CHAPTER

3

Matter Waves

3.0 Origin of quantum mechanics :

The phenomenon of ejection of electrons from the metal surface, when it is illuminated by light or any other radiations of suitable wavelength (or frequency) is known as photoelectric effect.

It was discovered by Heinrich Hertz in 1887 when the allowed ultra-violet light to fall on Zinc plate. Afterwards it was found that alkali metals like lithium, sodium, potassium, rubidum and cesium eject electrons when visible light falls on them. Millikan investingated the effect with a number of alkali metals over a wide range of light frequencies and we given Nobel Prize in 1923.

❖ Failure of the calssical Theory :

Three major features of the photoelectric effect could not be explained in terms of the way theory of light.

1) According to classical wave theory, larger amplitude of the light wave means higher intenisty ($I \alpha A^2$) and hence higher energies. Therefore kinetic energy of photoelectrons

Scanned by CamScanner

should increase with increase in the intensity of the light beam. However, the experimental observations are that the maximum kinetic energy as well as the stopping potentials are independent of the light intensity. They depend upon frequency instead of intensity.

2) According to wave theory, photoelectric effect should occur for any frequency of the light, provided only that the light beam is intense enough. However, it has been experimentally tested that each photo-cathode has a certain characteristic frequency called as threshold frequency or cut-off frequency below which photoelectric emission is not possible; for any intensity of incident light. Thus calssical theory can't predict the existence of threshold frequency.

3) The classical theory predicts that when light is incident on the metal surface, then it would take some time for the electrons to absorb energy from the waves and acquire enough energy to come out of the metal. If the intensity of the incident light is very low, then it would take several hours for electrons to acquire required energy for emission i.e. there is time lag between the incidence of light and the emission of electrons, according to classical theroy. However, according to experimental observations, the process of photoelectric emission is intantaneous.

◆ Einstein's Photoelectric Equation :

On the basis of Planck's quantum theory, Einstein could explain the photoelectric effect in 1905. Planck's quantum theory of thermal radiation was extended by Einstein to cover all radiation including light rays, γ - rays etc. Einstein postulated that-

(i) The monochromatic light of frequency υ consists of bundles or packets of energy called quanta.

(ii) These quanta which were then called as photons

Scanned by CamScanner

carry a definite amount of energy $E = h\upsilon$.

According to Einstein, when a monochromatic light is incident on a metal surface in photoelectric effect, a single photon is completely absorbed by a single electron. If this energy $h\upsilon$ of photon is greater than the work function W_0 of the metal, then the electron is emitted from the metal surface. This photon energy absorbed by an electron is utilised in the following two ways -

(i) a part of the photon energy is utilized to make the electron free from the metal surface. This amount of energy is known as photoelectric work function, W_0, of the metal.

(ii) The remaining part appears in the form of kinetic energy of the liberated electron.

Thus mathematically,

$$h\upsilon = W_0 + \frac{1}{2} mv^2_{max} \qquad ...(3.1)$$

This equation is known as Einstein's photoelectric equation. It explain all the laws of photoelectric effect as follows-

If the energy of incident photon is such that it is just sufficient to liberate electron with zero kinetic energy, then equation (3.1) reduces to -

$$h\upsilon_0 = W_0 \qquad ...(3.2)$$

where, υ_0 is the threshold frequency. It can be defined as the minimum frequency which can photoelectric emission.

For $\upsilon < \upsilon_0$, the photon energy will not be sufficient to overcome the work function and therefore photoemission will not take place.

From (3.1), the kinetic energy of photoelectron is given by-

$$\frac{1}{2} mv^2_{max} = h\upsilon - W_0$$

Scanned by CamScanner

or $\frac{1}{2}mv^2_{max} = h\upsilon - h\upsilon_0 = h(\upsilon - \upsilon_0)$ ...(3.3)

For a particular metal surface, the work function W_0 is constant and hence the maximum kinetic energy of the photoelectrons is directly proportional to the frequency of incident light.

If V_0 is the stopping potential then -

$$\frac{1}{2} mv^2_{max} = eV_0 \text{ (from defination)} \quad ...(3.4)$$

from (4),

$$V_{max} = \sqrt{\frac{2eV_0}{m}} \quad ...(3.5)$$

from (3.3) and (3.4), we get,

$$eV_0 = h(\upsilon - \upsilon_0)$$

or

$$V_0 = h(\upsilon - \upsilon_0) / e = h\upsilon / e - h\upsilon_0 / e \quad ...(3.6)$$

Comparision of this equation with $y = mx - c$, shows that the stopping potential is directly proportional to the frequency The graph between V_0 and υ will be a straight line having slope.

$$m = h/e.$$

Hence,

$$h = me \quad ...(3.7)$$

Thus, from the slope of experimental curve between V_0 and υ, the Planck's constant h can be calculated by using equation (3.7). The value of h obtained by this method agrees well with the value determined by other methods. Hence, we arrive at the conclusion that the Einstein's photoelectric equation is correct.

According to quantum theory, the intensity can be defined as the number of photons falling per second on unit area of the cathode surface. Therefore, as intensity increases the number of photons falling on the cathode surface will increase and hence the number of photoelectrons liberated will also increase.

Scanned by CamScanner

Thus photocurrent increases linearly with inrease in intensity of the incident light: provided $\upsilon > \upsilon_0$.

Since the increase in intensity does not change the energy hν of the individual photon, the K.E. of photoelectron is independent of the intensity of incident light.

Since energy is supplied in the form of photons (concentrated bundles of energy) and is not spread over a large area as in wave theory, the transfer of energy from photon to electron is in instantaneous i.e. there is no time lag between the incident of light on the metal surface and the emission of photoelectrons.

Thus, Einstein's theory based on the quantum hyphothesis explains all the experimental observations of photoelectric effect.

3.1 Dual nature of Matter :

Wave particle duality : We know that light phenomenon such as interference diffraction etc. were explained on the basis of wave theory of light. But wave theory of light failed to explain photoelectric effect. However photoelectric effect is explained very well using the particle nature of light. The phenomena of comption effect was also explained using the particle nature of light.

Thus we find that in some cases light exhitrits wave nature and in some other cases it exhibits particle nature. In other words light possesses dual nature: wave and particle. Light will not show both the properties simultaneously in a given experiment.

Thus we conclude that light has a dual nature and that is shows only one character in a given situation.

Scanned by CamScanner

- **De-Broglie hypothesis :**

According to De-Broglie hypothesis wave properties are associated with a moving particle, whatever its nature.

This hypothesis is based on the following observations :

i) Nature loves symmetry :

We observe many symmetries in nature. Leave of most of trees have symmetric shapes. Human body is also an excellent example of nature's love for symmetry.

De-Broglie argued that if matter and radiation are two forms of nature and if radiation shows dual nature, then matter must also show dual nature. Hence material particle must exhibit wave nature in addition to their particle nature.

ii) Similarities between mechanics and optics :

Mechanics and optics are the two major branches of physics concerning matter and light. There are some similarities between mechanics and optics as follows.

1) Mechanics : There is a principle of least action about the motion of the bodies.

2) Natural numbers occur in discussion of standing wave produced on stretched string. we have

$$L = \frac{n\lambda}{2} \qquad n = 1, 2, 3.....$$

1) Optics : There is a Fermat's principle of least time about the path of light.

2) Natural numbers occur in optics also, we have interference condition path difference.

$$= n\,\lambda \text{ or } (2\,n + 1)\,\lambda/2 \qquad n = 0, 1, 2, 3....$$

Due to many similarities between matter and light. De-Brogile believed that material particle must also have wave

Scanned by CamScanner

nature just as light has.

