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Complex - Analysis.

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Ans 3. (i) We can write $\ln(1+z)$ as

$$\ln(1+z) = \ln|1+z| + i\arg(1+z)$$

$$= (\text{Let's say}) \ \ln r + i\varphi$$

$$\therefore r = |1+z|, \quad \varphi = \arg(z).$$

Now in this form we can write :

$$\ln(1+z) = u(x,y) + iv(x,y)$$

where $u(x,y) = \ln(r) = \ln\sqrt{(1+x)^2+y^2}$ (z being $x+iy$)

$$= \frac{\ln((1+x)^2+y^2)}{2}$$

and $v(x,y) = \varphi.$

Now $\dfrac{\partial u}{\partial x} = \dfrac{1\cdot(1+x)}{(1+x)^2+y^2}$ — (1) ; $\dfrac{\partial u}{\partial y} = \dfrac{y}{(x+1)^2+y^2}$ — (2)

as we know that φ is an arguement of z, and we can easily write from complex no. properties.

$$\frac{\partial \varphi}{\partial y} = \frac{x+1}{(x+1)^2+y^2} \quad \text{— (3)}$$

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and $\dfrac{\partial \varphi}{\partial x} = \dfrac{-y}{(1+x)^2+y^2}$ — (4)

From eq^n (1) and (3).

$$\frac{\partial u}{\partial x} = \frac{\partial \varphi}{\partial y} = \frac{(x+1)}{(x+1)^2+y^2}$$

and from eq^n (2) & (4)

$$\frac{\partial u}{\partial y} = -\frac{\partial \varphi}{\partial x} = \frac{+y}{(x+1)^2+y^2}$$

★ They are satisfying the Cauchy-Reimann equations.

★ ⇒ ∴ it implies than $\ln(1+z)$ is differentiable. (except at $z=-1$)

and also it is not differentiable for $z=-1$ because denominator vanishes and limit doesn't exist.

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(ii) Taylor series expansion of $\ln(1+z)$ around point $z=0$.

In general :

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{f^{(n)}(z_0)}{n!} \quad \text{--- (1)}$$

here, $f'(z_0=0) = \left(\frac{1}{1+z}\right)\Big|_{z=0} = (1)$

$$f''(z_0=0) = \frac{-1}{(1+z)^2}\Big|_{z=0} = -1$$

$$f'''(z_0=0) = \frac{+2}{(1+z)^3}\Big|_{z=0} = 2$$

$$f^{(4)}(z_0=0) = \frac{-2\cdot 3}{(1+z)^4}\Big|_{z=0}$$

$$\therefore \boxed{f^{(n)}(z_0=0) = \frac{(-1)^{n-1}(n-1)!}{(1+z)^n}} = \boxed{(-1)^{n-1}\cdot(n-1)!} \quad \text{--- (2)}$$

Putting in eq^n (1).

$$f(z) = \sum_{n=0}^{\infty} (z-0)^n \frac{(-1)^{n-1}}{} \cdot \frac{(n-1)!}{n!}$$

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}\cdot z^n}{n}}$$

ans

Case 1.

$$\frac{df}{dz^*} = \lim_{\delta z^* \to 0} \frac{\delta u + i\,\delta v}{\delta x - i\,\delta y}$$

Now $\delta z^* \to 0$ means → (i) $\delta y = 0$, $\delta x \to 0$
(ii) $\delta x = 0$, $\delta y \to 0$.

for case (i)

$$\frac{df}{dz^*} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\delta x} \quad — (1)$$

for case (ii)

$$\frac{df}{dz^*} = \lim_{\delta y \to 0} \frac{\delta u + i\,\delta v}{-i\,\delta y}$$

$$= \lim$$

$$= i\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y}. \quad — (2)$$

∴ From eqn (1), (2)

$$\boxed{\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}} \quad — (3)$$

but from Cauchy-Reimann eqn $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

& $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$.

∴ it implies that all should be vanished.
means $\boxed{f(z) = \text{constant}}$ ans

Ans 2.

$$I = \oint_C \frac{dz}{z^2+z} = \oint_C \left(\frac{1}{z} - \frac{1}{z+1}\right) dz$$

A, B

Case I: $R < 1$. (ie. interior of C).

(a) For $\oint \frac{dz}{z}$, analycity fails at $z=0$

$\therefore$ Using Cauchy's integral formula.

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} d(z) = f(z_0)$$

here $z_0 = 0$.

$$\oint_C \frac{f(z)}{(z-z_0)} dz = 2\pi i f(z_0).$$

Comparing LHS with A.

$\therefore f(z) = 1$, $z_0 = 0$.

$$\therefore \oint_C \frac{f(z)}{(z-z_0)} dz = \oint_C \frac{1}{z} dz = (2\pi i) \quad -(1)$$

(b) for $\oint \frac{dz}{(z+1)}$, analycity fails at $z=-1$.

