Complex Analysis

in cartesian coordinates

A complex number is denoted by

$Z = x + iy$ $\quad x, y \in \mathbb{R}$ and $i = \sqrt{-1}$

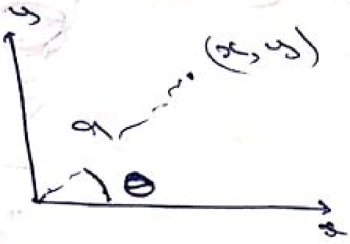
x Re, y Im

we can ~~also~~ denote

Z in ~~terms~~ of

cartesian coordinates

as



we can write

$x = r\cos\theta, \quad y = r\sin\theta.$

$\therefore$ in terms of polar coordinates we can write Z as

$Z = r(\cos\theta + i\sin\theta).$

x and y are real and imaginary parts of Z.

A complex function, $f(z)$, depending on Z can be resolved into real and imaginary parts.

$f(z) = u(x,y) + i\,v(x,y).$

For example, $f(z) = z^2$

$f(z) = (x+iy)^2 = x^2 - y^2 + 2ixy$

$\therefore u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy$

Complex functions can be constructed from functions of real variables.

Taylor's series expansion for real functions e^x, $\sin x$, $\cos x$ are as follows

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

We can define the complex functions e^z, $\sin z$, $\cos z$ as follows

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

For real θ, let us expand $e^{i\theta}$

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= \sum_{n=\text{even}} \frac{(i\theta)^n}{n!} + \sum_{n=\text{odd}} \frac{(i\theta)^n}{n!}$$

For $n = \text{even}$, $n = 2k$, $k = 0, 1, 2, \ldots$

For n = odd, $n = 2m+1$, $m = 0, 1, 2, \ldots$

$$\therefore e^{i\theta} = \sum_{l=0}^{\infty} \frac{(i\theta)^{2l}}{(2l)!} + \sum_{m=0}^{\infty} \frac{(i\theta)^{2m+1}}{(2m+1)!}$$

But ~~$(i)^{l}$~~ we have $(i)^{2l} = (i^2)^{l} = (-1)^{l}$

$$(i)^{2m+1} = i\,(i)^{2m} = i(-1)^{m}$$

$$\therefore e^{i\theta} = \sum_{l=0}^{\infty} (-1)^{l} \frac{\theta^{2l}}{(2l)!} + i \sum_{m=0}^{\infty} (-1)^{m} \frac{\theta^{2m+1}}{(2m+1)!}$$

$$= \cos\theta + i\sin\theta.$$

In the polar coordinates,

$$Z = r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta}$$

For a fixed Z, θ can have arbitrary values. Because for any integer n, we have

$$e^{i(\theta + 2n\pi)} = e^{i\theta}\, e^{i2n\pi}$$

~~$= \cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)$~~

~~$\cos\theta + i\sin\theta = e^{i\theta}$~~

$$= e^{i\theta}(\cos 2n\pi + i\sin 2n\pi)$$

$$= e^{i\theta}$$

Logarithm of a complex number can be written as

$$\ln Z = \ln\left[re^{i(\theta + 2n\pi)}\right]$$

$$= \ln r + \ln e^{i(\theta + 2n\pi)}$$

$$= \ln r + i(\theta + 2n\pi)$$

~~Since $e^{i\theta} = \cos\theta + i\sin\theta$, we can write~~

~~$\cos\theta$~~

we have $e^{i\theta} = \cos\theta + i\sin\theta$. ---(1)

~~its complex conjugate equation is~~

~~$(e^{i\theta})^* = (\cos\theta + i\sin\theta)^*$~~

$$\Rightarrow e^{-i\theta} = \cos(-\theta) + i\sin(-\theta)$$

$$= \cos\theta - i\sin\theta \quad (2)$$

using eqs (1) & (2), we can write

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

~~Cauchy-Riemann conditions for differentiation~~

~~Differentiation of a complex function~~

Let $f(z)$ be a complex function depending on the complex variable z. Differentiation of $f(z)$ with respect to z is defined as

$z+\delta z$

z

$$\frac{df}{dz} = f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{z+\delta z - z}$$

$$= \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z}$$

Provided the limit exist irrespective of the direction and $z = x+iy$.

For $f(z) = u(x,y) + i\,v(x,y)$ and

we can write

$$\delta f = \delta u + i\,\delta v \text{ and } \delta z = \delta x + i\,\delta y$$

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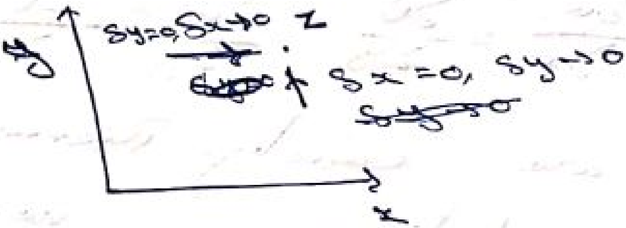
$$\therefore \quad \frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta u + i\,\delta v}{\delta x + i\,\delta y}$$

The limit $\delta z \to 0$ can be taken in two different ways, which is shown below.



We assume that the partial derivatives of $u(x,y)$ and $v(x,y)$ with respect to x, y exist.

For $\delta y = 0$ and $\delta x \to 0$, we have

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta f}{\delta z}$$

$$= \lim_{\delta x \to 0} \frac{\delta u + i\,\delta v}{\delta x}$$

$$= \lim_{\delta x \to 0} \frac{\delta u}{\delta x} + i\,\frac{\delta v}{\delta x}$$

$$= \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} \quad -(1)$$

For $\delta x = 0$ and $\delta y \to 0$, we have

$$\frac{df}{dz} = \lim_{\delta z \to 0} \frac{\delta f}{\delta z}$$

$$= \lim_{\delta y \to 0} \frac{\delta u + i\,\delta v}{i\,\delta y}$$

$$= \lim_{\delta y \to 0} -i\,\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y}$$

$$\frac{df}{dz} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \quad -(2)$$

Equating real and imaginary parts of Eqs (1) & (2), we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

These are Cauchy-Riemann conditions. These conditions are necessary if the derivative $\frac{df}{dz}$ exists.

Conversely, assume that Cauchy-Riemann conditions are satisfied and that the partial derivatives of $u(x,y)$, $v(x,y)$ are continuous. Then we can prove that the derivative $\frac{df}{dz}$ exists.

We have

$$\delta f = \delta u + i\delta v$$

$$= \frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + i\left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

Then,
$$\frac{\delta f}{\delta z} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y}{\delta x + i\delta y}$$

$$= \frac{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\frac{\delta y}{\delta x}}{1 + i\frac{\delta y}{\delta x}} \quad (1)$$

Using the Cauchy-Riemann conditions we can write

$$\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}$$

$$= i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) \quad (2)$$

Substituting Eq.(2) in (1), we can see that the dependence of $\delta y/\delta x$ cancels out ~~!~~ and we get

$$\therefore \frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Since the above relation doesn't depend on the direction of approach the derivative $\frac{df}{dz}$ exists.

Definition of **Analytic function**: A function $f(z)$ is analytic at $z = z_0$, if $f(z)$ is differentiable at $z = z_0$, (and in some small region around z_0). Otherwise, z_0 is singular point. ~~then $f(z)$ is~~

Example: $f(z) = z^2 = u(x,y) + i\,v(x,y)$

$\Rightarrow u(x,y) = x^2 - y^2, \quad v(x,y) = 2xy$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}; \quad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

$\therefore f(z) = z^2$ is analytic in the entire complex plane.

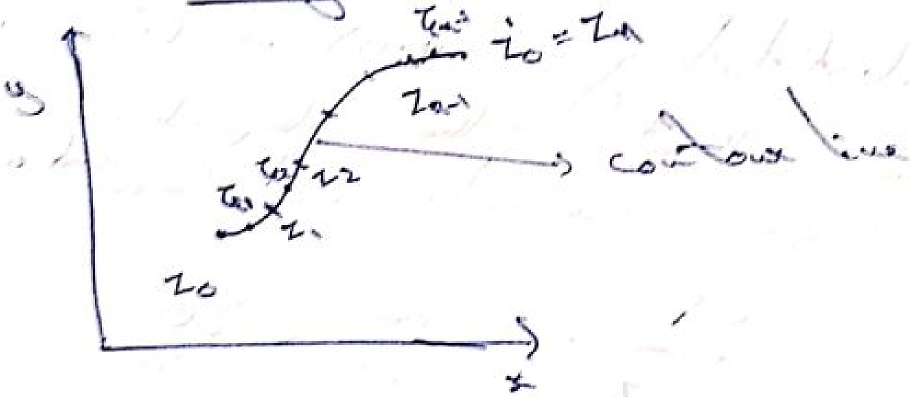
2. $f(z) = \bar{z} = u(x,y) + i\,v(x,y)$

$\Rightarrow u(x,y) = x, \quad v(x,y) = -y$

Now, $\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1$

$\therefore f(z) = \bar{z}$ is not an analytic function.

