Complex Analysis

11.2.1 show wheather or not the function $f(z) = Re(z) = x$ is analytic.

$\Rightarrow$ $f(z) = u + iv = Re(z)$.

$z = x + iy$, $\quad f(z) = x$,

so $u = x$, $\quad v = 0$, $\quad \frac{\partial u}{\partial x} = 1$, $\quad \frac{\partial v}{\partial y} = 0$,

$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$, f is not analytic

11.2.3 Find the analytic function $w(z) = u(x,y) + iv(x,y)$.

(a) if $u(x,y) = x^3 - 3xy^2$ (b) if $v = e^{-y}\sin x$.

$\Rightarrow$ (a) $w = f(z) = z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$

(b) $w = f(z) = e^{iz} = e^{i(x+iy)} = e^{-y}(\cos x + i\sin x)$.

11.2.4 If there is some common region in which $w_1 = u(x,y) + iv(x,y)$ and $w_2 = w_1^* = u(x,y) - iv(x,y)$ are both analytic, prove that $u(x,y)$ and $v(x,y)$ are constants.

$\Rightarrow$ $w_1 = u(x,y) + iv(x,y)$ $\rightarrow$ this to be analytic

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, ~~fo~~

f $w_2 = u(x,y) - i\,v(x,y)$ $\rightarrow$ this to be analytic

$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$.

from above $\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x}$ $\Rightarrow$ $\frac{\partial u}{\partial x} = 0$, $\quad \frac{\partial v}{\partial y} = 0$

therefore both u and v are constants and hence w_1 and w_2 = constant.

11.2.5 Starting from $f(z) = \frac{1}{(x+iy)}$ show that $\frac{1}{z}$ is analytic in the entire z plane except at the point $z=0$. This extends our discussion on the analyticity of z^n to negative integer powers of n.

$\Rightarrow$ $f(z) = \frac{1}{x+iy}$, $= \frac{x-iy}{x^2+y^2}$, $= u+iv$

$$u = \frac{x}{x^2+y^2}, \quad v = -\frac{y}{x^2+y^2}.$$

Now $$\frac{\partial u}{\partial x} = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{-x^2+y^2}{(x^2+y^2)^2}.$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2},$$

$$\frac{\partial v}{\partial x} = +\frac{2xy}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial y} = -\left[\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right]$$

$$= \frac{-x^2+y^2}{(x^2+y^2)^2}.$$

this proves the result

11.2.6 Show that given the Cauchy-Reimann equations, the derivative $f'(z)$ has the same value for $dz = a\,dx + ib\,dy$ $(a, b \neq 0)$ as it has for $dz = dx$.

$\Rightarrow$

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Write $f = u + iv$, the derivative in the direction (2) $a\,dx + ib\,dy$ is,

$$f' = \frac{a\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)dx + b\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)dy}{a\,dx + ib\,dy}.$$

Inserting the Cauchy - Reimann eq^{n}s to make all the derivatives w.r.t. x, we get,

$$f' = \frac{a\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)dx + b\left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)dy}{a\,dx + ib\,dy}$$

$$= \frac{\frac{\partial u}{\partial x}(a\,dx + ib\,dy) + i\frac{\partial v}{\partial x}(a\,dx + ib\,dy)}{a\,dx + ib\,dy}$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

This derivative has the same ~~direction~~ value as in the x direction.

11.27 Using $f(re^{i\theta}) = R(r,\theta)\,e^{i\Theta(r,\theta)}$, in which $R(r,\theta)$ and $\Theta(r,\theta)$ are differentiable real function of r and θ, show that the Cauchy - Riemann conditions in polar conditions became

(a) $\frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Theta}{\partial \theta}$ (b) $\frac{1}{rR}\frac{\partial R}{\partial \theta} = -R\frac{\partial \Theta}{\partial r}$

$\Rightarrow$ The real and imaginary parts of a analytic function must satisfy the Cauchy - Riemann eq^{n}s for an arbitrary orientation of the coordinate system.

take one coordinate system to be in the direction of $\hat{r}$ and the other in $\hat{\theta}$, and note that the derivatives of displacement in these directions are respectively $\frac{\partial}{\partial r}$ and $\frac{1}{r}\frac{\partial}{\partial \theta}$. Noting also that the real and imaginary parts of ~~$Re^{i\theta}$~~ $Re^{i\theta}$ are respectively $R\cos\theta$ and $R\sin\theta$, the Cauchy-Riemann eq^{n}s take the form

$$\frac{\partial R\cos\theta}{\partial r} = \frac{\partial R\sin\theta}{r\,\partial\theta}, \quad \frac{\partial R\cos\theta}{r\,\partial\theta} = -\frac{\partial R\sin\theta}{\partial r}$$

