## Matrices.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n}$$

no of rows — $m$, no. of columns — $n$

square matrix — $[A]_{m \times m}$

Diagonal matrix = $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ all are zero except main diagonal is non-zero

scalar matrix . $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ ~~only~~ main diagonal elements are non-zero & equal

Unit matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ main diagonal is non-zero & 1

Upper $\Delta$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{pmatrix}$$

Lower $\Delta$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 4 & 7 \end{pmatrix}$$

$A = A^T \longrightarrow$ Symmetric matrix

$A = -A^T \longrightarrow$ Skew symmetric matrix

$A \cdot A^T = I \longrightarrow$ Orthogonal matrix.

$A^2 = A \longrightarrow$ Idempotent

$A^2 = I \longrightarrow$ Involutary.

multiplication -

$[A]_{m \times n}$ $[B]_{n \times p}$
same ✓

Ans $[AB]_{m \times p}$

- if $AB = 0$, not necessary that $a = 0$ or $b = 0$

eg $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$.

$$A^{-1} = \frac{1}{\det(A)} (\operatorname{adj} A)$$

Q $A^{-1}$ if $A = \begin{pmatrix} 3 & 2 & -1 \\ 1 & 6 & 3 \\ 2 & -4 & 0 \end{pmatrix}$.

$\det(A) = 3(12) - 2(-6) - 1(-16)$
$= 36 + 12 + 16 = 64$

~~cofactor~~ $= \left[ \cancel{+3 \; -+2} \right]$

cofactor $= \begin{bmatrix} +12 & +6 & -16 \\ 4 & 2 & 16 \\ 12 & -10 & 16 \end{bmatrix}$

$= \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$

$A^{-1} = \frac{1}{64} \begin{bmatrix} 12 & 4 & 12 \\ 6 & 2 & -10 \\ -16 & 16 & 16 \end{bmatrix}$

Q. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ then find

$A^2 - 4A - 5I$

$A^2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$

$= \cancel{1+2+2} \; \cancel{2+4+4}$

$\cancel{\begin{pmatrix} 5 & 10 & 10 \\ 10 & 5 & 10 \\ 10 & 10 & 5 \end{pmatrix}} =$

$\begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$

Q. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ then show $A^2 - 4A - 5I = 0$ & also find $A^{-1}$

{multiply eq$^n$ with $A^{-1}$}

$A - 4I - 5A^{-1} = 0$

$A^{-1} = \frac{1}{5}(A - 4I)$

Q. Prove $A = \begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix}$ Idempotent.

$A^2 = A$

$\begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix}$

$5+15-25$

$-3+9+15$

$-5-15+25$

$-20+25$

$= \begin{pmatrix} -1 & +1 & -1 \\ 3 & -3 & 3 \\ 5 & -5 & 5 \end{pmatrix} = A.$

Q. If $A = \begin{pmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 0 & -4 \end{pmatrix}$ &

$B = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{pmatrix}$. Find AB & BA.

$$AB = \begin{pmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 0 & -4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} -2+6-3 & -6+6+0 & 2-3+1 \\ -1+4-3 & -3+4+0 & 1-2+1 \\ -6-12 & -18 & 6+4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -18 & -18 & 10 \end{pmatrix}$$

$$BA = \begin{pmatrix} 1 & 3 & -1 \\ 2 & 2 & -1 \\ 3 & 0 & -1 \end{pmatrix} \begin{pmatrix} -2 & 3 & -1 \\ -1 & 2 & -1 \\ -6 & 0 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -2-3+6 & 3+6 & -1-3+4 \\ -4-2+6 & 6+4 & -2-2+4 \\ -6+6 & 9 & -3+4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 9 & 0 \\ 0 & 10 & 0 \\ 0 & 9 & 1 \end{pmatrix}$$

continued..

## Gauss ~~Th^m~~ Divergence th^m

If $\vec{F} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$ is a vector point ft^n & defined at all points on a closed surface & in the region V enclosed by it & if $\vec{F}$ has continuous first partial derivatives at all points in V, then

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dV$$

where $\hat{n}$ is unit normal vector.

Q Using divergence th^m, evaluate $\iint \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken ~~using gauss~~ over the cube $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$

Gauss div. th^n is given by

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\, dv.$$

$$\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$$

$$\operatorname{div}\vec{F} = 4z - 2y + y$$
$$= 4z - y$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \int_0^1\int_0^1\int_0^1 4z - y\, dz\, dy\, dx$$

$$= \iint [2z^2 - yz]_0^1 = \iint 2 - y\, dy\, dx$$

$$= \int [2y - y^2/2]_0^1 = \int [2 - 1/2]\, dx$$

$$= \left[\frac{3}{2}x\, dx\right]_0^1 = \frac{3}{2}$$

Q Using divergence th$^m$, evaluate $\iint \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ & s is the surface of the sphere $x^2+y^2+z^2=a^2$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \text{div}\,\vec{F}\, dv$$

$$\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$$

$$\text{div}\,\vec{F} = 3x^2 + 3y^2 + 3z^2$$
$$= 3(x^2+y^2+z^2)$$
$$= 3a^2$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = \iiint 3a^2\, dv$$
$$= 3a^2 \iiint dv$$
$$= 3a^2(\text{vol. of sphere})$$
$$= 3a^2 \cdot \frac{4}{3}\pi a^3$$
$$= 4\pi a^5$$

Q Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ using divergence th$^m$ for $\vec{F} = y\hat{i} + x\hat{j} + z^2\hat{k}$ over the cylindrical region bounded by $x^2+y^2=9$, $z=0$, $z=2$

Ans: $36\pi$

(In matrices)

Linear equation

$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$

A ~~system~~ linear system of two ~~equan~~ equations & two unknowns is given by $a_1x + a_2y = c_1$
$b_1x + b_2y = c_2$

~~Cross~~ Gauss elemination method:

To find the sol$^n$ of system of linear eq$^n$

Q Solve $x + 2y + z = 3$, $2x + 5y - z = -4$, $3x - 2y - z = 5$ using ~~cross~~ gauss elemination method.

