Error Analysis

- Introduction
- Average error, r.m.s error, probable error and error propagation
- Significant digit and figures
- Uncertainties in Measurements: Measuring Errors, Uncertainties, Parent and Sample Distributions, Mean and Standard Deviation of Distributions,
- Probability Distributions: Binomial Distributions, Poisson distribution, Gaussian or Normal Error Distribution, Lorentzian Distribution.
- Numericals

What is an error?

- Error is a measure of the lack of certainty in a value.
- Error means the inevitable uncertainty that attends all measurements.

HOW ERRORS ARISE IN MEASUREMENT (SYSTEMS)?

A measurement under ideal conditions has no errors.

A systematic (clearly defined process) and systemic (all encompassing) approach is needed to identify every source of error that can arise in a given measuring system. It is then necessary to decide their magnitude and impact on the prevailing operational conditions.

Measurement (system) errors can only be defined in relation to the solution of a real specific measurement task.

If the errors of measurement (systems) given in technical documentation are specified, then one has to decide how that information relates to which

- measurand
- input
- elements of the measurement system
- auxiliary means
- measurement method
- output
- kind of reading
- environmental conditions.

Common Sources of Error in Physics Lab Experiments

This vague phrase does not describe the source of error clearly.

Incomplete definition (may be systematic or random) - One reason that it is impossible to make exact measurements is that the measurement is not always clearly defined. For example, if two different people measure the length of the same rope, they would probably get different results because each person may stretch the rope with a different tension. The best way to minimize definition errors is to carefully consider and specify the conditions that could affect the measurement.

Failure to account for a factor (usually systematic) - The most challenging part of designing an experiment is trying to control or account for all possible factors except the one independent variable that is being analyzed. For instance, you may inadvertently ignore air resistance when measuring free-fall acceleration, or you may fail to account for the effect of the Earth's magnetic field when measuring the field of a small magnet. The best way to account for these sources of error is to brainstorm with your peers about all the factors that could possibly affect your result. This brainstorm should be done before beginning the experiment so that arrangements can be made to account for the confounding factors before taking data. Sometimes a correction can be applied to a result after taking data, but this is inefficient and not always possible.

Environmental factors (systematic or random) - Be aware of errors introduced by your immediate working environment. You may need to take account for or protect your experiment from vibrations, drafts, changes in temperature, electronic noise or other effects from nearby apparatus.

Instrument resolution (random) - All instruments have finite precision that limits the ability to resolve small measurement differences. For instance, a meter stick cannot distinguish distances to a precision much better than about half of its smallest scale division (0.5 mm in this case).

Failure to calibrate or check zero of instrument (systematic) - Whenever possible, the calibration of an instrument should be checked before taking data. If a calibration standard is not available, the accuracy of the instrument should be checked by comparing with another instrument that is at least as precise, or by consulting the technical data provided by the manufacturer. When making a measurement with a micrometer, electronic balance, or an electrical meter, always check the zero reading first. Re-zero the instrument if possible, or measure the displacement of the zero reading from the true zero and correct any measurements accordingly. It is a good idea to check the zero reading throughout the experiment.

Physical variations (random) - It is always wise to obtain multiple measurements over the entire range being investigated. Doing so often reveals variations that might otherwise go undetected. If desired, these variations may be cause for closer examination, or they may be combined to find an average value.

Parallax (systematic or random) - This error can occur whenever there is some distance between the measuring scale and the indicator used to obtain a measurement. If the observer's eye is not squarely aligned with the pointer and scale, the reading may be too high or low (some analog meters have mirrors to help with this alignment).

Instrument drift (systematic) - Most electronic instruments have readings that drift over time. The amount of drift is generally not a concern, but occasionally this source of error can be significant and should be considered.

Lag time and hysteresis (systematic) - Some measuring devices require time to reach equilibrium, and taking a measurement before the instrument is stable will result in a measurement that is generally too low. The most common example is taking temperature readings with a thermometer that has not reached thermal equilibrium with its environment. A similar effect is hysteresis where the instrument readings lag behind and appear to have a "memory" effect as data are taken sequentially moving up or down through a range of values. Hysteresis is most commonly associated with materials that become magnetized when a changing magnetic field is applied.

Estimation of Error

- When attempting to estimate the error of a measurement, it is often important to determine whether the sources of error are systematic or random. A single measurement may have multiple error sources, and these may be mixed systematic and random errors.

- To identify a random error, the measurement must be repeated a small number of times. If the observed value changes apparently randomly with each repeated measurement, then there is probably a random error. The random error is often quantified by the standard deviation of the measurements. Note that more measurements produce a more precise measure of the random error.

- To detect a systematic error is more difficult. The method and apparatus should be carefully analysed. Assumptions should be checked. If possible, a measurement of the same quantity, but by a different method, may reveal the existence of a systematic error. A systematic error may be specific to the experimenter. Having the measurement repeated by a variety of experimenters would test this.