$\therefore$ Using Cauchy's integral formula.

$$\oint \frac{f(z)}{z-(-1)} dz = 2\pi i f(z_0).$$

On comparing above eqn with B,

$f(z) = 1$. (const.)

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$$\therefore \oint \frac{dz}{z+1} = (2\pi i) \quad \text{--- (2)}$$

$\therefore$ From eqn (1), (2).

for $R<1$.

$$\boxed{\oint \frac{dz}{(z)(z^2+1)} = (2\pi i - 2\pi i) = 0} \quad \text{ans}$$

Case II $R>1$.

$$\oint \frac{dz}{z} - \oint \frac{dz}{z+1} = \oint \frac{dz}{(z^2+z)}$$

For $\oint \frac{dz}{z}$ analycity fails at $z=0$, but $R>1$.

$\therefore$ Using Cauchy integral theorem

$$\oint_C f(z)\,dz = 0, \quad \therefore \oint \frac{dz}{z} = 0 \quad \text{--- (3)}$$

(outside)

For $\oint \frac{dz}{z+1}$, analycity fails at $z=-1$, but $R>1$.

$\therefore$ Using Cauchy integral theorem

$$\oint_C f(z)\,dz = 0 \quad \therefore \oint \frac{dz}{(z+1)} = 0 \quad \text{--- (4)}$$

(outside)

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from eq^n (3) (4).

For $R > 1$

$$\boxed{\oint \frac{dz}{(z+1)(z)} = 0 - 0 = 0}$$ as

$$\therefore \boxed{I = \oint_C \frac{dz}{(z)(z+1)} = \begin{cases} 0, & \text{for } z_0 \text{ exterior of } C \\ 0, & \text{for } z_0 \text{ interior of } C. \end{cases}}$$

as

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Ans 4. Consider the complex integral

$$\oint e^{iz} z^{\alpha} dz$$

$$\lim_{R\to\infty} \underbrace{\int_{0+\varepsilon}^{R} e^{ix} x^{\alpha} dx}_{1} + \underbrace{\int \text{inf arc}}_{2} + \underbrace{\int_{R}^{0+\varepsilon} e^{i(iy)} (iy)^{\alpha}\, i\,dy}_{3}$$

$$+ \underbrace{\int_{\pi/2}^{0} e^{iz} z^{\alpha} dz}_{4} = 2\pi i(\Sigma \text{ residues})$$

∵ No singular point enclosed within contour

∴ RHS $= 0$.

$$\lim_{R\to\infty} \int_{0+\varepsilon}^{R} e^{ix} x^{\alpha} dx + i^{\alpha+1} \int_{R}^{0+\varepsilon} e^{-y} y^{\alpha} dy$$

$$= -\int \text{inf arc} - \int_{\pi/2}^{0} e^{iz} z^{\alpha} dz \quad \text{—(A)}$$

Let's evaluate RHS.

$\int \text{inf arc} \to 0$ ∵ $|z| \to 0$ as $R\to\infty$, ∵ $z = Re^{i\theta}$

and $\int_{\pi/2}^{0} e^{iz} z^{\alpha} dz = -\frac{\pi}{2} i\, a_{(-1)}$

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residue at $z=0$ is $\int_{\pi/2}^{0} \left(1+\frac{z}{1!}+\frac{z^2}{2!}+\cdots\right) z^{\alpha}\, dz$

it is fraction power of z

$(\because -1<\alpha<0)$

$\therefore$ Coeff of $\frac{1}{z-0}$ term is 0.

($\because$ it doesn't exist)

$\therefore$ RHS of eqn (A) is equal to 0.

Similarly for LHS of eqn (A)

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} \int_{0+\varepsilon}^{R} e^{ix} x^{\alpha}\, dx + i^{\alpha+1} \int_{R}^{0+\varepsilon} e^{-y} y^{\alpha}\, dy = 0$$

replace $x \to t$, $y \to s$.

$$\lim_{R\to\infty} \int_0^R e^{it} t^{\alpha}\, dt + i^{\alpha+1} \int_R^0 e^{-s} s^{\alpha}\, ds = 0.$$

$$\therefore \lim_{R\to\infty} \int_0^R e^{it} t^{\alpha}\, dt = -i^{\alpha+1} \int_R^0 e^{-s} s^{\alpha}\, ds$$

$$= i^{\alpha+1} \int_0^R e^{-s} s^{\alpha}\, ds$$

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Hence Proved

$$\int_0^\infty e^{it} t^\alpha \, dt = i^{\alpha+1} \int_0^\infty e^{-s} s^\alpha \, ds$$

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