Recurrence Relations

by

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Outline

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Linear recurrence relations with constant coefficients

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Subject: MA253 Discrete Mathematics and Algebra

- Solutions of recurrence relation by direct methods
- Generating functions and solutions of recurrence relation.

Consider the sequence 0,1,1,2,3,5,8,13,..... This sequence of numbers is called Fibonacci sequence.
Let $a_0 = 1$ and $a_1 = 1$. Then $a_0 + a_1 = a_2 = 2$. In general, we have

$$a_{n+2} = a_{n+1} + a_n.$$

We get relation

$$a_{n+2} - a_{n+1} - a_n = 0, n \in \mathbb{N} \cup \{0\},$$

which is recurrence relation of Fibonacci sequence. Now consider general expression,

$$a_n = 3^n, n \geq 0.$$

Suppose, we have relation

$$a_n = 3a_{n-1}, a_0 = 1.$$

Taking $n = 1, 2, 3, ...$, we get
$a_1 = 3a_0 = 3, a_2 = 3a_1 = 9, a_3 = 3a_2 = 27, ..., a_n = 3^n$.
Hence 3^n is a solution of recurrence relation $a_n = 3a_{n-1}$.

Linear recurrence relations with constant coefficients

A recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k} = F(n), \qquad (2.1)$$

where c_i's $(1 \leq i \leq k)$ are constants is called a linear recurrence relation with constant coefficients. The recurrence relation (2.1) is known as k^{th}-order(or degree k) recurrence relation, provided that both c_0 and c_k are non-zero. For example

$$2a_n + 3a_{n-1} = 2^n,$$

which is first order recurrence relation and

$$3a_r - 5a_{r-1} + 2a_{r-2} = r^2 + 5$$

is second order recurrence relation.

We try to find solution of the form $a_n = r^n$, where r is constant.
$a_n = r^n$ is a solution of the recurrence relation
$a_n = c_1 a_{n-1} + c_2 a_{n-2} + ... + c_k a_{n-k}$ if and only if
$r^n = c_1 r^{n-2} + c_2 r^{n-2} + ... + c_k r^{n-k}$.
Divide both side by r^{n-k} and subtract the right-hand side from the left, we have

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - ... - c_{k-1} r - c_k = 0,$$

which is called the characteristics equation of the recurrence relation.

Distinct Roots

Theorem 1.

Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1 r - c_2 = 0$ have two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n, n = 0, 1, 2, ..,$ where α_1 and α_2 are constants.

Example 2.

What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution: The characteristic equation of the recurrence relation is

$$r^2 - r - 2 = 0.$$

Its roots are $r = 2$ and $r = -1$. Hence the sequence $\{a_n\}$ is a solution of the recurrence relation if and only if

$$a_n = \alpha_1 2^n + \alpha_2 (-1)^n,$$

for some constants α_1 and α_2

Given that $a_0 = 2$ and $a_1 = 7$. Hence

$$a_0 = 2 = \alpha_1 + \alpha_2,$$

and

$$a_1 = 7 = 2\alpha_1 - \alpha_2.$$

Solving these equations, we get $\alpha_1 = 3$ and $\alpha_2 = -1$. Therefore, the solution of the given recurrence solution is

$$a_n = 3 \times 2^n - (-1)^n.$$

Multiple Roots

Theorem 3.

Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0 which is repeated. Then the sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ if and only if $a_n = \alpha_1 {r_0}^n + \alpha_2 n {r_0}^n, n = 0, 1, 2, ..,$ where α_1 and α_2 are constants.

Example 4.