$\rightarrow$ Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 2 & 5 & -1 & -4 \\ 3 & -2 & -1 & 5 \end{array}\right]$$

$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & -8 & -4 & -4 \end{array}\right]$$

$R_3 \leftarrow R_3 + 8R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 1 & -3 & -10 \\ 0 & 0 & -28 & -84 \end{array}\right]$$

$x + 2y + z = 3$
$y - 3z = -10$
$-28z = -84$
$z = 3$
$y = -1$
$x = 2$

System has unique solⁿ, ∴ it is consistant.

Rank of matrix
The no. r is said to be rank of the matrix if
1) There is ~~atth~~ atleast one or zero minor of order ~~or~~ r.
2) Every minor of order r+1 is 0.

The rank of matrix A is denoted by $\rho(A)$

Nullity
If A is square matrix of order n, then $n - \rho(A)$ is called nullity of matrix A. & is denoted by $N(A)$

# A non-singular matrix of order n has rank = n & nullity of such a matrix is equal to 0.

Rank - Nullity Theorem

For any m×n matrix A ~~rank of A + Nullity of A~~

rank of(A) + Nullity(A) = n

Q Find rank of the following matrices.

1) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{pmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 3R_1$

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & 0 & -6 & 0 \end{pmatrix}$

$R_3 \to R_3 + R_2$

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & -6 & 0 \end{pmatrix}$

$R_4 \to R_4 - \frac{3}{4} R_3$

$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

In this matrix, no. of non-zero rows is 3. ∴ $\rho(A) = 3$
rank

2) $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 1 & 0 \\ -2 & -1 & 3 \\ -1 & 4 & 2 \end{pmatrix}$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 + 2R_1,\ R_4 \to R_4 + R_1$

$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 3 & -3 \\ 0 & 6 & -1 \end{pmatrix}$

$R_3 \to R_3 + R_2$

$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 0 & 3 \\ 0 & 6 & -1 \end{pmatrix}$

$R_4 \to R_4 + 2R_2$

$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 11 \end{pmatrix}$

$R_4 \to R_4 - \frac{11}{3} R_3$

$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -3 & 6 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}$

$\rho(A) = 3$

$n = 3$

Using rank nullity thm $\therefore$ Nullity $= -\rho(A) + n = 0$.

3) $A = \begin{pmatrix} 2 & 1 \\ 3 & -7 \\ -6 & 1 \\ 5 & -8 \end{pmatrix}_{4\times 2}$

$R_2 \to R_2 - 3/2\, R_1,\ R_3 \to R_3 + 3R_1,$

$R_4 \to R_4 - 5/2\, R_1$

$= \begin{pmatrix} 2 & 1 \\ 0 & -17/2 \\ 0 & 4 \\ 0 & -21/2 \end{pmatrix}$

$-7 - \frac{3}{2}(1)$

$\frac{-14-3}{2}$

$-8 - \frac{5}{2}(1)$

$\frac{-16-5}{2}$

$\frac{17}{2}\ \ \frac{17(4)}{28}$

$4 + \frac{17}{8}\cdot 4$

~~$R_3 \to R_2 + 17/2\, R_1$~~ $8/17$

$= R_3 \to R_3 + 17/8\, R_2$

$= \begin{pmatrix} 2 & 1 \\ 0 & -17/2 \\ 0 & 0 \\ 0 & -21/2 \end{pmatrix}$

$R_4 \to R_4 - \frac{21}{17} R_2$

$= \begin{pmatrix} 2 & 1 \\ 0 & -17/2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$

$\rho(A) = 2$

$n = 2$

$\therefore$ Nullity $= 0$

$N(A)$

Q. solve $x_1 + x_2 - 2x_3 + 3x_4 = 4$
$2x_1 + 3x_2 + 3x_3 - x_4 = 3$
$5x_1 + 7x_2 + 4x_3 + x_4 = 5$

$$\rightarrow \left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 2 & 3 & 3 & -1 & 3 \\ 5 & 7 & 4 & 1 & 5 \end{array}\right]$$

$R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 5R_1$

$$\left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 2 & 14 & -14 & -15 \end{array}\right]$$

$R_3 \rightarrow R_3 - 2R_2$

$$\left[\begin{array}{cccc:c} 1 & 1 & -2 & 3 & 4 \\ 0 & 1 & 7 & -7 & -5 \\ 0 & 0 & 0 & 0 & -5 \end{array}\right]$$

$x_1 + x_2 - 2x_3 + 3x_4 = 4$
$x_2 + 7x_3 - 7x_4 = -5$
$0 = -5$ X

$\therefore$ ~~Soln~~ Given system of $eq^n$ has no $sol^n$
$\therefore$ system is inconsistant.

Q. solve $-3x - 5y + 36z = 10$
$-x + 7z = 5$
$x + y - 10z = -4$

$$\left[\begin{array}{ccc:c} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{array}\right]$$

$R_3 \longleftrightarrow R_1$

$$+\left[\begin{array}{ccc:c} 1 & 1 & -10 & -4 \\ -1 & 0 & 7 & 5 \\ -3 & -5 & 36 & 10 \end{array}\right]$$

$R_3 \rightarrow R_3 + 3R_1$, $R_2 \rightarrow R_2 + R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & -2 & 6 & -2 \end{array}\right]$$

$R_3 \rightarrow R_3 + 2R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x + y - 10z = -4$
$y - 3z = 1$

let ~~y=~~ $z = t$
$y = 1 + 3t$
$x = -4 + 10t - 1 - 3t$
$= -5 + 7t$.

System has infinite $sol^n$ $\therefore$ it is consistant.

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~~Linear~~
Linearly Independent

If $s = \{v_1, v_2, v_3 \ldots v_n\}$ is a non-empty set of vectors then the vector equation $k_1v_1 + k_2v_2 + \ldots + k_nv_n = 0$ has only one solution $k_1 = 0, k_2 = 0 \ldots k_n = 0$, then s is called linearly independent set.

Eigen value & Eigen vectors

If A is $n \times n$ matrix, then a non-zero vector X in $R^n$ is called an Eigen vector of A if $AX = \lambda X$ for some scaler $\lambda$.