Best Estimate ± Uncertainty

We have seen that the correct way to state the result of measurement is to give a best estimate of the quantity and the range within which you are confident the quantity lies. For example, the result of the timings discussed in Section 1.6 was reported as

$$\text{best estimate of time} = 2.4 \text{ s}, \qquad \text{probable range: } 2.3 \text{ to } 2.5 \text{ s}. \tag{2.1}$$

Here, the best estimate, 2.4 s, lies at the midpoint of the estimated range of probable values, 2.3 to 2.5 s, as it has in all the examples. This relationship is obviously natural and pertains in most measurements. It allows the results of the measurement to be expressed in compact form. For example, the measurement of the time recorded in (2.1) is usually stated as follows:

$$\text{measured value of time} = 2.4 \pm 0.1 \text{ s}. \tag{2.2}$$

This single equation is equivalent to the two statements in (2.1).

In general, the result of any measurement of a quantity x is stated as

$$\boxed{(\text{measured value of } x) = x_{\text{best}} \pm \delta x.} \tag{2.3}$$

This statement means, first, that the experimenter's best estimate for the quantity concerned is the number x_{best}, and second, that he or she is reasonably confident the quantity lies somewhere between $x_{\text{best}} - \delta x$ and $x_{\text{best}} + \delta x$. The number δx is called the *uncertainty,* or *error,* or *margin of error* in the measurement of x. For convenience, the uncertainty δx is always defined to be positive, so that $x_{\text{best}} + \delta x$ is always the *highest* probable value of the measured quantity and $x_{\text{best}} - \delta x$ the *lowest.*

Discrepancy

How to use uncertainties in experimental reports?

First, if two measurements of the same quantity disagree, we say there is a discrepancy. Numerically, we define the discrepancy between two measurement as their difference:

$$\boxed{\text{discrepancy} = \text{difference between two measured values of the same quantity.}}$$

More specifically, each of the two measurements consists of a best estimate and an uncertainty, and we define the discrepancy as the difference between the two best estimates.

For example, if two students measure the same resistance as follows

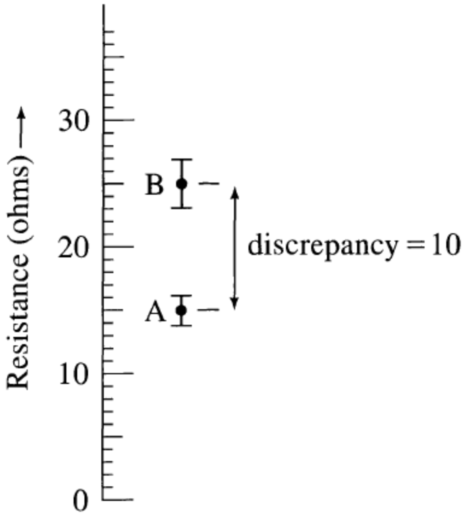
Student A: 15 ± 1 ohms

and

Student B: 25 ± 2 ohms,

their discrepancy is discrepancy = 25-15 = 10 ohms.

Recognize that a discrepancy may or may not be significant. The two measurements just discussed are illustrated in Figure (a), which shows clearly that the discrepancy of 10 ohms is significant because no single value of the resistance is compatible with both measurements. Obviously, at least one measurement is incorrect, and some careful checking is needed to find out what went wrong.



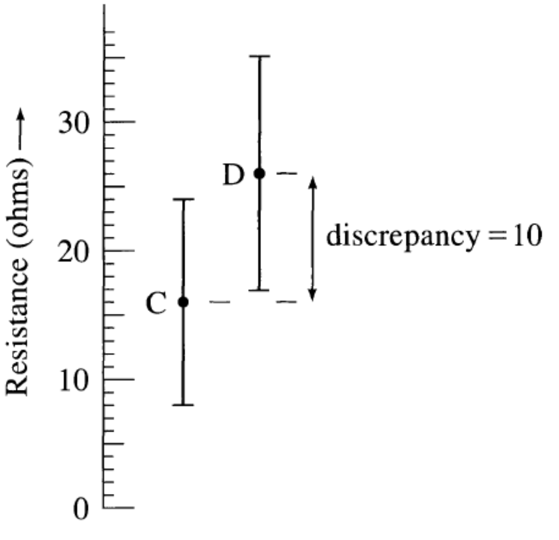
(a)

Figure (a): Two measurements of the same resistance. Each measurement includes a best estimate, shown by a black dot, and a range of probable values, shown by a vertical error bar. The discrepancy (difference between the two best estimates) is 10 ohms and is significant because it is much larger than the combined uncertainty in the two measurements. Almost certainly, at least one of the experimenters made a mistake.

Suppose, on the other hand, two other students had reported these results:

Student C: 16 ± 8 ohms and
Student D: 26 ± 9 ohms.