Expression for wavellength of De-Broglie wave :

Consider a radiation of frequency ν and wave length λ. The radiation behaves like a photon of energy

$$E = h\upsilon \quad ...(3.8)$$

According to Einstein's mass energy relation

$$E = mc^2 \quad ...(3.9)$$

$\therefore$ equating (3.8) and (3.9)

$$mc^2 = h\upsilon$$

$$\therefore \quad mc = \frac{h\upsilon}{c}$$

$$P = \frac{h\upsilon}{c} \qquad \therefore \quad p = mc$$

$$= \frac{h}{\lambda} \qquad \therefore \quad \upsilon = \frac{c}{\lambda}$$

$$\therefore \quad \lambda = \frac{h}{p} \quad ..(3.10)$$

From equation 3.10 we see that the wavelength λ of radiation is related to the momentum of photon. This equation is true for photons as well as for the material particle.

Material particles must also exhibit the wave particle duality. Thus it a particle of mass m is moving with a velocity V, then p = mv and the wavelength associated with the moving particle will be given by,

$$\lambda = \frac{h}{mv} = \frac{h}{p} \quad ...(3.11)$$

A moving body therefore must show wave characteristics. *The wave associated with a moving particle or with a body is called matter wave or De-Broglie wave.*

The De-Broglie formula is valid for all the moving

Scanned by CamScanner

bodies like earth, a ball, molecule, electron, proton etc.

Thus equation (3.11) is general equation giving the De-Borglie wavelength λ of matter waves.

If E is the K.E. of the material particle, then

$$E = \frac{1}{2} mv^2 = \frac{m^2v^2}{2m} = \frac{P^2}{2m} \quad \therefore \quad P = mv$$

or $$P = \sqrt{2mE}$$

Using this value of P in equation 3.11 we get

$$\therefore \quad \lambda = \frac{h}{\sqrt{2mE}} \qquad ...(3.12)$$

If a charged particle carrying charge 'e' is accelerated through a P.D. of V valts then K.E. = E = eV

$$\therefore \quad \lambda = \frac{h}{\sqrt{2meV}} \qquad ...(3.13)$$

For relativistic speeds total energy (E) is

$$E^2 = P^2C^2 + m_0^2 C^4 \qquad ...(3.14)$$

$$\text{K.E. } eV = E - m_0C^2 \qquad ...(3.15)$$

Using equation (3.14) in (3.15)

$$\therefore \quad eV = \sqrt{P^2C^2 + m_0^2 C^4} - m_0C^2$$

$$\therefore \quad eV + m_0C^2 = \sqrt{P^2C^2 + m_0^2 C^4}$$

squaring and rearranging we get

$$P = \sqrt{2 m_0 eV} \left[1 + \frac{eV}{2m_0C^2} \right]^{1/2}$$

$$\therefore \quad \lambda = \frac{h}{\sqrt{2m_0eV}} \left[1 + \frac{eV}{2m_0C^2} \right]^{-1/2}$$

Example 3.1 : Obtain expression for the De-Broglie wavelength associated with an electron accelerated through Vvolts.

When an electron is accelerated through V volts then K.E. acquired by an electron is

$$\frac{1}{2} mv^2 = eV$$

$$\frac{m^2v^2}{2m} = eV$$

$$\frac{p^2}{2m} = eV$$

$$\therefore \quad P = \sqrt{2meV}$$

but, $$\lambda = \frac{h}{p}$$

$\therefore$ Associated De-Broglie wavelength $= \lambda \dfrac{h}{\sqrt{2meV}}$

$$\therefore \quad \lambda = \frac{6.63 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19} \times V}}$$

$$= \frac{12.27}{\sqrt{V}} A^0$$

Let, $V = 54$ volt then $\lambda = \dfrac{12.27}{\sqrt{54}} A^0 = 1.67\ A^0$

$V = 100$ volt then $\lambda = \dfrac{12.27}{\sqrt{100}} A^0 = 1.227\ A^0$

3.2 Concept of wave group :

Phase - Speed : If ν is the frequency of radiation and λ is the wavelength of radiation then its speed C is given by.

$$C = \nu\lambda.$$

$\therefore$ Speed of wave = frequency $\times$ wavelength

saimilarly the speed with which the De - Borglie wave travel is called the phase speed (V_p) of De-Broglie wave.

phase speed $(V_p) = \nu\lambda$ ...(3.16)

Scanned by CamScanner

The phase velocity V_p is also called wave velocity. For a particle of mass m moving with velocity V the De-Broglie wavelength (λ) is given by.

Also we know $$\lambda = \frac{h}{mv} \qquad ...(3.17)$$

$$E = h\nu \qquad ...(3.18)$$

and $$E = mc^2 \qquad ...(3.19)$$

$$\therefore \quad h\nu = mc^2 \text{ using (3) and (4)}$$

$$\therefore \quad \nu = \frac{mc^2}{h} \qquad ...(3.20)$$

putting for v and in equation (3.16) we get.

$$V_p = \frac{mc^2}{h} \times \frac{h}{mv}$$

$$= \frac{c^2}{V} \qquad ...(3.21)$$

Thus, equation (3.21) indicates that phsre speed V_p must be grater than C; but for any particle V < C. This is unexpected result.

If the De-Broglie wave is associated with a moving particle then the wave must have same speed as that of the particle.

The difficulty indicated above regarding the phase speed (V_p) can be solved by understanding the wave group and group velocity.

Consider a simple form of a wave given by.

$$\psi = a \sin 2\pi\nu \left(t - \frac{x}{V_p} \right)$$

where ν is the frequency of the wave and V_p is two wave velocity

Scanned by CamScanner

$$\psi = a \sin\left(2\pi\upsilon t - \frac{2\pi\upsilon x}{V_p}\right)$$

$$= a \sin\left(2\pi\upsilon t - \frac{2\pi x}{\lambda}\right) \qquad ...(3.22)$$

but angular frequency $\omega = 2\pi\upsilon$ $\qquad ...(3.23)$

propagation constant $K = \dfrac{2\pi}{\lambda}$ $\qquad ...(3.24)$

$\therefore$ Equation (3.22) can be written as

$$\psi = a \sin(\omega t - kx) \qquad ...(3.25)$$

Equation (3.25) is the standard wave equation. From (3.23) and (3.24).

$$\frac{\omega}{k} = \upsilon\lambda$$

$$= V_p = \text{phase velocity or wave velocity}$$

$$\therefore \quad V_p = \frac{\omega}{k} \qquad ...(3.26)$$

Now if a wave given by equation (3.25) is associated with a moving body then the probabiltiy density $\psi\psi^*$ will be proportional to a^2 which is constant. But this is not correct. because if a body of finite size moves with a velocity V then the probability of finding it at a given istant will be maximum at its centre of mass and the probability should rapidly decrease on both sides of centre of mass. Hence De-Broglie wave associated with a moving body should not be like equation (3.25) of constant amplitude but its amplitude must be maximum at some point and should decrease rapidly on both sides. Such a wave shown in following figure is called a wave group or wave packet.

But the wave shown in figure 3.1 is formed in the phenomenon of beats where two sound wave slightly different in frequencies superpose to form a resultant wave similar to figure 3.1.

Scanned by CamScanner

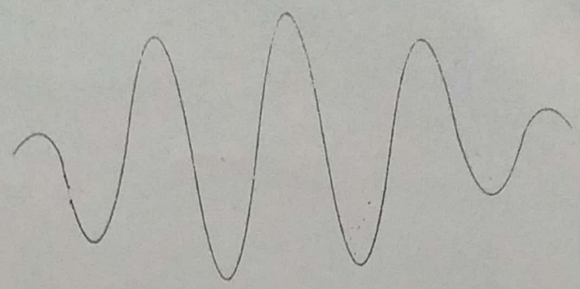

Figure 3.1 A Wage group

Thus a wave group or wave packet is formed by the superposition of two or number of waves slightly different in frequencies.