The scalar $\lambda$ is called eigen value of A and X is called eigen vector of A corresponding to $\lambda$.

Q. Find eigen values & eigen vectors of the following matrices.

1) $A = \begin{pmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{pmatrix}$

→ Consider

$|A - \lambda I| = 0$

$\begin{vmatrix} 4-\lambda & 6 & 6 \\ 1 & 3-\lambda & 2 \\ -1 & -4 & -3-\lambda \end{vmatrix} = 0$

~~$(4-\lambda)(3-\lambda)(3+\lambda)$~~

$(4-\lambda)[-9+\lambda^2+8] - 6[\lambda-1] + 6[\lambda-1] = 0$

$(4-\lambda)(\lambda^2-1) = 0$

which is characteristic Eqⁿ

$\lambda = -1, 1, 4$

which are eigen values of A.

for $\lambda = -1$;

$[A - \lambda I]X = 0$

$[A + I]X = 0$

$\left[\begin{array}{ccc|c} 5 & 6 & 6 & 0 \\ 1 & 4 & 2 & 0 \\ -1 & -4 & -2 & 0 \end{array}\right]$

$R_1 \leftrightarrow R_2$

$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 5 & 6 & 6 & 0 \\ -1 & -4 & -2 & 0 \end{array}\right]$

$R_2 \to R_2 - 5R_1$, $R_3 \to R_1 + R_3$

$\left[\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -14 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + 4x_2 - 2x_3 = 0$

$-14x_2 - 4x_3 = 0$

let $x_3 = t$

$x_2 = \frac{-4t}{14}$ $\quad x_1 = -4\left(\frac{-4t}{14}\right) + 2t$

$= -\frac{8t}{7} + 2t = \frac{6t}{7}$

Eigen vectors
for $\lambda = -1$

$$X = \begin{bmatrix} -6/7 \\ -2/7 \\ 1 \end{bmatrix} t = \begin{bmatrix} -6 \\ -2 \\ 7 \end{bmatrix}$$

Q. $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Consider

$|A - \lambda I| = 0$

$$\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = 0$$

$-\lambda(\lambda^2 - 1) - 1(-\lambda - 1) + 1(1 + \lambda) = 0$

$\cancel{-\lambda^3 - \lambda + \lambda + \lambda - 1 = \lambda}$

$\cancel{-\lambda(\lambda^2 + 1) = 0}$

$-\lambda^3 + \lambda + \lambda + 1 + 1 + \lambda = 0$

$\lambda^3 - 3\lambda - 2 = 0$

at $\lambda = -1$, $eq^n$ is satisfied so
$\lambda = -1$.

$$\begin{array}{r|rrrr} & -1 & 0 & -3 & -2 \\ -1 & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2, -1$

$\lambda = -1, 2, -1$

for $\lambda = 2$

$[A - \lambda I]X = 0$

$[A - 2I]X = 0$

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{rrr:r} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$R_2 \longleftrightarrow R_1$

$$\left[\begin{array}{rrr:r} 1 & -2 & 1 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right]$$

$R_3 \rightarrow R_3 - R_1$, $R_2 \rightarrow R_2 + 2R_1$

$$\left[\begin{array}{rrr:r} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$R_3 \rightarrow R_3 + R_2$

$$\left[\begin{array}{rrr:r} 1 & -2 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - 2x_2 + x_3 = 0$

$-3x_2 + 3x_3 = 0$

$x_2 = x_3 = t$

$x_1 = 2t - t = t$

Eigen vector

$$X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$$

for $\lambda = -1$

$[A - \lambda I]\, x = 0$

$[A + I]\, x = 0$

$$\left(\begin{array}{ccc:c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right)$$

$x_1 + x_2 + x_3 = 0$

$x_2 = t, \; x_3 = s$

$x_1 = -s - t$

$\therefore$ eigen vectors

$$X = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} s$$

Q. $$A = \begin{pmatrix} 1 & -6 & -4 \\ 0 & -4 & 2 \\ 0 & -6 & -3 \end{pmatrix}$$

consider $|A - \lambda I| = 0$

$$\begin{pmatrix} 1-\lambda & -6 & -4 \\ 0 & -4-\lambda & 2 \\ 0 & -6 & -3-\lambda \end{pmatrix} = 0$$

$\cancel{(1-\lambda)(12 + 3\lambda + 12 + 2\lambda) + 6(0) - 4(0) = 0}$

$\cancel{(1-\lambda)(5\lambda + 24) = 0}$

$\cancel{\lambda = -1, \; -\tfrac{24}{5}}$

$(1-\lambda)\left[(-4-\lambda)(-3-\lambda) + 12\right] = 0$

$[12 + 4\lambda + 3\lambda + \lambda^2 - 12]$

$\cancel{[\lambda^2 + 7\lambda + 24]}\; [\lambda^2 - \lambda] = 0$

$\lambda = 1, 1, 0.$

for $\lambda = 0$

$[A - \lambda I]\, X = 0$

$[A]\, X = 0$

$$\begin{pmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\left(\begin{array}{ccc:c} 1 & -6 & -4 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & -6 & -3 & 0 \end{array}\right)$$

$R_3 \longrightarrow R_3 + (3/2) R_2$

$-6 + \frac{3(4)}{2}$

$-3 + \frac{3(2)}{2}$

$$\left(\begin{array}{ccc:c} 1 & -6 & -4 & 0 \\ 0 & 4 & 2 & \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - 6x_2 - 4x_3 = 0$

$4x_2 + 2x_3 = 0$

$x_2 = t$

$x_3 = -2t$

$x_1 = 6t - 8t = -2t$

$$X = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} t$$

for $\lambda = 1$

$[A - 1]x = 0$

$$\left(\begin{array}{ccc|c} 0 & -6 & -4 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & -6 & -4 & 0 \end{array}\right)$$

$R_3 \to R_3 + 2R_2$

$$\left(\begin{array}{ccc|c} 0 & -6 & -4 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-6x_2 - 4x_3 = 0$

$+3x_2 + 2x_3 = 0$

$x_2 = s \qquad x_3 = -\frac{3}{2}s$

eigen vectors

$$X = \begin{bmatrix} 0 \\ 1 \\ -3/2 \end{bmatrix} s + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t$$

Properties of eigen values

1) If A is $n \times n$ matrix which is upper triangular, lower triangular or diagonal matrix then the eigen values of A are entries on the main diagonal of A.