Carrying out the differentiations and rearranging, these eq^{n}s become

$$\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial\theta}{\partial\theta} = \tan\theta\left[R\frac{\partial\theta}{\partial r} + \frac{1}{r}\frac{\partial R}{\partial\theta}\right]$$

$$\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial\theta}{\partial\theta} = -\cot\theta\left[R\frac{\partial\theta}{\partial r} + \frac{1}{r}\frac{\partial R}{\partial\theta}\right]$$

Multiplying together the left-hand sides of both these eq^{n}s and setting the result equal to the product of the right-hand sides, we get,

$$\left[\frac{\partial R}{\partial r} - \frac{R}{r}\frac{\partial\theta}{\partial\theta}\right]^2 = -\left[R\frac{\partial\theta}{\partial r} + \frac{1}{r}\frac{\partial R}{\partial\theta}\right]^2.$$

these quantities in square brackets are real, so the above eq^{n}s is equivalent to the requirement that they must vanish or

$$\frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial\theta}{\partial\theta}.$$

$$\frac{1}{r}\frac{\partial R}{\partial\theta} = -R\frac{\partial\theta}{\partial r}.$$

11.2.8 As per extension of exercise 11.2.7 ... show (3)
that $\theta(r,\theta)$ satisfies the 2-D Laplace equation
in polar co-ordinates,

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r}\frac{\partial \theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \theta}{\partial \theta^2} = 0.$$

⇒ Differentiating the first Cauchy Riemann eqn for previous problem w.r.t θ and rearranging, we get

$$\frac{1}{r^2}\frac{\partial^2 \theta}{\partial \theta^2} = \frac{1}{rR}\frac{\partial^2 R}{\partial r \partial \theta} - \frac{1}{r^2 R}\frac{\partial^2 R}{\partial r \partial \theta} - \frac{1}{r^2 R}\frac{\partial R}{\partial \theta}\frac{\partial \theta}{\partial \theta}$$

$$= \frac{1}{rR}\frac{\partial^2 R}{\partial r \partial \theta} + \frac{1}{R}\frac{\partial R}{\partial r}\frac{\partial \theta}{\partial r}$$

where we reached the last member of the above eqns by substituting from the polar Cauchy -Riemann eqns. Differentiating the second Cauchy -Riemann eqn w.r.t r and simplifying, we get after rearrangement

$$\frac{\partial^2 \theta}{\partial r^2} = -\frac{1}{R}\frac{\partial \theta}{\partial r}\frac{\partial R}{\partial r} + \frac{1}{r^2 R}\frac{\partial R}{\partial \theta} - \frac{1}{rR}\frac{\partial^2 R}{\partial r \partial \theta}.$$

We also need from the second Cauchy-Reimann eqn

$$\frac{1}{r}\frac{\partial \theta}{\partial r} = -\frac{1}{r^2 R}\frac{\partial R}{\partial \theta}.$$

Adding threee eqns LHS combine to give the Laplacian operator. while RHS give zero

11.2-9 For each of the following functions $f(z)$, find $f'(z)$ and identify the maximal region within which $f(z)$ is analytic.

(a) $f(z) = \frac{\sin z}{z}$, (b) $f(z) = \frac{1}{z^2+1}$, (c) $f(z) = \frac{1}{z(z+1)}$

(d) $f(z) = e^{-1/z}$, (e) $f(z) = z^2-3z+2$ (f) $f(z) = \tan z$.

(g) $f(z) = \tanh(z)$.

→ (a) $f'(z) = \frac{\cos z}{z} - \frac{\sin z}{z^2}$

Analytic everywhere except at infinity. Note that $f(z)$ approaches a finite limit at $z=0$ and the has the Taylor expansion, At $z=0$, $f' \to 0$.

(b) $f'(z) = \frac{-2z}{(z^2+1)^2}$.

Analytic everywhere except at $z=i$ and $z=-i$.

(c) $f'(z) = -\frac{1}{z^2} + \frac{1}{(z+1)^2}$.

