6. $\lim_{(x,y)\to(1,1)} \frac{xy - y - 2x + 2}{x-1}$

$\frac{y(x-1) - 2(x-1)}{x-1}$

$\frac{(y-2)(\cancel{x-1})}{\cancel{x-1}} = -1$

7. $\lim_{(x,y)\to(2,-3)} \left(\frac{1}{x} + \frac{1}{y}\right)^3$

$= (+1/6)^3 = 1/216$

→ Partial derivative

$z = f(x, y)$

$\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$

$z = x^2 y^3$

$\frac{\partial z}{\partial x} = 2xy^3$ $\qquad$ $\frac{\partial z}{\partial y} = 3x^2y^2$

Q. Find $\partial f/\partial y$ & $\partial f/\partial x$ for the following

1. $f(x, y) = y\sin(xy)$

$\frac{\partial f}{\partial y} = \sin(xy) + xy\cos(xy)$

$\frac{\partial f}{\partial x} = y^2 \cancel{\cos} \cos(xy)$

2. $f(x, y) = \frac{2y}{y + \cos x}$

$\frac{\partial f}{\partial x} = \frac{2y(0 \cancel{+} - \sin x) - (y + \cos x)(0)}{(y + \cos x)^2}$

$\frac{-2y\sin x}{(y + \cos x)^2}$

$\frac{\partial f}{\partial y} = \frac{-2y(1 - \cancel{\sin x}) + (y + \cos x)(2)}{(y + \cos x)^2}$

$= \frac{2y - 2y + 2\cos x}{(y + \cos x)^2}$

$= \frac{+2\cos x}{(y + \cos x)^2}$

3. Find value of $\partial f/\partial x$ & $\partial f/\partial y$ at point $(4, -5)$ if

$f(x, y) = x^2 + 3xy + y - 1$

$\frac{\partial f}{\partial x} = 2x + 3y = 8 - 15 = -7$

$\frac{\partial f}{\partial y} = 0 + 3x + 1 = 12 + 1 = 13$

4. If x, y, z are independent variables & $f(x, y, z) = x\sin(y + 3z)$ find $\partial f/\partial x$, $\partial f/\partial y$ & $\partial f/\partial z$

$\frac{\partial f}{\partial x} = \cancel{\sin}(y + 3z) + x(0) = \sin(y + 3z)$

$\frac{\partial f}{\partial y} = x\cos(y + 3z)[1]$

$\frac{\partial f}{\partial z} = x\cos(y + 3z)[3]$

A function, $f(x,y)$ can have partial derivatives with respect to both x & y at a point without the function being continuos at that point. this is different from functions of single variable, where the existance of derivates implies continuity.

Second Order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = f_{xy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = f_{yx}$$

Q. Find $\partial^2 f/\partial x^2$, $\partial^2 F/\partial y^2$, $\partial^2 f/\partial y \partial x$, $\partial^2 f/\partial x \partial y$ if

1. $f(x,y) = x\cos y + ye^x$

$$\frac{\partial f}{\partial x} = \cos y + ye^x \qquad \frac{\partial f}{\partial y} = -x\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = ye^x \qquad \frac{\partial^2 f}{\partial y^2} = -x\cos y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -\sin y + e^x \qquad \frac{\partial^2 f}{\partial x \partial y} = -\sin y + e^x$$

Q2. Find $\frac{\partial^2 w}{\partial x \partial y}$ & $\frac{\partial^2 w}{\partial y \partial x}$ $\qquad w = xy + \frac{e^y}{y^2+1}$

$$\frac{\partial w}{\partial y} = x + \frac{(y^2+1)e^y - e^y(2y)}{(y^2+1)^2} \qquad \frac{\partial w}{\partial x} = y$$

$$\frac{\partial^2 w}{\partial y \partial x} = 1 + 0 = 1 \qquad \frac{\partial^2 w}{\partial x \partial y} = 1$$

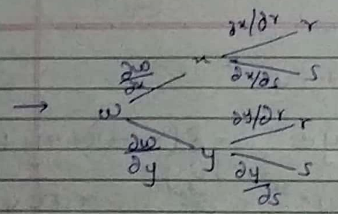
Chain Rule for ft^n of 2 independent var.