· Contour Integrals



Let the points z_0 and $\hat{z}_0$ are joined by a line, which is called contour line. we divide the contour line into n ~~segments~~ segments by picking $n-1$ intermediate points $z_1, z_2, \ldots, z_{n-1}$. τ_j is a point on the contour line which lies between z_{j-1} and z_j.

Now, consider the sum,

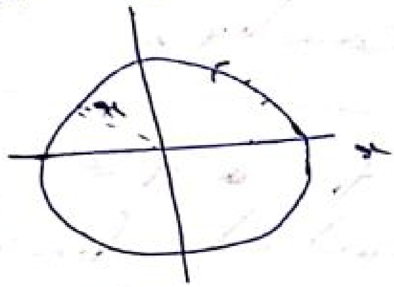
$$S_n = \sum_{j=1}^{n} f(\tau_j)(z_j - z_{j-1}).$$

In the limit $n \to \infty$ and $|z_j - z_{j-1}| \to 0$, if $\lim_{n\to\infty} S_n$ exists and is independent of the details of choosing

z_j and ζ_j, then we write the integral as

$$\int_{z_0}^{z_0} f(z)\,dz = \lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1})$$

<u>Example</u>: consider the contour integral $\int_c z^n\,dz$, where c is a circle of radius $r>0$ around the origin $z=0$. The contour is in counter-clockwise direction.



Any point on the contour can be written as

$$z = re^{i\theta}$$

$$\therefore\; dz = ire^{i\theta}\,d\theta$$

$$\therefore \int_c z^n\,dz = \int_0^{2\pi} r^n e^{in\theta}\, ire^{i\theta}\,d\theta$$

$$= i r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta}\,d\theta$$

if $n \neq -1$,

$$\int_c z^n\,dz = i r^{n+1} \left. \frac{e^{i(n+1)\theta}}{(n+1)} \right|_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1}\left[e^{i(n+1)2\pi} - 1\right]$$

$$\int_C z^n dz = 0$$

if $n=-1$,

$$\int_C z^n dz = i\int_0^{2\pi} d\theta = 2\pi i$$

∴ we have

$$\int_C \frac{dz}{z} = 2\pi i = \oint_C \frac{dz}{z}.$$

Cauchy's integral theorem

If a $f(z)$ is analytic and its partial derivatives are continuous throughout some simply connected region R, then for every closed path C in R. its integral is $\oint_C f(z)\,dz = 0$

Cauchy's integral theorem can be proved with Stokes's theorem.

Stokes's theorem states that

$$\oint_C \vec{A}\cdot d\vec{r} = \int_C (\vec{\nabla}\times\vec{A})\cdot d\vec{s}$$

For $\vec{A} = A_x\hat{i} + A_y\hat{j}$, $d\vec{r} = dx\hat{i} + dy\hat{j}$,
we get $d\vec{S} = dx\,dy\,\hat{k}$

$$(\vec{\nabla}\times\vec{A})_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

$\therefore$ Stokes's theorem becomes

$$\oint (A_x dx + A_y dy) = \iint\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) dx\,dy$$

For complex function $f(z)$, we take
$f(z) = u(x,y) + i\,v(x,y)$ and $dz = dx + i\,dy$

$$\therefore \oint_C f(z)\,dz = \oint (u + iv)(dx + i\,dy)$$

$$= \oint u\,dx - v\,dy + i\oint (v\,dx + u\,dy)$$

$$= \int\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$$+ i\int\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Since $f(z)$ is analytic, we have
have $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ we get

$$\oint f(z)\,dz = 0$$

~~A general version of~~ The Cauchy's integral theorem is generalized by Goursat where existence of partial derivatives of u and v are related.

According to Cauchy-Goursat theorem, if $f(z)$ is analytic in a region R then for any closed path c in it, the integral is

$$\oint_c f(z)\,dz = 0$$

$$\therefore \oint f(z)\,dz = 0 \Rightarrow \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz = 0$$

$$\Rightarrow \cancel{\int_{z_1}^{z_2} f(z)\,dz + \int_{z_2}^{z_1} f(z)\,dz = 0}$$

$$\Rightarrow \cancel{\int_{z_1}^{z_2} f(z)\,dz = -\int_{z_2}^{z_1} f(z)\,dz}$$

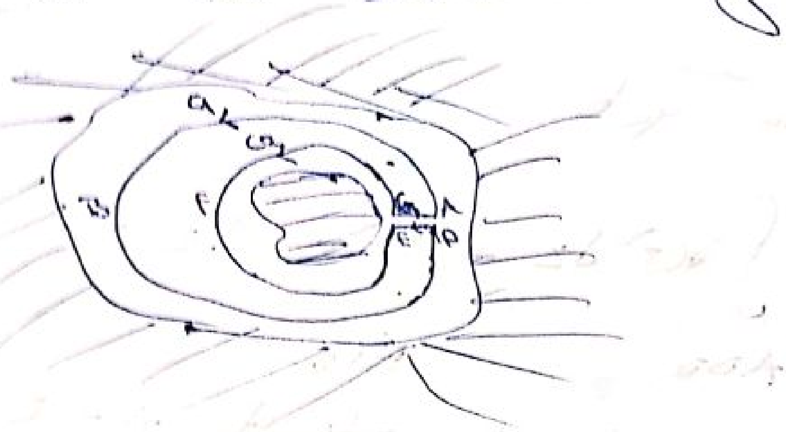
Since this relation is satisfied for any closed contour joining $z_1 z_2$, we can write

$$\int_{c_1} f(z)\,dz = \int_{z_1}^{z_2} f(z)\,dz = g(z_2) - g(z_1)$$

Then $$\int_{c_2} f(z)\,dz = \int_{z_2} f(z)\,dz = g(z_1) - g(z_2)$$

Multiply connected Regions:

Consider the following region in the complex plane, where $f(z)$ is analytic in the unshaded region.



In the unshaded region, let us consider a closed contour c' which is ABDEFGA. Since $f(z)$ is analytic within and on c', by ~~the~~ Cauchy's integral theorem, we have

$$\oint_{c'} f(z)\,dz = 0$$

$$\Rightarrow \int_{ABD} f(z)dz + \int_{DE} f(z)\,dz + \int_{EFG} f(z)\,dz$$

$$+ \int_{GA} f(z)\,dz = 0.$$

In the limit that the lines DE and GA are coinciding each other, we have

$$\int_{DE} f(z) = \int_{D}^{E} f(z)\,dz = -\int_{G}^{A} f(z)\,dz$$

$$= -\int_{GA} f(z)\,dz$$

$\therefore$ we get

$$\int_{ABD} f(z)\,dz + \int_{EFG} f(z)\,dz = 0$$

In the limit that $A \to D$ and $G \to E$, we ~~can~~ can represent ABD the with ~~closed~~ contour C_1 and ~~that~~ EFG with $-C_2$.

$$\therefore \oint_{C_1} f(z)\,dz + \oint_{-C_2} f(z)\,dz = 0$$

$$\Rightarrow \oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz \quad \text{---(1)}$$

<u>Cauchy's integral formula</u>

if a function $f(z)$ is analytic on a closed contour c and within the interior region bounded by c, then

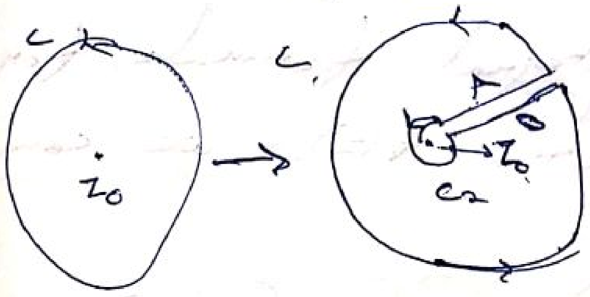
$$\frac{1}{2\pi i}\oint_{c} \frac{f(z)}{z-z_0}\,dz = f(z_0),$$

where z_0 is some point in the interior region bounded by c.

we can deform the contour c as follows

$c' = c + c_2 + A + B$

& $A \to B$



c_2 is a ~~unit~~ circular contour of radius r.