If we consider two waves with angular frequencies ω and $\omega + d\omega$ and propagation constants k and $k + dk$ then these waves superpose to form a wave group and *the velocity with which a wave group travels is called the group velocity (V_g).*

Expression for group velocity : Consider the two waves represented by,

$\psi_1 = a \sin(\omega t - kx)$ and

$\psi_2 = a \sin\{(\omega + d\omega)t - (k + dk)x\}$ are

superposed then the resultant displacement is given by,

$$\psi = \psi_1 + \psi_2$$

$$= a \sin(\omega t - kx) + a\sin\{(\omega + d\omega)t - (k + dk)x\}$$

$$= 2a \cos\left(\frac{d\omega}{2}t - \frac{dk}{2}x\right)\sin\left\{\left(\omega + \frac{d\omega}{2}\right)t - \left(k + \frac{dk}{2}\right)x\right\}$$

but $d\omega$ and dk are very small as compaired to ω and k

$$\therefore \quad \psi = 2a \cos\left(\frac{d\omega}{2}t - \frac{dk}{2}x\right)\sin\{\omega t - kx\}$$

This equation represents a wave travelling with phase velocity (V_p) whose amplitude is...

$$2A \cos\left(\frac{d\omega}{2} t - \frac{dk}{2} x\right) \qquad ...(3.26\ a)$$

This amplitude represents another wave travelling with velocity called group velocity (V_g) given by.

$$V_g = \frac{\text{coefficient of t in equation 3.26 a}}{\text{coefficient of x in equation 3.26 a}}$$

$$= \frac{d\omega/2}{dk/2}$$

$$\therefore \quad V_g = \frac{d\omega}{dk} \qquad ...(3.27)$$

Example 3.2 : Show that group velocity is equal to particle velocity.

Consider a particle of mass m is moving with velocity V then its K.E. is given by.

$$E = \frac{1}{2} mv^2$$

$$= \frac{m^2v^2}{2m}$$

$$\therefore \quad E = \frac{P^2}{2m} \qquad ... (1)$$

also

$$E = h\nu$$

$$= \frac{h}{2\pi} \times 2\pi\nu$$

$$= \hbar\omega \qquad ... (2)$$

we have $\lambda = \dfrac{h}{P}$

$$\therefore \quad P = \frac{h}{\lambda}$$

Scanned by CamScanner

$$= \frac{h}{2\pi} \times \frac{2\pi}{\lambda}$$

$$P = \hbar k \qquad \dots (3)$$

Using equation (2) and (3) in equation (1) we get

$$\hbar\omega = \frac{\hbar^2 k^2}{2m}$$

$$\therefore \quad \omega = \frac{\hbar k^2}{2m}$$

$$\therefore \quad \frac{d\omega}{dk} = \frac{2\hbar k}{2m}$$

$$= \frac{\hbar k}{m}$$

but, $$V_g = \frac{d\omega}{dk}$$

$$\therefore \quad V_g = \frac{\hbar k}{m}$$

$$= \frac{P}{m} \qquad \text{using equation (3)}$$

$$= \frac{mv}{m}$$

$$\therefore \quad V_g = V$$

$\therefore$ group velocity is equal to particle velocity

Example 3.3 : For a particle moving with non relativistic velocity show that group velocity is equal to the particle velocity.

Solution : We know that

$$V_g = \frac{d\omega}{dk}$$

$$= \frac{d(\hbar\omega)}{d(\hbar k)} \qquad \dots (1)$$

we have. $E = h\upsilon = \frac{h}{2\pi} 2\pi\upsilon = \hbar\omega$ ...(2)

$$P = \frac{h}{\lambda} = \frac{h}{2\pi} \times \frac{2\pi}{\lambda} = \hbar k \quad ...(3)$$

using (2) and (3) in equation (1) we get

$$V_g = \frac{dE}{dP} \quad ...(4)$$

K.E. is given by,

$$E = \frac{1}{2} mv^2 = \frac{m^2v^2}{2m} = \frac{P^2}{2m}$$

$$\therefore \quad \frac{dE}{dP} = \frac{2P}{2m} = \frac{P}{m} = \frac{mV}{m} = V \quad ...(5)$$

Comparing equation (4) and (5)

$V_g = V$.

$\therefore$ Group velocity is equal to particle velocity.

Example 3.4 : Show that phase velocity for waves associated with particle having non-relativistive velocity is equal to half the particle velocity.

Solution : We know that

$$V_p = \frac{\omega}{k}$$

$$= \frac{\hbar\omega}{\hbar k} \quad ...(1)$$

we have $E = hv = \frac{h}{2\pi} 2\pi v = \hbar\omega$ ...(2)

$$P = \frac{h}{\lambda} = \frac{h}{2\pi} \cdot \frac{2\pi}{\lambda} = \hbar k \quad ...(3)$$

using (2) and (3) equation (1) we get.

$$V_p = \frac{E}{P}$$

Scanned by CamScanner

$$= \frac{1/2\, mv^2}{mv}$$

$$\therefore \quad V_p = \frac{1}{2} V$$

Example 3.5 : Show that $V_g = V_p + k \frac{dV_p}{dk}$

and $V_g = V_p - \lambda \frac{dV_p}{d\lambda}$

Solution : We know that.

$$V_g = \frac{d\omega}{dk} \qquad ...(1)$$

and $V_p = \frac{\omega}{k}$

$$\therefore \quad \omega = kV_p \qquad ...(2)$$

using (2) in equation (1) we get

$$V_g = \frac{d(kV_p)}{dk}$$

$$V_g = V_p + k \frac{dV_p}{dk} \qquad ...(3)$$

But.

$$k = \frac{2\pi}{\lambda} \qquad ...(4)$$

$$\therefore \quad \frac{dk}{d\lambda} = \frac{-2\pi}{\lambda^2} \qquad ...(5)$$

Equation (3) can be written as...

$$V_g = V_p + k \frac{dV_p}{d\lambda} \cdot \frac{d\lambda}{dk}$$

$$= V_p + k \frac{dV_p}{d\lambda} \Big/ \frac{dk}{d\lambda}$$

using equation (4) and (5) we get,

Scanned by CamScanner

$$V_g = V_p + \frac{2\pi}{\lambda} \cdot \frac{dV_p}{d\lambda} / \left(\frac{-2\pi}{\lambda^2}\right)$$

$$= V_p - \frac{2\pi}{\lambda} \cdot \frac{\lambda^2}{2\pi} \frac{dV_p}{d\lambda}$$

$$V_g = V_p - \lambda \frac{dV_p}{d\lambda}$$

Example 3.6 : Calculate the de-Broglie wavelength of an electron moving with a velocity $\frac{1}{20}^{th}$ of the velocity of light

Solution : $m = 9.1 \times 10^{-31}$ kg $v = \frac{C}{20} = \frac{3 \times 10^8}{20}$

$= 1.5 \times 10^7$ m/s.

we know that, $\lambda = \frac{h}{P}$

$$= \frac{h}{mv}$$

$$\therefore \quad \lambda = \frac{6.63 \times 10^{-34}}{9.1 \times 10^{-31} \times 1.5 \times 10^7}$$

$$= 0.4845 \times 10^{-10} \text{ m} = 0.4845 \text{ A}^0$$

Example 3.7 : Calculate de-Broglie wavelength of an electron which has K.E. equal to 15 eV

Solution : $m = 9.1 \times 10^{-31}$ kg

$E = 15.\text{eV} = 15 \times 1.6 \times 10^{-19}$ J

we know that,

$$\lambda = \frac{h}{\sqrt{2mE}}$$

$$= \frac{6.63 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19}}}$$

$$= 0.312 \times 10^{-9} \text{ m} = 3.12 \text{ A}^0$$

Scanned by CamScanner

Example 3.8 : The electron beam in a TV receiver tube is accelerated by 10,000 volts, calculate the de-Broglie wavelength of the electron.

