Linear Algebra

* Defination

Vector Space: Let V be any non empty set together with the operations of addition and scalar multiplication is called vector space. Then V is called vactor space if it's satiesfies following conditions.

For any $u, v, w \in V$ and for all $\alpha, \beta \in \mathbb{R}$

1) $u + v \in V$ } [closure property of addition
2) $\alpha u \in V$ } and scalar multiplication]
3) $u + v = v + u$ [commutative property for addition]
4) $u + (v + w) = (u + v) + w$ [Associative law of addition]
5) There exist an element $0 \in V$ such that for every $u \in V$ $u + 0 = u = 0 + u$ [Existence of identity element for addition]
6) There exist an element $(-u) \in V$ such that for every $u \in V$ $u + (-u) = 0 = (-u) + u$ [Existence of additive inverse]
7) $\alpha \cdot (u + v) = \alpha u + \alpha v$ [distributive law for scalar multiplication over addition]
8) $(\alpha + \beta) u = \alpha u + \beta u$ [distributive law for scalar multiplication]
9) $(\alpha \beta) u = \alpha (\beta u)$ [Associative law for scalar multiplication]
10) $1 \cdot u = u$ [Identity property for scalar multiplication]

Ex: Show that $\mathbb{R}^3$, the set of 3-tuples (u_1, u_2, u_3) of real numbers is a vector space with usual vector addition and scalar multiplication.

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Sol^n. we we are given $V = \mathbb{R}^3$, the set of 3-tuples of real numbers (u_1, u_2, u_3)

Let us take $u, v, w \in V = \mathbb{R}^3$ Then

$u = (u_1, u_2, u_3)$; $v = (v_1, v_2, v_3)$; $w = (w_1, w_2, w_3)$

and Let $\alpha, \beta \in \mathbb{R}$

1) Here $u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3)$

$= (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

$\in \mathbb{R}^3$

$\therefore u + v \in \mathbb{R}^3$

2) $\alpha u = \alpha (u_1, u_2, u_3)$

$= (\alpha u_1, \alpha u_2, \alpha u_3)$

$\in \mathbb{R}^3$

$\therefore \alpha \cdot u \in V = \mathbb{R}^3$

3) $u + v = (u_1, u_2, u_3) + (v_1, v_2, v_3)$

$= (u_1 + v_1, u_2 + v_2, u_3 + v_3)$

$= (v_1 + u_1, v_2 + u_2, v_3 + u_3)$

$= (v_1, v_2, v_3) + (u_1, u_2, u_3)$

$= v + u$

4) $u + (v + w) = (u_1, u_2, u_3) + (v_1 + w_1, v_2 + w_2, v_3 + w_3)$

$= (u_1 + v_1 + w_1, u_2 + v_2 + w_2, u_3 + v_3 + w_3)$

$= (u_1 + v_1, u_2 + v_2, u_3 + v_3) + (w_1, w_2, w_3)$

$= (u + v) + w$

5) Here there exist an element $(0, 0, 0)$ in $\mathbb{R}^3$ such that for every $u \in \mathbb{R}^3$, we have $(u_1, u_2, u_3) + (0, 0, 0) = (u_1, u_2, u_3)$

i.e, $u + 0 = u = 0 + u$

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6) There exist $(-u) \in \mathbb{R}^3$ i.e, $(-u_1, -u_2, -u_3) \in \mathbb{R}^3$ such that for every $u \in \mathbb{R}^3$ we have,

$$(u_1, u_2, u_3) + (-u_1, -u_2, -u_3) = (0, 0, 0)$$

i.e, $u + (-u) = 0 = (-u) + u$

7) $$\begin{aligned}\alpha(u+v) &= \alpha(u_1+v_1,\ u_2+v_2,\ u_3+v_3)\\ &= (\alpha u_1 + \alpha v_1,\ \alpha u_2 + \alpha v_2,\ \alpha u_3 + \alpha v_3)\\ &= (\alpha u_1, \alpha u_2, \alpha u_3) + (\alpha v_1, \alpha v_2, \alpha v_3)\\ &= \alpha(u_1, u_2, u_3) + \alpha(v_1, v_2, v_3)\\ &= \alpha u + \alpha v.\end{aligned}$$

8) $$\begin{aligned}(\alpha+\beta)u &= (\alpha+\beta)(u_1, u_2, u_3)\\ &= ((\alpha+\beta)u_1,\ (\alpha+\beta)u_2,\ (\alpha+\beta)u_3)\\ &= (\alpha u_1 + \beta u_1,\ \alpha u_2 + \beta u_2,\ \alpha u_3 + \beta u_3)\\ &= (\alpha u_1, \alpha u_2, \alpha u_3) + (\beta u_1, \beta u_2, \beta u_3)\\ &= \alpha u + \beta u.\end{aligned}$$

9) $$\begin{aligned}(\alpha\beta)u &= (\alpha\beta)(u_1, u_2, u_3)\\ &= \alpha(\beta u_1, \beta u_2, \beta u_3)\\ &= \alpha[\beta(u_1, u_2, u_3)]\\ &= \alpha(\beta u)\end{aligned}$$

10) $$\begin{aligned}1u &= 1(u_1, u_2, u_3)\\ &= (u_1, u_2, u_3)\end{aligned}$$

$1 \cdot u = u$

Hence, $V = \mathbb{R}^3$ forms a vector space under usual vector addition & scalar multiplication

→ Note: The set $\mathbb{R}^n$ of n-tuples of real number $(u_1, u_2, \ldots u_n)$ forms a vector space under usual vector addition & scalar multiplication over $\mathbb{R}^3$

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* check that $P_4(x)$, is a vector space under usual addition and scallar multiplication or not.

$$P_4(x) = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 / a_i \in R, 0 \le i \le 4\}$$

$$x^4+1 \in P_4(x)$$

$$P_4(x) \neq \phi$$

(1) closure under addition

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4$$

$$f(x) + g(x) = a_0+b_0 + x(a_1+b_1) + x^2(a_2+b_2) + x^3(a_3+b_3) + x^4(a_4+b_4) \in P_4(x)$$

(2) $\alpha \in \mathbb{R}$,

$$\alpha f(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \alpha a_3 x^3 + \alpha a_4 x^4 \in P_4(x)$$

(3) $f(x) + g(x) = (a_0+b_0) + x(a_1+b_1) + x^2(a_2+b_2) + x^3(a_3+b_3) + x^4(a_4+b_4)$

$$= (b_0+a_0) + x(b_1+a_1) + x^2(b_2+a_2) + x^3(b_3+a_3) + x^4(b_4+a_4)$$

$$= g(x) + f(x)$$

(4) $h(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4$

$$h(x) + [f(x)+g(x)] = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + [(a_0+b_0) + (a_1+b_1)x + x^2(b_2+a_2) + x^3(b_3+a_3) + x^4(b_4+a_4)]$$

$$= [(c_0+a_0) + (c_1+a_1)x + (c_2+a_2)x^2 + (c_3+a_3)x^3 + (c_4+a_4)x^4] + b_0 + b_1x + b_2x^2 + b_3x^3 + b_4x^4$$

$$= [h(x) + f(x)] + g(x)$$

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(5) Existance of identity element for addition