If $w = f(x,y)$ has continuous partial derivates f_x, f_y if $x = x(t)$, $y = y(t)$ are differentiable ftns of t then

$$\frac{\partial w}{\partial t} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

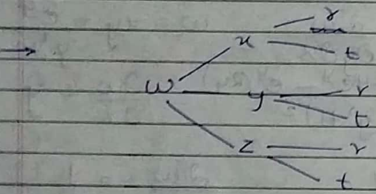
$$\begin{tikzcd} & x \arrow[r, "\frac{dx}{dt}"] & t \\ w \arrow[ur, "\frac{\partial w}{\partial x}"] \arrow[dr, "\frac{\partial w}{\partial y}"'] & & \\ & y \arrow[r, "\frac{dy}{dt}"'] & t \end{tikzcd}$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial t}\right).$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial t}$$

Q. Use chain rule to find derivate of $w = xy$ wrt t along the path $x = \cos t$, $y = \sin t$. Find derivative at $t = \pi/2$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\left(\frac{dx}{dt}\right) + \frac{\partial w}{\partial y}\left(\frac{dy}{dt}\right)$$
$$= y(-\sin t) + x(\cos t)$$
$$= -y\sin t + x\cos t$$
$$= -y + 0 = -y = -1$$

Q. Express $\partial w/\partial r$ & $\partial w/\partial s$ in terms of r & s if $w = x + 2y + z^2$, $x = \frac{r}{s}$, $y = r^2 + \ln s$, $z = 2r$

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial r}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial r}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial r}\right)$$

$\frac{\cancel{r} - s(0)}{s^2}$

$$= 1\left(\frac{1}{s}\right) + 2\left(2r\right) + 2z(2)$$
$$= \frac{1}{s} + 4r + 4z$$
$$= 1/s + 4r + 4r = 1/s + 12r.$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\left(\frac{\partial x}{\partial s}\right) + \frac{\partial w}{\partial y}\left(\frac{\partial y}{\partial s}\right) + \frac{\partial w}{\partial z}\left(\frac{\partial z}{\partial s}\right)$$
$$= 1\left(\frac{+r}{s^2}\right) + 2\left(\frac{1}{s}\right) + 2(0)$$

$\frac{r}{s}$

$$= \frac{+r}{s^2} + \frac{2}{s^2} = \frac{2s - r}{s^2}$$

Application of partial derivative

Scalar point f^{t^n}

A scalar pt. f^{t^n} that assigns a real no (i.e. scalar) to each point of some region of space. If to each point (x, y, z) of a region R in space, there is a sign a real no $u = \phi(x, y, z)$ then ϕ is called scalar pt. f^{t^n}.

Ex:
- The temp distribution within some body at particular point in time
- The density distribution within some fluid at a particular point in time.

Directional derivative

Let $f(x, y, z)$ be a scalar point f^{t^n} defined over some region R of space.
A f^{t^n} $f(x, y, z)$ could for example represent the temp distribution within some body at specified point $P(x, y, z)$ of R we wish to know the rate of change of f in particular direction, then the rate of change of f at point p in specific direction is called directional derivative of f at P.

A derivative of f at point $P_0(x_0, y_0)$ in the direction of unit vector $u = u_1\hat{i} + u_2\hat{j}$ is given by

$$\left.\frac{df}{ds}\right|_{u, P_0} = \lim_{s \to 0} \frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}$$

Gradient vector.

Gradient vector of $f(x, y)$ is given by

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$$

Th^n

If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$ then

$$\left.\frac{df}{ds}\right|_{u, P_0} = \nabla f\Big|_{P_0} \cdot u$$

u → unit normal.

