (1)^4 \, d\theta$$

$$\Rightarrow \frac{1}{4}\, 24 \int_{\theta=0}^{2\pi} d\theta$$

$$\Rightarrow 6\,[2\pi - 0]$$

$$\Rightarrow \boxed{12\pi}$$

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Q.2) Find $\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dA$, by changing into polar co-ordination

Sol$^n$: We are given $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$ & the region bounded by the curves $x=0$, $x=2$, $y=0$

& $y = \sqrt{2x-x^2}$

$y^2 = 2x - x^2$ $\therefore$ $x^2+y^2 = 2x$.

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∴ Here we want to convert given problem in polar form

∴ $x = r\cos\theta$ & $y = r\sin\theta$ & $dA = r\,dr\,d\theta$.

∴ $f(r,\theta) = \dfrac{\cancel{r}\cos\theta}{\cancel{r}}$

Here $x = 2$, polar form is

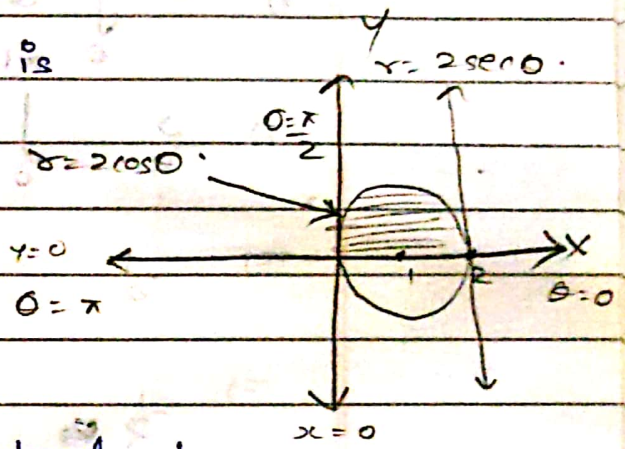
$r\cos\theta = 2$

$r = \underline{2\sec\theta}$.

$r^2 + y^2 = 2x$

$r^2 = 2\cancel{r}\cos\theta$

$\underline{r = 2\cos\theta}$




which represents a circle centred at $(1,0)$ with radius $1$.

∴ We have

limits of $r$ is $r = 0$ to $r = 2\sec\theta$.

$\theta$ is $\theta = 0$ to $\theta = \pi/2$.

∴ We have.

$$\int_0^{\pi/2}\int_0^{2\cos\theta} (\cos\theta)\, r\, dr\, d\theta.$$

$$\Rightarrow \int_0^{\pi/2} \frac{\cos\theta}{2}\left[r^2\right]_0^{2\cos\theta} d\theta.$$

$$\Rightarrow \frac{\cos\theta}{2} \cdot \int_0^{\pi/2} 4\cos^2\theta \, d\theta.$$

$$\Rightarrow 2 \int^{\pi/2} \cos^3\theta \, d\theta.$$

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$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta.$$

$$\frac{\cos 3\theta + 3\cos\theta}{4}.$$

$$\Rightarrow 2\int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right)\cos\theta \, d\theta.$$

$$\Rightarrow \frac{2}{2}\int_0^{\pi/2} \frac{\cos\theta}{\cancel{2}} - \cos\theta \cdot \cos 2\theta \, d\theta.$$

$$\Rightarrow 2\int_0^{\pi/2} \cancel{\cos\theta} + - \frac{\cos 3\theta + 3\cos\theta}{4} \, d\theta.$$

$$\Rightarrow \frac{2}{4}\int_0^{\pi/2} \cos 3\theta + 3\cos\theta \, d\theta.$$

$$\Rightarrow \frac{1}{2}\left[\frac{\sin 3\theta}{3}\right]_0^{\pi/2} + 3\left[\sin\theta\right]_0^{\pi/2}$$

$$\Rightarrow \frac{1}{2}\left[\frac{-1}{3}\right] + 3\left[1\right].$$

$$\Rightarrow \frac{1}{2}\left[\frac{-1}{3}+3\right] \Rightarrow \frac{1}{(2)}\frac{8}{3} \Rightarrow \frac{4}{3}.$$

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)\, dA$ by changing into polar co-ordinate system.