Analytic everywhere except at $z=0$, and $z=-1$.

(d) $f'(z) = \frac{e^{-1/z}}{z^2}$. Analytic everywhere except at $z=0$

(e) $f'(z) = 2z-3$, Analytic everywhere except at $z=\infty$.

(f) $f(z) = \frac{1}{\cos^2 z}$,

Analytic everywhere except at ∞ and at the poles of $\cos z \Rightarrow (n+\frac{1}{2})\pi$.

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(9) $f'(z) = \frac{1}{\cosh^2 z}$. (4)

Analytic everywhere except at ∞ and at the poles of $\cosh z$ which are $(n+\frac{1}{2})i\pi$.

11.2.11 Two dimensional irrotational fluid flow is described by a complex potential $f(z) = u(x,y) + i v(x,y)$. We label the real part, $u(x,y)$, the velocity potential and the imaginary part $v(x,y)$, the stream function. The fluid velocity V is given by $V = \nabla u$.

If $f(z)$ is analytic,

(a) Show that $\frac{df}{dz} = V_x - iV_y$

(b) Show that $\nabla \cdot V = 0$

(c) Show $\nabla \times V = 0$.

⇒ (a) Since $f'(z)$ is independent of direction, compute it for an infinitesimal displacement in the x direction, we have

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}.$$

identify $\frac{\partial u}{\partial x} = (\nabla u)_x = V_x$, $\frac{\partial u}{\partial y} = (\nabla u)_y = V_y$.

$$f' = V_x - iV_y.$$

(b) Use the fact that the real and imaginary parts of an imaginary parts of an analytic function each satisfy Laplace eq^n

& $\nabla \cdot V = 0 \quad = \nabla \cdot \nabla u = 0$

(c) $\nabla \cdot V = \frac{\partial}{\partial x} V_y - \frac{\partial}{\partial y} V_x = \frac{\partial^2 \psi}{\partial x \partial y} = \frac{\partial^2 \psi}{\partial y \partial x} = 0$

11.2.12 The function $f(z)$ is analytic. Show that the derivative of $f(z)$ w.r.t z^* does not exist unless $f(z)$ is a constant.

→ Equate the derivatives of $f(z) = u + iv$ w.r.t z^* in the x and y direction:

$$\frac{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}{dx} = \frac{\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}}{-i\,dy}$$

11.3.1 Show that $\int_{z_1}^{z_2} f(z)\,dz = -\int_{z_2}^{z_1} f(z)\,dz$

→ $\int_{z_2}^{z_1} f(z)\,dz = -\int_{z_1}^{z_2} \left[(u\,dx - v\,dy) + i(v\,dx + u\,dy)\right]$

11.3.3 Show that the integral $\int_{3-4i}^{4-3i} (4z^2 - 3iz)\,dz$ has the same value on the two paths, (a) the straight line connecting the integration limit (b) an arc on the circle $|z| = 5$

⑤ $F = 4z^2 - 3iz = 4(x^2-y^2) + 3y + (8xy - 3x)i$; ⑤

On the straight line path x and y are related by $y = -7x + 25$, so F has two representation

$$F_1(x) = -192x^2 + 1379x - 2425 + (-56x^2 + 197x)i$$

$$F_2(x) = \frac{-192y^2 - 53y + 2500}{49} + \frac{(-8y^2 + 203y - 75)i}{7}$$

Integrating

$$\int_{3+4i}^{4-3i} F(z)\,dz = \int_{3+4i}^{4-3i} F(z)\,(dx + i\,dy)$$

$$= \int_3^4 F_1(x)\,dx + \int_4^{-3} F_2(y)\,dy$$

$$= \left(\frac{67}{2} - \frac{7i}{6}\right) + \left(-\frac{49}{6} - \frac{469i}{2}\right) = \frac{76 - 707i}{3}$$

⑥ To integrate on the circle $|z| = 5$, use the polar representation $z = 5e^{i\theta}$. The starting point of the integral is at $\theta_1 = \tan^{-1}(4/3)$ and its end point is at $\theta_2 = \tan^{-1}(-3/4)$. F can now be written as $F_3(\theta) = 4(5^2 e^{2i\theta}) - 3i(5e^{i\theta})$