Here again, the discrepancy is 10 ohms, but in this case the discrepancy is insignificant because, as shown in Figure (b), the two students' margins of error overlap comfortably and both measurements could well be correct.



(b)

Figure (b): Two different measurements of the same resistance. The discrepancy is again 10 ohms, but in this case it is insignificant because the stated margins of error overlap. There is no reason to doubt either measurement (although they could be criticized for being rather imprecise).

Accuracy and Precision:

- Accuracy refers to the closeness of a measured value to a standard or known value.

    For example, if in lab you obtain a weight measurement of 3.2 kg for a given substance, but the actual or known weight is 10 kg, then your measurement is not accurate. In this case, your measurement is not close to the known value.

- Precision refers to the closeness of two or more measurements to each other.

    Using the example above, if you weigh a given substance five times, and get 3.2 kg each time, then your measurement is very precise. Precision is independent of accuracy. You can be very precise but inaccurate, as described above. You can also be accurate but imprecise.

It is important to distinguish between the terms *accuracy* and *precision*. The accuracy of an experiment is a measure of how close the result of the experiment is to the true value; the precision is a measure of how well the result has been determined, without reference to its agreement with the true value. The precision is also a measure of the reproducibility of the result in a given experiment.

- For example, if on average, your measurements for a given substance are close to the known value, but the measurements are far from each other, then you have accuracy without precision.

- A good analogy for understanding accuracy and precision is to imagine a basketball player shooting baskets. If the player shoots with accuracy, his aim will always take the ball close to or into the basket. If the player shoots with precision, his aim will always take the ball to the same location which may or may not be close to the basket. A good player will be both accurate and precise by shooting the ball the same way each time and each time making it in the basket.

	Accuracy	Precision
Definition	The degree of conformity and correctness of something when compared to a true or absolute value.	A state of strict exactness — how often something is strictly exact.
Measurements	Single factor or measurement	Multiple measurements or factors are needed
Relationship	Something can be accurate on occasion as a fluke. For something to be consistently and reliably accurate, it must also be precise.	Results can be precise without being accurate. Alternatively, results can be precise AND accurate.
Uses	Physics, chemistry, engineering, statistics, and so on.	Physics, chemistry, engineering, statistics, and so on.

Rules for Rounding Off Numbers

Rule One. Determine what your rounding digit is and look to the right side of it. If the digit is 0, 1, 2, 3, or 4 do not change the rounding digit. All digits that are on the right-hand side of the requested rounding digit will become 0.

Rule Two. Determine what your rounding digit is and look to the right of it. If the digit is 5, 6, 7, 8, or 9, your rounding digit rounds up by one number. All digits that are on the right-hand side of the requested rounding digit will become 0.

Error bar

When representing data as a graph, we represent uncertainty in the data points by adding error bars. We can see the uncertainty range by checking the length of the error bars in each direction.

Error bars only need to be used when the uncertainty in one or both of the plotted quantities are significant. Error bars are not required for trigonometric and logarithmic functions.

To add error bars to a point on a graph, we simply take the uncertainty range (expressed as "± value" in the data) and draw lines of a corresponding size above and below or on each side of the point depending on the axis the value corresponds to.

What is an error bar and how can you estimate one?

An error bar tells you how closely your measured result should be matched by true result.

An error bar estimates how confident you are in your own measurement or result.

When plotting points (x, y) with known uncertainties on a graph, we plot the average, or mean, value of each point and indicate its uncertainty by means of "error bars." Figure 1 shows how the upper error bar at y_{max} and the lower error bar at y_{min} are plotted. If the quantity x also has significant uncertainty, one adds horizontal error bars (a vertical error bar rotated 90°) with the rightmost error bar at position x_{max} and the leftmost error bar at position x_{min}.

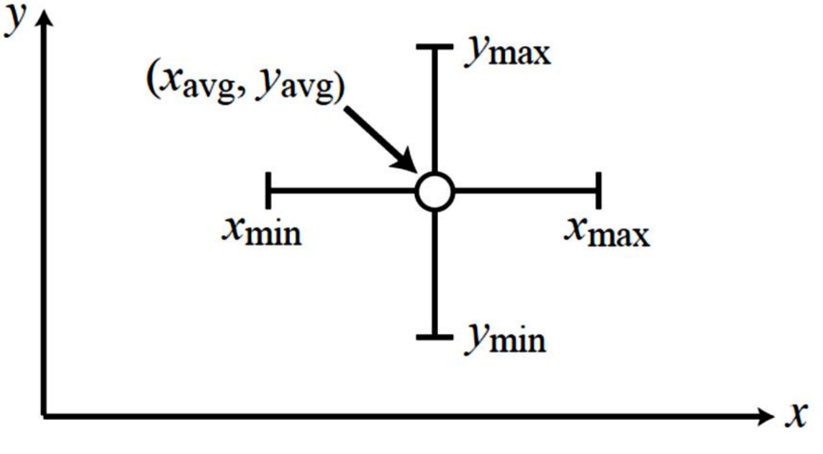


Figure 1 Diagram of error bars showing uncertainties in the value of the x- and y-coordinates at point (x_{avg} , y_{avg}).