In the limit that line A approaching line B, we can write, by the result of Eq.(1)

$$\oint_{c} \frac{f(z)}{z-z_0}\,dz = \oint_{c_2} \frac{f(z)}{z-z_0}\,dz.$$

For points on contour c_2, we can write $z = z_0 + re^{i\theta}$.

$$\therefore \oint_{c_2} \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\, re^{i\theta}\, i\,d\theta$$

In the limit $r \to 0$, we get

$$\oint_{c_2} \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} f(z_0)\, i\,d\theta$$

$$= 2\pi i\, f(z_0)$$

For any point z_0, Cauchy's integral theorem states that

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = \begin{cases} f(z_0), & \text{for } z_0 \text{ is interior of } C \\ 0, & \text{for } z_0 \text{ exterior of } C \end{cases}$$

Derivatives.

Using Cauchy's integral formula, for an analytic function $f(z)$, we can write

$$\frac{f(z_0+\delta z_0)-f(z_0)}{\delta z_0} = \frac{1}{\delta z_0}\left[\frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0-\delta z_0}\,dz - \frac{1}{2\pi i}\oint \frac{f(z)}{z-z_0}\,dz\right]$$

$$= \frac{1}{2\pi i\,\delta z_0}\oint\left(\frac{f(z)}{z-z_0-\delta z_0} - \frac{f(z)}{z-z_0}\right)dz$$

$$= \frac{1}{2\pi i\,\delta z_0}\oint \frac{\delta z_0\, f(z)}{(z-z_0-\delta z_0)(z-z_0)}\,dz.$$

The derivative of $f(z)$ at z_0 is

$$f^{(1)}(z_0) = \lim_{\delta z_0 \to 0} \frac{f(z_0+\delta z_0)-f(z_0)}{\delta z_0}$$

$$= \frac{1}{2\pi i}\oint \frac{f(z)}{(z-z_0)^2}\,dz$$

Second derivative of $f(z)$ at z_0 is

$$f^{(2)}(z_0) = \lim_{\delta z_0 \to 0} \frac{f^{(1)}(z_0+\delta z_0)-f^{(1)}(z_0)}{\delta z_0}$$

$$= \frac{2}{2\pi i} \oint \frac{f(z)}{(z-z_0)^3} dz$$

By method of induction, for any positive integer n, one can show that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz.$$

If $f(z)$ is analytic on and within a closed contour c, then derivatives of all orders of $f(z)$ ~~of all orders~~ can exist at an interior point z_0. Moreover, all these derivatives are analytic at z_0.

Morera's theorem:

If a function $f(z)$ is continuous in a simply connected region R and $\oint_c f(z)\,dz = 0$ for every closed contour c within R, then $f(z)$ is analytic throughout R.

Morera's theorem is the converse of Cauchy's integral theorem.

Since $\oint_c f(z)\,dz = 0$ for every closed contour c within R, we can state that the integral $\int_{z_1}^{z_2} f(z)\,dz$ depends only on the end points z_1 and z_2.

$\therefore$ we can write

$$\int_{z_1}^{z_2} f(z)\, dz = g(z_2) - g(z_1),$$

where $g(z)$ is some continuous function.

Now, consider

$$\frac{g(z_2) - g(z_1)}{z_2 - z_1} - f(z_1) = \frac{\int_{z_1}^{z_2} (f(z') - f(z_1))\, dz'}{z_2 - z_1} \quad (1)$$

we can show the following result

$$\left| \int_c f(z)\, dz \right| \le |f|_{max} L,$$

where $|f|_{max}$ is the maximum value of $f(z)$ on the contour and L is the length of the contour. ~~in the limit $z_2 \to z_1$~~

Now, ~~in the limit $z_2 \to z_1$~~, we have

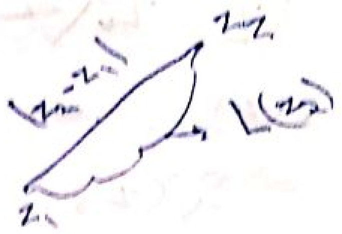
$$\cancel{\lim_{z_2 \to z_1}} \left| \frac{\int_{z_1}^{z_2} (f(z') - f(z_1))\, dz'}{z_2 - z_1} \right|$$

$$\le \cancel{\lim_{z_2 \to z_1}} \frac{|f(z') - f(z_1)|_{max} L(z_2)}{|z_2 - z_1|}$$

$\therefore$ in the limit $z_2 \to z_1$,

$$|f(z') - f(z_1)|_{max} \to f(z_1) - f(z_1) = 0$$

$$\frac{L(\gamma_0)}{|z_0 - z_1|} \geq 1$$



$\therefore$ we get

$$\lim_{z_0 \to z_1} \frac{\int_{z_1}^{z_0} (f(z) - f(z_1))\, dz}{z_0 - z_1} = 0$$

From eq. (1), we get

$$\lim_{z_0 \to z_1} \frac{g(z_0) - g(z_1)}{z_0 - z_1} = f(z_1).$$

$$\Rightarrow \quad g'(z)\Big|_{z=z_1} = f(z_1)$$

$\therefore \Rightarrow g(z)$ is analytic at $z = z_1$
z_1 is any point in the region R and $g(z)$ is analytic in R
Using Cauchy's integral formula, we can state that its first derivative $g'(z)$ is also analytic at any point $z = z_1$ in R. Hence, $f(z)$ is analytic in R.

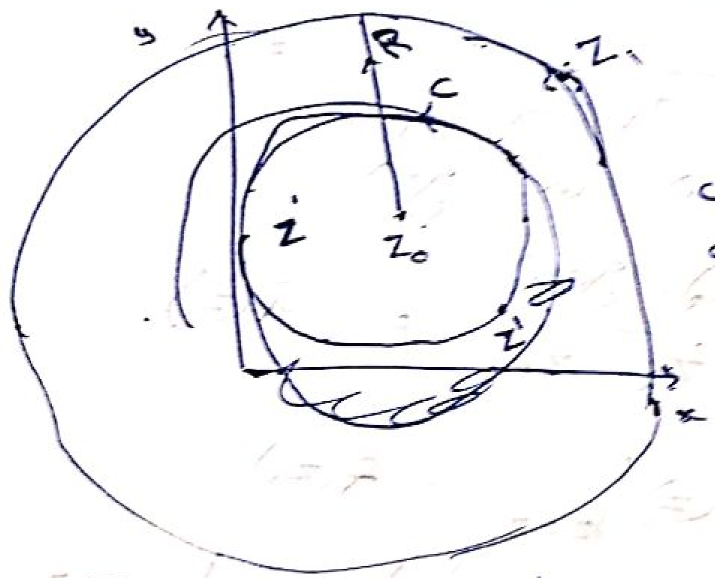
Taylor Expansion

~~In the complex plane, suppose we want to expand $f(z)$ at $z = z_0$.~~
Consider ~~suppose that~~ a function $f(z)$, which is analytic at $z = z_0$. Suppose we know ~~that the nearest point to~~ z_1 is a nearest point to z_0 at which $f(z)$ is not analytic. Then for any point in the region

$|z - z_0| < |z_1 - z_0|$, we can write

$f(z)$ as

$$f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{f^{(n)}(z_0)}{n!}$$



c is any circular contour centered at z_0 with radius r

Using cauchy's integral formula, for any point z within C, we can write

$$f(z) = \frac{1}{2\pi i} \oint_c \frac{f(z')}{z' - z} dz'$$

$$= \frac{1}{2\pi i} \oint_c \frac{f(z')}{z' - z_0 - (z - z_0)} dz'$$

$$= \frac{1}{2\pi i} \oint_c \frac{f(z')\, dz'}{(z'-z_0)\left[1 - \frac{z-z_0}{z'-z_0}\right]} \quad (1)$$

we can prove the following identity

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n.$$

The above series converges for $|t| < 1$.