Ex. $A = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 5 & 0 \\ 3 & 8 & 7 \end{pmatrix} \to$ E. values $= 2, 5, 7$

& $B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 8 \end{pmatrix} \to$ E. values $= 1, 4, 8$

2) The square matrix A & A transpose & $(A^T)$ have same eigen values.

3) If $\lambda_1, \lambda_2, \ldots \lambda_n$ are eigen values of A then eigen values of $A^{-1}$ are $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots \frac{1}{\lambda_n}$

$\to$ Also $A^{-1}$ exists if & only if 0 is not an eigen value of A.

4) The matrix KA has eigen values $K\lambda_1, K\lambda_2, \ldots K\lambda_n$

5) The matrix $A^m$ where m is the integer has the eigen values $\lambda_1^m, \lambda_2^m, \ldots \lambda_n^m$.

Cayley-Hamilton Th$^m$

Every square matrix satisfies its characteristic eq$^n$

Q. Verify cayley-hamilton th$^m$ for

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$\to |A - \lambda I| = 0$

$$\begin{pmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)[(1-\lambda)(2-\lambda)] - 1[0] + (-1+\lambda) = 0$$

$$(2-\lambda)[2-3\lambda+\lambda^2] - 1 + \lambda = 0$$

$$4 - 6\lambda + 2\lambda^2 - 2\lambda + 3\lambda^2 - \lambda^3 - 1 + \lambda = 0$$

$$-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

To verify cayley-hamilton th$^m$, prove that

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$A^2 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix}$$

$$A^3 = A^2 . A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}\begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{pmatrix}.$$

$$A^3 - 5A^2 + 7A - 3I = 0$$

$$\begin{pmatrix} 14 & 13 & 13 \\ 0 & 1 & 0 \\ 13 & 13 & 14 \end{pmatrix} - \begin{pmatrix} 25 & 20 & 20 \\ 0 & 5 & 0 \\ 20 & 20 & 25 \end{pmatrix} + \begin{pmatrix} 14 & 7 & 7 \\ 0 & 7 & 0 \\ 7 & 7 & 14 \end{pmatrix} - \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

Q. Use cayley-hamilton th$^m$ to find $A^{-1}$ if $A = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}$

$$\longrightarrow |A - \lambda I| = 0$$

$$\begin{pmatrix} 1-\lambda & 3 & 7 \\ 4 & 2-\lambda & 3 \\ 1 & 2 & 1-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)((2-\lambda)(1-\lambda) - 6) - 3(4 - 4\lambda - 3) + 7(8 - 2 + \lambda) = 0$$

$$(1-\lambda)(2 - 3\lambda + \lambda^2 - 6) - 3(1 - 4\lambda) + 7(6+\lambda) = 0$$

$$-3\lambda + \lambda^2 - 4 + 3\lambda^2 - \lambda^3 + 4\lambda - 3 + 12\lambda + 42 + 7\lambda = 0$$

$$-\lambda^3 + 4\lambda^2 + 20\lambda + 35 = 0$$

$$\lambda^3 - 4\lambda^2 - 20\lambda - 35 = 0$$

$$A^3 - 4A^2 - 20A - 35 = 0$$

$$A^2(A.A^{-1}) - A(A.A^{-1}).A - 20(A.A^{-1}) - 35I(A^{-1}) = 0$$

$$A^2 - 4A - 20I - 35A^{-1} = 0$$

$$A^2 = \begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 7 \\ 4 & 2 & 3 \\ 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix}$$

$$A^2 - 4A - 20I = 35A^{-1}$$

$$\begin{pmatrix} 20 & 23 & 23 \\ 15 & 22 & 37 \\ 10 & 9 & 14 \end{pmatrix} - \begin{pmatrix} 4 & 12 & 28 \\ 16 & 8 & 12 \\ 4 & 8 & 4 \end{pmatrix} - \begin{pmatrix} 20 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 20 \end{pmatrix} = 35A^{-1}$$

$$\begin{pmatrix} -4 & +11 & -5 \\ -1 & -6 & +25 \\ 6 & +1 & -10 \end{pmatrix} = 35A^{-1}$$

$$A^{-1} = \frac{1}{35}\begin{pmatrix} -4 & 11 & -5 \\ -1 & -6 & 25 \\ 6 & 1 & -10 \end{pmatrix}$$

## Diagonalization

A square matrix $A$ is called diagonalizable if there is an invertible matrix $P$ such that $P^{-1}AP$ is a diagonal matrix.

Q. Find a matrix $P$ that diagonalizes

$$A = \begin{pmatrix} 10 & 2 \\ 2 & 7 \end{pmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} 10-\lambda & 2 \\ 2 & 7-\lambda \end{vmatrix} = 0$$

$(10-\lambda)(7-\lambda) - 4 = 0$

$70 - 17\lambda + \lambda^2 - 4 = 0$

$\lambda^2 - 17\lambda + 66 = 0$

$\lambda^2 - 11\lambda - 6\lambda + 66 = 0$

$\lambda = 11, 6$

For $\lambda = 6$

$[A - \lambda I]X = 0$

$[A - 6I]X = 0$

$$\left(\begin{array}{cc|c} 4 & 2 & 0 \\ 2 & 1 & 0 \end{array}\right) = 0$$

$R_2 \longrightarrow R_2 - R_1/2$

$$\left(\begin{array}{cc|c} 4 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$2x_1 + x_2 = 0$

