Rosenzweig-MacArthur Model of Predator-Prey

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In this project, we model the Rosenzweig-MacArthur model, which is a simulation of the classic Lotka-Volterra predator prey with some changes in modelling the interactions. The purpose of the project is to analyze the non linear system and study the behaviour of the interaction between Predator and Prey. Further, the goal would be to introduce a paradox and explain the circumstances leading to that condition.

I. INTRODUCTION

The original model of Predator prey given by Lotka-Volterra takes into account number of assumption which makes it less realistic. The assumptions taken into account are

1. Food resource for prey is unlimited. This means the relation is exponential. Prey is diminished only through predation.

2. Prey is only food for the predator and does not have a natural death. Growth depends on the prey populations.[1]

The assumption of the above model makes it unrealistic. Rosenzweig-MacArthur model extends this idea to create a more realistic model. Analysis of the above equation leads to startling conclusions. We intend to show that the model follows Poincare-Bendixson's theorem and hence follows Hopf Bifurcation. Further this model shows the presence of the Paradox of enrichment.

II. MODEL

This model describes an extended version of the classical Lotka Volterra predator prey. Let us denote x and y to be the prey and predator density respectively.[3] Here density denotes, the population of any one model over the total population.

A. Assumptions

- Logistic growth and death is the best choice to model the prey population.

- Predators assume a linear death rate. This is because we assume there is an inherent competition between the predators for searching the prey.

- Predator reproduction rate is proportional to the predator kill rate.[3]

Based on the above assumption we now draw the compartment model.

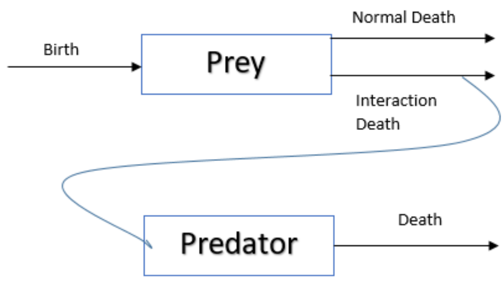

FIG. 1: Compartment Model

Now the equation for the model is

$$\frac{dx}{dt} = bx(1-x) - \frac{axy}{0.5+x} \tag{1}$$

$$\frac{dy}{dt} = \frac{xy}{0.5+x} - dy \tag{2}$$

where a captures interaction between predators and prey, b is prey growth rate and d is predator death rate.

B. Holling Response function

The Functional response is an intake rate for the consumption of resource. The important types used in the model are

- Type I functional response assumes a linear increase in intake rate with food density. It is used in the Lotka-Volterra model.[4]

- Type II functional response is characterized by a decelerating intake rate, which follows from the assumption that the consumer is limited by its capacity to process food [4]. The equation is a rectangular hyperbola given by

$$y = \frac{ax}{0.5+x} \tag{3}$$

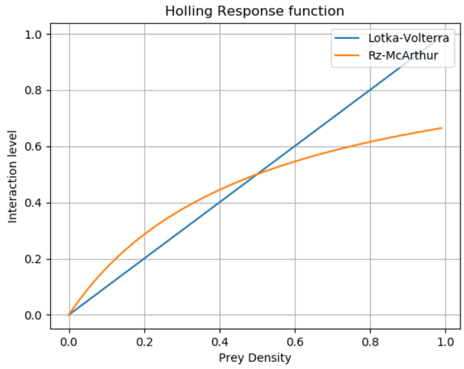

FIG. 2: Holling functions of 2 types

III. ANALYSIS

The first step in the analysis of a 2-D system is the presence of the equilibrium points(fixed points). The fixed points are created when the derivative of 1 and 2 are 0. So the equation for the fixed points is

$$bx(1-x) - \frac{axy}{0.5+x} = 0 \tag{4}$$

$$\frac{xy}{0.5+x} - cy = 0 \tag{5}$$

Now using 4

$$y = \frac{bx(1-x)*(0.5+x)}{ax} \tag{6}$$

So the equilibrium points are $(0,0)$ and $(1,0)$. There is also a third equilibrium point which leads to the coexistence of the two species. This is given by $(\frac{0.5d}{1-d}, \frac{b}{a}(1-x^*)(0.5+x^*))$. After finding the fix points, the next step is to check the stability of the points. For this we find the Jacobian. It is given by

$$J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} \tag{7}$$

Now analyzing the system at the fixed point $(0,0)$

$$J = \begin{bmatrix} b & 0 \\ 0 & -d \end{bmatrix} \tag{8}$$

The trace of the matrix $\tau = b - d$ and determinant of the matrix $\Delta = -bd$. As the parameters are positive, the determinant is negative implying the presence of a saddle.