Significant Figures (digits):

Significant figures are the digits in a value that are known with some degree of confidence. As the number of significant figures increases, the more certain the measurement. As precision of a measurement increases, so does the number of significant figures.

RULES FOR SIGNIFICANT FIGURES

1	All non zero digits are significant.	549 has three significant figures 1.892 has four significant figures
2	Zeros between non zero digits are significant.	4023 has four significant figures 50014 has five significant figures
3	Zeros to the left of the first non zero digit are not significant.	0.000034 has only two significant figures. (This is more easily seen if it is written as 3.4×10^{-5}) 0.001111 has four significant figures.
4	Trailing zeros (the right most zeros) are significant when there is a decimal point in the number. For this reason it is important to give consideration to when a decimal point is used and to keep the trailing zeros to indicate the actual number of significant figures.	400. has three significant figures 2.00 has three significant figures 0.050 has two significant figures
5	Trailing zeros are not significant in numbers without decimal points.	470,000 has two significant figures 400 or 4×10^2 indicates only one significant figure. (To indicate that the trailing zeros are significant a decimal point must be added. 400. has three significant digits and is written as 4.00×10^2 in scientific notation.)
6	Exact numbers have an infinite number of significant digits but they are generally not reported. Defined numbers also have an infinite number of significant digits.	If you count 2 pencils, then the number of pencils is 2.000... The number of centimeters per inch (2.54) has an infinite number of significant digits, as does the speed of light (299792458 m/s).

Fractional Uncertainties

The uncertainty δx in a measurement,

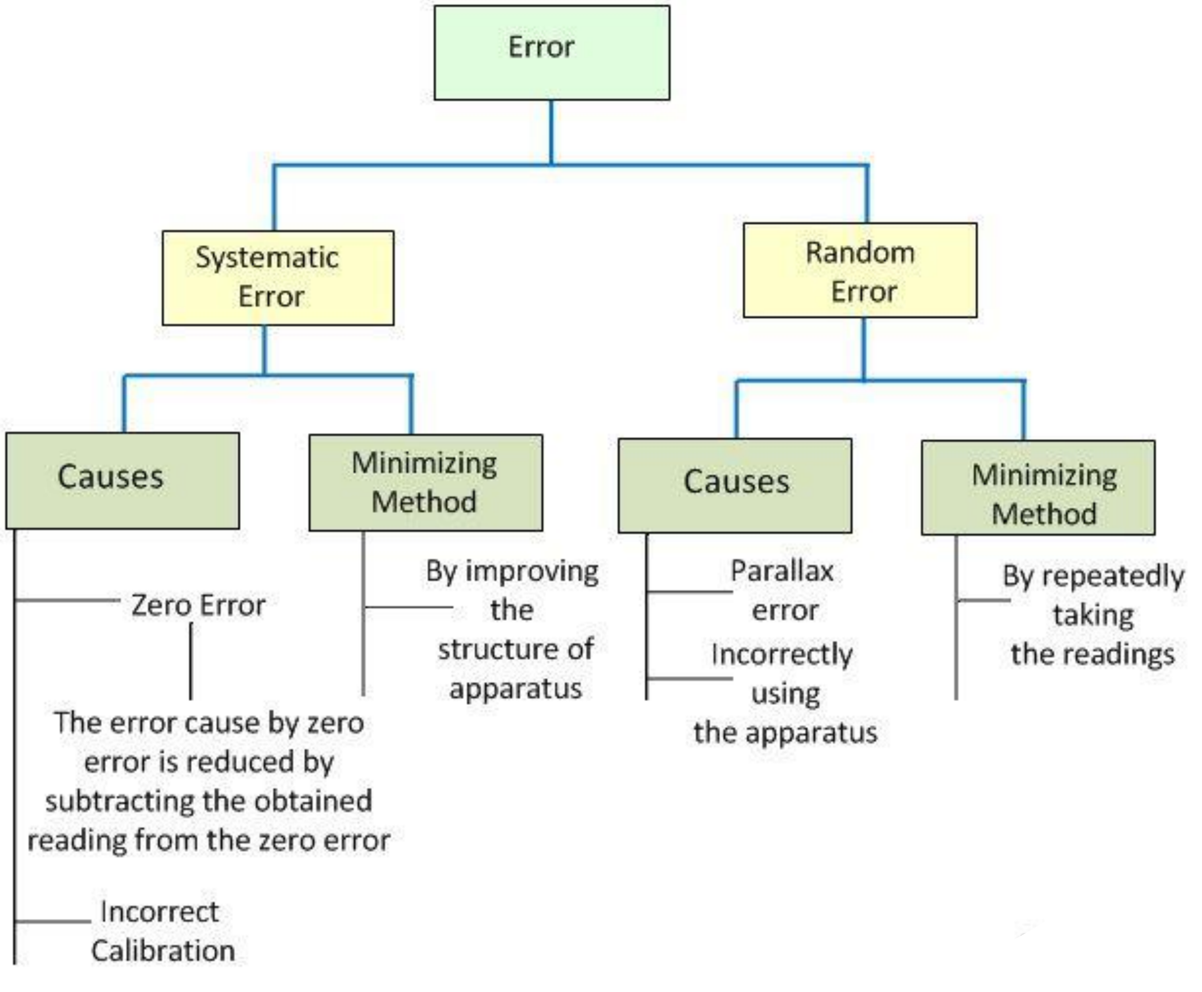
$$(\text{measured } x) = x_{best} \pm \delta x,$$