Now, we have $\left|\frac{z-z_0}{z'-z_0}\right| < 1$.

$\therefore$ we can write

$$\frac{1}{1-\frac{z-z_0}{z'-z_0}} = \sum_{n=0}^{\infty} \left(\frac{z-z_0}{z'-z_0}\right)^n$$

substituting this in eq. (1), we get

$$f(z) = \frac{1}{2\pi i} \oint_c \sum_{n=0}^{\infty} \frac{(z-z_0)^n f(z')\, dz'}{(z'-z_0)^{n+1}}.$$

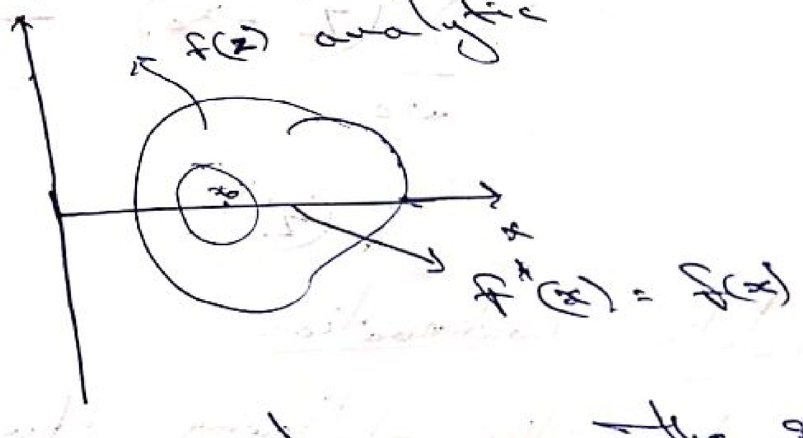
$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z-z_0)^n \oint_c \frac{f(z')}{(z'-z_0)^{n+1}} dz'$$

$$= \sum_{n=0}^{\infty} (z-z_0)^n \frac{f^{(n)}(z_0)}{n!}$$

<u>Schwarz reflection principle</u>:

If a function $f(z)$ is (1) analytic over some region including the real axis and (2) real when z is real, then

$$f^*(z) = f(z^*).$$



For ~~any~~ some point x_0 on the real axis, we can make Taylor expansion for $f(z)$ as

$$f(z) = \sum_{n=0}^{\infty} (z-x_0)^n \frac{f^{(n)}(x_0)}{n!}$$

Since $f(x_0)$ is real, we can argue that $f^{(n)}(x_0)$ must be real for all n.

$$f^{(1)}(x_0) = \lim_{\delta z_0 \to 0} \frac{f(x_0+\delta z_0) - f(x_0)}{\delta z_0}$$

$$= \lim_{\delta x_0 \to 0} \frac{f(x_0+\delta x_0) - f(x_0)}{\delta x_0}$$

$\therefore f^{(1)}(x_0)$ is real

$$f^{(2)}(x_0) = \lim_{\delta z_0 \to 0} \frac{f^{(1)}(x_0+\delta z_0) - f^{(1)}(x_0)}{\delta z_0}$$

Now, consider

$$f^*(z) = \left[\sum_{n=0}^{\infty} (z - x_0)^n \frac{f^{(n)}(x_0)}{n!}\right]^*$$

$$= \sum_{n=0}^{\infty} \left[(z - x_0)^n\right]^* \frac{f^{(n)}(x_0)}{n!}$$

$$= \sum_{n=0}^{\infty} (z^* - x_0)^n \frac{f^{(n)}(x_0)}{n!}$$

$$= f(z^*).$$

<u>Analytic continuation</u>

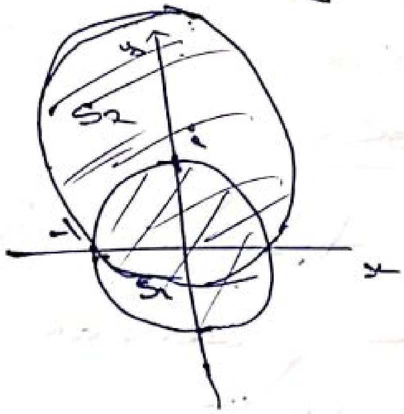
Consider the following function

$$f(z) = \frac{1}{1+z}.$$

This function has singularity at $z=-1$. For $|z|<1$, we expand $f(z)$ as

$$f(z) = \frac{1}{1+(-z)} = \sum_{n=0}^{\infty} (-z)^n. \quad (1)$$

This series expansion for $f(z)$ is valid in the region S_1 of the below figure.



Suppose we want to expand $f(z)$ around $z = i$. Then, we can write

$$f(z) = \frac{1}{1+z} = \frac{1}{1+i+z-i}$$

$$= \frac{1}{1+i} \frac{1}{1+\frac{z-i}{1+i}}$$

For $\left|\frac{z-i}{1+i}\right| < 1$, i.e. $|z-i| < |1+i| = \sqrt{2}$,

we can write

$$f(z) = \frac{1}{1+i} \sum_{n=0}^{\infty} \left(-\frac{z-i}{1+i}\right)^n$$

$$= \frac{1}{1+i} \left[1 - \frac{z-i}{1+i} + \left(\frac{z-i}{1+i}\right)^2 - \cdots\right] \quad (2)$$

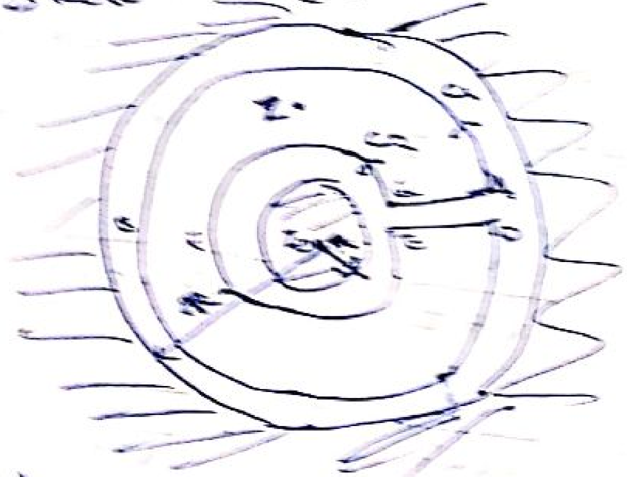
This series expansion is convergent in the region S_2 of the above figure.

The series expansion of $f(z)$

in eq.(1) is convergent in region S_1. But, with eq.(2) we have extended the series expansion function into the region S_2. This extension is called analytic continuation.

Laurent Series

Suppose a function $f(z)$ is analytic in the annular region, which is shown below.



We want to expand $f(z)$ around z_0. We construct two circular contours C_1 and C_2 in the annular region. In the region between C_1 & C_2, Cauchy integral formula is applicable. Hence, we can write

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')\,dz'}{z'-z} - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')\,dz'}{z'-z}$$

$$= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')\,dz'}{z'-z_0-(z-z_0)} - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')\,dz'}{z'-z_0-(z-z_0)}$$

On C_1, we have $|z'-z_0| > |z-z_0|$. Whereas, on C_2, we have $|z'-z_0| < |z-z_0|$.