Solve the recurrence relation
$a_n = 6a_{n-1} - 9a_{n-2}, a_0 = 1, a_1 = 6.$

Solution: The characteristic equation is

$$r^2 - 6r + 9 = 0.$$

Its roots are $r_1 = r_2 = r_0 = 3$. Hence, the solution of the recurrence relation is

$$a_n = \alpha_1 3^n + \alpha_2 n 3^n,$$

for some constants α_1 and α_2.
Since $a_0 = 1, a_1 = 6$, we have $a_0 = 1 = \alpha_1$ and
$a_1 = 6 = 3\alpha_1 + 3\alpha_2$, which yields $\alpha_1 = 1$ and $\alpha_2 = 1$.
The solution of the given recurrence relation is

$$a_n = 3^n + n3^n.$$

Example 5.

Solve the recurrence relation

$$a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}, a_0 = 1, a_1 = -2, a_2 = -1.$$

Solution: The characteristic equation is

$$r^3 + 3r^2 + 3r + 1 = 0.$$

Its roots are $r_1 = r_2 = r_3 = r_0 = -1$. Hence, the solution of the recurrence relation is

$$a_n = \alpha_1(-1)^n + \alpha_2 n(-1)^n + \alpha_3 n^2(-1)^n,$$

The given initial conditions are $a_0 = 1, a_1 = -2,\ a_2 = -1$, from which we obtain

$$a_0 = \alpha_1 = 1,$$
$$a_1 = (\alpha_1 + \alpha_2 + \alpha_3)(-1) = -2,$$
$$a_2 = \alpha_1 + 2\alpha_2 + 4\alpha_3 = -1.$$

Hence

$$\alpha_1 = 1, \alpha_2 = 3, \alpha_3 = -2.$$

The solution of the given recurrence relation is

$$a_n = (-1)^n(1 + 3n - 2n^2).$$

Example 6.

Find an explicit formula for the Fibonacci numbers.

Solution: The Fibonacci numbers satisfy the recurrence relation

$$a_n = a_{n-1} + a_{n-2}$$

with initial conditions $a_0 = 0$ and $a_1 = 1$. The characteristic equation is

$$r^2 - r - 1 = 0,$$

which has two distinct roots

$$r_1 = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1-\sqrt{5}}{2}.$$

Hence, the solution of the recurrence relation

$$a_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for some constants α_1 and α_2.

Given that $a_0 = 0$ and $a_1 = 1$. Hence

$$a_0 = 0 = \alpha_1 + \alpha_2,$$

and

$$a_1 = 1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2} \right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2} \right).$$

Solving these equations, we get $\alpha_1 = \frac{1}{\sqrt{5}}$ and $\alpha_2 = -\frac{1}{\sqrt{5}}$. Therefore, the solution of the given recurrence solution is

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

Exercise.

Solve the following recurrence relations.

1. $a_n + 5a_{n-1} + 6a_{n-2} = 0, a_0 = 1, a_1 = 2$
2. $a_n - 7a_{n-1} + 10a_{n-2} = 0, a_0 = 0, a_1 = 3$
3. $a_n - 13a_{n-1} + 36a_{n-2} = 0, a_0 = 2, a_1 = 1$

Mixed Roots

Example 7.

Suppose that the roots of the characteristic equation of a linear homogeneous recurrence relation are 2, 2, 2, 5, 5, and 9 . What is the form of the general solution?

Solution:

$$a_n = (\alpha_1 2^n + \alpha_2 n 2^n + \alpha_2 n^2 2^n) + (\alpha_3 5^n + \alpha_4 n 5^n) + \alpha_5 9^n,$$

for some constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5.

Exercise.

Solve the following recurrence relations

1. $a_r - 4a_{r-1} + 4a_{r-2} = 0, a_0 = 1, a_1 = 6$
2. $a_r - 10a_{r-1} + 25a_{r-2} = 0, a_0 = 2, a_1 = 3$
3. $a_n - 8a_{n-1} + 21a_{n-2} - 18a_{n-3} = 0$

Linear Non-homogeneous Recurrence Relations with Constant Coefficients

The general form of linear non homogeneous recurrence relation with constant coefficients is

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} + F(n). \tag{3.1}$$

For example,

$$a_n = 3a_{n-1} + 3n^2.$$

Theorem 8.