Solution : de-Broglie wavelength is given by,

$$\lambda = \frac{h}{\sqrt{2meV}} \quad \text{or} \quad \lambda = \frac{12.2}{\sqrt{v}}$$

$$= \frac{6.63 \times 10^{-34}}{\sqrt{2 \times 9.1 \times 10^{-31} \times 1.6 \times 10^{-19} \times 10.000}}$$

$$= \frac{12.2}{\sqrt{10.000}}$$

$$= 0.122 \times 10^{-10}\ m$$

$$= 0.122\ A^{0}$$

Example 3.9 : Neglecting surface tension, the velocity of deep waves in sea is given by.

$$u = \sqrt{\frac{g\lambda}{2\pi}} \qquad \text{Then show that } V_g = \frac{u}{2}$$

Solution : We know that,

$$V_g = V_p - \lambda \frac{dV_p}{d\lambda}$$

Here, $V_p = u = \sqrt{\frac{g\lambda}{2\pi}}$

$$\therefore \quad \frac{dV_p}{d\lambda} = \frac{d}{d\lambda}\left(\frac{g\lambda}{2\pi}\right)^{1/2} = \left(\frac{g}{2\pi}\right)^{1/2} \frac{d\lambda^{1/2}}{d\lambda}$$

$$= \frac{1}{2}\left(\frac{g}{2\pi}\right)^{1/2} . \lambda^{-1/2}$$

$$\therefore \quad V_g = \sqrt{\frac{g\lambda}{2\pi}} - \lambda\lambda^{-1/2}\sqrt{\frac{g}{2\pi}}$$

Scanned by CamScanner

$$= \sqrt{\frac{g}{2\pi}} - \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}$$

$$= \frac{1}{2}\sqrt{\frac{g\lambda}{2\pi}}$$

$$= \frac{1}{2} V_p$$

$\therefore$ group velocity $= \frac{1}{2} \times$ phase velocity.

Example 3.10 : The velocity of ocean waves is $\sqrt{g/k}$. Show that the group velocity of these waves is half the phare velocity.

Solution : We know that,

$$V_g = V_p + k\frac{dV_p}{dk} \qquad ...(1)$$

Here, $V_p = \sqrt{g/k} = k^{-1/2}\sqrt{g}$

$$\therefore \frac{dV_p}{dk} = \frac{d(k^{-1/2}\sqrt{g})}{dk} = -\frac{1}{2}k^{-3/2}\sqrt{g}$$

$$= -\frac{1}{2}\frac{1}{k}\sqrt{g/k}$$

$$= -\frac{1}{2}\frac{V_p}{k} \qquad ...(2)$$

using equation (2) in (1) we get.

$$V_g = V_p + k\left(-\frac{1}{2}\frac{V_p}{k}\right)$$

$$= V_p - \frac{1}{2}V_p$$

$$= \frac{1}{2}V_p$$

Thus group velocity is equal to half the phase velocity.

Scanned by CamScanner

Example 3.11 : The phase velocity of ripples on a liquid surface of surface tension S and density ρ is given by.

$$W = \sqrt{\frac{2\pi S}{\rho\lambda}}$$

Solution : Here, $V_p = W = \sqrt{\frac{2\pi S}{\rho\lambda}} = \sqrt{\frac{kS}{\rho}}$

$$\therefore \quad k = \frac{2\pi}{\lambda}$$

We know that $V_g = V_p + k\frac{dV_p}{dk}$ ...(1)

$$\frac{dV_p}{dk} = \frac{d}{dk}\left(\frac{ks}{\rho}\right)^{1/2}$$

$$= + \frac{1}{2}\, k^{-1/2}\left(\frac{S}{\rho}\right)^{1/2}$$

using in equation (1) we get

$$V_g = V_p + k \times \frac{1}{2}\, k^{-1/2}\left(\frac{S}{\rho}\right)$$

$$= V_p + \frac{1}{2}\left(\frac{kS}{\rho}\right)^{1/2}$$

$$= V_p + \frac{1}{2} V_p$$

$$\therefore \quad V_g = \frac{3}{2} V_p$$

3.3 Davisson and Germer Experiment :

The existance of matter waves was confirmed by Davisson and Germer experiment.

The apperatus consists of electron Gun, nickel crystal and collector (C). These are all placed in an evacuated chamber. A galvanometer connected to the collector is used to record the current. Electrons are emitted from the electron gun.

Scanned by CamScanner

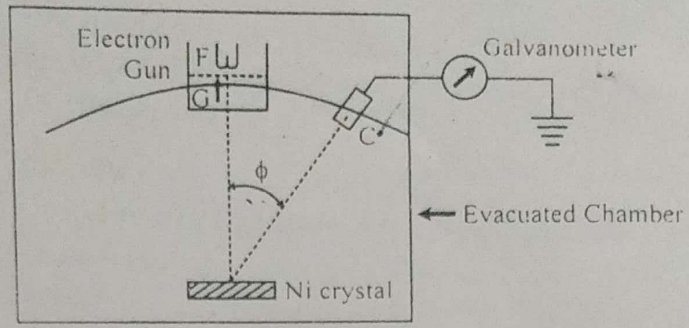

Figure 3.2

They are accelerated by potential V provided by grid G. These electrons in the form of fine beam are allowed to incident normally on the nickel crystal. The electrons get scattered in different directions. The scattered electrons can be collected by collector and scattering current is measured by the galvanometer. The position of collector can be adjusted at any desired angle ϕ.

The collector current I is measured for various angles ϕ for a given accelerating potential. Then a polar graph is plotted as shown in figure 3.3.

It was found that electron intensity was maximum at certain angle. There is a hump in the curve at this angle ϕ. For a graph with $V = 54V$ the $\phi = 50^o$.

The occurance of maxima and minima is due to diffraction of matter waves associated with electron. The diffraction phenomenon arises due to scattering of electron waves from various atomic planes.

It is to be nated that the principle maxima occurs at different values of ϕ for different values of accelerating potential V. It means that for a constant accelerating voltage the constructive intereference of the rays in the reflected beam occurs only in certain directions.

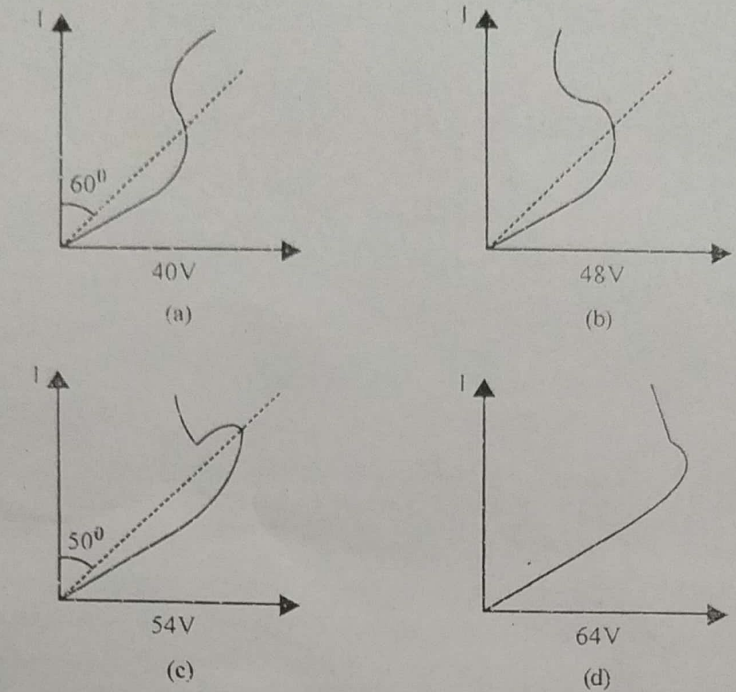

Figure 3.3 Variation of I w.r.t. angle at various potentials

For Figure 3.3 C $\phi = 50^0$ for V = 54 V. The glancing angle $\theta = 65^0$. The spacing between the atomic planes is $d = 0.91\ A^0$. According to Bragg's law

$$2d \sin\theta = n\lambda$$

$$2 \times 0.91 \times \sin 65^0 = \lambda \qquad n = 1 \text{ for 1st maxima}$$

$$\therefore \qquad \lambda = 1.65\ A^0$$

Also theoretical $\lambda = \dfrac{12.2}{\sqrt{54}} = 1.66\ A^0$

Thus there is a close agreement between the two values which confirms the de-Broglie hypothesis of matter waves.