indicates the reliability or precision of the measurement.

However, the uncertainty δx by itself does not tell the whole story. An uncertainty of one inch in a distance of one mile would indicate an unusually precise measurement, whereas an uncertainty of one inch in a distance of three inches would indicate a rather crude estimate.
Obviously, the quality of a measurement is indicated not just by the uncertainty δx but also by the ratio of dx x_{best}, which leads us to consider the fractional uncertainty,

$$\text{fractional uncertainty} = \frac{\delta x}{|x_{best}|}$$

(The fractional uncertainty is also called the relative uncertainty.) In this definition, the symbol $|x_{best}|$ denotes the absolute value x_{best}.



Random Error and Systematic Error

All experimental uncertainty is due to either random errors or systematic errors.

Random errors are statistical fluctuations (in either direction) in the measured data due to the precision limitations of the measurement device.

Random errors usually result from the experimenter's inability to take the same measurement in exactly the same way to get exact the same number.

Or

Random errors in experimental measurements are caused by unknown and unpredictable changes in the experiment.

These changes may occur in the measuring instruments or in the environmental conditions.
Examples of causes of random errors are:

- electronic noise in the circuit of an electrical instrument,
- irregular changes in the heat loss rate from a solar collector due to changes in the wind.

Systematic errors, by contrast, are reproducible inaccuracies that are consistently in the same direction.

Or

Systematic errors in experimental observations usually come from the measuring instruments.

They may occur because:

- There is something wrong with the instrument or its data handling system, or
- The instrument is wrongly used by the experimenter.

Systematic errors are often due to a problem which persists throughout the entire experiment.

1.4.1 Systematic errors

The systematic errors are those errors that tend to be in one direction, either positive or negative. Some of the sources of systematic errors are:

(a) Instrumental errors:

It arises from the errors due to imperfect design or calibration of the measuring instrument, zero error in the instrument, etc. For example, the temperature graduations of a thermometer may be inadequately calibrated (it may read 104 °C at the boiling point of water at STP whereas it should read 100 °C); in a vernier callipers the zero mark of vernier scale may not coincide with the zero mark of the main scale, or simply an ordinary metre scale may be worn off at one end.

(b) Imperfection in experimental technique or procedure

To determine the temperature of a human body, a thermometer placed under the armpit will always give a temperature lower than the actual value of the body temperature. Other external conditions (such as changes in temperature, humidity, wind velocity, etc.) during the experiment may systematically affect the measurement.

(c) Personal errors

It arise due to an individual's bias, lack of proper setting of the apparatus or individual's carelessness in taking observations without observing proper precautions, etc. For example, if you, by habit, always hold your head a bit too far to the right while reading the position of a needle on the scale, you will introduce an error due to parallax.

Systematic errors can be minimised by improving experimental techniques, selecting better instruments and removing personal bias as far as possible. For a given set-up, these errors may be estimated to a certain extent and the necessary corrections may be applied to the readings.

Random errors

The random errors are those errors, which occur irregularly and hence are random with respect to sign and size. These can arise due to random and unpredictable fluctuations in experimental conditions (e.g. unpredictable fluctuations in temperature, voltage supply, mechanical vibrations of experimental set-ups, etc), personal (unbiased) errors by the observer taking readings, etc. For example, when the same person repeats the same observation, it is very likely that he may get different readings every time.

The main **differences** between these two error types are:

Random errors are (like the name suggests) completely random. They are unpredictable and can't be replicated by repeating the experiment again.

Systematic Errors produce consistent errors, either a fixed amount (like 1 lb) or a proportion (like 105% of the true value). If you repeat the experiment, you'll get the same error.

Preventing Errors

Random error can be reduced by:

- Using an average measurement from a set of measurements, or
- Increasing sample size.

It's difficult to detect — and therefore prevent — **systematic error.** In order to avoid these types of error, know the limitations of your equipment and understand how the experiment works. This can help you identify areas that may be prone to systematic errors.