3) Find the area of a region enclosed by the cardioid $r = 1 + \sin\theta$

Here we want to find $\iint_R 1 \, dA$; where

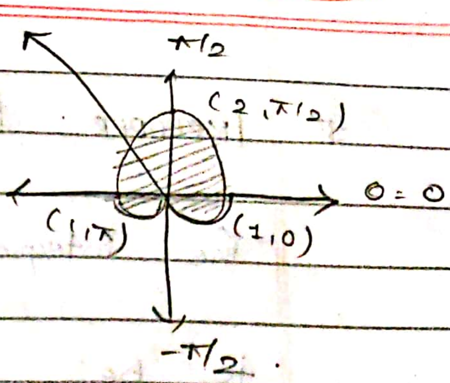
R is the region bounded by the cardioid $r = 1 + \sin\theta$.

$\therefore$ We have.

limits of $r$ is as

$r = 0$ to $r = 1 + \sin\theta$

$\theta = 0$ to $\theta = 2\pi$.




$$\int_{\theta=0}^{2\pi} \int_{r=0}^{1+\sin\theta} 1 \, r \, dr \, d\theta.$$

$$\Rightarrow \frac{1}{2} \int_{\theta=0}^{2\pi} \left[r^2\right]_0^{1+\sin\theta} d\theta.$$

$$\Rightarrow \frac{1}{2} \int_{\theta=0}^{2\pi} (1+\sin\theta)^2 \, d\theta.$$

$$\Rightarrow \frac{1}{2} \int_{\theta=0}^{2\pi} 1 + 2\sin\theta + \sin^2\theta \, d\theta.$$

$$\Rightarrow \frac{1}{2} \left[\theta\right]_0^{2\pi} - 2\left[\cos\theta\right]_0^{2\pi} + \left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}.$$

$$\Rightarrow \frac{1}{2} \left[2\pi\right] - 2\left[1-1\right] + \left[2\pi - 0\right]$$

$$\Rightarrow \frac{1}{2} \left[2\pi\right] - 0 + \pi \Rightarrow \boxed{3\pi/2}$$

Ans. Find the area of region enclosed by the cardioid $r = 1 + \cos\theta$ $(3\pi/2)$

# IMPROPER INTEGRAL :

The integral of the form $\int_a^b f(x)\,dx$ is said to be an improper integral if

i) one or both limits of integration are infinite &

ii) $f(x)$ becomes infinite at end points of interval or at a point within the interval of integration.

① $\int_{-\infty}^{0} e^{x}\,dx$ ② $\int_0^{\infty} \frac{1}{x^2+1}\,dx$.

③ $\int_2^3 \frac{1}{x-2}\,dx$. ④ $\int_1^4 \frac{x}{x^2-9}\,dx$.

/03/19.

For the solution of improper integrals, we will use the foll. special functions:

① Gamma function
② Beta function

+P) only. Gamma function : Let n be any positive real number. Then the gamma funⁿ of small n is denoted by '$\Gamma(n)$' & defined

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$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx \quad ; n > 0$$

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Properties :

$e^{-\infty} = 0$

$\frac{3}{2}+1 \qquad \frac{3}{2} \cdot \Gamma(1/2) \cdot \Gamma(1)$

$5/2 \qquad \sqrt{\pi} \quad (1)$

① $\Gamma(1) = 1$

② $\Gamma(n+1) = n\Gamma(n) = n!$ $\qquad \boxed{\Gamma(n) = (n-1)!}$

Also we can observe that

$= n(n-1)\,\Gamma(n-1)$

$= n(n-1)(n-2)\,\Gamma(n-2) \qquad n \in R^+$

$\vdots$

$= n(n-1)(n-2)\cdots 1 = n!$

e.g. $\Gamma(10)$ so $= 9!$

③ $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Beta function : Let m & n be any two positive real numbers. Then the beta funⁿ of ~~sma~~ m & n is denoted by $\beta(m,n)$ & defined as

$$\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1}\, dx \quad ; m,n \in R^+$$

Properties :

① $\beta(m,n) = \beta(n,m)$

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② $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}x \cdot \cos^{2n-1}x \, dx.$

# Some important results :

① Relation b/w $\beta$ & $\Gamma$ funⁿ is given by

$$\beta(m,n) = \frac{\Gamma(m) \cdot \Gamma(n)}{\Gamma(m+n)} \quad m,n > 0.$$

kind of mcq's

e.g. Find value of $\beta(2,3)$

$$\beta(2,3) = \frac{\Gamma(2)\ \Gamma(3)}{\Gamma(5)} = \beta(3,2)$$

$$= \frac{1 \times \cancel{2}}{\cancel{2} \times 3 \times 4} \Rightarrow \frac{1}{12}.$$

② From 2nd prop. of $\beta$ funⁿ we have

$$\int_0^{\pi/2} \sin^p x \cdot \cos^q x \, dx = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

NOTE :

If the improper integral of the form $\int_0^\infty$ or it contains the log funⁿ Ex. as integrand funⁿ, then we always use $\Gamma$ funⁿ. otherwise use $\beta$ funⁿ.