∴ we can write

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')\,dz'}{(z'-z_0)\left[1-\frac{z-z_0}{z'-z_0}\right]}$$

$$+ \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')\,dz'}{(z-z_0)\left[1-\frac{z'-z_0}{z-z_0}\right]}$$

$$= \frac{1}{2\pi i}\oint_{C_1} \sum_{n=0}^{\infty} \frac{f(z')}{(z'-z_0)}\left(\frac{z-z_0}{z'-z_0}\right)^{n} dz'$$

$$+ \frac{1}{2\pi i}\oint_{C_2} \sum_{m=0}^{\infty} \frac{f(z')}{(z-z_0)}\left(\frac{z'-z_0}{z-z_0}\right)^{m} dz'$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty} (z-z_0)^{n} \oint_{C_1} \frac{f(z')}{(z'-z_0)^{n+1}}\,dz'$$

$$+ \frac{1}{2\pi i}\sum_{m=0}^{\infty} (z-z_0)^{-(m+1)} \oint_{C_2} f(z')(z'-z_0)^{m}\,dz'$$

$$= S_1 + S_2 \quad (2)$$

Now, consider

$$S_2 = \frac{1}{2\pi i}\sum_{m=0}^{\infty} (z-z_0)^{-(m+1)} \oint_{C_2} f(z')(z'-z_0)^{m}\,dz'$$

Put $m+1 = l$, then

$$S_2 = \frac{1}{2\pi i}\sum_{l=1}^{\infty} (z-z_0)^{-l} \oint_{C_2} f(z')(z'-z_0)^{l-1}\,dz'$$

Now, put $l = -n$, then

$$S_2 = \frac{1}{2\pi i}\sum_{n=-\infty}^{-1} (z-z_0)^n \oint_{C} \frac{f(z')}{(z'-z_0)^{n+1}}\, dz'$$

Substituting this in eq. (3), we get

$$f(z) = \frac{1}{2\pi i}\sum_{n=0}^{\infty} (z-z_0)^n \oint_{C_1} \frac{f(z')}{(z'-z_0)^{n+1}}\, dz'$$

$$+ \frac{1}{2\pi i}\sum_{n=-\infty}^{-1} (z-z_0)^n \oint_{C_2} \frac{f(z')}{(z'-z_0)^{n+1}}\, dz'$$

$$= \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n .$$

This is the Laurent series of $f(z)$ around z_0. Here,

$$a_n = \frac{1}{2\pi i}\oint_{C} \frac{f(z')}{(z'-z_0)^{n+1}}\, dz',$$

where c is any closed contour in the annular region.

Example: consider $f(z) = \frac{1}{z(z-1)}$

we can write

$$f(z) = \frac{1}{z-1} - \frac{1}{z}$$

$$= \frac{-1}{z} - \frac{1}{1-z}$$

for $|z|<1$, we can write

$$f(z) = -z^{-1} - [1 + z + z^2 + \dots]$$

$$= -\sum^{\infty} z^n$$

$f(z)$ has singularities at $z=0$ and $z=1$.

~~From the results~~

we can write a Laurent series for $f(z)$ around $z=0$ as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')}{(z')^{n+1}} dz'$$

$$= \frac{1}{2\pi i} \oint \frac{1}{(z')^{n+2}(z'-1)} dz'$$

$$= -\frac{1}{2\pi i} \sum_{m=0}^{\infty} \oint_C \frac{(z')^m}{(z')^{n+2}} dz'$$

Now, $z' = re^{i\theta}$

$$\therefore a_n = \frac{-1}{2\pi i} \sum_{m=0}^{\infty} \int_0^{2\pi} r^{m-n-2} e^{i(m-n-2)\theta} re^{i\theta}\, i\, d\theta$$

$$= \frac{-1}{2\pi} \sum_{m=0}^{\infty} \int_0^{2\pi} r^{m-n-1} e^{i(m-n-1)\theta} d\theta$$

if $m-n-1 \neq 0$, then

$$\int_0^{2\pi} e^{i(m-n-1)\theta} d\theta = \left. \frac{e^{i(m-n-1)\theta}}{i(m-n-1)} \right|_0^{2\pi}$$

$$= \frac{e^{i(m-n-1)2\pi} - 1}{i(m-n-1)} = 0$$

if $m-n-1 = 0$, then

$$\int_0^{2\pi} e^{i(m-n-1)\theta} d\theta = \int_0^{2\pi} d\theta = 2\pi.$$

$$\therefore \int_0^{2\pi} e^{i(m-n+1)\theta}\, d\theta = 2\pi\, \delta_{m,n-1}$$

$$\therefore a_n = \frac{1}{2\pi} \sum_{m=0}^{\infty} z^{(m-n+1)}\; 2\pi\, \delta_{m,n-1}$$

$$= \frac{1}{2\pi}\, z^{(n-1)}\, 2\pi = -1 \text{ for } n \geq 1$$

$$= 0 \text{ for } n < 1$$

$$f(z) = \sum_{n=1}^{\infty} a_n z^n = \sum_{n=1}^{\infty} (-1) z^n$$

Mapping

For a function $f(z)$, we have

$$f(z) = u(x, y) + i\,v(x, y)$$

Functional values of $f(z)$ can be denoted in a w-plane, where the x- and y-axis can be denoted by the values of $u(x,y)$ and $v(x,y)$.

i) Translation:

$$f(z) = z + z_0$$

$$\Rightarrow u(x,y) = x + x_0,\quad v(x,y) = y + y_0$$

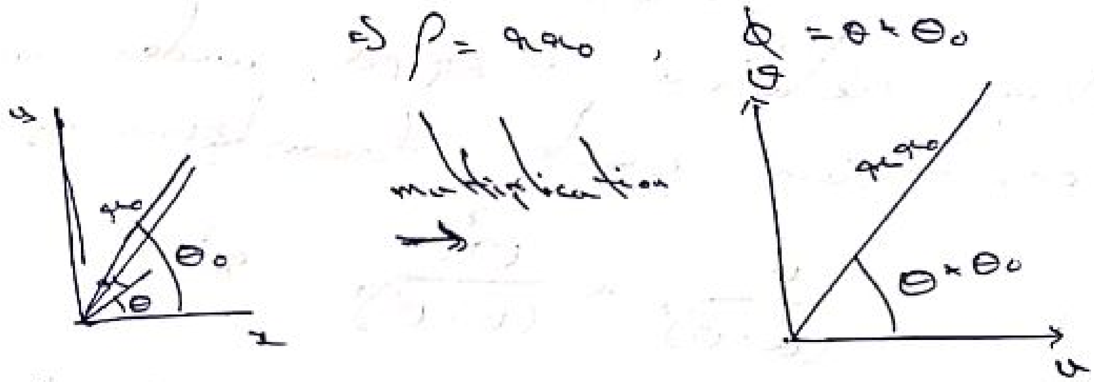
ii) Rotation:

$$f(z) = z z_0$$

In polar representation, we can write $f(z) = \rho e^{i\phi}$

then
$$\rho e^{i\phi} = r e^{i\theta} \, r_0 e^{i\theta_0}$$
$$= r r_0 e^{i(\theta + \theta_0)}$$

$$\Rightarrow \rho = r r_0 \, , \quad \phi = \theta + \theta_0$$

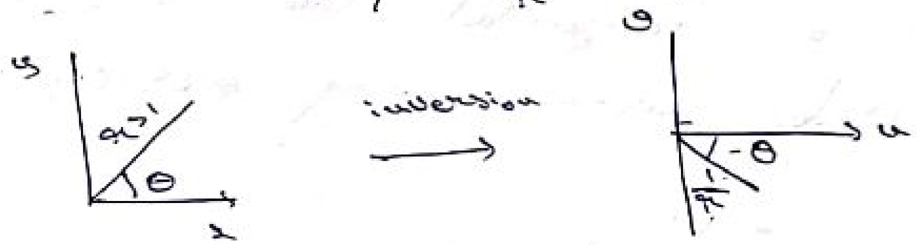


iii) inversion:

$$f(z) = \frac{1}{z}$$

$$\Rightarrow \rho e^{i\phi} = \frac{1}{r e^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

$$\Rightarrow \rho = \frac{1}{r} \, , \quad \phi = -\theta$$



In cartesian coordinates we can express $f(z) = \frac{1}{z}$ as

$$u + iv = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$$
$$= \frac{x-iy}{x^2+y^2}$$

$$\Rightarrow u = \frac{x}{x^2+y^2} \, , \quad v = \frac{-y}{x^2+y^2}$$

But, $u^2 + v^2 = \frac{x^2}{(x^2+y^2)^2} + \frac{y^2}{(x^2+y^2)^2} = \frac{1}{x^2+y^2}$

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$\therefore$ we can write

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{-v}{u^2+v^2} \quad (1)$$

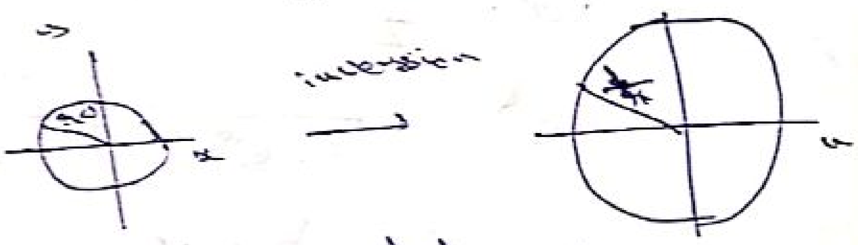
A circle centered at origin in z-plane has the form

$$x^2+y^2 = r^2.$$

Using relations in Eq.(1), under inversion, this circle transforms as

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} = r^2$$

$$\Rightarrow \quad \frac{1}{u^2+v^2} = r^2 \quad \text{or} \quad u^2+v^2 = \frac{1}{r^2}$$