$\therefore x_1 = t, \; x_2 = -2t$

$$X = \begin{bmatrix} 1 \\ -2 \end{bmatrix} t$$

For $\lambda = 11$

$[A - 11I]X = 0$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} X = 0$$

$$\left(\begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array}\right)$$

~~$x_1 = -1, x_2 =$~~

$-x_1 + 2x_2 = 0$

$x_1 = t$

$x_2 = +t/2$

$$X = \begin{bmatrix} 1 \\ +1/2 \end{bmatrix} t = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t$$

$$P = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \qquad P^{-1}AP = \begin{bmatrix} 6 & 0 \\ 0 & 11 \end{bmatrix}$$

$\det(P) = 1 + 4 = 5$

$$\text{adj}(P) = \cancel{\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}}$$

$$\cancel{PA = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 10 & 2 \\ 2 & 7 \end{pmatrix}}$$

$$\cancel{= \begin{pmatrix} 14 & 16 \\ -18 & 3 \end{pmatrix}}$$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \text{adj}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\text{adj}(P) = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

$P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$

$P^{-1}A = \frac{1}{5}\begin{bmatrix} 1 & -2 \\ +2 & 1 \end{bmatrix}\begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 6 & -12 \\ 22 & 11 \end{bmatrix}$

$(P^{-1}A)P = \frac{1}{5}\begin{bmatrix} 6 & -12 \\ 22 & 11 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$

$= \frac{1}{5}\begin{bmatrix} 30 & 0 \\ 0 & 55 \end{bmatrix}$

$= \begin{bmatrix} 6 & 0 \\ 0 & 11 \end{bmatrix}$

$\therefore$ P diagonizes A.

Q. Show that matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ is diagonalizable.

$(A - \lambda I) = 0$

$\begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = 0$

$4 + \lambda^2 - 4\lambda - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$\lambda^2 - 3\lambda - 1\lambda + 3 = 0$

$\lambda = 3, 1$

For $\lambda = 1$

$(A - 1I)X = 0$

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} X = 0$

$\begin{pmatrix} 1 & 1 & : & 0 \\ 1 & 1 & : & 0 \end{pmatrix}$

$x_1 + x_2 = 0$

$x_1 = t, \; x_2 = -t$ $\quad X = \begin{bmatrix} 1 \\ -1 \end{bmatrix} t$

For $\lambda = 3$

$(A - 3I)X = 0$

$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} X = 0$

$\begin{pmatrix} -1 & 1 & : & 0 \\ 1 & -1 & : & 0 \end{pmatrix}$

$-x_1 + x_2 = 0$

$x_1 = t, \; x_2 = t$ $\quad X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$

$P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$\det(P) = 1 + 1 = 2$

$\text{Adj}(P) = \begin{bmatrix} 1 & -1 \\ +1 & 1 \end{bmatrix}$

$P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$P^{-1}A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix}$

$(P^{-1}A).P = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$\therefore$ matrix A is diagonizable.

## LU Decomposition

LU Decomposition of a matrix A is a product of a lower triangular matrix & an upper triangular matrix that is equal to A.

$$A = L.U$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}.U_{11} & L_{21}.U_{12} + U_{22} & L_{21}.U_{13} + U_{23} \\ L_{31}.U_{11} & L_{31}.U_{12} + L_{32}.U_{22} & L_{31}.U_{13} + L_{32}.U_{23} + U_{33} \end{bmatrix}$$

Q. Find LU Decomposition of matrix

$$A = \begin{pmatrix} 1 & 2 & 14 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix}.$$

→ Let us consider $L = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix}$ &

$$U = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

∴ $A = LU$

$$\begin{pmatrix} 1 & 2 & 14 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{22} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 14 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{11}.L_{21} & U_{12}.L_{21} + U_{22} & L_{21}.U_{13} + U_{23} \\ L_{31}.U_{11} & L_{31}.U_{12} + U_{22}.L_{32} & L_{31}.U_{13} + L_{32}.U_{23} + U_{33} \end{pmatrix}$$

$U_{11} = 1$, $U_{12} = 2$, $U_{13} = 14$

$U_{11}.L_{21} = 3$
$1(L_{21}) = 3$
$L_{21} = 3$

$U_{12}.L_{21} + U_{22} = 8$
$(2)(3) + U_{22} = 8$
$U_{22} = 2$

$L_{21}.U_{13} + U_{23} = 14$
$3(14) + U_{23} = 14$
$U_{23} = -28$

$L_{31}.U_{11} = 2$
$L_{31}.(1) = 2$
$L_{31} = 2$

$L_{31}.U_{12} + U_{22}.L_{32} = 6$
$(2)(2) + (2)L_{32} = 6$
$L_{32} = 1$

$L_{31}.U_{13} + L_{32}.U_{23} + U_{33} = 13$
$2(14) + 1(-28) + U_{33} = 1$
~~$U_{33} = -1+28$~~
~~$U_{33} = 27$~~
$U_{33} = 13$

$$\begin{pmatrix} 1 & 2 & 14 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 14 \\ 0 & 2 & -28 \\ 0 & 0 & 13 \end{pmatrix}$$

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Q. Find LU Decomposition of

$$A = \begin{pmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ L_{21} U_{11} & L_{21}.U_{12} + U_{22} & L_{21}.U_{13} + U_{23} \\ L_{31}.U_{11} & L_{31}.U_{12} + L_{32}.U_{22} & L_{31} U_{13} + L_{32} U_{23} + U_{33} \end{pmatrix}$$

$U_{11} = 3$ , $U_{12} = 1$ , $U_{13} = 6$

$L_{21}.U_{11} = -6$ $\quad$ $L_{21}.U_{12} + U_{22} = 0$ $\quad$ $L_{21}.U_{13} + U_{23} = -16$

$L_{21}(3) = -6$ $\quad$ $2(1) + U_{22} = 0$ $\quad$ $-2(6) + U_{23} = -16$

$L_{21} = -2$ $\quad$ $U_{22} = -2$ $\quad$ $U_{23} = -4$

$L_{31}.U_{11} = 0$ $\quad$ $L_{31}.U_{12} + L_{32}.U_{22} = 8$

$L_{31}(3) = 0$ $\quad$ $0(1) + L_{32}(-2) = 8$

$L_{31} = 0$ $\quad$ $L_{32} = -4$

Dolittle's

for each $k = 0, 1, 2, 3, \ldots n$ ,

$$U_{k,m} = a_{k,m} - \sum_{j=1}^{k} L_{k,j} U_{j,m} \quad \text{for}$$

$m = k, k+1, \ldots, n$

&

$$L_{i,k} = \frac{a_{i,k} - \sum_{j=1}^{k-1} L_{i,j} U_{j,k}}{U_{kk}},$$

$i = k+1, k+2, k+3 \ldots n$

& $L_{i,i} = 1$

LU Decomposition to solve system of linear equation

Suppose we want to solve $AX = B$.