Q1) Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

∴ We are given $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 (1-x^4)^{-1/2}\, dx.$

Let us take $x^4 = t$

$\therefore \quad x = t^{1/4}$

$\therefore \quad \frac{dx}{dt} = \frac{1}{4} t^{-3/4} \Rightarrow \frac{1}{4t^{3/4}}$

$\therefore \quad dx = \frac{1}{4} t^{-3/4}\, dt.$

if $x = 0 \quad t = 0$

$x = 1 \quad t = 1$

$\Rightarrow \int_0^1 (1-t)^{-1/2} \cdot \frac{1}{4} t^{-3/4}\, dt.$

$\Rightarrow \frac{1}{4} \int_0^1 t^{-3/4} \cdot (1-t)^{-1/2}\, dt.$

Now compare with $\beta(m,n) = \int_0^1 t^{m-1} (1-t)^{n-1}\, dt.$

we get,

$m-1 = -\frac{3}{4} \qquad n-1 = \frac{-1}{2}$

$m = -\frac{3}{4} + 1 \Rightarrow \frac{1}{4} \qquad n = -\frac{1}{2} + 1 \Rightarrow \frac{1}{2}.$

$\therefore \frac{1}{4} \beta\left(\frac{1}{4}, \frac{1}{2}\right). \Rightarrow \frac{1}{4} \frac{\Gamma(1/4) \cdot \Gamma(1/2)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} \Rightarrow \boxed{\frac{\Gamma(1/4)\sqrt{\pi}}{4\,\Gamma(3/4)}}$

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Q-2). Find $\int_0^1 \frac{dx}{\sqrt{1-x^n}}$

Soln: let us take $x^n = t$

$x = t^{1/n}$

$\therefore dx = \frac{1}{n} t^{1/n - 1} dt$

$dx = \frac{1}{n} t^{\frac{1-n}{n}} dt$.

if $x = 0 \quad t = 0$

$x = 1 \quad t = 1$.

$\therefore \int_0^1 (1-t)^{-1/2} \cdot \frac{1}{n} \cdot t^{\frac{(1-n)}{n}} dt$.

$\Rightarrow \frac{1}{n} \int_0^1 t^{\frac{1-n}{n}} \cdot (1-t)^{-1/2} dt$.

Comparing it with $\beta(M,N)$

$M-1 = \frac{1-n}{n}$. $\quad$ ~~$n = \frac{1}{2}$~~ $\quad N-1 = \frac{-1}{2}$.

~~$n = 1+n$~~ $\quad \boxed{N = \frac{1}{2}}$.

$M-1 = \frac{1-n}{n}$

$M = \frac{1-n+1}{n}$

$= \frac{1-n+n}{n} \quad \boxed{M = \frac{1}{n}}$

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$\therefore \frac{1}{n} \beta\left(\frac{1}{n}, \frac{1}{2}\right) \Rightarrow \frac{1}{n} \frac{\Gamma(1/n) \cdot \Gamma(1/2)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$

$\Rightarrow \boxed{\frac{1}{n} \frac{\Gamma(1/n) \cdot \sqrt{\pi}}{\Gamma\left(\frac{n+2}{2n}\right)}.}$

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Q.3). $\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$, $(n > 0)$.

We are given $\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$;

let us take $\log\left(\frac{1}{y}\right) = t$

$\therefore \frac{1}{y} = e^t.$ $\therefore y = e^{-t}.$

$\therefore dy = -e^{-t} dt.$

$\Rightarrow \int_0^{\infty} (t)^{n-1} \cdot t\, e^{-t}\, dt.$

$y = 0 \quad t = \infty$

$y = 1 \quad t = 0.$

$\Rightarrow \boxed{\Gamma(n)}$

$\Rightarrow \boxed{(n-1)!}$ if $n$ is int then only.

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$\left(\frac{10x!}{x}\right)^{10}$ $= 11 - 1$

$(11-1)!$

$\Rightarrow 10!$

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$(a-x)^{10}$ then $x = at$.

Q.4) $\int_0^n x^n (n-x)^p \, dx$.

$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx$

$(n-x)$

$n(1-x/n)$

$x/n = t$

$\therefore \boxed{x = nt}$

Ans: We are given $\int_0^n x^n (n-x)^p \, dx$.

let us take $x = nt$.

$\therefore dx = n\,dt$.