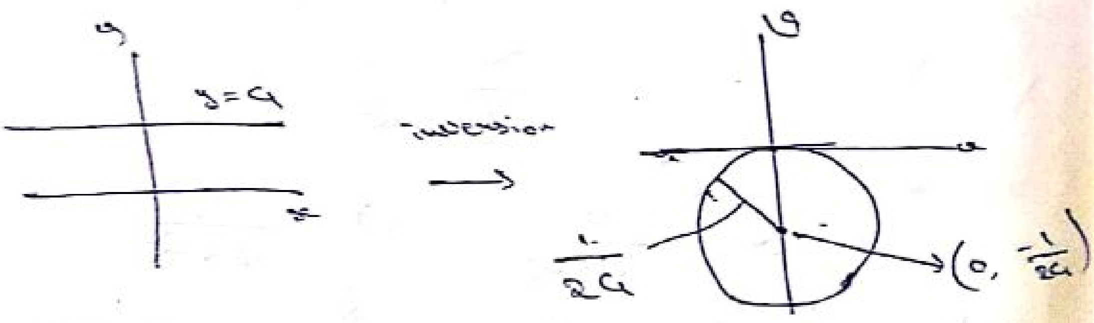


consider a horizontal line $y = c_1$ in z-plane under inversion, using Eq.(1), we get

$$\frac{-v}{u^2+v^2} = c_1$$

$$\Rightarrow \quad u^2+v^2+\frac{1}{c_1}v = 0$$

$$\Rightarrow \quad u^2+\left(v+\frac{1}{2c_1}\right)^2 = \left(\frac{1}{2c_1}\right)^2$$



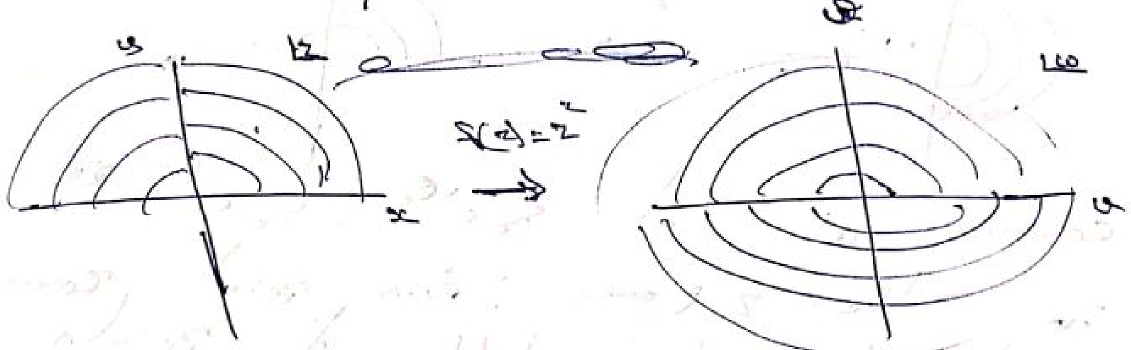
Branch points and multivalent functions

consider the function

$$f(z) = z^2$$

$$\Rightarrow \rho e^{i\phi} = (re^{i\theta})^2$$

$$\Rightarrow \rho = r^2, \quad \phi = 2\theta$$



The upper half-plane in z-plane maps into entire plane in the w-plane. The lower half-plane will cover the w-plane, for a second time. Here, two points in z-plane are mapped to one point in w-plane.

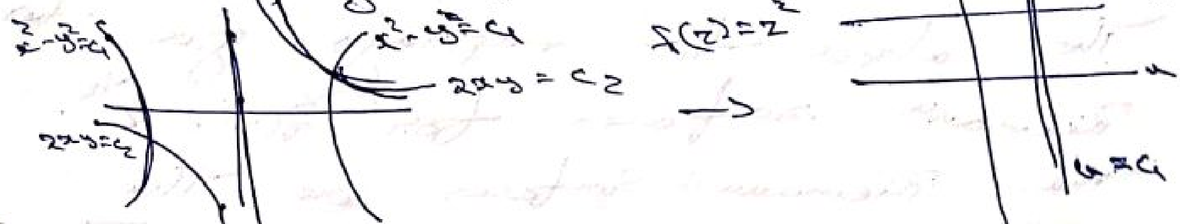
In cartesian coordinates we can write as

$$f(z) = z^2$$

$$\Rightarrow u + iv = (x+iy)^2$$

$$\Rightarrow u = x^2 - y^2, \quad v = 2xy.$$

Consider two lines $u = c_1$ and $v = c_2$ in the w-plane. The corresponding lines in z-plane are $x^2 - y^2 = c_1$ and $2xy = c_2$

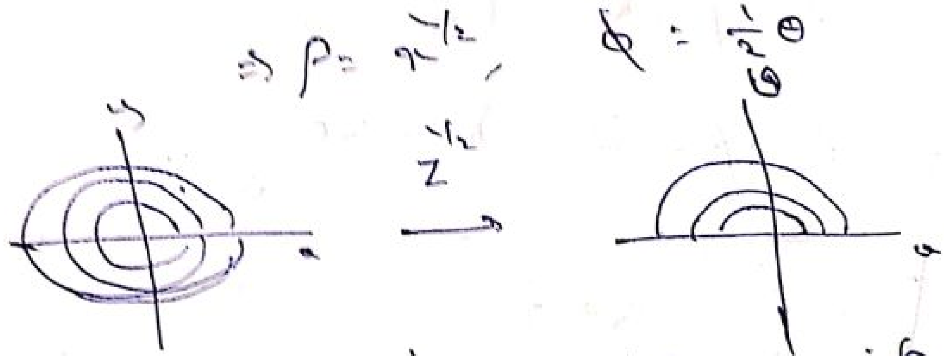


Now, consider

$$f(z) = z^{1/2}$$

$$\rho e^{i\phi} = (r e^{i\theta})^{1/2}$$

$$\Rightarrow \rho = r^{1/2}, \quad \phi = \frac{1}{2}\theta$$



Consider a point $r_0 e^{i\theta_0} = r_0 e^{i(\theta_0 + 2\pi)}$ in the z-plane. This point corresponds to two different points in the w-plane: $r_0^{1/2} e^{i\frac{\theta_0}{2}}$, $r_0^{\frac{1}{2}} e^{i\left(\frac{\theta_0}{2}+\pi\right)} = -r_0^{1/2} e^{i\theta_0/2}$

So the function $f(z) = z^{1/2}$ gives a mapping of ~~mor~~ one-to-many. To make ~~the~~ $f(z) = z^{1/2}$ as single valued function, we define a cut line which is a straight line from $z=0$ to infinity. For example, we can choose positive x-axis as cut line.

Points in the z plane, under $f(z) = z^{1/2}$ are mapped as follows

$0 \le r < \alpha, \quad 0 \le \theta < 2\pi \rightarrow$ upper half-plane in w-plane

$0 \le r < \alpha, \quad 2\pi \le \theta < 4\pi \rightarrow$ lower half-plane in w-plane

The above two mappings give two Riemann surfaces for $f(z) = z^{1/2}$. The Riemann surfaces are the complex z-planes, glued along the cut lines.

Consider $f(z) = e^z$

$\Rightarrow \rho e^{i\phi} = e^{x+iy}$

$\Rightarrow \rho = e^x, \quad \phi = y$

For $0 \le x < \infty$ and $0 \le y < 2\pi$, covers the whole w-plane.

Further ~~two~~ different points $x_0 + iy_0$ and $x_0 + i(y_0 + 2n\pi)$ in z-plane correspond ~~of ways to~~ the same point in the w-plane.

$\therefore f(z) = e^z$ gives a many-to-one mapping.

Now, consider $f(z) = \ln z$

$\Rightarrow u + i\theta = \ln re^{i\theta}$

$= \ln[r\, e^{i(\theta + 2n\pi)}]$

$= \ln r + i(\theta + 2n\pi)$

$f(z) = \ln z$ gives a one-to-many mapping.