Q. Find derivative of $f(x, y) = e^y x + \cos(xy)$ at $(2, 0)$ in direction $V = 3\hat{i} - 4\hat{j}$

→ Derivative in given direction is

$$\frac{df}{ds} = \nabla f \cdot u$$

$$V = 3\hat{i} - 4\hat{j}$$

$$u = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

$$f = xe^y + \cos xy$$

$$\nabla f = \left[e^y - y\sin(xy)\right]\hat{i} + \left[xe^y - x\sin(xy)\right]\hat{j}$$

$D_u f = \frac{df}{ds} = [(e^y - \sin(xy)y)\hat{i} + [xe^y - x\sin(xy)]\hat{j}]$.
$[3/5\,\hat{i} - 4/5\,\hat{j}]$
$= (\hat{i} + 2\hat{j})(3/5\,\hat{i} - 4/5\,\hat{j})$
$= 3/5 - 8/5 = -1$.

Evaluating dot product in formula

$D_u f = |\nabla f| . |u| . \cos\theta$
$= |\nabla f| . \cos\theta$

Properties of directional derivatives

1. The function f increases most rapidly when $\cos\theta = 1$ i.e. f increases most rapidly in the direction of gradient vector ∇f at P.

2. The function decreases most rapidly in direction of $-\nabla f$ i.e. $\cos\theta = -1$, i.e. ~~θ=π~~

3. Any direction u orthogonal to the gradient is direction of 0 change in f.

Q. Find the directions in which $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$

1) Increases most rapidly at pt. (1, 1)
2) Decreases most rapidly at pt. (1, 1)
3) What are the directions of 0 change in f at (1, 1)

→ 1) The ftn increases most rapidly in the direction of ∇f at (1, 1)

$\nabla f = x\hat{i} + y\hat{i}$
$u = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}$

2) Ftn decreases most rapidly in direction of
$-\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}$

3) The direction of 0 change at (1, 1) are

$v_1 = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j}$

$v_2 = \frac{-1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}$

Q. Find derivative of $f(x,y,z) = x^3 - xy^2 - z$ at $(1,1,0)$ in direction $v = 2\hat{i} - 3\hat{j} + 6\hat{k}$.

$$v = 2\hat{i} - 3\hat{j} + 6\hat{k}$$

$$u = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{4+9+36}}$$

$$= \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{\sqrt{49}}$$

$$= \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

$$\nabla f = [3x^2 - y^2]\hat{i} + [-2xy]\hat{j} + [-1]\hat{k}$$

$$= 2\hat{i} + (-2)\hat{j} - \hat{k}$$

$$\nabla f \cdot u = (2\hat{i} - 2\hat{j} - \hat{k}) \cdot \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right)$$

$$= 2\left(\frac{2}{7}\right) + 2\left(\frac{3}{7}\right) - \frac{6}{7}$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7}$$

$$= \frac{4}{7}$$

Tangent plane & Normal line.

A tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x,y,z) = c$ of a differentiable ft^n f is the plane through P_0, normal to ∇f at P_0.

A normal line of the surface at P_0 is the line through P_0 & parallel to $\nabla f|_{P_0}$

$\Rightarrow$ Tangent plane to $f(x,y,z) = c$ at point $P_0(x_0, y_0, z_0)$ is given by

$$0 = \frac{\partial f}{\partial x}\bigg|_{P_0}(x - x_0) + \frac{\partial f}{\partial y}\bigg|_{P_0}(y - y_0) + \frac{\partial f}{\partial z}\bigg|_{P_0}(z - z_0)$$

$\Rightarrow$ Normal line to $f(x,y,z) = c$ at point $P_0(x_0, y_0, z_0)$ is

$$x = x_0 + \frac{\partial f}{\partial x}\bigg|_{P_0} t$$

$$y = y_0 + \frac{\partial f}{\partial y}\bigg|_{P_0} t$$

$$z = z_0 + \frac{\partial f}{\partial z}\bigg|_{P_0} t$$

Q. Find the tangent plane & normal line of the surface $f(x,y,z) = x^2 + y^2 + z - 7$ at point $P(1,2,4)$

Ans. ~~Tangent plane~~ $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 1$

$$2(x-1) + 4(y-2) + 1(z-4)$$

$$2x + 4y + z - 14 = 0.$$

Nomal $x = 1+2(2x+4y+z-14)$

$= 4x+8y+2z-27$

$y = 2+4(2x+4y-z-14)$

$= 8x+16y-4z-54$

$z = 4+2x+4y-z-14$

$z = 2x+4y-2z-10$

$x = 1+2t$

$y = 2+4t$

$z = 4+t.$

Total differentiation

If we move from (x_0, y_0) to a point (x_0+dx, y_0+dy) the resultant change $\partial f = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} dx + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} dy$ is called total differential of f.