$x = 0 \quad t = 0$

So $\int_0^1 (nt)^n \, n(1-t)^p \, n\,dt$. $x = n \quad t = 1$

$\Rightarrow n^{n+p+1} \int_0^1 (t^n) \, (1-t)^p \, dt$.

$\Rightarrow$ Now comparing.

$M-1 = n \qquad N-1 = p$

$\therefore M = n+1 \qquad N = p+1$.

So $n^{n+p+1} \, (n+1, p+1)$

$$\Rightarrow \boxed{n^{n+p+1} \frac{\Gamma(n+1)\,\Gamma(p+1)}{\Gamma(n+p+2)}}$$

$$= n^{n+p+1} \frac{\Gamma(p+1)}{\Gamma(n+p+2)}$$

HW $\int_0^4 x^4 (4-x)^3 \, dx$.

Q.5) Find $\int_0^\infty \frac{x^c}{c^x} dx$ ; c is any arbitary constant.

Sol$^n$: We are given $\int_0^\infty \frac{x^c}{c^x} dx = \int_0^\infty x^c c^{-x} dx$.

let us take $c^{-x} = e^{-t}$.

$\log c^{-x} = \log e^{-t}$.

$-x \log c = -t$ (1).

$$\boxed{x = \frac{t}{\log c}}$$

$dx = \frac{dt}{\log c}$. $x = 0$ $t = 0$; $x = \infty$ $t = \infty$.

$\Rightarrow \int_0^\infty \left(\frac{t}{\log c}\right)^c \cdot e^{-t} \cdot \frac{dt}{\log c}$

$\Rightarrow \frac{1}{(\log c)^{c+1}} \int_0^\infty (t)^c e^{-t} dt$.

Now compare with def$^n$

$n-1 = c$ $\therefore$ $n = c+1$.

$$\Rightarrow \boxed{\frac{\Gamma(c+1)}{(\log c)^{c+1}}}$$

Ans.

changing value of c we get answers.

e.g. $\int_0^\infty x^7 \cdot 7^{-x} dx$.

Q.6) $\int_0^\infty a^{-bx^2}\,dx$.

Sol^n: let us take $a^{-bx^2} = e^{-t}$.

$-bx^2 \log(a) = -t \quad (1)$

$x = \sqrt{\dfrac{t}{b\log a}}$ $\quad \therefore \; x^2 = \dfrac{t}{b\log(a)}$.

$\therefore \; \cancel{2x\,dx = \dfrac{dt}{b\log(a)}}$

$dx = \dfrac{t^{-1/2}}{2(b\log a)^{1/2}}\,dt$. $\quad \therefore \; \cancel{dx = \dfrac{dt}{2bx\cdot\log(a)}}$.

$\cancel{x = 0} \quad \cancel{t = 0}$

$\cancel{x = \infty} \quad t = \infty$.

So $\displaystyle\int_0^\infty \cancel{\frac{e^{-t}\,dt}{2bx\cdot\log(a)}}$

$\Rightarrow \dfrac{1}{2b\log(a)}\displaystyle\int_0^\infty x^{-1}\,e^{-t}\,dt$.

comparing $n-1 = -1$

$n = 0$

$\dfrac{1}{2(b\log a)^{1/2}}\displaystyle\int_0^\infty t^{-1/2}\,e^{-t}\,dt$.

$\Rightarrow \quad n-1 = \dfrac{-1}{2}$

$\displaystyle\int_0^\infty 2^{-3x^2}\,dx$. $\quad n = -1/2+1$

H.W. $\boxed{n = 1/2}$ $\Rightarrow \dfrac{\sqrt{\pi}}{2(b\log a)^{1/2}}$

Q.7) $\int_0^1 x^4 \left(\log \frac{1}{x}\right)^3 dx.$

Sol$^n$: let us take $\log \frac{1}{x} = t.$

$\therefore \frac{1}{x} = e^t \quad \therefore x = e^{-t}$

$dx = -e^{-t}\, dt.$

$\Rightarrow \int_0^\infty (e^{-t})^4 \cdot (t)^3 \cdot e^{-t}\, dt.$ $\quad x = 0 \quad t = \infty$

$x = 1 \quad t = 0.$

$\Rightarrow \int_0^\infty e^{-5t} \cdot (t)^3\, dt.$

let us take $5t = u.$

~~Edtte~~ $\therefore t = u/5$

$\therefore dt = \frac{du}{5}.$

$\Rightarrow \int_0^\infty e^{-u} \cdot (u/5)^3 \cdot \frac{du}{5}.$ $\quad$ if $t = 0 \quad u = 0$

$t = \infty \quad u = \infty$

$\Rightarrow \frac{1}{5^4} \int (u)^3 \cdot e^{-u}\, du.$

$n - 1 = 3$

$\boxed{n = 4}$

$\Rightarrow$ So $\frac{\Gamma(4)}{5^4} \Rightarrow \frac{3!}{5^4}$

$\Rightarrow \boxed{\frac{6}{625}}$ //.