The integral then takes the form

$$\int_{\theta_1}^{\theta_2} F_3(\theta)\,(5ie^{i\theta})\,d\theta = \int_{\theta_1}^{\theta_2} (500i\,e^{3i\theta} + 75e^{2i\theta})\,d\theta$$

$$= \frac{500}{3}\left(e^{3i\theta_2} - e^{3i\theta_1}\right) - \frac{75i}{2}\left(e^{2i\theta_2} - e^{2i\theta_1}\right)$$

$$e^{3i\theta_1} = \frac{-117 + 44i}{125}, \quad e^{3i\theta_2} = \frac{-44 - 117i}{125}$$

$e^{2i\theta_1} = \frac{-7+24i}{25}$, $e^{2i\theta_2} = \frac{7-24i}{25}$.

$$\int_{\theta_1}^{\theta_2} F_3(\theta)(5ie^{i\theta})d\theta = \frac{76-707i}{3}.$$

11.3-5 Evaluate $\oint_c (x^2-iy^2)dz$, where the integration is (a) clockwise around the unit circle, (b) on a square with vertices at $\pm 1 \pm i$. Explain why the results of parts (a) and (b) are or are not identical.

⇒ (a) To integrate around the unit circle, set $x = \cos\theta$, $y = \sin\theta$, $dz = ie^{i\theta}d\theta = i(\cos\theta + i\sin\theta)\,d\theta$, and the integral will integrate from $\theta = 0$ to $\theta = 2\pi$. The integral will vanish because every term contains one odd power of either $\sin\theta$ or $\cos\theta$ and the integral is over the interval of length 2π.

(b) For the square, taking first as the horizontal lines at $y = 1$ and $y = -1$, we note that for any given x the integrand has the same value on both lines, but dz values are equal and opposite, these portions of the contour integral add to zero. Similar remarks apply to the vertical line segments at $x = \pm 1$, giving an overall result of zero.

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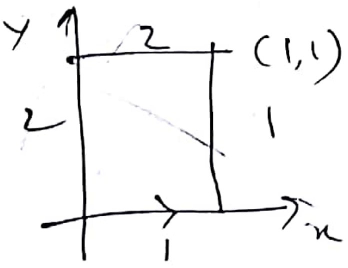
11.3.6 verify that $\int_0^{1+i} z^* dz$ depends on the path by evaluating the integral for the two paths shown in figure.



$$\Rightarrow \int_C z^* dz = \int_0^1 x\,dx + \int_0^1 (1-iy)\, i\,dy$$

$$= \frac{1}{2} + i + \frac{1}{2} = 1+i$$

whereas, $$\int_{C'} z^* dz = \int_0^1 -iy\,dy + \int_0^1 (-i+x)\,dx$$

$$= -\frac{i}{2} + \frac{1}{2} - i = \frac{1}{2} - \frac{3}{2} i.$$

11.4.1 Show that $\frac{1}{2\pi i} \oint z^{m-n-1} dz$, m and n integers is a representation of δ_{mn}.

$$\Rightarrow \frac{1}{2\pi i} \oint z^{m-n-1} dz$$

for $m = n$: $\frac{1}{2\pi i} \oint \frac{1}{z} dz = 1.$

$m \neq n$: $\frac{1}{2\pi i} \oint z^{m-n-1} = 0.$

$\Bigg\} \rightarrow \delta_{mn}.$

11.4.2 Evaluate $\oint_C \frac{dz}{z^2-1}$ where c is the circle $|z-1| = 1$.

$$\Rightarrow \oint \frac{1}{(z^2-1)} dz = \frac{1}{2}\left[\oint \frac{1}{z-1} dz - \oint \frac{1}{z+1} dz\right]$$

$$= 0.$$

11.4.3 Assuming that $f(z)$ is analytic on and within a closed contour C and that the point z_0 is within C, show that

$$\oint_c \frac{f'(z)}{(z-z_0)} dz = \oint_c \frac{f(z)}{(z-z_0)^2} dz$$

⇒ We know $\oint \frac{f'(z)dz}{(z-z_0)} = 2\pi i f'(z_0)$.

where the contour surrounds around z_0.

This formula is legitimate since f' must be analytic because f is. Now apply —

$$f'(z_0) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^2} dz.$$

14.4.6 Evaluate $\oint \frac{e^{iz}}{z^3} dz$, for the contour a square with sides of length $a>1$, centered at $z=0$.

⇒ The detailed description of the contour is irrelevent, what is important is that it encloses the point $z=0$.

$$\oint \frac{e^{iz}}{z^3} dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2}\left(e^{iz}\right)\Big|_{z=0} = -\pi i$$

11.4.7 Evaluate $\oint_{c} \frac{\sin^2 z - z^2}{(z-a)^3} dz$.

where the contour encircles the point $z=a$.