Mean / Average

The mean is the average of all numbers and is sometimes called the arithmetic mean. To calculate mean, add together all of the numbers in a set and then divide the sum by the total count of numbers.

Suppose, we make *N* measurements of the quantity *x* (all using the same equipment and procedures) and find the *N* values

$$x_1, x_2, \dots \dots, x_N$$

Once again, the best estimate for x is usually the average of $x_1, x_2, \dots \dots, x_N$. That is,

$$x_{best} = \bar{x},$$

where

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_N}{N} = \frac{\sum x_i}{N}$$

In the last line, I have introduced the useful sigma notation, according to which

$$\sum_{i=1}^{N} x_i = \sum x_i = x_1 + x_2 + \cdots + x_N;$$

Standard deviation (SD)

The standard deviation of the measurements is an estimate of the average uncertainty of the individual measurements.

The standard deviation is given by

$$\sigma_x = \sqrt{\frac{1}{N-1}\sum d_i^2} = \sqrt{\frac{1}{N-1}\sum (x_i - \bar{x})^2}\,,$$

The mean $\bar{x}$ is our best estimate of the quantity x, it is natural to consider the difference $x_i - \bar{x} = d_i$.

This difference, often called the deviation (or residual) of x_i from x, tells us *how much the i^{th} measurement x_i differs from the average $\bar{x}$*. If the deviations $d_i = x_i - \bar{x}$ are all very small, our measurements are all close together and presumably very precise. If some of the deviations are large, our measurements are obviously not so precise.

Notice that the deviations are not (of course) all the same size; d_i is small if the i^{th} measurement x_i happens to be close to $\bar{x}$, but d_i is large if x_i is far from $\bar{x}$. Notice also that some of the d_i are positive and some negative because some of the x_i are bound to be higher than the average $\bar{x}$, and some are bound to be lower.

To estimate the average reliability of the measurements, we might naturally try averaging the deviations d_i. The definition of the average $\bar{x}$ ensures that $d_i = x_i - \bar{x}$ is sometimes positive and sometimes negative in just such a way that $\bar{d}$ may be zero.

The average of the deviations is not a useful way to characterize the reliability of the measurements.

The best way to avoid this annoyance is to *square* all the deviations, which will create a set of positive numbers, and then average these numbers.

Now, we obtain average of the d_i^2,

$$\sigma_x^2 = \frac{1}{N}\sum d_i^2$$

σ_x^2 is known as the variance of the measurements.

If we then take the *square root* of the result, we obtain a quantity with the same units as x itself. This number is called the standard deviation σ_x.

The Standard Deviation of the Mean (SDOM) / Standard Error

If $x_1, x_2, \dots\dots, x_N$ are the results of measurements of the same quantity then, as we have seen, our best estimate for the quantity x is their mean $\bar{x}$. We have also seen that the standard deviation σ_x characterizes the average uncertainty of the separate measurements $x_1, x_2, \dots\dots, x_N$.

The uncertainty in the final answer $x_{best} = \bar{x}$ is given by the standard deviation σ_x divided by $\sqrt{N}$. This quantity is called the *standard deviation* of the mean (SDOM), or standard error, and is denoted $\sigma_{\bar{x}}$:

$$\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}.$$

Thus, based on the N measured values $x_1, x_2, \dots\dots, x_N$, we can state our final answer for the value of x as

$$(value\ of\ x) = x_{best} \pm \delta x,$$

where $x_{best} = \bar{x}$ mean of $x_1, x_2, \dots\dots, x_N$, and δx is the standard deviation of the mean,

$$\delta x = \sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{N}}.$$

1.4.2 Absolute (average or mean) Error, Relative Error and Percentage Error

Suppose the values obtained in several measurements are $a_1, a_2, a_3, \dots, a_n$. The arithmetic mean of these values is taken as the best possible value of the quantity under the given conditions of measurement as:

$$= \qquad a_{mean} = \sum_{i=1}^{n} a_i / n \qquad (a_1+a_2+a_3+\dots+a_n)/n$$

The magnitude of the difference between the true value of the quantity and the individual measurement value is called the absolute error of the measurement. This is denoted by $|\Delta a|$. In absence of any other method of knowing true value, we considered arithmetic mean as the true value. Then the errors in the individual measurement values are

$$\Delta a_1 = a_{mean} - a_1,$$
$$\Delta a_2 = a_{mean} - a_2,$$
$$\dots \qquad \dots \qquad \dots$$
$$\dots \qquad \dots \qquad \dots$$
$$\Delta a_n = a_{mean} - a_n$$

The Δa calculated above may be positive in certain cases and negative in some other cases. But absolute error $|\Delta a|$ will always be positive.

The arithmetic mean of all the absolute errors is taken as the final or mean absolute error of the value of the physical quantity a. It is represented by Δa_{mean}.