# Error function :

The error function of 'x' is denoted by erf(x) and defined as

IMP

$$\boxed{\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt.}$$

# Complementary error function :

The complementary error function of x is denoted by erfc (x) and defined as

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$$\boxed{\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt.}$$

# Properties of error function :

(1) $\operatorname{erf}(0) = 0.$

(2) $\operatorname{erf}(\infty) = 1.$

Example

By defⁿ of error function, $\operatorname{erf}(\infty)$

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-t^2}\, dt$$

let us take

$$t^2 = u$$

$$t = \sqrt{u}$$

$$dt = \frac{1}{2} u^{-1/2}\, du$$

$$\Rightarrow \frac{2}{\sqrt{\pi}} \cdot \frac{1}{2} \int_0^\infty u^{-1/2} \cdot e^{-u} \, du$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \qquad n-1 = \frac{-1}{2} \qquad n = \frac{1}{2}.$$

$$\Rightarrow \frac{1}{\sqrt{\pi}} \cdot \Gamma(1/2) \Rightarrow \frac{\sqrt{\pi}}{\sqrt{\pi}} \Rightarrow 1.$$

(3) $\underset{1}{\underline{erf(x)}} \underset{+}{+} \underset{0}{\underline{erf_c(x)}} = \underline{\underline{1}}.$

(4) $erf(-x) = -erf(x)$. i.e. error $fun^n$ is $\boxed{odd}$ $fun^n$

By $def^n$ of error $fun^n$ :

$$erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^{-x} e^{-t^2} \, dt.$$

let us take $t = -u$.

$\therefore$ $dt = -du$.

if $t = 0$, then $u = 0$

$t = -x$ $\quad u = x$.

$$\therefore erf(-x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} (-du)$$

$$\underline{\underline{=}} \; -erf(x).$$

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Q.1) Prove that $\int_{-a}^{a} e^{-(x+a)^2} dx = \frac{\sqrt{\pi}}{2}[erf(2a)]$

$e^{-t^2}$.

Sol^n: let us take $x + a = t$.

$dx = dt$.

$$\int_{0}^{2a} e^{-t^2} . dt.$$

$$\Rightarrow \frac{2}{\sqrt{\pi}} \quad \frac{\sqrt{\pi}}{2} \frac{2}{\sqrt{\pi}} \int_{0}^{2a} e^{-t^2} dt.$$

$$\Rightarrow \boxed{\frac{\sqrt{\pi}}{2} [erf(2a)].}$$

// Hence proved.

Q.2) Show that $\int_{a}^{\infty} e^{-(2x-a)^2} dx = \frac{\sqrt{\pi}}{4}[1 - erf(a)]$

Sol^n:

$2x - a = t$

$2dx = dt$ if $x = a$ $t = a$

$x = \infty$ $t = \infty$.

$$\Rightarrow \int_{a}^{\infty} e^{-t^2} . \frac{dt}{2}.$$

$$\Rightarrow \frac{\sqrt{\pi}}{2(2)} . \frac{2}{\sqrt{\pi}} \int_{a}^{\infty} e^{-t^2} dt.$$

$$= \frac{\sqrt{\pi}}{4} erfc(a)$$

$$= \frac{\sqrt{\pi}}{4} \left[ 1 - erf(a) \right]$$

13) Show that $erf(x) = \alpha(x\sqrt{2})$

where $\alpha(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-t^2/2} dt$.

Sol: We are given that

$$\alpha(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{\frac{-t^2}{2}} dt$$

then $\alpha(x\sqrt{2}) = \sqrt{\frac{2}{\pi}} \int_0^{x\sqrt{2}} e^{-t^2/2} dt$.

So let us take $\frac{t^2}{2} = u^2$

$t^2 = 2u^2$

$\cancel{2t\,dt = 4u\,du}$

$t = \sqrt{2}u$

$dt = \sqrt{2}\,du$.

$$\Rightarrow \sqrt{2}\cdot\sqrt{\frac{2}{\pi}} \int_0^{x} e^{-u^2}\cdot du.$$

$t = 0 \quad u = 0$

$t = x\sqrt{2} \quad u = x$.

$$\Rightarrow \frac{2}{\sqrt{\pi}} \int_0^{x} e^{-u^2} du.$$

$\Rightarrow erf(x)$

Hence proved.