Ex: Suppose that a cylindrical can is designed to have a radius of 1 inch & a height of 5 in. but the radius & height are off by the amounts $dr = 0.03$ & $dh = -0.1$. Estimate the resulting absolute change in the volume of can.

Ans. vol of cylinder $= \pi r^2 h$

To estimate the absolute change in volume, we use

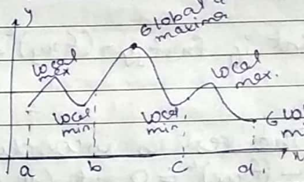
$$dV = \left.\frac{\partial V}{\partial r}\right| \partial r + \frac{\partial V}{\partial h} \cdot \partial h$$

$$\partial V = 2\pi r h(\partial r) + (\pi r^2)(dh)$$
$$= 2\pi(1)(5)(0.03) + \pi(1)(-0.1)$$
$$= 0.3\pi - 0.1\pi$$
$$= 0.2\pi \text{ inch}^3$$

Q. Find total differential of $w = x^3yz+xy+z+3$ at point $(1,2,3)$

$$dw = \frac{\partial w}{\partial x} \cdot dx + \frac{\partial w}{\partial y} \cdot dy + \frac{\partial w}{\partial z}(dz)$$
$$= (3x^2yz+y)dx + (x^3z+x)dy + (x^3y+1)dz$$
$$= (18+2)dx + 4dy + 3dz$$
$$= 20dx + 4dy + 3dz$$

Extreme values. (local, global)



Continuous functions of 2 variables assume extreme values on closed & bounded domain.

Let $f(x,y)$ be defined on a region R containing the point (a,b) then

1) $f(a,b)$ is a local maximum value of f if $f(a,b) \geq f(x,y)$ for all points (x,y)
2) $f(a,b)$ is a local minimum value of f if $f(a,b) \leq f(x,y)$ for all pts. (x,y)

3

→ Second derivative test
(for local extreme values)

Suppose that $f(x,y)$ & its first & second partial derivatives are continuous throughout a disc centered at (a,b) &
$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$ at (a,b)
then f has
1) Local maximum if $f_{xx} < 0$ & $f_{xx}f_{yy} - (f_{xy})^2 > 0$ at (a,b).

2) Local minimum if $f_{xx} > 0$ & $f_{xx}f_{yy} - (f_{xy})^2 > 0$ at (a,b)

3) Saddle point if $f_{xx}f_{yy} - (f_{xy})^2 < 0$ at (a,b).

4) ~~Rest is in~~ The test is inconclusive if $f_{xx} \cdot f_{yy} - (f_{xy})^2 = 0$ at (a,b)

$r = \frac{\partial^2 f}{\partial x^2} = f_{xx}$

$t = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

$s = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$

Q. Find local extreme values of the fn
$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$

$\frac{\partial f}{\partial x} = y - 2x - 2 = 0$ $\quad y - 2x = 2$ —①

$\frac{\partial f}{\partial y} = x - 2y - 2 = 0$ $\quad x - 2y = 2$ —②

$r = \frac{\partial^2 f}{\partial x^2} = -2$

$t = \frac{\partial^2 f}{\partial y^2} = -2$

$s = \frac{\partial f}{\partial y \partial x} = \frac{\partial}{\partial y}(y - 2x - 2)$
$= 1$

From ① & ②
$x = -2$
$y = -2$
$(x,y) = (-2,-2)$
∴ extreme pt. is $(-2,-2)$

For minima or maxima
$rt - s^2$
$(-2)(-2) - (1)^2$
$4 - 1 = 3 > 0$ & $r < 0$

∴ at pt. $(-2,-2)$ we get local ~~minima~~ maximum.