Suppose that

$i(x) = i_0 + i_1 x + i_2 x^2 + i_3 x^3 + i_4 x^4$

is an identity element in $P_4(x)$

$i(x) + f(x) = f(x)$

$i_0 + a_0 + (i_1 + a_1)x + (i_2 + a_2)x^2 + (i_3 + a_3)x^3 + (i_4 + a_4)x^4 = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$\therefore i_0 = i_1 = i_2 = i_3 = i_4 = 0$

$i(x) = 0 \in P_4(x)$

(6) Existance of inverse element under addition

$f(x) \in P_4(x) \; \exists \; g(x) \in P_4(x) \ni f(x) + g(x) = i(x) = 0$

$(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 + (a_4 + b_4)x^4$

$a_i = -b_i \; ; \; 0 \leq i \leq 4$

$\therefore g(x) = -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - a_4 x^4 \in P_4(x)$

(7) distributive law for scalar multiplication over addition : Let $\alpha \in \mathbb{R}$

$\alpha[f(x) + g(x)] = \alpha[(a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 + (a_4 + b_4)x^4]$

$= (\alpha a_0 + \alpha b_0) + (\alpha a_1 + \alpha b_1)x + (\alpha a_2 + \alpha b_2)x^2 + (\alpha a_3 + \alpha b_3)x^3 + (\alpha a_4 + \alpha b_4)x^4$

$= [\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \alpha a_3 x^3 + \alpha a_4 x^4] + [\alpha b_0 + \alpha b_1 x + \alpha b_2 x^2 + \alpha b_3 x^3 + \alpha b_4 x^4]$

$= \alpha f(x) + \alpha g(x)$

(8) distributive law for scalar multiplication

$(\alpha + \beta) f(x) = (\alpha + \beta)[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4]$

$= [(\alpha + \beta)a_0 + (\alpha + \beta)a_1 x + (\alpha + \beta)a_2 x^2 + (\alpha + \beta)a_3 x^3 + (\alpha + \beta)a_4 x^4]$

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$= [(\alpha a_0 + \beta a_0) + (\alpha a_1 + \beta a_1)x + (\alpha a_2 + \beta a_2)x^2 + (\alpha a_3 + \beta a_3)x^3 + (\alpha a_4 + \beta a_4)x^4]$

$= [\alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \alpha a_3 x^3 + \alpha a_4 x^4] + [\beta a_0 + \beta a_1 x + \beta a_2 x^2 + \beta a_3 x^3 + \beta a_4 x^4]$

$= \alpha(f(x)) + \beta(f(x))$

(9) $(\alpha\beta)f(x) = (\alpha\beta)[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4]$

$= \alpha[\beta a_0 + \beta a_1 x + \beta a_2 x^2 + \beta a_3 x^3 + \beta a_4 x^4]$

$= \alpha[\beta(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4)]$

$= \alpha[\beta f(x)]$

$= \alpha(\beta f(x))$

(10) $f(x) \in P_4(x)\ \exists\ e \in R$

Let $e \cdot f(x) = f(x)$

$\therefore\ ea_0 + ea_1 x + ea_2 x^2 + ea_3 x^3 + ea_4 x^4 =$

$a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$e\cancel{a_i} = \cancel{a_i} \quad 0 \le i \le 4$

$e = 1 \in R$

* Show that $P_2(x)$ is not a vector space under usual addition and scalar multiplication as defined by $a \cdot b = a^b$

$P_2(x) = \{a_0 + a_1 x + a_2 x^2 \mid a_i \in R\ \ 0 \le i \le 2\}$

$x^2 + 1 \in P_2(x)$

$P_2(x) \neq 0$

1) closure under addition

$f(x) =$

$g(x) =$

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(2) Let $\alpha \in \mathbb{R}$,

$\alpha f(x) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 \in P_2(x)$

$= a_0^{\alpha} + a_1^{\alpha} x + a_2^{\alpha} x^2$

take $\alpha = \frac{1}{2}$ and $a_0 = a_1 = a_2 = -1$

than $\frac{1}{2} f(x) = i + ix + ix^2 \notin P_2(x)$

Note: The set P_n of all polynomials of degree $\leq n$ is a vector space with usual addition and scalar multiplication of polynomials over $\mathbb{R}$

$P_n = \{ a_0 + a_1 x + \dots + a_n x^n \mid a_0, a_1, \dots, a_n \in \mathbb{R} \}$

* The set M_{33} of all matrices of order 3×3 is a vector space with addition and scalar multiplication of matrices.

Soln. Let $V = M_{33}$ Let $A, B, C \in M_{33}$ Then

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$$ and Let $\alpha, \beta \in \mathbb{R}$.

(1) $$A + B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

$\therefore A + B \in M_{33}$

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(2) $\alpha A = \alpha \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{bmatrix}$

$\therefore \alpha A \in M_{33}$

(3) $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$

$= \begin{bmatrix} b_{11}+a_{11} & b_{12}+a_{12} & b_{13}+a_{13} \\ b_{21}+a_{21} & b_{22}+a_{22} & b_{23}+a_{23} \\ b_{31}+a_{31} & b_{32}+a_{32} & b_{33}+a_{33} \end{bmatrix}$

$= B + A$

(4) $A + (B+C) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11}+c_{11} & b_{12}+c_{12} & b_{13}+c_{13} \\ b_{21}+c_{21} & b_{22}+c_{22} & b_{23}+c_{23} \\ b_{31}+c_{31} & b_{32}+c_{32} & b_{33}+c_{33} \end{bmatrix}$

$= \begin{bmatrix} a_{11}+b_{11}+c_{11} & a_{12}+b_{12}+c_{12} & a_{13}+b_{13}+c_{13} \\ a_{21}+b_{21}+c_{21} & a_{22}+b_{22}+c_{22} & a_{23}+b_{23}+c_{23} \\ a_{31}+b_{31}+c_{31} & a_{32}+b_{32}+c_{32} & a_{33}+b_{33}+c_{33} \end{bmatrix}$

$= \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

$= (A+B) + C$

(5) Let $O_3 \in M_{33}$ i.e, $O_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ such that

$A + O_3 = A = O_3 + A$

Hence, 'O_3' is an identity element for M_{33}

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(6) Let $-A \in M_{33}$ i.e; $-A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} \\ -a_{21} & -a_{22} & -a_{23} \\ -a_{31} & -a_{32} & -a_{33} \end{bmatrix}$ such that

$$A + (-A) = O_3 = (-A) + A$$

Hence '$-A$' is an inverse element of M_{33}.