Scanned by CamScanner

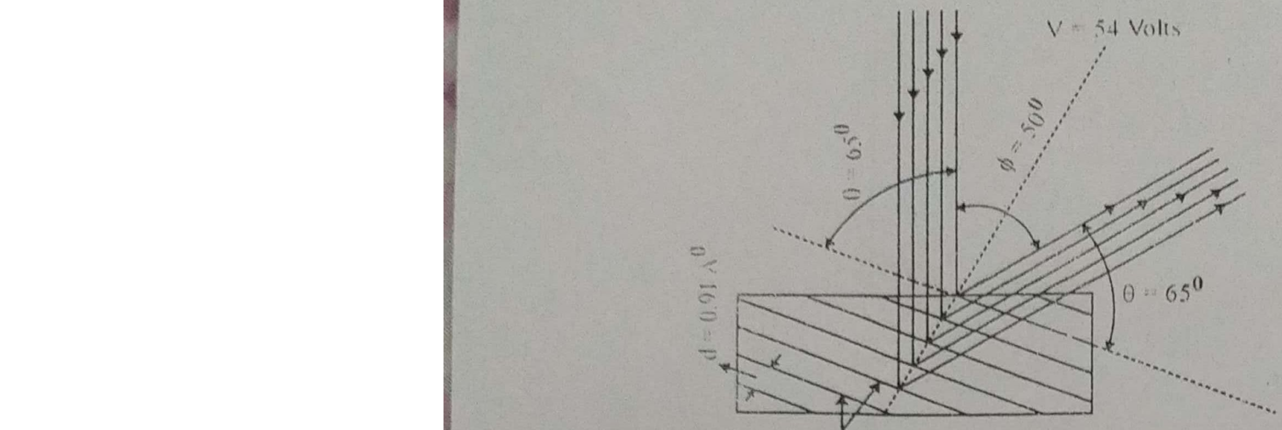

Figure 3.4 Schematic of reflection at $\phi = 50^0$ in Davisson - Germer experiment

When are the matter waves 'seen' ?

According to de-Broglie hypothesis, matter waves are associated with all moving bodies. These waves were seen in case of electrons through the diffraction phenomenon. Then the question arises : If this is the property for all moving bodies, do these waves exist in case of macroscopic bodies such as marble ball, cricket ball etc. ? If yes, why were they not observed earlier ? Answer to this question in simple terms is that wave properties are seen only when the wavelength 'λ' associated with moving bodies is comparable with the characteristic length 'd' associated with the apparatus with which a body or a particle interacts.

In Davisson-Germer experiment the characteristic length 'd' is the lattice constant d = 0.91 A°. Wavelength for electron accelerated by 54 V is λ = 1.66 A°.

Thus, 'λ' and 'd' are comparable. Therefore, electrons exhibit wave nature.

The wavelength associated with 20 g marble ball moving at 10 m/s is 3.3×10^{-33} m. The characteristic length associated with the ball may be taken as the diameter say d = 1 cm = 10^{-2} m. Thus $\lambda << d$. This wavelength is too small to be observed experimentally.

3.4 Heisenberg's Uncertainty Principle :

The uncertainty principle stated by Heisenberg in 1927 is the postulate of quantum mechanics. It introduces totally new ideas about the measurement of physical quantities. The principle is stated below and its consequences are discussed later in detail.

Statement : The principle states that one cannot measure the position co-ordinate and corresponding momentum of the particle **simultaneously with arbitrary accuracy.**

If Δx and ΔP_x are the uncertainties in the simultaneous measurement of x-co-ordinate and corresponding momentum, then according to Heisenberg's uncertainty principle, the product of uncertainties is always greater than or equal to Planck's constant (h).

$$\left.\begin{array}{ll} \Delta x \, \Delta P_x \geq h & \text{..(1)} \\ \text{Similarly, } \Delta Y, \Delta P_y \geq h & \text{...(2)} \\ \text{and } \Delta Z \, \Delta P_z \geq h & \text{...(3)} \end{array}\right\} \quad \text{...(3.28)}$$

The principle implies that if one tries to determine position co-ordinate (x) more accurately i.e. $\Delta x \rightarrow 0$, then the momentum beomes more and more uncertain i.e. $\Delta Px \rightarrow \infty$ and vice versa.

One can never determine both x and Px as accurate as one whishes i.e. we can never have $\Delta x \rightarrow 0$ and $\rightarrow \Delta Px \rightarrow 0$ simultaneously.

Scanned by CamScanner

This, the uncertantly product $\Delta x \Delta P_x$ can never be made less than h. This is a totally new concept about the accuracy of measurement. It was belived earlier that any physical quantity can be measured with any desired accuracy at least in principle.

Precise mathematical statement of the principle states that,

$$\Delta x \Delta P_x \geq \hbar/2$$

$$\Delta y \Delta P_y \geq \hbar/2$$

and $$\Delta Z \Delta P_z \geq \hbar/2$$

We shall establish the uncertainty relation with the help of some simple 'thought experiments'. A thought experiment is an experiment such that it may not be possible to actually perfrom the experiment in the laboratory but even then the various steps in the experiments are such that no physical principle is violated at any state of the experiment. For example, suppose we say that the intensity of the light bulb is so weak that it emits just one photon at a time. We cannot have a bulb of such a weak intensity but there is no violation of any physical principle in imagining such thing.

1. γ - Ray Microscope Experimental : Let us consider the experiment of measurement of position of electrons with the help of γ - Ray micro scope. To observe the electron, it must be illuminated by light of very weak intensity so that the electron is not lost due to large momentum transfer. Let us suppose that the intensity is so weak that just one photon in incident on the electron. We

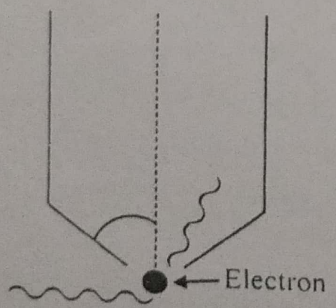

Figure 3.5 Electron

assume that the photon gets scattered into the cone of the microscope and finally it enters into the eye of the observer. Only then we can say that the electron is 'seen.'

In case of microscope. we define a quantity called resolving power. It is the minimum distance between two objects so that they are distincity observed. This means that if the distance of seperation is less than resolving power then we cannot distinguish between the two objects. Alternatively. if we consider a single object then its position is uncertain by a distance at least equal to the resolving power (R.P.).

Therefore, if Δx is the uncertainty in the x co-ordinate of the position of electron then,

$$\Delta x = R.P.$$

But. R.P. of microscope $= \dfrac{\lambda}{\sin\theta}$

$$\therefore \quad \Delta x = \frac{\lambda}{\sin\theta} \qquad \ldots(3.29)$$

The scattered photon may enter the eye by travelling from any direction within the cone of the microscope.

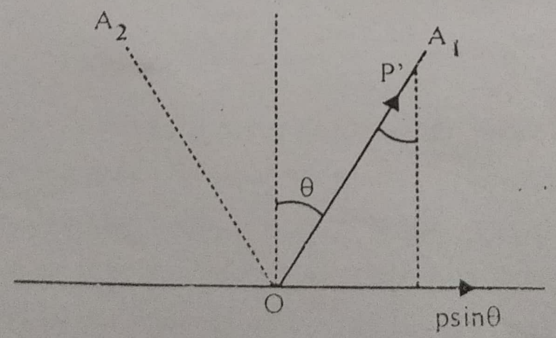

Figure 3.6

Let P is the momentum of photon incident on electron and p' be the momentum of scattered photon. It is clear that the x-momentum of scattered photon may have any value between P'sinθ to P'sinθ. It is P' sinθ when photon comes along OA_1 and -P'sinθ when it comes along OA_2. (See figure 3.6.). The total uncertainty in the x-momentum of scattered photon is :

$$P'\sin\theta - (-p' \sin \theta) = 2p' \sin\theta$$

Now, momentum lost by photon in scattering = momentum gained by electron.