Q. Examin $f(x,y) = x^3 + y^3 - 3xy$ for maximum & minimum values.

$\frac{\partial f}{\partial x} = 0$
$3x^2 - 3y = 0$
$x^2 = y$

$\frac{\partial f}{\partial y} = 0$
$3y^2 - 3x = 0$
$y^2 = x$

$(x = 1, y = 1)$, $(x = 0, y = 0)$ →

$\frac{\partial^2 f}{\partial x^2} = 6x = 6$

$\frac{\partial^2 f}{\partial y^2} = 6y = 6$

$\frac{\partial f}{\partial y \partial x} = -3$

solving eq^n

$x = y^2 \quad x^2 = y$

$x = (x^2)^2$

$x = x^4$

$(x^3-1)x = 0$

$x.(x-1)(x^2+x+1) = 0$

$x = 0, \quad x = 1$

$\therefore$ pts are $(0,0)$ & $(1,1)$.

for $(0,0)$

$r = 0, \quad t = 0$

$s = -3$

$rt - s^2 < 0$

$0 - 9 < 0$

$\therefore$ saddle pt

for $(1,1)$.

$r = 6, \quad t = 6$

$s = -3$

$36 - 9$

$27 > 0$.

$r > 0$

$\therefore$ minima

Q $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$

$\frac{\partial f}{\partial x} = 0$

$3x^2 - 63 + 12y = 0$

$x^2 - 21 + 4y = 0 \quad -(1)$

$\left(\frac{21-y^2}{4}\right)^2 - 21 + 4y = 0$

$(21)^2 + y^4 - 42y^2 - 16(21) + 64y = 0$

$y^4 - 42y^2 + 64y + 105 = 0$

$\frac{\partial f}{\partial y} = 0$

$3y^2 - 63 + 12x = 0$.

$y^2 - 21 + 4x = 0 \quad -(2)$

$x = \frac{21-y^2}{4}$

$(1) - (2)$

$x^2 - y^2 + 4(y-x) = 0$.

$(x+y)(x-y) - 4(x-y)$

$(x-y)(x+y-4) = 0$.

$x - y = 0$ & $x + y - 4 = 0$

$x = y \qquad x + y = 4 \quad -(1)$

Replacing in (1)

$x^2 + 4x - 21 = 0$

~~$x = +7 \quad x = -3$~~

$x = -7, \ +3$

$y = -7, \ +3$.

$x^2 + 3x + 7x - 21$

$x(x-3) + 7(x-3)$

$(x+7)(x-3)$

$x = -7, +3$

$\frac{49-28-21}{9+12-21}$

Replacing $x = 4 - y$ in (1),

$x^2 - 21 + 4y = 0$.

$(4-y)^2 - 21 + 4y = 0$

$16 + y^2 - 8y - 21 + 4y = 0$

$y^2 - 4y - 5 = 0$

$y^2 - 5y + y - 5 = 0$

$y(y-5) + 1(y-5) = 0$

$(y+1)(y-5) = 0$

$y = -1, \quad y = +5$

$x = 5 \quad x = -1$

Points $(-7,-7)$, $(+3,+3)$, $(-1,5)$, $(5,-1)$.

$r = \frac{\partial^2 f}{\partial x^2} = 2x \qquad t = \frac{\partial^2 f}{\partial y^2} = 2y$

$s = \frac{\partial^2 f}{\partial y \partial x} = 4$.

at $(-7,-7)$
$r=-14$, $t=-14$
$s=4$
$rt-s^2>0$
$r<0$
$\therefore$ maxima

at $(+3,+3)$
$r=6$ $t=6$
$s=4$
$rt-s^2>0$
$r>0$
$\therefore$ minima

at $(5,-1)$
$r=10$, $t=-2$
$s=4$
$rt-s^2<0$
saddle pt.

at $(-1,5)$
$r=-2$, $t=10$
$s=4$
$rt-s^2<0$
saddle pt.

Lagrange Method of Undetermined multipliers

Let $f(x,y,z)$ be a function of 3 variables ~~&~~ x, y, z & the variables connected by the relation ~~$\phi(x,y,z)$~~

$\phi(x,y,z)=0$ —— (1)

Find $\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0$ —— (2)

$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0$ —— (3)

$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$ —— (4)

On solving equation (1), (2), (3), & (4), we can find values of x, y, z & λ, for which $f(x,y,z)$ has stationary values.