(7) $$\alpha(A+B) = \alpha \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_{11}+\alpha b_{11} & \alpha a_{12}+\alpha b_{12} & \alpha a_{13}+\alpha b_{13} \\ \alpha a_{21}+\alpha b_{21} & \alpha a_{22}+\alpha b_{22} & \alpha a_{23}+\alpha b_{23} \\ \alpha a_{31}+\alpha b_{31} & \alpha a_{32}+\alpha b_{32} & \alpha a_{33}+\alpha b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{bmatrix} + \begin{bmatrix} \alpha b_{11} & \alpha b_{12} & \alpha b_{13} \\ \alpha b_{21} & \alpha b_{22} & \alpha b_{23} \\ \alpha b_{31} & \alpha b_{32} & \alpha b_{33} \end{bmatrix}$$

$$= \alpha A + \alpha B$$

(8) $$(\alpha+\beta)A = (\alpha+\beta) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} (\alpha+\beta)a_{11} & (\alpha+\beta)a_{12} & (\alpha+\beta)a_{13} \\ (\alpha+\beta)a_{21} & (\alpha+\beta)a_{22} & (\alpha+\beta)a_{23} \\ (\alpha+\beta)a_{31} & (\alpha+\beta)a_{32} & (\alpha+\beta)a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_{11}+\beta a_{11} & \alpha a_{12}+\beta a_{12} & \alpha a_{13}+\beta a_{13} \\ \alpha a_{21}+\beta a_{21} & \alpha a_{22}+\beta a_{22} & \alpha a_{23}+\beta a_{23} \\ \alpha a_{31}+\beta a_{31} & \alpha a_{32}+\beta a_{32} & \alpha a_{33}+\beta a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \alpha a_{21} & \alpha a_{22} & \alpha a_{23} \\ \alpha a_{31} & \alpha a_{32} & \alpha a_{33} \end{bmatrix} + \begin{bmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{bmatrix}$$

$$= \alpha A + \beta A$$

(10) For any $\alpha, \beta \in R$ $(\alpha\beta)A$

$$= (\alpha\beta)\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= \alpha\begin{bmatrix} \beta a_{11} & \beta a_{12} & \beta a_{13} \\ \beta a_{21} & \beta a_{22} & \beta a_{23} \\ \beta a_{31} & \beta a_{32} & \beta a_{33} \end{bmatrix}$$

$$= \alpha(\beta A)$$

(10) For any $1 \in R$

$$1 \cdot A = 1\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$$

Hence M_{33} is a vactor space.

* Note: The set $M_{m\times n}$ of all matrices of order $m\times n$ is a vactor space with usual addition and scalar multiplication of matrices

→ Determine whether the set IR^+ is a vactor space with usual addition and scalar multiplication or not?

Sol^n: We know that IR^+ is the set of all positive real numbers

Therefore $0 \notin IR^+$

Hence there does not exists identity element in $\mathbb{R}^+$. Therefore $\mathbb{R}^+$ is not a vector space.

Ex: Determine whether the set $\mathbb{R}^+$ of all positive real numbers with the operations

$x + y = xy$ and

$kx = x^k$ for any scalar k

is a vector space or not.

Sol^n^: Let $V = \mathbb{R}^+$. Let $x, y, z \in \mathbb{R}^+$ and $\alpha, \beta \in \mathbb{R}$

(1) $x + y = xy \in \mathbb{R}^+$ $\left\{\begin{array}{l}\text{as } x \text{ and } y \text{ both are positive} \\ \text{real numbers}\end{array}\right.$

$\therefore x + y \in \mathbb{R}^+$

(2) $\alpha x = x^\alpha \in \mathbb{R}^+$

$\therefore \alpha x \in \mathbb{R}^+$

(3) $x + y = xy = yx = y + x$

$\therefore x + y = y + x$

(4) $x + (y+z) = x \cdot (y+z)$

$= x(yz)$

$= (xy) \cdot z$

$= (xy) + z$

$= (x+y) + z$

(5) Let $a \in \mathbb{R}^+$ such that $x + a = x$

$\therefore xa = x$

$\therefore \boxed{a = 1}$

Similarly $a + x = x$

Hence '1' is an identity element of $\mathbb{R}^+$ with given operation

(6) Let $a \in \mathbb{R}^+$ such that $x + a = 1$

$xa = 1$

$\therefore \boxed{a = 1/x}$

Similarly $a + x = 1$

Hence '$1/x$' is an inverse element of $\mathbb{R}^+$ with given operations.

(7) $\alpha(x+y) = \alpha(xy)$
$= (xy)^\alpha$
$= x^\alpha \cdot y^\alpha$
$= (\alpha x)(\alpha y)$
$= \alpha x + \alpha y$

(8) $(\alpha+\beta)x = x^{\alpha+\beta}$
$= x^\alpha \cdot x^\beta$
$= (\alpha x)(\beta x)$
$= \alpha x + \beta x$

(9) $(\alpha\beta)x = x^{\alpha\beta}$
$= x^{\beta\alpha} = (x^\beta)^\alpha = (\beta x)^\alpha = \alpha(\beta x)$

(10) For any $1 \in \mathbb{R}$, we have
$1 \cdot x = x^1 = x$

Hence $\mathbb{R}^+$ is vactor space with given operation

* Defination:

→ Subspace: A non empty subset S of a vactor space V is said to be a subspace of V if S itself a vactor space under the operations defined on V.

* Note: Every vactor space has atleast two subspaces, vactor space itself and $\{0\}$. Here the subspace $\{0\}$ is called the zero subspace containing only zero as element.

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* Condition to check subspace:

Theorem: A non-empty subset S is a subspace of V if and only if

(i) $u + v \in S$ for all $u, v \in S$

(ii) $\alpha u \in S$, for all $u \in S$ and $\alpha \in \mathbb{R}$

i.e, closour property for addition and scalar multiplication of 'S'.

Ex: Show that $\{(x, y) \mid x = 3y\}$ is a subspace of $\mathbb{R}^2$.

Sol.n Let $S = \{(x, y) \mid x = 3y\}$

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ be in S

Such that $x_1 = 3y_1$ and $x_2 = 3y_2$

Then $u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

But here $x_1 = 3y_1$; $x_2 = 3y_2$

$x_1 + x_2 = 3y_1 + 3y_2$

$= 3(y_1 + y_2)$

There fore $u + v = \{(x_1 + x_2, y_1 + y_2 \mid x_1 + x_2 = 3(y_1 + y_2)\}$

$\therefore u + v \in S$

Now for any $\alpha \in \mathbb{R}$,

$\alpha u = \alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$

But here $x_1 = 3y_1$

$\alpha x_1 = \alpha(3y_1) = 3(\alpha y_1)$

There fore $\alpha u = \{(\alpha x_1, \alpha y_1) \mid \alpha x_1 = 3(\alpha y_1)\} \in S$

Hence set S is a subspace of $\mathbb{R}^2$.

* check that whether the following sets are subspaces of respective real vector space under the standard operation or not.

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1) $S = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 = 0 \}$; $V = P_3$

Sol^n: we are given $S = \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_0 = 0 \}$

Let $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ and

$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3$ be in S such

that $a_0 = 0$ and $b_0 = 0$

Then $f(x) + g(x)$

$= (a_0 + a_1 x + a_2 x^2 + a_3 x^3) + (b_0 + b_1 x + b_2 x^2 + b_3 x^3)$

$= (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + (a_3 + b_3) x^3$

But Here $a_0 = 0$; $b_0 = 0$

$\therefore a_0 + b_0 = 0$

Therefore $f(x) + g(x) = \{ (a_0 + b_0) + (a_1 + b_1) x + (a_2 + b_2) x^2 + (a_3 + b_3) x^3 \mid a_0 + b_0 = 0 \}$

$\therefore f(x) + g(x) \in S$

(ii) For any $\alpha \in \mathbb{R}$;

$\alpha f(x) = \alpha (a_0 + a_1 x + a_2 x^2 + a_3 x^3)$

$= \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \alpha a_3 x^3$

But Here $a_0 = 0$

$\therefore \alpha a_0 = 0$

Therefore $\alpha f(x) = \{ \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 + \alpha a_3 x^3 \mid \alpha a_0 = 0 \}$

$\therefore \alpha f(x) \in S$

→ Hence S is a subspace of Vector Space of P_3

2) $S = \{ (x, y) \mid x^2 = y^2 \}$; $V = \mathbb{R}^2$

Sol^n: we are given $S = \{ (x, y) \mid (x^2 = y^2) \}$ such

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ be in S

such that $x_1^2 = y_1^2$; $x_2^2 = y_2^2$

Then, $u + v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

But Here $x_1^2 = y_1^2$; $x_2^2 = y_2^2$

$\therefore$ $x_1^2 + x_2^2 = y_1^2 + y_2^2$

Therefore Here we have

$(x_1+x_2)^2 \neq (y_1+y_2)^2$

$\therefore$ $u+v \notin S$

Hence S is not a subspace of vector space.