To make this a single valued function, we can take positive x-axis as cutline.

For $f(z) = \ln z$, we get several Riemann surfaces.

<u>Branch Points</u>: To make function

like $z^{1/2}$, $\ln z$, we define cut. The cut line could be any straight line from $z=0$ to infinity. The intersection of these cut lines is the branch point.

<u>Examples</u>: for $f(z) = z^{1/2}$, the branch point is $z=0$

ii) For $f(z) = (z-1)^{1/2}$; the branch point is $z=1$

cut, $z = 1 + re^{i\theta}$, then

$$f(z) = r^{1/2} e^{i\theta/2}$$

iii) For $f(z) = (z-1)^{1/2}(z+1)^{1/2}$, there are two branch points: $z=1$ and $z=-1$

<u>isolated singularity</u>: suppose that $f(z)$ is not analytic at $z=z_0$, but is analytic at neighboring points, then z_0 is an isolated singularity.

<u>Poles</u>: suppose $f(z)$ is not analytic at z_0, we can express $f(z)$ by Laurent expansion as

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n = \cdots + a_{-3}$$

in this expansion if $a_n = 0$

for $n < -m < 0$, then z_0 is called a pole of order m.

$f(z) = \dots + \frac{a_{-(m+1)}}{(z-z_0)^{m+1}} + \frac{a_{-(m-1)}}{(z-z_0)^{m-1}} + \frac{a_{-m}}{(z-z_0)^{m}} + \dots + a_0 + \dots$

if the first non-vanishing term of $f(z)$ is $\frac{a_{-1}}{z-z_0}$, then z_0 is called a simple pole. ~~and~~ a_{-1} is the residue of $f(z)$.

if, in the Expansion of $f(z)$, the non-vanishing terms are upto $n = -\infty$, then z_0 is called essential singularity.

The nature of singularity of $f(z)$ at $z \to \infty$ can be determined with $f(1/t)$ as $t = \frac{1}{z} \to 0$.

Consider,

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Now, replace z with $1/t$.

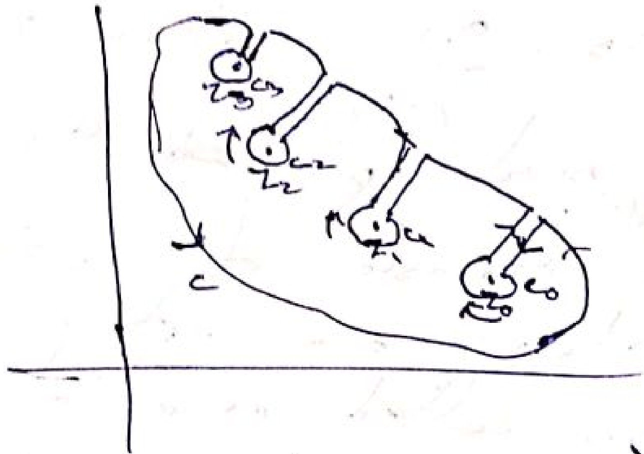
Then,

$$\sin \frac{1}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{t^{2n+1}}$$

$\sin \frac{1}{t}$ has essential singularity at $t = 0$. $\therefore$ $\sin z$ has essential singularity at $z \to \infty$.

<u>Residue Theorem</u>: Suppose $f(z)$ has some isolated singularities in the complex plane. For a contour encircling these ~~[illegible]~~ singularities, the integral is

$$\oint_C f(z)\,dz = 2\pi i \,(\text{sum of enclosed residues})$$



We deform the original contour so that each isolated singular point is ~~enclo~~ closed by a infinitesimally small circular contours, which is shown above.

Around isolated singular point z_i, we can expand $f(z)$ as

$$f(z) = \sum_{n=-\alpha}^{\alpha} a_n (z-z_i)^n$$

For a circular contour c, centered at z_i we have

$$\oint_c (z-z_i)^n\,dz = \begin{cases} 0, & \text{if } n \neq -1 \\ 2\pi i, & \text{if } n=-1 \end{cases}$$

($\therefore$ For the contour c_0,

$$\oint_{c_0} f(z)\,dz = -2\pi i\, a_{-1 z_0} \,)$$

Similarly, for the contour c_i,

$$\oint_{c_i} f(z)\,dz = -2\pi i\, a_{-1 z_i}$$

$\therefore$ Here $a_{-1 z_i}$ ~~at~~ is residue of $f(z)$ at $z = z_i$

Applying the cauchy's integral theorem to the deformed contour & we have and in the limit st. lines approaching

$$\cancel{\oint_c f(z)\,dz = 0}$$

$$\oint_c f(z)\,dz + \oint_{c_0} f(z)\,dz + \oint_{c_1} f(z)\,dz + \dots = 0$$

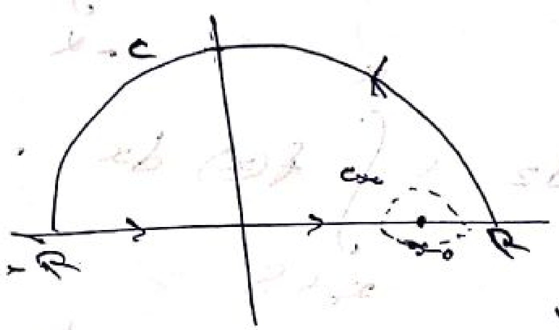
In the limit that the radii of $c_0, c_1, c_2,$ etc goes to zero, we get

$$\oint_c f(z)\,dz = 2\pi i\left(a_{-1z_0} + a_{-1z_1} + \dots\right)$$

$$= 2\pi i \text{ (sum of enclosed residues).}$$

Cauchy principle value

Suppose $f(z)$ has isolated singular point x_0, (simple pole) lying on the contour as shown below



we deform the original contour by making a semicircle around x_0. $f(z)$ around x_0 can be written as

$$f(z) = \frac{a_{-1}}{z - x_0} + a_0 + a_1(z - x_0) + \dots$$

For the radius of c_{x_0} approaching zero, we have.

$$\int_{c_{x_0}} f(z)\,dz = a_{-1}\int_{c_{x_0}} \frac{dz}{z - x_0}.$$

~~we can write~~ circ. $z = x_0 + \delta e^{i\phi}$

if ~~cont~~ is ~~counter~~ clockwise,

$$\therefore \int_{C_{x_0}} f(z)dz = a_{-1}\int_{C_{x_0}} \frac{dz}{z-x_0} = a_{-1}\int_{\pi}^{0} \frac{\delta e^{i\phi}\, i\, d\phi}{\delta e^{i\phi}}$$

$= -a_{-1}\pi i$

if C_{x_0} is ^counter clockwise,

$$\int_{C_{x_0}} f(z)dz = a_{-1}\int_{C_{x_0}} \frac{dz}{z-x_0} = a_{-1}\int_{\pi}^{2\pi} \frac{\delta e^{i\phi}\, i\, d\phi}{\delta e^{i\phi}} = a_{-1}\pi i$$

In the limit $R \to \infty$, ~~apply~~ fr. Residue theorem, we have

$$\oint_C f(z)\, dz = 2\pi i \sum \text{enclosed res.}$$

$$\Rightarrow \int_C \text{infinite semicircle} + \int_{-\infty}^{x_0-\delta} f(x)\, dx$$

$$+ \int_{C_{x_0}} f(z)dz + \int_{x_0+\delta}^{\infty} f(x)\, dx = 2\pi i \sum \text{encl. res.}$$

The cauchy principle value of $\int_{-\infty}^{\infty} f(x)\, dx$ is defined as

$$P\int_{-\infty}^{\infty} f(x)dx = \lim_{\delta\to 0} \int_{-\infty}^{x_0-\delta} f(x)\, dx + \int_{x_0+\delta}^{\infty} f(x)\, dx$$

if C_{x_0} is clockwise, then x_0 is ~~incl~~ excluded ~~of~~ by the contour

$\therefore \int_C$ inf. semicircle $+ P\int_{-\alpha}^{\alpha} f(x)\,dx$

$- a_{-1}\pi i = 0$

$\Rightarrow \int_C$ inf. semicircle $+ P\int_{-\alpha}^{\alpha} f(x)\,dx = a_{-1}\pi i$

If C is counterclockwise, then z_0 is included by the contour

$\therefore \int_C$ inf. semicircle $+ P\int_{-\alpha}^{\alpha} f(x)\,dx$

$+ a_{-1}\pi i = 2\pi i\, a_{-1}$

$\Rightarrow \int_C$ inf. semicircle $+ P\int_{-\alpha}^{\alpha} f(x)\,dx = \pi i a_{-1}$

Integrals of the form $\int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$.