If $\{a_n^p\}$ is a particular solution of (3.1), then every solution of the form $\{a_n^p + a_n^h\}$, where a_n^h is solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

is a general solution of (3.1).

Note 3.1.

We can solve equation (3.1) for special cases. In particular, if $F(n)$ is

- a polynomial function
- exponential function
- the product of a polynomial and exponential functions

Example 9.

Solve the recurrence relation $a_n = 3a_{n-1} + 2n, a_1 = 3$.

Solution: The characteristic equation is

$$r - 3 = 0.$$

Its root is $r = 3$. The solution of the associated homogeneous recurrence relation is

$$a_n^h = \alpha \times 3^n,$$

for some constant α.

Let $a_n^p = cn + d$, where $c, d \in \mathbf{R}$. If $a_n^p = cn + d$ is a solution of $a_n = 3a_{n-1} + 2n$, then

$$cn + d = 3(c(n-1) + d) + 2n,$$

from which

$$(2c+2)n + (2d-3c) = 0n + 0.$$

We get

$$c = -1, d = -\frac{3}{2}.$$

Hence

$$a_n^p = -n - \frac{3}{2}.$$

So

$$a_n = a_n^h + a_n^p = \alpha \times 3^n - n - \frac{3}{2}.$$

Given that $a_1 = 3$, we obtain

$$a_1 = 3\alpha - 1 - \frac{3}{2} = 3,$$

therefore

$$\alpha = \frac{11}{6}.$$

Finally, the general solution of the given recurrence relation is

$$a_n = \frac{11}{6}3^n - n - \frac{3}{2}.$$

Example 10.

Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2} + 7^n.$$

Solution: The characteristic equation is

$$r^2 - 5r + 6 = 0.$$

Its roots are $r_1 = 3$ and $r_2 = 2$. The solution of the associated homogeneous recurrence relation is

$$a_n^h = \alpha_1 3^n + \alpha_2 2^n,$$

for some constants α_1 and α_2.

Let $a_n^p = c7^n$, where $c \in \mathbf{R}$. If $a_n^p = c7^n$ is a solution of $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$, then

$$c7^n = 5c7^{n-1} - 6c7^{n-2} + 7^n,$$

form which

$$c7^2 = 5c7 - 6c + 7^2.$$

We obtain $c = \frac{49}{20}$. Hence

$$a_n^p = \frac{49}{20}7^n$$

Finally, the general solution of the given recurrence relation is

$$a_n = a_n^h + a_n^p = \alpha_1 3^n + \alpha_2 2^n + \frac{49}{20}7^n.$$

Example 11.

What form does a particular solution of the linear non-homogeneous recurrence relation

$$a_n = 6a_{n-1} - 9a_{n-2} + F(n)$$

have when $F(n) = 3^n, F(n) = n3^n, F(n) = n^2 2^n$ and $F(n) = (n^2 + 1)3^n$?

Solution: The characteristic equation is

$$r^2 - 6r + 9 = 0.$$

Its roots are $r_1 = 3$ and $r_2 = 3$.
If $F(n) = 3^n$ and 3 is a root, then particular solution is

$$a_n^p = \alpha n^2 3^n,$$

for some constant α.
If $F(n) = n3^n$ and 3 is a root, then particular solution is

$$a_n^p = n^2(\alpha_1 n + \alpha_2)3^n,$$

for some constants α_1 and α_2.

If $F(n) = n^2 2^n$ and 2 is not a root, then particular solution is

$$a_n^p = (\alpha_2 n^2 + \alpha_1 n + \alpha_0)2^n,$$

for some constants α_0, α_1 and α_2.
If $F(n) = (n^2 + 1)3^n$ and 3 is a root, then particular solution is

$$a_n^p = n^2(\alpha_2 n^2 + \alpha_1 n + \alpha_0)3^n,$$

for some constants α_0, α_1 and α_2.

Exercise.