Note: for e.g. Let $u, v \in S$

$u = (-1, 1)$, $v = (2, 2)$

$\therefore$ $u+v = (-1,1) + (2,2) = (1,3)$

$1^2 \neq 3^2$

3) $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big/ a+b+c+d = 0 \right\}$ $V = M_{22}$

Sol^n Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ and $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ be in S such

that $a_1+b_1+c_1+d_1 = 0$ and $a_2+b_2+c_2+d_2 = 0$

Then $A+B = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$

$= \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$

But Here $a_1+b_1+c_1+d_1 = 0$ and $a_2+b_2+c_2+d_2 = 0$

$\therefore$ $a_1+b_1+c_1+d_1+a_2+b_2+c_2+d_2 = 0$

$\therefore$ $(a_1+a_2)+(b_1+b_2)+(c_1+c_2)+(d_1+d_2) = 0$

Hence $A+B = \left\{ \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \Big/ \begin{matrix} (a_1+a_2)+(b_1+b_2)+ \\ (c_1+c_2)+(d_1+d_2) = 0 \end{matrix} \right\}$

$\therefore$ $A+B \in S$

(ii) For any $\alpha \in \mathbb{R}$, we have

$\alpha A = \alpha \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \alpha a_1 & \alpha b_1 \\ \alpha c_1 & \alpha d_1 \end{bmatrix}$

But Here $a_1+b_1+c_1+d_1=0$

$\alpha a_1+\alpha b_1+\alpha c_1+\alpha d_1=0$

Therefore $\alpha A=\left\{\begin{bmatrix}\alpha a_1 & \alpha b_1\\ \alpha c_1 & \alpha d_1\end{bmatrix} \Big/ \alpha a_1+\alpha b_1+\alpha c_1+\alpha d_1\right.$

Hence set is a Subspace of M_{22}.

4) $S=\{(x,y,z)/x^2+y^2+z^2\leq 1\}$ $V=\mathbb{R}^3$

Sol^n Let $u=(x_1,y_1,z_1)$ and $v=(x_2,y_2,z_2)$ such that $x_1^2+y_1^2+z_1^2\leq 1$; $x_2^2+y_2^2+z_2^2\leq 1$

Then $u+v=(x_1,y_1,z_1)+(x_2,y_2,z_2)$

$=(x_1+x_2,\ y_1+y_2,\ z_1+z_2)$

$u+v\notin S$

e.g. Let $u=(1,0,0)$ $v=(0,1,0)$ be in S

Then $u+v=(1,0,0)+(0,1,0)$

$=(1,1,0)$

Here, $1^2+1^2+0^2\leq 1$

$\therefore 2\nleq 1$

5) $S=\{A_{nn}/AB=BA$ for fixed $B_{nn}\}$, $V=M_{nn}$

Sol^n we are given $S=\{A_{nn}/AB=BA$ for fixed $B_{nn}\}$

Let A_1 and A_2 be in S such that

$A_1B=BA_1$ and $A_2B=BA_2$

Then $(A_1+A_2)B=A_1B+A_2B$

$=BA_1+BA_2$

$=B(A_1+A_2)$

ii) For any $\alpha\in\mathbb{R}$,

$(\alpha A_1)B=B(\alpha A_1)$

$=\alpha(A_1B)$

$=\alpha(BA_1)$

$=B(\alpha A_1)$

Hence S is a subspace of M_{mn}

6) $S = \{(x, y, z) / y = x+z+1\}$, $V = \mathbb{R}^3$

we are given $S = \{(x, y, z) / y = x+z+1$

Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be in S

$$u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

But Here $x_1 + z_1 + 1 = y_1$; $x_2 + z_2 + 1 = y_2$

$$y_1 + y_2 = (x_1 + z_1 + 1) + (x_2 + z_2 + 1)$$

$$= (x_1 + x_2) + (z_1 + z_2) + 2$$

$$\therefore u + v \notin S$$

Hence S is not a subspace of $\mathbb{R}^3$

→ Linear combination: Let S be any non empty subset of a vector space V. Than a vector $v \in V$ is a linear combination of the vectors in S if there exists a finite subset $\{v_1, v_2 \dots v_n\}$ in S such that

$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$, for real numbers $\alpha_1, \alpha_2, \dots \alpha_n$

Ex. Note: The method to check if a vector v is a linear combination of given vectors $v_1, v_2, \dots v_n$ is as follows.

(1) Express v as a linear combination of $v_1, v_2 \dots v_n$ and form a system of equation i.e, $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n$

(2) If the system of equation is consistent then v is a linear combination of $v_1, v_2 \dots v_n$ otherwise it is not a linear combination of $v_1, v_2 \dots v_n$ (i.e. for inconsistent system)

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Ex. Express a vector ~~(2,-2,3)~~ (2,-2,3) as a linear combination of $\{(0,1,-1), (2,0,1), (-3,2,5)\}$ $\mathbb{R}^3$

Soln Let $v = (2,-2,3)$ and $v_1 = (0,1,-1)$ $v_2 = (2,$
$v_3 = (-3, 2, 5)$

Let us consider $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ For scalars $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

Then

$$(2,-2,3) = \alpha_1(0,1,-1) + \alpha_2(2,0,1) + \alpha_3(-3,2,5)$$
$$= (0, \alpha_1, -\alpha_1) + (2\alpha_2, 0, \alpha_2) + (-3\alpha_3, 2\alpha_3$$
$$(2,-2,3) = (2\alpha_2 - 3\alpha_3, \alpha_1 + 2\alpha_3, -\alpha_1 + \alpha_2 + 5\alpha_3)$$

By equating the components on both side we

$$2\alpha_2 - 3\alpha_3 = 2$$
$$\alpha_1 + 2\alpha_3 = -2$$
$$-\alpha_1 + \alpha_2 + 5\alpha_3 = 3$$

Therefore the augmented matrix from above System is

$$[A \mid B] = \left[\begin{array}{ccc|c} 0 & 2 & -3 & 2 \\ 1 & 0 & 2 & -2 \\ -1 & 1 & 5 & 3 \end{array}\right]$$

R_{12} i.e., $R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 2 & -3 & 2 \\ -1 & 1 & 5 & 3 \end{array}\right]$$

$R_3 \to R_3 + R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 2 & -3 & 2 \\ 0 & 1 & 7 & 1 \end{array}\right]$$

R_{23}

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & 7 & 1 \\ 0 & 2 & -3 & \end{array}\right]$$

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$R_3 \to R_3 + (-2) R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & -17 & 0 \end{array}\right]$$

$R_3 \to -(1/17) R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & -2 \\ 0 & 1 & 7 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \quad -(1)$$

$\therefore \gamma[A] = 3 = \gamma[A|B] =$ no of ~~solution~~ unknowns.