Thus,

$$\Delta a_{mean} = (|\Delta a_1| + |\Delta a_2| + |\Delta a_3| + ... + |\Delta a_n|)/n$$

$$= \sum_{i=1}^{n} |\Delta a_i|/n$$

If we do a single measurement, the value we get may be in the range $a_{mean} \pm \Delta a_{mean}$

i.e. $a = a_{mean} \pm \Delta a_{mean}$

or,

$$a_{mean} - \Delta a_{mean} \leq a \leq a_{mean} + \Delta a_{mean}$$

This implies that any measurement of the physical quantity a is likely to lie between
$(a_{mean} + \Delta a_{mean})$ and $(a_{mean} - \Delta a_{mean})$.

Relative error and the percentage error (δa)

The relative error is the ratio of the mean absolute error Δa_{mean} to the mean value a_{mean} of the quantity measured.

$$\text{Relative error} = \Delta a_{mean}/a_{mean}$$

When the relative error is expressed in per cent, it is called the percentage error (δa).

Thus, Percentage error

$$\delta a = (\Delta a_{mean}/a_{mean}) \times 100\%$$

1.4.3 Propagation of Errors

Error of a sum or a difference

Suppose two physical quantities A and B have measured values $A \pm \Delta A$, $B \pm \Delta B$ respectively where ΔA and ΔB are their absolute errors. We wish to find the error ΔZ in the sum $Z = A + B$.

We have by addition, $Z \pm \Delta Z$

$$= (A \pm \Delta A) + (B \pm \Delta B).$$

The maximum possible error in Z

$$\Delta Z = \Delta A + \Delta B$$

For the difference $Z = A - B$,

We have

$$Z \pm \Delta Z = (A \pm \Delta A) - (B \pm \Delta B)$$
$$= (A - B) \pm \Delta A \pm \Delta B$$

or,

$$\pm \Delta Z = \pm \Delta A \pm \Delta B$$

The maximum value of the error ΔZ is again $\Delta A + \Delta B$.

Hence, when two quantities are added or subtracted, the absolute error in the final result is the sum of the absolute errors in the individual quantities.

Error of a product or a quotient

Suppose $Z = AB$ and the measured values of A and B are $A \pm \Delta A$ and $B \pm \Delta B$. Then

$$Z \pm \Delta Z = (A \pm \Delta A)(B \pm \Delta B)$$
$$= AB \pm B\,\Delta A \pm A\,\Delta B \pm \Delta A\,\Delta B.$$

Dividing LHS by Z and RHS by AB we have,

$$1 \pm (\Delta Z/Z) = 1 \pm (\Delta A/A) \pm (\Delta B/B) \pm (\Delta A/A)(\Delta B/B).$$

Since ΔA and ΔB are small, we shall ignore their product.

Hence the maximum relative error

$$\Delta Z/Z = (\Delta A/A) + (\Delta B/B).$$

Same result will follow for division also.

Hence, when two quantities are multiplied or divided, the relative error in the result is the sum of the relative errors in the multipliers.

Or

PRODUCTS AND QUOTIENTS

Section 2.9 discussed the uncertainty in the product $q = xy$ of two measured quantities. We saw that, provided the fractional uncertainties concerned are small, the fractional uncertainty in $q = xy$ is the sum of the fractional uncertainties in x and y. Rather than review the derivation of this result, I discuss here the similar case of the quotient $q = x/y$. As you will see, the uncertainty in a quotient is given by the same rule as for a product; that is, the fractional uncertainty in $q = x/y$ is equal to the sum of the fractional uncertainties in x and y.

Because uncertainties in products and quotients are best expressed in terms of fractional uncertainties, a shorthand notation for the latter will be helpful. Recall that if we measure some quantity x as

$$\text{(measured value of } x) = x_{\text{best}} \pm \delta x$$

in the usual way, then the fractional uncertainty in x is defined to be

$$\text{(fractional uncertainty in } x) = \frac{\delta x}{|x_{\text{best}}|}.$$

(The absolute value in the denominator ensures that the fractional uncertainty is always positive, even when x_{best} is negative.) Because the symbol $\delta x/|x_{\text{best}}|$ is clumsy to write and read, from now on I will abbreviate it by omitting the subscript "best" and writing

$$\text{(fractional uncertainty in } x) = \frac{\delta x}{|x|}.$$

The result of measuring any quantity x can be expressed in terms of its fractional error $\delta x/|x|$ as

$$\text{(value of } x) = x_{\text{best}}(1 \pm \delta x/|x|).$$

Therefore, the value of $q = x/y$ can be written as

$$\text{(value of } q) = \frac{x_{\text{best}}}{y_{\text{best}}} \frac{1 \pm \delta x/|x|}{1 \pm \delta y/|y|}.$$

Our problem now is to find the extreme probable values of the second factor on the right. This factor is largest, for example, if the numerator has its largest value, $1 + \delta x/|x|$, and the denominator has its *smallest* value, $1 - \delta y/|y|$. Thus, the largest probable value for $q = x/y$ is

$$\text{(largest value of } q) = \frac{x_{\text{best}}}{y_{\text{best}}} \frac{1 + \delta x/|x|}{1 - \delta y/|y|}. \tag{3.5}$$