Step 1: Decompose matrix A into two matrices L & U

Step 2: Solve $LY = B$ & find matrix Y

Step 3: Solve $UX = Y$ to find matrix X.

Q. Find solution of $x_1 + 2x_2 + 4x_3 = 3$
$3x_1 + 8x_2 + 14x_3 = 13$, $2x_1 + 6x_2 + 13x_3 = 4$
using LU decomposition method.

Sol

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 13 \\ 4 \end{pmatrix}$$

Here $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}.U_{11} & L_{21}.U_{12}+U_{22} & L_{21}.U_{13}+U_{23} \\ L_{31}.U_{11} & L_{31}.U_{12}+L_{32}.U_{22} & L_{31}U_{13}+L_{32}U_{23}+U_{33} \end{pmatrix}$

$U_{11} = 1$, $U_{12} = 2$, $U_{13} = 4$

$L_{21}.U_{11} = 3$, $L_{21}.U_{12}+U_{22} = 8$, $L_{21}.U_{13}+U_{23} = 14$

$L_{21} = 3$, $3(2)+U_{22} = 8$, $(3)(4)+U_{23} = 14$

$U_{22} = 2$, $U_{23} = 2$

$L_{31}.U_{11} = 2$, $L_{31}.U_{12}+L_{32}.U_{22} = 6$, $L_{31}.U_{13}+L_{32}.U_{23}+U_{33} = 13$

$L_{31} = 2$, $2(2)+L_{32}(2) = 6$, $2(4)+1(2)+U_{33} = 13$

$L_{32} = 1$, $U_{33} = 3$

$L = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{pmatrix}$

$U = \begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

$AX = B$

$LY = B$

$UX = Y$

Consider $LY = B$.

$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 13 \\ 4 \end{bmatrix}$

$y_1 = 3$, $3y_1 + y_2 = 13$, $2y_1 + y_2 + y_3 = 4$

$y_1 = 3$, $y_2 = 4$, $y_3 = -6$.

Consider $UX = Y$, :.

$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -6 \end{bmatrix}$

$x_1 + 2x_2 + 4x_3 = 3$

$2x_2 + 2x_3 = 4$

$3x_3 = -6$

$x_3 = -2$,

$x_2 = 4$

$x_1 = 3$

$X = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}$.

Q. Use LU decomposition to solve following system of equation.

$$3x_1 + x_2 + 6x_3 = 0$$
$$-6x_1 - 16x_3 = 4$$
$$8x_2 - 17x_3 = 17$$

$$\begin{pmatrix} 3 & 1 & 6 \\ -6 & 0 & -16 \\ 0 & 8 & -17 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}.U_{11} & L_{21}.U_{12} + U_{22} & L_{21}.U_{13} + U_{23} \\ L_{31}.U_{11} & L_{31}.U_{12} + L_{32}.U_{22} & L_{31}.U_{13} + L_{32}.U_{23} + U_{33} \end{pmatrix}$$

$3 = U_{11}$, $U_{12} = 1$, $U_{13} = 6$

$L_{21}.U_{11} = -6$ $\quad$ $L_{21}.U_{12} + U_{22} = 0$ $\quad$ $L_{21}.U_{13} + U_{23} = -16$

$L_{21} = -2$ $\quad$ $(-2)(1) + U_{22} = 0$ $\quad$ $(-2)(6) + U_{23} = -16$

$U_{22} = 2$ $\quad$ $U_{23} = -16 + 12$

$= -4$

$L_{31}$

$L_{31}.U_{11} = 0$ $\quad$ $L_{31}.U_{12} + L_{32}.U_{22} = 8$ $\quad$ $L_{31}.U_{13} + L_{32}.U_{23} + U_{33} = -17$

$L_{31} = 0$ $\quad$ $L_{32}.(2) = 8$

$L_{32} = 4$ $\quad$ $4(-4) + U_{33} = -17$

$U_{33} = -17 + 16$

$= -1$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{pmatrix}$$

$AX = B$

$LY = B$

$UX = Y$.

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 17 \end{pmatrix}$$

$y_1 = 0$

$-2y_1 + y_2 = 4$, $4y_2 + y_3 = 17$

$y_1 = 0$, $y_2 = 4$, $y_3 = 1$

$UX = Y$

$$\begin{pmatrix} 3 & 1 & 6 \\ 0 & 2 & -4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}$$

$3x_1 + x_2 + 6x_3 = 0$

$2x_2 - 4x_3 = 4$

$-x_3 = 1$

$x_3 = -1$

$x_2 = 0$, $x_1 = 2$

$$X = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

Q Using LU decomposition solve.