Therefore given system is consistance.

$\therefore$ v can be expressed as linear combination of v_1, v_2 and v_3

from eq. (1) we have,

$\alpha_1 + 2\alpha_3 = -2$ -(i) and $\boxed{\alpha_3 = 0}$ -(iii)

$\alpha_2 + 7\alpha_3 = 1$ -(ii)

from eq. (2) $\boxed{\alpha_2 = 1}$

from eq. (i) $\boxed{\alpha_1 = -2}$

Hence $(2, -2, 3)$ can be written as

$(2, -2, 3) = -2(0, 1, -1) + 1(2, 0, 1) + 0(-3, 2, 5)$

* Span of a Set: This span of a non empty subset S of a vector space V is the set of all linear combinations of any finite number of element of S.

→ It is denoted by 'span(S)' or 'L[S]'

i.e, $\text{Span}(S) = \{ \alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n \mid v_1, v_2, \dots v_n \in S \; \& \; \alpha_1, \alpha_2, \dots \alpha_n \in \mathbb{R} \}$

* Determine the span of a vector $(4,2)$ of $\mathbb{R}^2$

Let $v \in \mathbb{R}^2$ and $v_1 = (4,2)$, Then $S = \{(4,2)\}$;

Then for any $\alpha_1 \in \mathbb{R}$, we have

$$v = \alpha_1 v_1$$

$$v = \alpha_1 (4,2) = (4\alpha_1, 2\alpha_1)$$

Therefore $\text{Span}(S) = \{ v \in \mathbb{R}^2 \mid v = (4\alpha_1, 2\alpha_1),\ \alpha_1 \in \mathbb{R} \}$

* Determine the span of $\{(1,0,0), (0,1,0), (0,0,1)\}$ of $\mathbb{R}^3$.

Soln → Let $v \in \mathbb{R}^3$ and $S = \{(1,0,0), (0,1,0), (0,0,1)\}$

Let $v_1 = (1,0,0)$, $v_2 = (0,1,0)$, $v_3 = (0,0,1)$

Then for any $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$, we have

$$v = \alpha_1 (1,0,0) + \alpha_2 (0,1,0) + \alpha_3 (0,0,1)$$

$$v = (\alpha_1, \alpha_2, \alpha_3)$$

Therefore $\text{Span}(S) = \{ v \in \mathbb{R}^3 \mid v = (\alpha_1, \alpha_2, \alpha_3),\ \alpha_1, \alpha_2, \alpha_3 \ldots$

* Determine Span of $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} \right\}$ of M_{22}

Soln. Let $A \in M_{22}$ and $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix} \right\}$

Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$,

$$A = \alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3$$

$$= \alpha_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2\alpha_2 \\ 3\alpha_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2\alpha_3 \\ -\alpha_3 & \alpha_3 \end{bmatrix}$$

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$$A = \begin{bmatrix} \alpha_1 & 2\alpha_2 + 2\alpha_3 \\ 3\alpha_2 - \alpha_3 & \alpha_3 \end{bmatrix}$$

$$\therefore \text{Span}(S) = \left\{ A \in M_{2\times2} \;\middle|\; A = \begin{bmatrix} \alpha_1 & 2\alpha_2 + 2\alpha_3 \\ 3\alpha_2 - \alpha_3 & \alpha_3 \end{bmatrix}, \alpha_1, \alpha_2, \alpha_3 \in \mathbb{R} \right\}$$

* Determine span of a set $\{1+3x, x+x^2\}$ of P_2

→ Let $p(x) \in P_2$ and $S = \{1+3x, x+x^2\}$ and

Let $p_1(x) = 1+3x$ and $p_2(x) = x+x^2$

Then for $\alpha_1, \alpha_2 \in \mathbb{R}$ we have

$$p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x)$$
$$= \alpha_1 [1+3x] + \alpha_2 [x+x^2]$$
$$= \alpha_1 + 3\alpha_1 x + \alpha_2 x + \alpha_2 x^2$$
$$= \alpha_1 + (3\alpha_1 + \alpha_2)x + \alpha_2 x^2$$

$$\therefore \text{Span}(S) = \{ p(x) \in P_2 \mid \alpha_1 + (3\alpha_1 + \alpha_2)x + \alpha_2 x^2, \alpha_1, \alpha_2 \in \mathbb{R} \}$$

* **Linear dependence and independence of a set:**

→ A Finite set $\{v_1, v_2, \dots v_n\}$ of a vector space V is said to be linearly dependent (L.D.) if there exists scalar's $\alpha_1, \alpha_2, \dots \alpha_n \in \mathbb{R}$, not all zero. (i.e, at least one of them must be non zero) such that $\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n = 0$

→ A finite set $\{v_1, v_2 \dots v_n\}$ of a vector space V is said to be linearly independent (L.I.) If there exists $\alpha_1, \alpha_2, \dots \alpha_n \in \mathbb{R}$, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n \Longrightarrow \alpha_1 = \alpha_2 = \dots \alpha_n = 0$$

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Note: Method to check a set to be L.I. or L.D.

If the system of equations $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$ has trivial solution then the set of vector is L.I. otherwise L.D. (i.e, the system has non-trivial solution.

*) Check whether the following set of vectors are L.I. or L.D.

1) $\{(1,2,3), (0,2,1), (0,1,3)\}$ in $\mathbb{R}^3$

Soln: Let $v_1 = (1,2,3)$, $v_2 = (0,2,1)$, $v_3 = (0,1,3)$

Then for any scalers $\alpha_1, \alpha_2, \alpha_3$ in $\mathbb{R}$, we consider

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\alpha_1(1,2,3) + \alpha_2(0,2,1) + \alpha_3(0,1,3) = (0,0,0)$$

$$\therefore (\alpha_1, 2\alpha_1, 3\alpha_1) + (0, 2\alpha_2, \alpha_2) + (0, \alpha_3, 3\alpha_3) = (0,0,0)$$

$$\therefore (\alpha_1, 2\alpha_1 + 2\alpha_2 + \alpha_3, 3\alpha_1 + \alpha_2 + 3\alpha_3) = (0,0,0)$$

Now equating the components on both side we get,

$$\boxed{\alpha_1 = 0}$$

$$2\alpha_1 + 2\alpha_2 + \alpha_3 = 0$$

$$3\alpha_1 + \alpha_2 + 3\alpha_3 = 0$$

from above system the matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

$R_2 \to R_2 + (-2)R_1$; $R_3 \to R_3 + (-3)R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

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$R_{23} \to (1/1)\ R_2 \leftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_3 \to R_3 + (-2) R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & -5 \end{bmatrix}$$

$R_3 \to (-1/5) R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Therefore $r(A) = 3 =$ no. of unknows

$\therefore$ given system has trivial solutions.

Hence $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$

$\therefore$ given set of vector is linearly independent.