The last factor in expression (3.5) has the form $(1 + a)/(1 - b)$, where the numbers a and b are normally small (that is, much less than 1). It can be simplified by two approximations. First, because b is small, the binomial theorem[2] implies that

$$\frac{1}{(1 - b)} \approx 1 + b. \tag{3.6}$$

Therefore,

$$\begin{aligned} \frac{1 + a}{1 - b} \approx (1 + a)(1 + b) &= 1 + a + b + ab \\ &\approx 1 + a + b, \end{aligned}$$

where, in the second line, we have neglected the product ab of two small quantities. Returning to (3.5) and using these approximations, we find for the largest probable value of $q = x/y$

$$(\text{largest value of } q) = \frac{x_{\text{best}}}{y_{\text{best}}}\left(1 + \frac{\delta x}{|x|} + \frac{\delta y}{|y|}\right).$$

A similar calculation shows that the smallest probable value is given by a similar expression with two minus signs. Combining these two, we find that

$$(\text{value of } q) = \frac{x_{\text{best}}}{y_{\text{best}}}\left(1 \pm \left[\frac{\delta x}{|x|} + \frac{\delta y}{|y|}\right]\right).$$

Comparing this equation with the standard form,

$$(\text{value of } q) = q_{\text{best}}\left(1 \pm \frac{\delta q}{|q|}\right),$$

we see that the best value for q is $q_{\text{best}} = x_{\text{best}}/y_{\text{best}}$, as we would expect, and that the fractional uncertainty is

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \frac{\delta y}{|y|}. \tag{3.7}$$

We conclude that when we divide or multiply two measured quantities x and y, the fractional uncertainty in the answer is the sum of the fractional uncertainties in x and y, as in (3.7). If we now multiply or divide a series of numbers, repeated application of this result leads to the following provisional rule.

Uncertainty in Sums and Differences
(Provisional Rule)

If several quantities $x, \ldots, w$ are measured with uncertainties $\delta x, \ldots, \delta w$, and the measured values used to compute

$$q = x + \cdots + z - (u + \cdots + w),$$

then the uncertainty in the computed value of q is the sum,

$$\delta q \approx \delta x + \cdots + \delta z + \delta u + \cdots + \delta w,$$

of all the original uncertainties.

Uncertainty in Products and Quotients
(Provisional Rule)

If several quantities $x, \ldots, w$ are measured with small uncertainties $\delta x, \ldots, \delta w$, and the measured values are used to compute

$$q = \frac{x \times \cdots \times z}{u \times \cdots \times w},$$

then the fractional uncertainty in the computed value of q is the sum,

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \cdots + \frac{\delta z}{|z|} + \frac{\delta u}{|u|} + \cdots + \frac{\delta w}{|w|},$$

of the fractional uncertainties in $x, \ldots, w$.

(3.8)

Two Important Special Cases

MEASURED QUANTITY TIMES EXACT NUMBER

Suppose we measure a quantity x and then use the measured value to calculate the product $q = Bx$, where the number B has *no uncertainty*. For example, we might measure the diameter of a circle and then calculate its circumference, $c = \pi \times d$; or we might measure the thickness T of 200 identical sheets of paper and then calculate the thickness of a single sheet as $t = (1/200) \times T$. According to the rule (3.8), the fractional uncertainty in $q = Bx$ is the sum of the fractional uncertainties in B and x. Because $\delta B = 0$, this implies that

$$\frac{\delta q}{|q|} = \frac{\delta x}{|x|}.$$

That is, the fractional uncertainty in $q = Bx$ (with B known exactly) is the same as that in x. We can express this result differently if we multiply through by $|q| = |Bx|$ to give $\delta q = |B|\delta x$, and we have the following useful rule:[3]

Measured Quantity Times Exact Number

If the quantity x is measured with uncertainty δx and is used to compute the product

$$q = Bx,$$

where B has no uncertainty, then the uncertainty in q is just $|B|$ times that in x,

$$\delta q = |B|\delta x.$$

POWERS

The second special case of the rule (3.8) concerns the evaluation of a power of some measured quantity. For example, we might measure the speed v of some object and then, to find its kinetic energy $\frac{1}{2}mv^2$, calculate the square v^2. Because v^2 is just $v \times v$, it follows from (3.8) that the fractional uncertainty in v^2 is *twice* the fractional uncertainty in v. More generally, from (3.8) the general rule for any power is clearly as follows.

Uncertainty in a Power

If the quantity x is measured with uncertainty δx and the measured value is used to compute the power

$$q = x^n,$$

then the fractional uncertainty in q is n times that in x,

$$\frac{\delta q}{|q|} = n\frac{\delta x}{|x|}.$$