For the fix point at $(1,0)$ we have

$$J = \begin{bmatrix} -b & \frac{-a}{1.5} \\ 0 & \frac{2-3d}{3} \end{bmatrix} \tag{9}$$

The trace is given by $\frac{2-3(b+d)}{3}$ and determinant is given by $\frac{-b(2-3d)}{3}$

This results is dependent on the parameter value d because based on different value of it we see different behaviour.

1. When d is less than 2/3, the behaviour of the system is saddle.
2. When d is equal to 2/3, we see that determinant is 0 indicating the presence of center, but with some changes.
3. When d is greater than 2/3 we see that system has a stable solution.

For the third fixed point given by $(\frac{0.5d}{1-d}, \frac{b}{a}(1-x^*)(0.5+x^*))$. The simplified Jacobian is given by

$$J = \begin{bmatrix} \frac{1-2bd}{1-d} + ab(1-d)^2 & -0.5ad \\ -b(1-d)^2(1-1.5d) & -0.5d \end{bmatrix} \tag{10}$$

After the matrix creation, we now need to find the eigen values of the matrix. From the Jacobian we can further simplify by taking each element as a constant. So the matrix is given by

$$J = \begin{bmatrix} \alpha & \beta \\ \gamma & \kappa \end{bmatrix} \tag{11}$$

So solving the equation for eigen values we get

$$\lambda = \frac{(\alpha+\kappa) \pm \sqrt{(\alpha+\kappa)^2 - 4(\alpha\kappa - \gamma\beta)}}{2\alpha} \tag{12}$$

The square root part in the above equation can be negative as well as positive. For the negative values we have 2 complex conjugate roots. This satisfy the condition of presence of Hopf bifurcation.

The real part of the eigen values obtained from this Jacobian is given by

$$Re(\lambda) = \frac{1-2bd}{1-d} + ab(1-d)^2 - 0.5d \tag{13}$$

This value can have both negative and positive values, implying that the stable spiral changes into unstable spiral.

So in the above model, changing the parameters of the system, it goes from stable solution to nonlinear oscillatory behaviour. In the stable state, the system tends to have fixed positive values for predator and prey (both species coexist)

IV. RESULTS

Based on the above mentioned model, analyses of specific cases with different values of parameter.

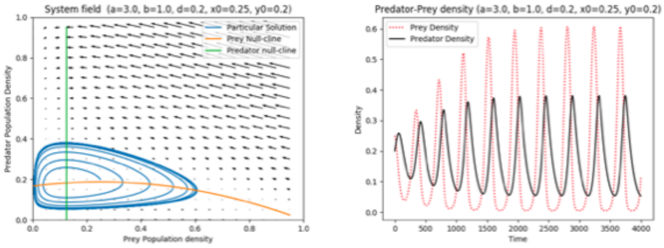

FIG. 3: (left)Phase plot with a = 3.0, b = 1.0, d = 0.2 and initial conditions (0.25,0.2), (right) Density of populations w.r.t. time

Figure 3 shows us that the populations of prey and predator show oscillatory behavior as time forwards. This shows us that both the species will live and doesn't tend to extinction. This happens because at these values of a, b and d the real part of the eigen values for the third fixed point have positive value. So this fixed point is a unstable spiral near the fixed point and there exists a limit cycle around this fixed point to which the system tends. I addition these initial value shows us the example of converging to limit cycle from inside the limit cycle.