Therefore, uncertainty (ΔP_x) in x-momentum of electron = uncertainty in x-momentum of photon. We assume that momentum transfer by photon is small. Then P' = P

$$\therefore \quad \Delta P_x = 2\, p \sin\theta$$

In writing approximate results, we often drop between 1 to 10 occuring in the numerator or decominator. Thus,

$$\Delta P_x \cong p \sin\theta \qquad ...(3.30)$$

From equation (3.29) and (3.30) we get,

$$\Delta x\ \Delta P_x \cong \frac{\lambda}{\sin\theta}\ p \sin\theta = \lambda p \qquad ...(3.31)$$

But, wavelength 'λ' and momentum 'p' of photon are related by,

$$\lambda = \frac{h}{p} \qquad \text{i.e. } \lambda . p = h \qquad ...(3.32)$$

Therefore, equation 3.32 becomes,

$$\Delta x\ .\ \Delta P_x = h$$

The above result gives the product of uncertainties when they are minimum. In the actual experiment, there are many factors such as personal error of judgement, instrumental error, etc. which may contribute in increasing error in Δx and ΔP_x. Hence, $\Delta x\ .\ \Delta P_x$ will always be greater than h. Thus, we get,

$$\Delta x\ .\ \Delta P_x \geq h$$

similarly,

$$\Delta y\ .\ \Delta P_y \geq h$$

and

$$\Delta Z\ .\ \Delta P_z \geq h$$

Significance of Uncertainty Principle : As stated earlier, according to Heisenberg's uncertainty principle we cannot determine both the position co-ordinate and the corresponding momentum simultaneously with arbitrary accuracy. Accurate measurement of 'x' destroy's the accuracy in 'p_x' and viceversa. We can achieve the optimum occuracy in both 'x' and 'P_x' such that,

$$(\Delta x.\Delta P_x)\min = \frac{\hbar}{2}$$

This is the nature's limit on the accuracy of measurement. The act of measurement such as looking at the object invoies some interaction (such as collision with photon) and this gives rise to uncertainties. This non-zero limit on the uncertainty product is not due to limitations on observer's capacity of accurate observation or due to limitations of available technology in the instruments. Even under ideal conditions in these regards $\Delta x.\Delta P_x \geq \frac{\hbar}{2}$. Thus, we may consider Heisenberg's uncertanity principle to be the law of nature.

According to the classical physics, both position and momentum could be measured simultaneously and with any desired accuracy i.e. $\Delta x.\Delta P_x$ can be made zero. However, Heiseinberg's uncertainty principle says that it is not possible to make both $\Delta x \to 0$ and $\Delta Px \to 0$ at the same time.

It is due to extremely small value of 'h' that uncertainty principle does not become important in everyday phenomenon. The principle does not become relevant and needs attention only in microscopic phenomens.

• Time Energy Uncertainty relation :

The energy E of a moving particle along x-axis is...

Scanned by CamScanner

$$E = K.E. + P.E.$$

$$= \frac{P^2}{2m} + V_{(x)}$$

$$\therefore \quad \frac{\delta E}{\delta P} = \frac{2P}{2m}$$

$$= \frac{P}{m}$$

$$= \frac{mV}{m}$$

$$= V$$

$$\therefore \quad \frac{\delta E}{\delta P} = \frac{\delta x}{\delta t}$$

$$\therefore \quad \frac{\Delta E}{\Delta P} = \frac{\Delta x}{\Delta t} \quad \text{replacing } \delta \text{ by } \Delta$$

$$\therefore \quad \Delta x \Delta P = \Delta E \, \Delta t \qquad \text{...(3.33)}$$

But, $\Delta x \Delta P \geq \frac{\hbar}{2}$

$\therefore$ Using equation (3.33) we get,

$$\Delta E \, \Delta t \geq \frac{\hbar}{2}$$

Example 3.12 : What is the smallest possible uncertainty in the position of the electron moving with velocity 10^6 m/s ?

Solution : Let is Δx is the minimum uncertainty in the determination of position then

$$\Delta x \,.\, \Delta P = \hbar$$

$$\therefore \quad \Delta x = \frac{\hbar}{P}$$

Scanned by CamScanner

$$= \frac{\hbar}{mv} = \frac{1.055 \times 10^{-34}}{9.1 \times 10^{-31} \times 10^{6}} = \frac{1.055}{9.1} \times 10^{-9}$$

$= 0.1159 \times 10^{-9}$ m

$= 1.159$ A°

Example 3.13 : A 10 gm bullet shoots through a cyllendrical tunnel of 5 cm diameter. What would be the uncertainty in the velocity of bullet ?

Solution : $\Delta x = 5$ cm $= 5 \times 10^{-2}$ m

$m = 10$ gm $= 10 \times 10^{-3}$ kg

$\Delta x \Delta P = h$

$\therefore \quad \Delta x \,.\, m \Delta V_x = h$

$$\therefore \quad \Delta V_x = \frac{h}{m\Delta x} = \frac{6.63 \times 10^{-34}}{10^{-2} \times 5 \times 10^{-2}}$$

$$= \frac{6.63}{5} \times 10^{-30}$$

$= 1.326 \times 10^{-30}$ m/s

Example 3.14 : Suppose the instantaneous position of 1gm particle is measured with maximum possible error of 10^{-2} cm. What is the error in corresponding velocity ?

Solution : $m = 1$gm $= 1 \times 10^{-3}$ kg

$\Delta x = 10^{-2}$ cm $= 10^{-4}$ m

$\Delta x \Delta P_x = h$

$\therefore \quad \Delta x \, m \Delta V_x = h$

$$\therefore \quad \Delta V_x = \frac{h}{m\Delta x} = \frac{6.63 \times 10^{-34}}{10^{-3} \times 10^{-4}}$$

$= 6.63 \times 10^{-27}$ m/s

Example 3.15 : The electrons are allowed to pass through a crystal with lattice constant 1A°. What is the uncertainty in its velocity ?

Scanned by CamScanner

Solution : $\Delta x = 1 A^0 = 1 \times 10^{-10}$ m

$\Delta x \, \Delta P_x = h$

$\therefore \quad \Delta x \, m \Delta V_x = h$

$$\therefore \quad \Delta V_x = \frac{h}{m\Delta x} = \frac{6.63 \times 10^{-34}}{9.1 \times 10^{-31} \times 10^{-10}}$$

$$= \frac{6.63}{9.1} \times 10^7$$

$$= 0.728 \times 10^7$$

$$= 7.28 \times 10^6 \text{ m/s}$$

Example 3.15 : Show that $\Delta x . \Delta \lambda = \lambda^2$

Solution : We have $P = \frac{h}{\lambda}$

$$\therefore \quad \frac{\Delta P}{\Delta \lambda} = -\frac{h}{\lambda^2}$$

$$\therefore \quad \Delta P = -\frac{h}{\lambda^2} \Delta \lambda$$

$$\Delta \lambda = \frac{\lambda^2}{h} \Delta P \quad \text{considering magnitude only}$$

$$\therefore \quad \Delta_x . \Delta \lambda = \Delta_x . \frac{\lambda^2}{h} \Delta P$$

$$= \frac{\lambda^2}{h} (\Delta_x . \Delta P)$$

$$= \frac{\lambda^2}{h} \times h \qquad \therefore \Delta_x . \Delta P = h$$

$\therefore \Delta_x . \Delta \lambda = \lambda^2$

Example 3.16 : The average lifetime for which electron stays in a given excited state before it jumps lower energy state is about 10^{-8} S. What is the uncertainty in emitted spectral lines ?

Solution : $\Delta t = 10^{-8}$ S

Scanned by CamScanner

$$\Delta E \cdot \Delta t = h$$

But. $E = h\upsilon$

$\therefore \quad \Delta E = h\Delta\upsilon$

$\therefore \quad h\Delta\upsilon \, \Delta t = h$

$\therefore \quad \Delta\upsilon \, \Delta t = 1$

$$\therefore \quad \Delta\upsilon = \frac{1}{\Delta t}$$

$$= \frac{1}{10^{-8}}$$

$\therefore \quad \Delta\upsilon = 10^{8}$ Hz

Example 3.17 : Electron has a speed of 300 cm/sec accurate to 0.01% with what fundamental accuracy can we locate the position of the electron ?