⇒ Here we need the ~~sec~~ second derivative of $\sin^2 z - z^3 \Big|_{at\, z=a}$.

$$\frac{d^2}{dz^2}\left(\sin^2 z - z^3\right)\Big|_{z=a} = 2\cos 2a - 2.$$

so the result is $\frac{2\pi i}{2!}(2\cos 2a - 2) = \underline{2\pi i(\cos 2a - 1)}$.

11.4.8 Evaluate $\oint_{c} \frac{dz}{z(2z+1)}$ for the contour unit circle.

⇒ $\frac{1}{z(2z+1)} = \frac{1}{z} - \frac{2}{2z+1} = \frac{1}{z} - \frac{1}{z+\frac{1}{2}}$

Both denominator are of form $(z-a)$ with a within the unit circle, and the integrals of the partial fractions are case of Cauchy's formula with the respective function $f(z)=1$ and $f(z)=-1$.

so total integral is $\underline{zero}$.

11.4.9 Evaluate $\oint_{c} \frac{f(z)}{z(2z+1)^2} dz$ for unit circle contour.

⇒

$$\oint f(z)\frac{1}{z(2z+1)^2}dz = \oint f(z)\left[\frac{1}{z} - \frac{2}{2z+1} - \frac{2}{(2z+1)^2}\right]dz$$

$$= \oint \frac{f(z)\,dz}{z} - \oint \frac{f(z)\,dz}{z+\frac{1}{2}} - \oint \frac{\frac{1}{2}f(z)\,dz}{(z+\frac{1}{2})^2}$$

$$= 2\pi i\, f(0) - 2\pi i\, f(-\tfrac{1}{2}) - \pi i\, f'(-\tfrac{1}{2})$$

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11.5.2 ~~Develop the Taylor expansion of $\ln(1+z)$~~ (8)

⇒ Derive the binomial expansion

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1\cdot 2} z^2 + \dots = \sum_{n=0}^{\infty} \binom{m}{n} z^n$$

⇒ From $\frac{d}{dz}(1+z)^m\Big|_0 = m(1+z)^{m-1}\Big|_0 = m.$

$$\frac{d^2}{dz^2}(1+z)^m\Big|_0 = m(m-1).$$

$$\frac{d^r}{dz^r}(1+z)^m\Big|_0 = m(m-1) \dots (m-r+1)$$

Taylor theorem yields for $|z|<1$

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1\cdot 2} z^2 + \dots$$

11.5.8 Obtain the Laurent expansion of $\frac{e^z}{z^2}$ about $z=0$

⇒ $$\frac{e^z}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!} = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty}\frac{z^n}{(n+2)!}$$

11.5.7 Obtain the Laurent expansion of $\frac{ze^z}{z-1}$ about $z=1$

⇒ One way to proceed is to write $z=(z-1)+1$ and $e^z = e\cdot e^{z-1}$. Expanding the exponential we have

$$e\left(1+\frac{1}{z-1}\right)\sum_{n=0}^{\infty}\frac{(z-1)^n}{n!} = \frac{e}{z-1} + e\sum_{n=0}\left(\frac{n+2}{n+1}\right)\frac{(z-1)^n}{n!}.$$

11.5.8 Obtain the Laurent exp. of $(z-1)e^{1/z}$ about $z=0$

⇒ Expanding $e^{1/z}$ in powers of $\frac{1}{z}$ and then multiply by $(z-1)$

$$(z-1)e^{1/z} = (z-1)\sum_{n=0}^{\infty}\frac{z^{-n}}{n!} = z - \sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)\frac{z^{-n}}{n!}$$

11.6.1 As an example of an essential singularity consider $e^{1/z}$ as z approaches zero. For any complex number z_0, $z_0 \neq 0$, show $e^{1/z} = z_0$ has an infinite number of solution.

→ $$e^{1/z} \simeq 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots + \frac{1}{n!\,z^n} = z_0 .$$

multiplying this by z^n and solving the resulting nth-order polynomial yields n different solutions $z = z_j$, $j = 1, 2, \ldots, n$. Then we let $n \to \infty$.