2) $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix} \right\}$ in M_{22}

Let $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix}$, $A_4 = \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix}$

Then for any scalar's $\alpha_1, \alpha_2, \alpha_3, \& \alpha_4$ in $\mathbb{R}$, we consider

$$\alpha_1 A_1 + \alpha_2 A_2 + \alpha_3 A_3 + \alpha_4 A_4 = 0$$

$$\therefore \alpha_1 \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} + \alpha_4 \begin{bmatrix} 2 & 6 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_1 & 2\alpha_1 \end{bmatrix} + \begin{bmatrix} \alpha_2 & 0 \\ 0 & 2\alpha_2 \end{bmatrix} + \begin{bmatrix} 0 & 3\alpha_3 \\ \alpha_3 & 2\alpha_3 \end{bmatrix} + \begin{bmatrix} 2\alpha_4 & 6\alpha_4 \\ 4\alpha_4 & 6\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \alpha_1 + \alpha_2 + 2\alpha_4 & \alpha_1 + 3\alpha_3 + 6\alpha_4 \\ \alpha_1 + \alpha_3 + 4\alpha_4 & 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 6\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

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Now equating the components on both si[des] we get,

$\alpha_1 + \alpha_2 + 2\alpha_4 = 0$ — (1)

$\alpha_1 + 3\alpha_3 + 6\alpha_4 = 0$ — (2)

$\alpha_1 + \alpha_3 + 4\alpha_4 = 0$ — (3)

$2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 6\alpha_4 = 0$ — (4)

from the above system the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 1 & 0 & 3 & 6 \\ 1 & 0 & 1 & 4 \\ 2 & 2 & 2 & 6 \end{bmatrix}$$

$R_2 \to R_2 + (-1)R_1$; $R_3 \to R_3 + (-1)R_1$; $R_4 \to R_4 + (-2)R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$R_2 \to (-1)R_1$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & -1 & 1 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

$R_3 \to R_3 / -2$; $R_4 \to R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$\therefore$ $r(A) = 3 < 4 =$ no. of unknowns

$\therefore$ given system has non-trivial solutions.

$\therefore$ given set is linearly dependent.

H.W. $\left\{ \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix} \right\}$ in M_{22} $\boxed{\text{Ans. L.I.}}$

* **Basis:** A subset 'S' of vector of a vector space V is said to be basis for V if

i) set S is L.I.

ii) S spans V

Note:

i) Basis for a vector space is not unique.

ii) Standard Basis of vector spaces:

1) $\{(1,0), (0,1)\}$ for $\mathbb{R}^2$

2) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ for M_{22}

3) $\{1, x, x^2\}$ for P_2

Ex. Prove that the set $\{(1,0,0), (2,2,0), (3,3,3)\}$ is a basis of $\mathbb{R}^3$

Soln Let $S = \{(1,0,0), (2,2,0), (3,3,3)\}$

Let $x = (x_1, x_2, x_3)$ be in $\mathbb{R}^3$ and

$v_1 = (1,0,0)$, $v_2 (2,2,0)$, $v_3 (3,3,3)$

Then for any $\alpha_1, \alpha_2, \alpha_3$ we can write a linear combination

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$(x_1, x_2, x_3) = \alpha_1 (1,0,0) + \alpha_2 (2,2,0) + \alpha_3 (3,3,3)$$

$$= (\alpha_1, 0, 0) + (2\alpha_2, 2\alpha_2) + (3\alpha_3, 3\alpha_3, 3\alpha_3)$$

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$(x_1, x_2, x_3) = (\alpha_1 + 2\alpha_2 + 3\alpha_3,\ 2\alpha_2 + 3\alpha_3,\ 3\alpha_3)$

now equating the components on both si[des]

we get,

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = x_1$$
$$2\alpha_2 + 3\alpha_3 = x_2$$
$$3\alpha_3 = x_3$$

Then from above system the augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 2 & 3 & x_2 \\ 0 & 0 & 3 & x_3 \end{array}\right]$$

$R_2 \to R_2(1/2)$; $R_3 \to (1/3)R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & x_1 \\ 0 & 1 & 3/2 & x_2/2 \\ 0 & 0 & 1 & x_3/3 \end{array}\right] \quad \text{--- (1)}$$

$\therefore$ we can observe that for any value of x_3 (i.e, $x_3 \in \mathbb{R}$)

$r(A) = 3 = r([A|B]')$

$\therefore$ given system is consistant

$\therefore$ S spans $\mathbb{R}^3$

now to prove set S is L.I. we consider linear combination $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

from (1), we have

$r(A) = 3 =$ no. of unknowns

$\therefore$ System has trivial solution.

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore$ S is L.I.

Hence set S spans $\mathbb{R}^3$ and S is L.I.,

S is a basis for $\mathbb{R}^3$.

* check whether the set
$S = \{(0,1,0), (0,2,3), (1,2,2)\}$ is Basis for $\mathbb{R}^3$ or not.

Let x be in $\mathbb{R}^3$ and $v_1 = (0,1,0)$, $v_2 (0,2,3)$, $v_3 (1,2,2)$

Then for any $\alpha_1, \alpha_2, \alpha_3$ in $\mathbb{R}$ we can write a linear combination

$$x = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$(x_1, x_2, x_3) = \alpha_1 (0,1,0) + \alpha_2 (0,2,3) + \alpha_3 (1,2,2)$

$(x_1, x_2, x_3) = (0, \alpha_1, 0) + (0, 2\alpha_2, 3\alpha_2) + (\alpha_3, 2\alpha_3, 2\alpha_3)$

$(x_1, x_2, x_3) = (\alpha_3, \alpha_1 + 2\alpha_2 + 2\alpha_3, 3\alpha_2 + 2\alpha_3)$

now equating the components on both side we get,

$$\alpha_3 = x_1$$
$$\alpha_1 + 2\alpha_2 + 2\alpha_3 = x_2$$
$$3\alpha_2 + 2\alpha_3 = x_3$$

Then from above system the augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 0 & 0 & 1 & x_1 \\ 1 & 2 & 2 & x_2 \\ 0 & 3 & 2 & x_3 \end{array}\right]$$

R_{12}, $R_1 \longleftrightarrow R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & x_2 \\ 0 & 0 & 1 & x_1 \\ 0 & 3 & 2 & x_3 \end{array}\right]$$

$R_3 \longleftrightarrow R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & x_2 \\ 0 & 3 & 2 & x_3 \\ 0 & 0 & 1 & x_1 \end{array}\right]$$

$R_2 \rightarrow (1/3) R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & x_2 \\ 0 & 1 & 2/3 & x_3/3 \\ 0 & 0 & 1 & x_1 \end{array}\right] \quad -(1)$$

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$\therefore$ we can observe that for any value of x_1 (i.e. $x_1 \in \mathbb{R}$)

$r(A) = 3 = r[A|B]$

$\therefore$ given system is consistent.

$\therefore$ S spans $\mathbb{R}^3$

now to prove set S is L.I. we consider linear combination $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$

from (1), we have

$r(A) = 3 =$ no. of unknowns.

$\therefore$ system has trivial solution.

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$

$\therefore$ S is L.I.

Hence set S spans $\mathbb{R}^3$ and S is L.I., so S is a basis for $\mathbb{R}^3$

* Determine whether the set $\{(1,0), (0,1), (2,-1)\}$ is a basis for $\mathbb{R}^2$ or not.