Solution : $\Delta V_x = \dfrac{300 \times 0.01}{100} = 3 \times 10^{-2}$ cm/sec

$= 3 \times 10^{-4}$ m/sec

$\Delta_x \, \Delta P = h$

$\Delta_x . m \Delta V_x = h$

$$\therefore \quad \Delta X = \frac{h}{m\Delta V_x} = \frac{6.63 \times 10^{-34}}{9.1 \times 10^{-31} \times 3 \times 10^{-4}}$$

$$= \frac{6.63}{27.3} \times 10^{+1}$$

$= 0.242 \times 10^{+1}$ m

$= 2.42 \times 10^{+2}$ m

Exampe 3.18 : Using the uncertainty relation

$$\Delta P \Delta X \geq \frac{\hbar}{2}$$

show that for a free particle the uncertainty relation can be written as...

Scanned by CamScanner

$$\Delta X\,\Delta\lambda = \frac{\lambda^2}{4\pi}$$

where λ is the de-Broglie wavelength.

Solution : Now the de-Broglie wavelength is given by,

$$\lambda = \frac{h}{p}$$

$$\therefore \quad \frac{\Delta\lambda}{\Delta p} = \frac{-h}{p^2}$$

$$\therefore \quad \Delta P = \frac{p^2}{h}\Delta\lambda \qquad ...(1)$$

considering the magnitude only

we are given $\Delta P\,\Delta X \geq \frac{\hbar}{2}$ ...(2)

$$\frac{p^2}{h}\Delta\lambda\,\Delta X \geq \frac{\hbar}{2} \qquad \text{using equation (1)}$$

but, $p = \frac{h}{\lambda}$

$$\therefore \quad \frac{h}{\lambda^2}\Delta\lambda\,\Delta X \geq \frac{\hbar}{2}$$

$$\therefore \quad \Delta X\Delta\lambda \geq \frac{\lambda^2}{4\pi} \qquad \therefore \quad \hbar = \frac{h}{2\pi}$$

Example 3.19 : Show that the uncertainty relation can be expressed as $\Delta L\Delta\theta \geq \hbar$ where ΔL is the uncertainty in angular momentum and $\Delta\theta$ is the uncertainty in angular displacement.

Solution : For a rotating body, the energy

$$E = \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{(I\omega)^2}{I} = \frac{L^2}{2I} \qquad \therefore L = I\omega$$

Scanned by CamScanner

$$\therefore \quad \frac{\Delta E}{\Delta L} = \frac{2L}{2I}$$

$$\therefore \quad \Delta E = \frac{L}{I} \Delta L$$

$$\therefore \quad \Delta E \, \Delta t = \frac{L}{I} \Delta L \, \Delta t$$

$$= \frac{I\omega}{I} \Delta L \, \Delta t \qquad \therefore L = I\omega$$

$$= \omega \, \Delta L \, \Delta t$$

$$= \Delta L \, \Delta \theta \qquad \therefore \Delta\theta = \omega \Delta t$$

But, $\Delta E \, \Delta t \geq \hbar$

$\therefore \quad \Delta L \, \Delta \theta \geq \hbar$

Example 3.20 : A satellite of mass 100 kg is revalving in circular orbit close to the earths surface with a period 90 minutes. Calculate the uncertainty in the measurement of velocity

Solution : Centripetal force is balnced by gravitatioal force.

$$\therefore \quad \frac{mV^2}{r} = mg$$

$$\therefore \quad V^2 = gr$$

$$\frac{\Delta V^2}{\Delta r} = g$$

$$\therefore \quad 2V \frac{\Delta V}{\Delta r} = g$$

$$\therefore \quad 2V \, \Delta V = g \Delta r \quad ...(1)$$

$\Delta V \rightarrow$ uncertainty in the measurement of velocity

$\Delta r \rightarrow$ uncertainty in the measurement of position

Therefore the uncertainty relation is,

Scanned by CamScanner

$$\Delta P \, \Delta r = \frac{\hbar}{2}$$

$$m \Delta V \Delta r = \frac{\hbar}{2}$$

$$\therefore \quad \Delta r = \frac{\hbar}{2m \Delta V} \qquad ...(2)$$

using (2) in equation (1) we get,

$$2V \Delta V = g \times \frac{\hbar}{2m \Delta V}$$

$$\therefore \quad (\Delta V)^2 = \frac{g\hbar}{4mV}$$

$$\therefore \quad \Delta V = \sqrt{\frac{g\hbar}{4mV}}$$

But, $V = \dfrac{2\pi r}{T}$

$\therefore$ $r \longrightarrow$ radius of earth $= 6400$ km $= 6400 \times 10^3$ m

$T = 90$ min $= 90 \times 60$ sec.

$= 5400$ sec.

$$= \frac{2 \times 3.142 \times 64 \times 10^5 \text{ m}}{5400}$$

$= 7.4477 \times 10^3$ m

$\Delta V = 18.63 \times 10^{-16}$ m/s

Example 3.21 : The average time that an atom retains excess excitation energy before reemitting it in the form of electromagnetic radiation is 10^{-8} sec. Calculate the limit of accuracy with which the excitation energy of the radiation can be determined and the frequency of light.

Solution : $\Delta E \, . \, \Delta t = h$

Scanned by CamScanner

$$\therefore \quad \Delta E = \frac{h}{\Delta t} = \frac{6.63 \times 10^{-34}}{10^{-8}} = 6.63 \times 10^{-26} \text{ J}$$

$$= \frac{6.63 \times 10^{-26}}{1.6 \times 10^{-19}} = 4.12 \times 10^{-7} \text{ eV}$$

The frequency of radiation is uncertain by.

$$\Delta \upsilon = \frac{\Delta E}{h} = \frac{6.63 \times 10^{-26}}{6.63 \times 10^{-34}} = 10^{8} \text{ H}_z$$

Example 3.22 : Calculate the de-Broglie wavelength associated with neutrons at 27°C.

Soltuion : $E = KT = 1.38 \times 10^{-23} \times 300$

$\therefore \quad K = 138 \times 10^{-23} \text{ J/°K}$

$$\lambda = \frac{h}{\sqrt{2mE}} \qquad \therefore m_n = 1.67 \times 10^{-27} \text{ kg}$$

$$= 1.78 \text{ A}^0$$

3.5 Application of Uncertainty Principle :

1) Calculate the size of the hydrogen atom using Heisenberg's uncertainty relation :

Solution : We assume that the spread in the position of the electron in a hydrogen atom is = a. Hence an electron can be found at a list of about 'a' from the necleus.

$\therefore \quad \Delta X = a$

using uncertainty relation

$\Delta X \, \Delta P = \hbar$

$$\Delta P = \frac{\hbar}{\Delta x}$$

$$\therefore \quad \Delta P = P = \frac{\hbar}{a}$$

$$\therefore \quad K.E. = \frac{1}{2} mV^2 = \frac{P^2}{2m} = \frac{\hbar^2}{2ma^2}$$

Scanned by CamScanner

Electrostatic P.E. = $\dfrac{-e^2}{4\pi\varepsilon_0 a}$

$\therefore$ Total Energy E = K.E. + P.E.

$$= \frac{\hbar^2}{2ma^2} + \frac{-e^2}{4\pi\varepsilon_0 a}$$

for ground state of atom, energy will be minimum if,

$$\frac{dE}{da} = 0$$

$$\therefore \quad \frac{dE}{da} = \frac{-2\hbar^2}{2ma^3} + \frac{e^2}{4\pi\varepsilon_0 a^2} = 0$$

$$\therefore \quad \frac{\hbar^2}{ma^3} = \frac{e^2}{4\pi\varepsilon_0 a^2}$$

$$\therefore \quad a = \frac{4\pi\varepsilon_0 \hbar^2}{me^2} \qquad \therefore \frac{1}{4\pi\varepsilon_0} = 9 \times 10^9 \frac{Nm^2}{C^2}$$

$$\therefore \hbar = 1.054 \times 10^{-34}$$

$$= \frac{1}{9 \times 10^9} \frac{(1.054 \times 10^{-34})^2}{9.1 \times 10^{-31} \times (1.6 \times 10^{-19})^2}$$

$$= 0.502 \times 10^{-10} \text{ m}$$

$$= 0.502 \text{ A}^0$$

2) Use uncertainty relation $\Delta X \,.\, \Delta P = \hbar = \dfrac{h}{2\pi}$ to estimate ground state energy of hydrogen atom.