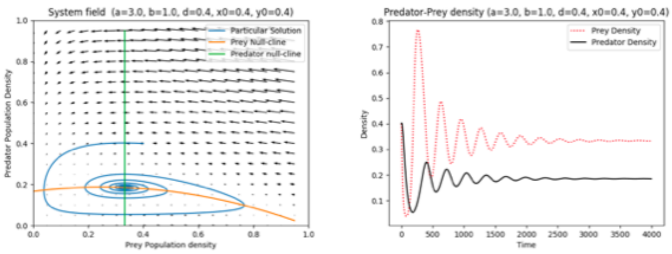

FIG. 4: (left)Phase plot with a = 3.0, b = 1.0, d = 0.4 and initial conditions (0.4,0.4), (right) Density of populations w.r.t. time

Figure 4 shows that at any given initial point the system spirals down to the fixed point. This means that after a long time the predator and prey population are converging to the values corresponding to the third fixed point.

Figure 5 shows that as prey's population increases predator's population decreases and it goes to zero. This is known as paradox of Enrichment proposed by Rosenzweig. He argued that an increase in food availability leads to extinction of species. The presence of paradox is strongly dependent on the assumption that the prey population follows functional response.

Figure 6 shows that whenever a limit cycle exists if the point is outside the limit cycle then also the system converges to the limit cycle. This is the example of converging to the limit cycle from outside the limit cycle.

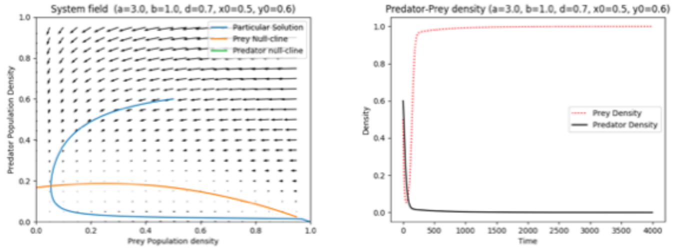

FIG. 5: (left)Phase plot with a = 3.0, b = 1.0, d = 0.7 and initial conditions (0.5,0.6), (right) Density of populations w.r.t. time

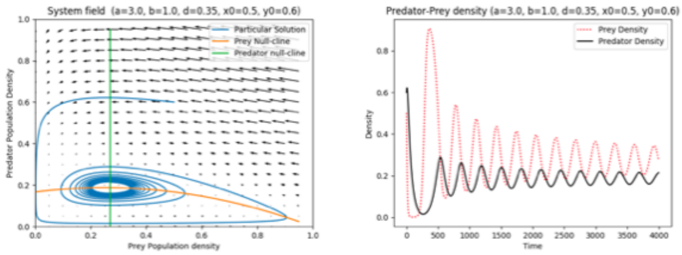

FIG. 6: (left)Phase plot with a = 3.0, b = 1.0, d = 0.35 and initial conditions (0.5,0.6), (right) Density of populations w.r.t. time

V. CONCLUSIONS

From the above model we conclude the following points:

1. The Rosenweig and R. H. MacArthur model depicts the predator prey model realistically than Lotka-Voltera Model.

2. This model shows Hopf Bifurcation and we can see that by changing the value of death rate of predators we can see that the stable spiral in the phase space changes to an unstable spiral with a limit cycle.

3. This model is an example of the "Paradox of Enrichment" because in some cases this model shows us that as the prey density increases the predator density decreases which is unacceptable by intuition because if food increases then the consumer should benefit by that. But in some cases this model doesn't follow this rule and the by any means the predator tends to extinction. This happens because of the death term introduced in the predator rate and the Holling function that decreases the propensity to consume of the predator.

[1] M.k. Hasan *Graphical Analysis of Rosenzweig-MacArthur model.*

[2] M. L. Rosenzweig and R. H. MacArthur *Graphical representations and stability conditions of Predator-Prey model*

[3] Hal L. Smith *The Rosenzweig model for Predator prey.*

[4] Pablo Elias Fuentes Sommer *Analysis of Rosenzweig-MacArthur model with Bifurcation structures and stochastic process*

$(\frac{0.5d}{1-d}, \frac{b}{a}(1-x^*)(0.5+x^*))$.