Sol^n → Let $S = \{(1,0), (0,1), (2,-1)\}$

Let $x = (x_1, x_2) \in \mathbb{R}^2$ and $v_1 = (1,0)$, $v_2 = (0,1)$, $v_3 = (2,-1)$

Then for any $\alpha_1, \alpha_2, \alpha_3$ we can write a linear combination

$x = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

$(x_1, x_2) = \alpha_1(1,0) + \alpha_2(0,1) + \alpha_3(2,-1)$

$(x_1, x_2) = (\alpha_1 + 2\alpha_3, \alpha_2 - \alpha_3)$

now equating the components on both sides, we get

$\alpha_1 + 2\alpha_3 = x_1$

$\alpha_2 - \alpha_3 = x_2$

$\therefore$ Then for above

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & x_1 \\ 0 & 1 & -1 & x_2 \end{array}\right] \quad \text{--- (1)}$$

Then for any $x_2 \in \mathbb{R}^2$, we have

$r(A) = 2 = r[A|B]$

$\therefore$ given system is consistent

$\therefore$ S spans $\mathbb{R}^2$

now, to Prove the set S is L.I,

consider $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$ (i.e, $(x_1, x_2) = (0,0)$)

Then From (1) we can observe that

$\therefore$ we can observe that

$r(A) = 2 < 3 =$ no. of unknows $(\alpha_1, \alpha_2, \alpha_3)$

$\therefore$ given system has non-trivial solution.

$\therefore$ S is L.D.

Hence S is not basis for $\mathbb{R}^2$.

Ex. Determine whether the set $\{1-3x+2x^2, 1+x+4x^2, 1-7x\}$ Forms a basis for P_2 or not.

$\rightarrow$ Let $S = \{1-3x+2x^2, 1+x+4x^2, 1-7x\}$

then $p_1(x) = 1-3x+2x^2$

$p_2(x) = 1+x+4x^2$

$p_3(x) = 1-7x$

Let $p(x) = a_0 + a_1 x + a_2 x^2$ be in P_2

Then ~~p(x)~~ For any scaler's α_1, α_2 and α_3

$p(x)$ is a linear combination as,

$p(x) = \alpha_1 p_1(x) + \alpha_2 p_2(x) + \alpha_3 (p_3)(x)$

$$a_0 + a_1 x + a_2 x^2 = \alpha_1 (1-3x+2x^2) + (1+x+4x^2)\alpha_2 + (1-7x)$$

$$= \alpha_1 - 3\alpha_1 x + 2\alpha_1 x^2 + \alpha_2 + \alpha_2 x + 4\alpha_2 x^2$$

$$+ \alpha_3 - 7\alpha_3 - x$$

$$= (\alpha_1 + \alpha_2 + \alpha_3) + (-3\alpha_1 + \alpha_2 - 7\alpha_3)x +$$

$$(2\alpha_1 + 4\alpha_2)x^2$$

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Now, by equating the components on both sides, we get

$$\alpha_1 + \alpha_2 + \alpha_3 = a_0$$
$$-3\alpha_1 + \alpha_2 - 7\alpha_3 = a_1$$
$$2\alpha_1 + 4\alpha_2 = a_2$$

From above the augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ -3 & 1 & -7 & a_1 \\ 2 & 4 & 0 & a_2 \end{array}\right]$$

$R_2 \to R_2 + 3(R_1)$; $R_3 \to R_3 + (-2)R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ 0 & 4 & -4 & a_1 + 3a_0 \\ 0 & 2 & -2 & a_2 - 2a_0 \end{array}\right]$$

$R_2 \to R_2/4$; $R_3 \to R_3/2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ 0 & 1 & -1 & \frac{a_1+3a_0}{4} \\ 0 & 1 & -1 & \frac{a_2-2a_0}{4} \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & a_0 \\ 0 & 1 & -1 & \frac{a_1+3a_0}{4} \\ 0 & 0 & 0 & \frac{a_2-2a_0}{2} - \frac{a_1+3a_0}{4} \end{array}\right]$$

Here we can observe that for any value of a_0, a_1 and a_2 (i.e; in $\mathbb{R}$) the system is inconsistent.

$\therefore$ S cannot spans P_2.

Hence S not forms a basis for P_2

H.W. Determine whether the set

$$\left\{ \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & 2 \end{bmatrix} \right\}$$

* Dimension :

→ The number of vectors in a basis of a non-zero vector space V is called the Dimension V. It is denoted by dim(V)

→ $\dim(\{0\}) = \underline{0}$ it is L.D. set.

Note: - dimensions of some standard vector spaces can be obtain directly from their standard basis.

1) $\dim(\mathbb{R}^n) = \underline{n}$
2) $\dim(M_{mn}) = \underline{m \cdot n}$
3) $\dim(P_n) = \underline{n+1}$
4) $\dim(\{0\}) = \underline{0}$
($\because$ '0' is a L.D. set therefore it not forms a basis)

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* Linear Tranformations :

-> Let V and W be two vector spaces. then a linear transformation is a function from V to W i.e, $T : V \to W$ such that

(i) $T(u+v) = T(u) + T(v)$
(ii) $T(\alpha u) = \alpha T(u)$

$\Big\}$ (linearity Property)

For all u, v in V and for all scalar α

* Determine whether the following functions are linear transformation or not.

1) $T : \mathbb{R}^2 \to \mathbb{R}^2$, where $T(x,y) = (x+2y, 3x-y)$

Soln. we are given a $T : \mathbb{R}^2 \to \mathbb{R}^2$ where

$T(x,y) = (x+2y, 3x-y)$

Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ be in $\mathbb{R}^2$

i) Then $T(u) = T(x_1, y_1)$

$= (x_1 + 2y_1, 3x_1 - y_1)$

and $T(v) = T(x_2, y_2)$

$= (x_2 + 2y_2, 3x_2 - y_2)$

Here $u+v = (x_1, y_1) + (x_2, y_2)$

$= (x_1 + x_2, y_1 + y_2)$

Now $T(u+v) = (x_1 + x_2, y_1 + y_2)$

$= (x_1 + x_2 + 2y_1 + 2y_2, 3x_1 + 3x_2 - y_1 - y_2)$

$= (x_1 + 2y_1 + x_2 + 2y_2, 3x_1 - y_1 + 3x_2 - y_2)$

$= (x_1 + 2y_1, 3x_1 - y_1) + (x_2 + 2y_2, 3x_2 - y_2)$

$T(u+v) = T(u) + T(v)$

Now, $\alpha u = \alpha(x_1, y_1) = (\alpha x_1, \alpha y_1)$

Then $T(\alpha u) = T(\alpha x_1, \alpha y_1)$

$= (\alpha x_1 + 2\alpha y_1, 3\alpha x_1 - \alpha y_1)$

$= \alpha(x_1 + 2y_1, 3x_1 - y_1)$

Hence T is a linear Transformation.

2) $T: M_{nn} \to \mathbb{R}$, $T(A) = \det(A)$

Sol.^{n} we are given $T: M_{nn} \to \mathbb{R}$ where

$$T(A) = \det(A)$$

Let A_1 and A_2 be in M_{nn}

Then $T(A_1) = \det(A_1)$ and

$T(A_2) = \det(A_2)$

Now $T(A_1 + A_2) = \det(A_1 + A_2)$

$\neq \det(A_1) + \det(A_2)$

$\neq T(A_1) + T(A_2)$

$\therefore$ T is not a linear Transformation.