Drawback in lagrange's method is that nature of stationary point can't be determined

Q. Find the point upon the plane $ax+by+cz=p$ at which the f^n $f=x^2+y^2+z^2$ has a minimum value & find this minimum f.

→ we have $f=x^2+y^2+z^2$ —— (1)

& $ax+by+cz=p$ — (*)

$ax+by+cz-p=0$

$\phi=ax+by+cz-p$ —— (2)

Using lagrange's method,

$\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}=0$

$2x+\lambda a=0$ —— (3)

$\frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}=0$

$2y+\lambda b=0$ —— (4)

$\frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}=0$

$2z+\lambda c=0$ —— (5)

$x=-\lambda a/2$, $y=-\lambda b/2$, $z=-\lambda c/2$

Replacing in (*) $a\left(\frac{-\lambda a}{2}\right)+b\left(\frac{-\lambda b}{2}\right)+c\left(\frac{-\lambda c}{2}\right)=p$

$\frac{\lambda}{2}(-a^2-b^2-c^2)=p$

$\lambda=\frac{-2p}{(a^2+b^2+c^2)}$

min val of f

$$f = x^2 + y^2 + z^2$$

$$= \left[\frac{\cancel{2}pa}{\cancel{2}(a^2+b^2+c^2)}\right]^2 + \left[\frac{\cancel{2}pb}{\cancel{2}(a^2+b^2+c^2)}\right]^2 + \left[\frac{pc}{(a^2+b^2+c^2)}\right]^2$$

$$= \frac{p^2(a^2+b^2+c^2)}{(a^2+b^2+c^2)^2}$$

$$= \frac{p^2}{a^2+b^2+c^2}.$$

Q. A rectangular box which is open at the top has a capacity of 256 cubic feet. Determine the dimensions of the box such that the least material is required for the construction of the box.

Ans Let x, y, z be the length, ~~breadth~~ width & height of the box.
Given capacity is 256 i.e.

$vol^m = 256$

$xyz = 256$ —— (1)

$\therefore \phi(x,y,z) = xyz - 256$ —— (2)

Let s be the material surface of the box $\quad s = \cancel{2}xy + 2yz + 2zx$ —— (3)

Using lagrange's method, we have

$$\frac{\partial s}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$$

$y + 2z + (\lambda)(yz) = 0$ —— (4)

$$\frac{\partial s}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$$

$2z + x + \lambda(xz) = 0$ —— (5)

$$\frac{\partial s}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$$

$2x + 2y + \lambda(xy) = 0$ —— (6)

multiply (4) by $x \rightarrow xy + 2zx + \lambda xyz = 0 \rightarrow (7)$

multiply (5) by $y \rightarrow 2zy + xy + \lambda xyz = 0 \rightarrow (8)$

multiply (6) by $z \rightarrow 2xz + 2yz + \lambda xyz = 0 \rightarrow (9)$

subtract (7) − (8)

$2z(x-y) = 0$

$z \neq 0, \quad x = y$

subtract (8) − (9)

$x(y-2z) = 0$

$x \neq 0, \quad y = 2z$

$x = 2z$

Replacing in (1)

$(2z)(2z)(z) = 256$

$\cancel{4}z^3 = \cancel{256}\ 64$

$z = 4. \quad \therefore x = 8, \quad y = 8$

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Q1 Divide 24 into 3 parts such that continued product of first, sqr of second & cube of third is max.

$$\phi(x,y,z) = x+y+z-24$$

$$\& \quad S = x(y^2)(z^3)$$

$$y^2z^3 + \lambda(1) = 0$$

$$2xyz^3 + \lambda(1) = 0$$

$$3xy^2z^2 + \lambda(1) = 0$$

$$y^2 z^3 = 2xyz^3 \qquad 2xyz^3 = 3xy^2z^2$$

$$2z = 3y$$

$$x + y + z = 24$$

$$y/2 + y + 3/2\, y = 24$$

$$6y/2 = 24$$

$$y = 8, \quad x = 4, \quad z = 12$$

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Q2 Show that the rectangle solid of max. vol that can be inscribed in a sphere is a cube.

$$\phi(x,y,z) = x^2+y^2+z^2-r^2$$

$$S = 8xyz$$