$x_1 + 2x_2 + x_3 = 3$, $2x_1 + 5x_2 - x_3 = -4$, $3x_1 - 2x_2 - 3x_3 = 5$

$\rightarrow$
$$\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & -1 \\ 3 & -2 & -1 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ L_{21}.U_{11} & L_{21}.U_{12} + U_{22} & L_{21}.U_{13} + U_{23} \\ L_{31}.U_{11} & L_{31}.U_{12} + L_{32}.U_{22} & L_{31}.U_{12} + L_{32}.U_{23} + U_{33} \end{pmatrix}$$

$U_{11} = 1$, $U_{12} = 2$, $U_{13} = 1$, $L_{21}.U_{11} = 2$, $L_{21} = 2$

$L_{21}.U_{12} + U_{22} = 5$ $\quad$ $L_{21}.U_{13} + U_{22} = -1$

$2(2) + U_{22} = 5$ $\quad$ $(2)(1) + U_{22} = -1$

$U_{22} = 6$ $\quad$ $U_{22} = -3$

(2, -1, 3)

$U_{11} = 1$, $U_{12} = 2$, $U_{13} = 1$
$L_{21} = 2$, $U_{22} = 7$, $U_{23} = -3$
$L_{31} = 3$, $L_{32} = -4$, $U_{33} = -16$

$$\cancel{A}\ L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & 1 \end{pmatrix} \quad U = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & -16 \end{pmatrix}$$

$LY = B$

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -4 \\ 5 \end{pmatrix}$$

$y_1 = 3$
$2y_1 + y_2 = -4$
$y$

~~Metric Space~~

~~Definition~~

Metric Space

Let $X$ be a non-empty set, a mapping $d : X \times X \to R$ is said to be metric on $X$ if it satisfies the following properties, for any $x, y, z \in X$

1) Distance $> 0$ i.e $d(x, y) > 0$ &
$d(x, y) = 0 \Rightarrow x = y$

2) $d(x, y) = d(y, x)$

3) $d(x, y) \le d(x, z) + d(z, y)$

If $d$ is metric on $X$ then $(X, d)$ is called Metric space.

Example:
For $X = \mathbb{R}$ define $d(x, y) = |x - y|$, $\forall x, y \in \mathbb{R}$

$\rightarrow$ 1) $d(x, y) > 0$ (Definition)
$d(x, y) = 0 \iff |x - y| = 0$
$x - y = 0$
$x = y$

2) $d(x, y) = |x - y|$
$= |y - x|$
$= d(y, x)$.

3) $d(x y) \le d(x, z) + d(z, y)$
$|x - y| \le |x - z| + |z - y|$
$|x + z - z - y|$
$|x - z + z - y|$
$|x - y| = |(x - z) + (z - y|$
$\le |x - z| + |z - y|$ $\rightarrow$ Triangular inequality
$\le d(x, z) + d(z, y)$
$d(x, y) \le d(x, z) + d(z, y)$

$\therefore d(x, y)$ is a ~~not~~ metric space over $\mathbb{R}$.

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Q. Let $R$ be the set of real numbers, show that the function $d : R \times R \to R$ defined by $d(x,y) = |x^2 - y^2|$, $x, y \in R$ is not a metric space on $R$

→ 1) $d(x,y) \geq 0$ (Definition)

$d(x,y) = 0 \Leftrightarrow |x^2 - y^2| = 0$

$x^2 - y^2 = 0$

$(x-y)(x+y) = 0$

$x = y, -y$

∴ ~~not~~ property is not satisfied.

∴ $d$ is not a metric space

Q. $d(x,y) = (x-y)^2$, $x, y \in R$.

→ $d(x,y) \geq 0$

$d(x,y) = 0 \Leftrightarrow (x-y)^2 = 0$

$x = y$

2) $d(x,y) = (x-y)^2$

$= (y-x)^2$

$= d(y,x)$

3) $d(x,y) \leq d(x,z) + d(z,y)$

fractions will not be satisfied

$(x-y)^2$

$= ((x-z) + (z-y))^2$

$= (x-z)^2 + (z-y)^2 + 2(x-z)(z-y)$ ~~$\geq (x-y)^2$~~

$(x-z)^2 + (z-y)^2 + 2(x-z)(z-y) > (x-z)^2 + (z-y)^2$

∴ it doesnt satisfy the condition.

∴ $d$ is not a metric space.

## Line Integral

$\int_a^b f(x)\,dx$

In case of line integral, we shall integrate $f(x)$ along the curve C in space. ~~The~~ ~~integrant~~ $\int f(x)$ will be a $f^n$ defined at each point of curve.

Evaluation of line integral

Find a curve C : $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

& replace $ds$ by $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$

& evaluate $\int_C f(x,y,z)\,ds$.

Q. Evaluate $\int_C (xy^3)\,ds$ where $C: y = 2x$ in $xy$ plane from $(-1,-2)$ to $(1,2)$

Let $x = t \longrightarrow \frac{dx}{dt} = 1$

$y = 2t \longrightarrow \frac{dy}{dt} = 2$

$x = -1$ to $x = 1$ ∴ $t = -1$ to $t = 1$

$\int_C (xy^3)\,ds$

$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{1^2 + 2^2} = \sqrt{5}\,dt$

$= \int_{-1}^{1} t \cdot (2t)^3 \cdot \sqrt{5}\,dt$

$$8\sqrt{5}\int_{-1}^{1} t^4\,dt$$

$$= 8\sqrt{5}\left[\frac{t^5}{5}\right]_{-1}^{1} = 8\sqrt{5}\left[\frac{2}{5}\right] = \frac{16}{\sqrt{5}}$$

Q Evaluate ~~integration~~ $\int_c (x+y)\,ds$ where c is the line $x=t,\ y=-t+1$ from (0,1) to (1,0)

$x=t,\ y=1-t$ $\qquad ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$

$\frac{dx}{dt}=1,\ \frac{dy}{dt}=-1$ $\qquad = \sqrt{(1)^2+(-1)^2}$

$\qquad = \sqrt{2}\,dt$

t from 0 to 1

$$\int_c (x+y)\,ds$$

$$= \int_0^1 (\cancel{t}+1-\cancel{t})(\sqrt{2})\,dt$$

$$= \int_0^1 \sqrt{2}\,dt = \sqrt{2}\,[t]_0^1 = \sqrt{2}$$

Q $\int_c (x-3y^2+z)\,ds$ where c is a line segment joining origin and point (1,1,1)

Ans ~~x=t~~

(0,0,0) to (1,1,1)

$x_1, y_1, z_1)$ $\quad x_2, y_2, z_2$

~~$x_1 x_2$~~ $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$

$$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1} = t$$

$$x = y = z = t$$

$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = 1$ $\qquad ds = \sqrt{1^2+1^2+1^2}\,dt$