3) $T: \mathbb{R}^3 \to \mathbb{R}^2$, where $T(x, y, z) = (2x - y + z, y - 4z)$

we are given $T: \mathbb{R}^3 \to \mathbb{R}^2$ where

$$T(x, y, z) = (2x - y + z, y - 4z)$$

Let $u = (x_1, y_1, z_1)$ and $v = (x_2, y_2, z_2)$ be in $\mathbb{R}^3$

Then $T(u) = T(x_1, y_1, z_1)$

$= (2x_1 - y_1 + z_1, y_1 - 4z_1)$

$T(v) = T(x_2, y_2, z_2)$

$= (2x_2 - y_2 + z_2, y_2 - 4z_2)$

Here $u + v = (x_1, y_1, z_1) + (x_2, y_2, z_2)$

$= (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

Now $T(u+v) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$= (2x_1 + 2x_2 - y_1 - y_2 + z_1 + z_2, y_1 + y_2 - 4z_1 - 4z_2)$

$= (2x_1 - y_1 + z_1 + 2x_2 - y_2 + z_2, y_1 - 4z_1 + y_2 - 4z_2)$

$= (2x_1 - y_1 + z_1, y_1 - 4z_1) + (2x_2 - y_2 + z_2, y_2 - 4z_2)$

$T(u+v) = T(u) + T(v)$

Also, $\alpha u = \alpha(x_1, y_1, z_1) = (\alpha x_1, \alpha y_1, \alpha z_1)$

$\therefore T(\alpha u) = T(\alpha x_1, \alpha y_1, \alpha z_1)$

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$T(\alpha u) = (2\alpha x_1 - \alpha y_1 + \alpha z_1 , \alpha y_1 - 4\alpha z_1)$

$T(\alpha u) = \alpha(2x_1 - y_1 + z_1 , y_1 - 4z_1)$

$T(\alpha u) = \alpha T(u)$

→ Hence T is linear Transformation.

4) $T : P_2 \to P_2$, $T(a_0 + a_1 x + a_2 x^2) = (a_0+1) + (a_1+1)x + (a_2+1)x^2$

Sol^n we are given $T : P_2 \to P_2$ where

$T(a_0 + a_1 x + a_2 x^2) = (a_0+1) + (a_1+1)x + (a_2+1)x^2$

Let $P_1(x) = a_0 + a_1 x + a_2 x^2$ and $P_2(x) = b_0 + b_1 x + b_2 x^2$ in P^2

Then $T(P_1(x)) = T(a_0 + a_1 x + a_2 x^2)$

$= (a_0+1) + (a_1+1)x + (a_2+1)x^2$ and

$T(P_2(x)) = T(b_0 + b_1 x + b_2 x^2)$

$= (b_0+1) + (b_1+1)x + (b_2+1)x^2$

Here $P_1(x) + P_2(x) = (a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2$

$\therefore$ $T(P_1(x) + P_2(x)) = (a_0+b_0+1) + (a_1+b_1+1)x + (a_2+b_2+1)x^2$

$= (a_0+1) + (a_1+1)x + (a_2+1)x^2 +$

$b_0 + b_1 x + b_2 x^2$

$= T(P_1(x)) + P_2(x)$

$\neq T(P_1(x)) + T(P_2(x))$

Hence T is not a linear Transformation.

* Matrix Representation of a linear Transformation:

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear Transformation. There exists a matrix A of order $m \times n$ such that

$T(X) = AX$

for e.g. consider a linear Transformation

$T : \mathbb{R}^3 \to \mathbb{R}^3$, where $T(x, y, z) = ~~(x+y~~$

$(x-y+2z, 2x+y-z, -x-2y)$

then its matrix representation is

$$T(x,y,z) = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

i.e, $T(X) = AX$ where $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & -1 \\ -1 & -2 & 0 \end{bmatrix}$

* Range and Kernal of a linear Transformation:

→ Let V and W be two vector spaces and $T : V \to W$ be a linear Transformation. Then the set $\{T(x) \mid x \in V\}$ is called the Range of T.

→ It is denoted by '$R(T)$'

→ The dimension of $R(T)$ is known as rank of a Transformation.

→ It is denoted by $\text{rank}(T) = \rho(T) = \dim(R(T))$

→ The set $\{x \in V \mid T(x) = 0\}$ is called the Kernal or (null space) of a linear Transformation T.

→ It is denoted by '$\ker(T)$' or '$N(T)$'

→ The dimension of '$\ker(T)$' or '$N(T)$' is called nullity of a Transformation.

→ It is denoted by $\text{nullity}(T) = n(T) = \dim(N(T))$

Note: If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear Transformation such that $T(X) = AX$, where A is A matrix of linear Transformation. Then $r(A) = r(T)$ and $n(A) = n(T)$

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* Rank- Nullity theorem : (Dimension Theorem)

-> statement: Let V and W be two vector spaces. If $T: V \to W$ is a linear Transformation then $\text{rank}(T) + \text{nullity}(T) = \dim(V)$

i.e, $\rho(T) + \eta(T) = \dim(\text{domain of a L.T.})$

* Find rank & nullity of the following Transformation & verify the rank nullity theorem:

(1) $T: \mathbb{R}^2 \to \mathbb{R}^2$, where $T(x, y) = (2x - y, -8x + 4y)$

Sol.^{n} we are given, $T: \mathbb{R}^2 \to \mathbb{R}^2$, where

$T(x, y) = (2x - y, -8x + 4y)$

Then its matrix representation is

$$T(x, y) = \begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\therefore$ $T(X) = AX$, where $A = \begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix}$

we know that for a linear Transformation $T: \mathbb{R}^n \to \mathbb{R}^m$

$$\rho(A) = \rho(T) \quad \& \quad \eta(A) = \eta(T)$$

Here $A = \begin{bmatrix} 2 & -1 \\ -8 & 4 \end{bmatrix}$

$R_1 \to R_1/2$; $R_2 \to R_2/4$

$$\sim \begin{bmatrix} 1 & -1/2 \\ -2 & 1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$$\sim \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

Therefore $r(A) = 1$ & Hence $\rho(T) = 1$

Now $\eta(A) =$ no. of columns of $A - \rho(A)$

$= 2 - 1 = \boxed{1}$ & Hence $\rho(T) = 1$

Verification:

we know that by rank - nullity theorem

$\rho(T) + n(T) = \dim$ (domain of a L.T.)

Here $\rho(A) + n(T) = 1+1 = 2 = \dim(\mathbb{R}^2)$

Hence, Theorem is verified

2) $T : \mathbb{R}^3 \to \mathbb{R}^2$ where, $T(x, y, z) = (x+y+z, x-y-z)$

Sol.ⁿ we are given $T : \mathbb{R}^3 \to \mathbb{R}^3$, where

$T(x, y, z) = (x+y+z, x-y-z)$

Then its matrix representation is

$$T(x,y) = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore T(X) = AX$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

we know

$\rho(A) = \rho(T)$ & $n(A) = n(T)$

Here $A = \begin{bmatrix} 1 & 1 & 1 \\ +1 & -1 & -1 \end{bmatrix}$