Solve the following recurrence relations

1. $a_r - 7a_{r-1} + 10a_{r-2} = 3^r, a_0 = 0, a_1 = 1$
2. $a_r + 6a_{r-1} + qa_{r-2} = 3, a_0 = 0, a_1 = 1$
3. $a_r + 5a_{r-1} + 6_{r-2} = 3r^2 - 2r + 1$

Definition 12.

The generating function for the sequence $a_0, a_1, a_2, ...$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + a_2 x^2 + ... = \sum_{k=0}^{\infty} a_k x^k \tag{4.1}$$

Note 4.1.

1. $\sum\limits_{k=0}^{\infty} r^k = \dfrac{1}{1-r}, |r| < 1$
2. $\sum\limits_{k=0}^{\infty} (-1)^k r^k = \dfrac{1}{1+r}, |r| < 1$

Example 13.

Solve the recurrence relation $a_n = 3a_{n-1}$ for $n = 1, 2, 3, ...$ and the initial condition $a_0 = 2$ using generating function.

Solution: Let

$$G(x) = a_0 + a_1 x + a_2 x^2 + ... = \sum_{k=0}^{\infty} a_k x^k$$

be the generating function for $\{a_k\}$. Note that $a_k = 3a_{k-1}$. We have

$$\sum_{k=1}^{\infty} a_k x^k = 3\sum_{k=1}^{\infty} a_{k-1} x^k = 3x\sum_{k=1}^{\infty} a_{k-1} x^{k-1} = 3x\sum_{k=0}^{\infty} a_k x^k.$$

We obtain

$$G(x) - a_0 = 3xG(x).$$

Since $a_0 = 2$,

$$G(x) - 3xG(x) = 2,$$

which yield

$$G(x) = \frac{2}{1-3x} = 2\sum_{k=0}^{\infty} (3x)^k = 2\sum_{k=0}^{\infty} 3^k x^k.$$

Hence $a_n = 2 \times 3^n$.

Example 14.

Solve the recurrence relation $a_n = 8a_{n-1} + 10^{n-1}$, $n = 1, 2, 3, ...$ and $a_0 = 1, a_1 = 9$.

Solution: Let

$$G(x) = a_0 + a_1 x + a_2 x^2 + ... = \sum_{k=0}^{\infty} a_k x^k$$

be the generating function for $\{a_k\}$.

$$\begin{aligned} G(x) - 1 = \sum_{k=1}^{\infty} a_k x^k &= \sum_{k=1}^{\infty} (8a_{k-1} + 10^{k-1}) x^k \\ &= 8 \sum_{k=1}^{\infty} a_{k-1} x^k + \sum_{k=1}^{\infty} 10^{k-1} x^k \\ &= 8x \sum_{k=0}^{\infty} a_k x^k + x \sum_{k=0}^{\infty} 10^k x^k \\ &= 8xG(x) + \frac{x}{1-10x}, \end{aligned}$$

which gives

$$(1-8x)G(x) = 1 + \frac{x}{1-10x} = \frac{1-9x}{1-10x}.$$

Therefore,

$$G(x) = \frac{1-9x}{(1-10x)(1-8x)} \tag{4.2}$$

Using method of partial fraction, we get

$$\begin{aligned} G(x) &= \frac{1}{2}\left[\frac{1}{1-10x} + \frac{1}{1-8x}\right] \\ &= \frac{1}{2}\left(\sum_{k=0}^{\infty} 10^k x^k + \sum_{k=0}^{\infty} 8^k x^k\right) \\ &= \frac{1}{2}\sum_{k=0}^{\infty}(10^k + 8^k)x^k. \end{aligned}$$

Hence, $a_n = \frac{1}{2}(10^n + 8^n)$.

Exercise.

Solve the following recurrence relations.

1. $a_n = 3a_{n-1} + 2, a_0 = 1$
2. $a_n = a_{n-1} + a_{n-2}, a_1 = 2, a_2 = 3$

Thank you