$\qquad = \sqrt{3}\,dt$

$$\int_c (x-3y^2+z)\,ds$$

$$= \int_0^1 (t-3t^2+t)\,dt\,(\sqrt{3})$$

$$= \sqrt{3}\int_0^1 2t-3t^2\,dt$$

$$= \sqrt{3}\left[t^2-t^3\right]_0^1 = \sqrt{3}\,[0] = 0$$

## Work done by line integration.

work done by the force F is given by

$$W = \int_a^b \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt$$

Q. Find the work done by force field $F = (y-x^2)\hat{i} + (z-y^2)\hat{j} + (x-z^2)\hat{k}$ over the curve $\vec{r} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$, $0 \le t \le 1$

Ans. Given $x=t,\ y=t^2,\ z=t^3$

Also

$$\overline{F} = (y-x^2)\hat{i} + (z-y^2)\hat{j} + (x-z^2)\hat{k}$$

$$= (t^2-t^2)\hat{i} + (t^3-t^4)\hat{j} + (t-t^6)\hat{k}$$

$$= (t^3-t^4)\hat{j} + (t-t^6)\hat{k}$$

$$r = t\,\hat{i} + t^2\,\hat{j} + t^3\,\hat{k}$$

$$\frac{dr}{dt} = \hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k}$$

$$\vec{F}\cdot\frac{dr}{dt}$$

$$= \left((t^3 - t^4)\hat{j} + (t - t^6)\hat{k}\right)\cdot\left(\hat{i} + 2t\hat{j} + 3t^2\hat{k}\right)$$

$$= 2t^4 - 2t^5 + 3t^3 - 3t^8$$

$$W = \int_0^1 (2t^4 - 2t^5 + 3t^3 - 3t^8)\,dt$$

$$= \left[\frac{2t^5}{5} - \frac{2t^6}{6} + \frac{3t^4}{4} - \frac{3t^9}{9}\right]_0^1$$

$$= \left[\frac{2}{5} - \frac{1}{3} + \frac{3}{4} - \frac{1}{3}\right]$$

$$= \frac{2}{5} + \frac{3}{4} - \frac{2}{3} = \frac{24 + 45 - 40}{60}$$

$$= \frac{29}{60}$$

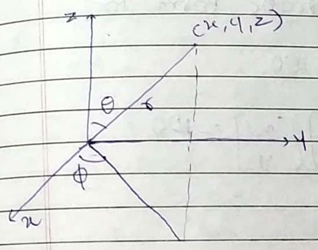
## Curvilinear coordinates

It is often convenient to work with variables other than the cartesian coordinates $(x, y, z)$

We have two types of curvilinear coordinates
1) spherical polar
2) cylindrical coordinates

### 1) spherical polar coordinates

In this system a pt. is referred to by its distance from the origin $r$ & two angles $\phi$ & $\theta$




The relationship b/w the coordinate system is given by

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

### 2) Cylindrical polar coordinates.

The relationship b/w the coordinate system is given by

$x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$.

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Page No.
Date

Orthogonal transformation of symmetric matrix to diagonal form.

A square matrix A with real elements is said to be orthogonal if
$AA' = A'A = I$

That is A is orthogonal if $A' = A^{-1}$

Diagonalization by orthogonal transformation is possible only for a real symmetric matrix. If A is real symmetric matrix then eigen vectors of A will be linearly independent and orthogonal

Q. Reduce given quadratic form to canonical form
$10x^2 + 4xy + 7y^2 = 100$

Ans. $\begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = 100$

$A = \begin{bmatrix} 10 & 2 \\ 2 & 7 \end{bmatrix}$

Eigen values are $\lambda = 11, 6$

For $\lambda = 11$, eigen vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

For $\lambda = 6$, eigen vectors $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Page No.
Date

Normalized form of eigen vectors are :
$\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ or $\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$

we $H = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$

now
$H^T A H = \begin{bmatrix} 11 & 0 \\ 0 & 6 \end{bmatrix}$

consider
$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} 11 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} = 100$

$11x'^2 + 6y'^2 = 100$

Q $\int_0^1 \int_0^2 \int_0^{1-z} dy\, dx\, dz$

$= \int [y]_0^{1-z} \, dx\, dz$

$= \int_0^1 1-z \, dx\, dz \quad = [x - zx]_0^2 \, dz$

$= \int 2 - 2z \, dz$

$= [2z - z^2]_0^1$

$= 2 - 1 = 1$

Q. Evaluate $\int_0^1 \int_0^{1-x} \int_{2y}^{1+y^2} x \, dz \, dy \, dx$

$= \int \int [xz]_{2y}^{1+y^2} dy \, dx$

$= \int \int (x(1+y^2) - x(2y)) \, dy \, dx$

$= \int \left[xy + \frac{xy^3}{3} - xy^2\right]_0^{1-x} dx$

$= x - x^2 + \frac{x(1-x)^3}{3} - x(1-x)^2$

$= x(1-x) + \frac{x(1-x)^3}{3} - x(1-x)^2$

$= \int \left[x - x^2 + \frac{x}{3}(1 - x^3 - 3x + 3x^2) - x(x^2 + 1 - 2x)\right]_0^1 dx$

$= \left[\frac{x^2}{2} - \frac{x^3}{3} + \frac{x^2}{6} - \frac{x^5}{15} - \frac{x^3}{3} + \frac{x^4}{4} - \frac{x^4}{4} - \frac{x^2}{2} + \frac{2x^3}{3}\right]_0^1$

$= \frac{1}{2} - \frac{1}{3} + \frac{1}{6} - \frac{1}{15} - \frac{1}{3} + 1 - \frac{1}{4} - \frac{1}{2} + \frac{2}{3}$

Ans $\frac{1}{10}$

$= 2 - \frac{4}{3} + \frac{1}{6} - \frac{1}{15} - \frac{1}{4}$

$= 2 \quad \frac{1}{6} - \frac{1}{15} + 1 - \frac{1}{4}$

$= \frac{5-2}{30} + 1 - \frac{1}{4} = \frac{1}{10}$

Volume by triple integration

Volume of close bounded region D in space is given by $V = \iiint_D dV$

$V = \iint_R f(x,y) \, dy \, dx$