Solution : There is a single electron revolving around the nucleus at a distant 'a'. As the distance between the nucleus and electron is 'a', the uncertainty Δa in the measurement of 'a' should be of the order of 'a'

$\therefore \quad \Delta a = a$ and $\Delta P = P$

$$\therefore \quad P = \frac{\hbar}{a}$$

Scanned by CamScanner

$$K.E. = \frac{1}{2} mv^2 = \frac{P^2}{2m} = \frac{\hbar^2}{2ma^2}$$

$$P.E. = \frac{-Ze^2}{4\pi\varepsilon_0 a}$$

$\therefore$ Total energy E is given by,

$$E = K.E. + P.E.$$

$$= \frac{\hbar^2}{2ma^2} + \frac{-Ze^2}{4\pi\varepsilon_0 a} \quad ...(1)$$

Ground state energy is the minimum energy. For the energy to be minimum $\frac{dE}{da} = 0$

$$\therefore \quad \frac{dE}{da} = \frac{-2\hbar^2}{2ma^3} + \frac{Ze^2}{4\pi\varepsilon_0 a^2} = 0$$

$$\therefore \quad \frac{\hbar^2}{2ma^3} = \frac{Ze^2}{4\pi\varepsilon_0 a^2}$$

$$\therefore \quad a = \frac{4\pi\varepsilon_0 \hbar^2}{mZe^2}$$

Using this value in equation (1)

$$E_{min} = \frac{\hbar^2}{2m} \times \frac{Z^2e^4m^2}{(4\pi\varepsilon_0\hbar^2)^2} \frac{Ze^2}{4\pi\varepsilon_0} \cdot \frac{mZe^2}{4\pi\varepsilon_0\hbar^2}$$

$$= \frac{\hbar^2}{2} \frac{Z^2e^4m}{(4\pi\varepsilon_0)^2\hbar^4} \frac{-Z^2e^4m}{(4\pi\varepsilon_0)^2\hbar^2}$$

$$= \frac{1}{2} \frac{mZ^2e^4}{(4\pi\varepsilon_0)^2\hbar^2} + \frac{-mZ^2e^4}{(4\pi\varepsilon_0)^2\hbar^2}$$

$$= -\frac{1}{2} \frac{mZ^2e^4}{(4\pi\varepsilon_0)^2\hbar^2} + -\frac{1}{2} \frac{mZ^2e^4}{(4\pi\varepsilon_0)^2} \frac{4\pi^2}{h^2}$$

$$E_m = -\frac{mZ^2e^4}{8\varepsilon_0^2 h^2}$$

This is the ground state energy of Hydrogen atom.

Scanned by CamScanner

3) Show that electrons can not exist inside the nucleus.

Solutions : The radius of nucleus of any atom is of the order of 10^{-14} m. If an electron is assumed to exist in the nucleus, then the uncertainty in the position of electron is equal to the diameter of nucleus.

i.e. $\Delta X \simeq 2 \times 10^{-14}$ m.

We have the uncertainty relation

$$\Delta X . \Delta P = \hbar$$

$$\therefore \quad \Delta P = \frac{\hbar}{\Delta X}$$

$$= \frac{h}{2\pi \Delta X}$$

$$= \frac{6.63 \times 10^{-34}}{2 \times 3.14 \times 2 \times 10^{-14}}$$

$$= 5.278 \times 10^{-21} \text{ kg m/s}$$

Thus for the electron to exist in the nucleus, then its minimum momentum will be

$$P_{min} = 5.278 \times 10^{-21} \text{ kg m/s}$$

Since mass of electron is 9.1×10^{-31} kg, its velocity will be very large and hence relativistic formula for its energy to be used.

$$E^2 = P^2C^2 + m_0^2 C^4$$

As P is very large the second term on RHS is negligible as it is very small.

$$\therefore \quad E^2 = P^2C^2$$

$$\therefore \quad E = PC$$

$$= 5.278 \times 10^{-21} \times 3 \times 10^8$$

$$= 1.582 \times 10^{-12} \text{ J}$$

Scanned by CamScanner

$$= \frac{1.582 \times 10^{-12}}{1.6 \times 10^{-19}}$$

$$= 9.875 \text{ meV}$$

This means that if the electrons exists inside the nucleus then their energy must be of the order of 9.875 meV. However the electronsd emitted by some radioactive nuclei during B-decay have energies only of the order of 4 meV. Hence electrons can not exist in the nucleus.

For protons and neutrons of mass 1.67×10^{-27} kg

$K.E. = \frac{P^2}{2m_0} = 52$ kev. This energy is smaller than the energies current by there particles emitted by nuclei so both can exist inside the nucleus.

•••

Scanned by CamScanner

Questions and Problems :

1. State and explain de-Broglie's hypothesis of matters. Show that $\lambda = h/p$, where the terms have usual significane.
2. Obtain the expression for de-Broglie wavelength in terms of kinetic energy of the particle.
3. Show that the wavelength associated with electron accelerate from rest by potential difference of 'V' volts is given by,

$$\lambda = \frac{12.2}{\sqrt{V}} \quad \text{(measured in } A^0\text{)}$$

4. Distinguish clearly between wave velocity (V_p) and group velocity (V_g). Show that velocity of wave-group is given by.

$$Vg = Vp - \lambda \frac{dV_p}{d\lambda}$$

5. Show that the group velocity is always equal to the particle velocity.
6. Describe, in detail Davison and Germer Experiment. Discuss its importance.
7. Calculate in A^0, the wavelength associated with electron whose kinetic energy equal to its rest mass energy.
8. If the velocity of ocean waves is equal to $\sqrt{g\lambda/2\pi}$, show that group velocity is half the phase velocity.
9. Velocity of ripple waves in a water tank is equal to $\sqrt{\frac{2\pi S}{p\lambda}}$ where 'S' is the surface tension. 'λ' is the wavelength and 'p' is the density of the liquid. Find out the group velocity.
10. State and explain Heisenberg's uncertainty principle. With the help of suitable thought experiment, prove the

Scanned by CamScanner

uncertainty relation : $\Delta X \Delta P = h$.

11. Using the uncertainty principle, show that electron cannot be a particle inside the nucleus.

12. If $\Delta X = 1$ A° for electron, what could be the minimum energy of electrons ?

13. Calculate de-Broglie wavelength associated with electron accelerated from rest by potential difference of 200 V.

14. What is the energy of electrons having wavelength 2A° = ?

15. Suppose that electron is located within 0.2A°. Then what will be uncertainty in the velocity of electron ?

16. The average life time of excited state of nucleus is 10^{-12} seconds. What is the uncertainty in the energy and frequency of emiteed photon ?

17. Determine the wavelength associated with electron having kinetic energy equal to 1MeV. **Hint :** Use relativistic formula. 1 MeV = 1.6×10^{-15} J, $m_0 C^2 = 0.5$ MeV for electron.

18. Calculate de-Broglie wavelength associated with a thermal neutron. **Hint :** Kinetic energy of thermal neutron is equal to 3/2 kT, where 'k' is Bolzmann constant and 'T' is room temperature measured in K. For thermal neutron : T = 300 K.

19. What is the velocity of α - particles having de-Broglie wavelength equal to that of 1 KeV X-ray photon ?
Hint : energy of photon $\varepsilon = h\nu$

$\therefore$ $\lambda =$ where $\varepsilon = 1$ KeV = 1000eV.

Scanned by CamScanner