(i) Consider $I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta$

For $z = e^{i\theta}$, $dz = e^{i\theta}\, i\, d\theta = z\, i\, d\theta$

$\therefore d\theta = -i\dfrac{dz}{z}$, $\sin\theta = \dfrac{e^{i\theta} - e^{-i\theta}}{2i} = \dfrac{z - z^{-1}}{2i}$,

$\cos\theta = \dfrac{z + z^{-1}}{2}$

$\therefore$ integral becomes

$$I = -i \oint_{\text{unit circle}} f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{z}$$

By residue theorem

$I = -i(2\pi i) \sum$ residues within unit circle.

Example: $I = \int_0^{2\pi} \frac{d\theta}{1+\epsilon\cos\theta}, \quad |\epsilon|<1$

Using previous method, we can write

$$I = \oint_{\text{unit circle}} \frac{1}{1+\epsilon\frac{z+z^{-1}}{2}}\left(-i\frac{dz}{z}\right)$$

$$= -i\frac{2}{\epsilon}\oint \frac{dz}{z^2+\frac{2}{\epsilon}z+1}$$

$z^2+\frac{2}{\epsilon}z+1=0$ has the roots:

$z_- = \frac{-1}{\epsilon} - \frac{1}{\epsilon}\sqrt{1-\epsilon^2}$ and $z_+ = \frac{-1}{\epsilon}+\frac{1}{\epsilon}\sqrt{1-\epsilon^2}$

$\therefore$ z_- lies outside unit circle and z_+ lies inside unit circle.

$$\therefore f(z) = \frac{1}{z^2+\frac{2}{\epsilon}z+1} = \frac{1}{(z-z_-)(z-z_+)}$$

$$= \frac{1}{(z-z_+)}\frac{1}{[z-z_+-(z_--z_+)]}$$

~~for $|z-z_+|$~~

$$= \frac{1}{z_+-z_-}\frac{1}{(z-z_+)}\frac{1}{\left[1+\frac{z-z_+}{z_+-z_-}\right]}$$

$$= \frac{1}{z_+-z_-}\frac{1}{(z-z_+)}\left[1-\frac{z-z_+}{z_+-z_-}+\left(\frac{z-z_+}{z_+-z_-}\right)^2-\cdots\right]$$

$\therefore$ The residue of $f(z)$ at $z=z_+$ is

$$\frac{1}{z_+-z_-}.$$

$$\therefore I = -i\frac{2}{\epsilon}2\pi i\frac{1}{z_+-z_-}$$

$$= \frac{2\pi}{\sqrt{1-\epsilon^2}}$$

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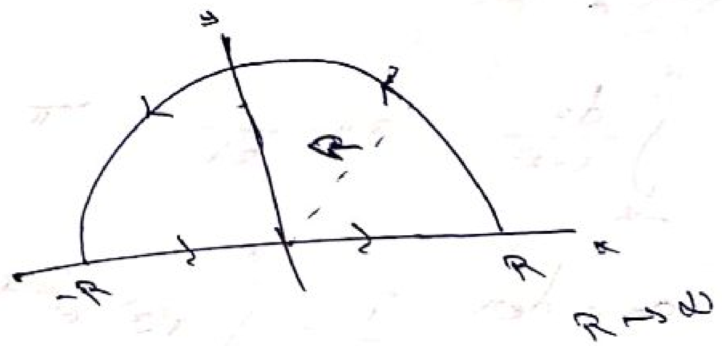
(ii) Consider $I = \int_{-\infty}^{\infty} f(x)\,dx$,

where the function satisfies the two conditions

1. $f(z)$ has finite number of poles.
2. $f(z)$ vanishes as $\frac{1}{z^2}$ for $|z| \to \infty$;

$0 \leq \arg(z) \leq \pi$.

Let us ~~say $f(z)$ has not~~ choose a semicircular contour in the upper half-plane.



From residue theorem,

$$\oint f(z)\,dz = 2\pi i \sum \text{residues in upper half-plane}$$

$$\Rightarrow \lim_{R\to\infty} \int_0^{\pi} f(Re^{i\theta})\, Re^{i\theta}\, i\,d\theta$$

$$+ \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx = 2\pi i \sum \text{residues in upper half-plane}$$

Since for $R \to \infty$, $f(Re^{i\theta})R \sim \frac{1}{R^2} R \to 0$

$\therefore$ we get

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum \text{residues}$$

Example: $I = \int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

consider the ~~integral~~ function

$\cancel{\int \frac{1}{1+z^2}\,dz}$ $\quad f(z) = \frac{1}{1+z^2}$

$$f(z) = \frac{1}{z^2 - i^2} = \frac{1}{(z-i)(z+i)}$$

$f(z)$ has simple poles at i and $-i$ with residues $\frac{1}{2i}$ and $\frac{-1}{2i}$, respectively. Applying the previous method,

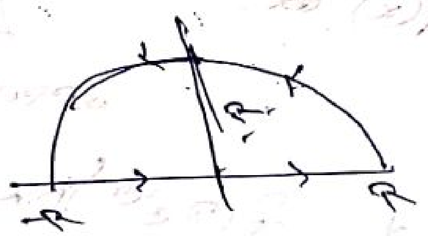
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = 2\pi i \, \frac{1}{2i} = \pi$$

(iii) consider $I = \int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx$,

with a real and positive. The function $f(z)$ has to satisfy the following conditions:

$f(z)$ has finite number of poles and

$$\lim_{|z|\to\infty} f(z) = 0, \quad 0 \le \arg(z) \le \pi$$



$$\oint f(z)\, e^{iaz} = 2\pi i \sum \text{residues}$$

$$\Rightarrow \lim_{R\to\infty} \int_{-R}^{R} f(x)\, e^{iax}\, dx + \lim_{R\to\infty} \int_0^{\pi} f(Re^{i\theta})\, e^{iaR\cos\theta - aR\sin\theta}\, Re^{i\theta}\, i\, d\theta = 2\pi i \sum$$

$$I_R = \int_0^{\pi} f(Re^{i\theta})\, e^{iaR\cos\theta - aR\sin\theta}\, Re^{i\theta}\, i\, d\theta$$

Hence, using $\left|\int f(\theta)\, d\theta\right| \le \int |f(\theta)|\, d\theta$,

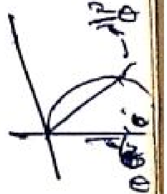
$$|I_R| \le \int_0^{\pi} |f(Re^{i\theta})|\, e^{-aR\sin\theta}\, R\, d\theta$$

Since $\lim_{|z|\to\infty} f(z) = 0$, we can expect $|f(z)|$ be bounded as $R \to \infty$.

Say, $|f(Re^{i\theta})| < \epsilon$ as $R \to \infty$

$$|I_R| \le \epsilon R \int_0^{\pi} e^{-aR\sin\theta}\, d\theta$$

$$\le 2\epsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta$$



For $\theta \in \left[0, \frac{\pi}{2}\right]$, $\frac{2}{\pi}\theta \le \sin\theta$

$$\therefore |I_R| \le 2\epsilon R \int_0^{\pi/2} e^{-aR\frac{2\theta}{\pi}}\, d\theta$$

$$\le 2\epsilon R\, \frac{1 - e^{-aR}}{aR\,2/\pi}$$

$$\therefore \lim_{R\to\infty} |I_R| \le \frac{\pi}{a}\epsilon \to 0$$

$\therefore$ we get

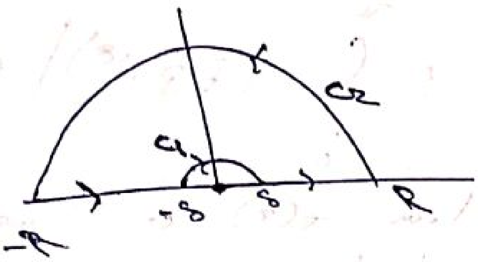
$$\int_{-\infty}^{\infty} f(x)\, e^{iax}\, dx = 2\pi i \sum \text{residues}$$

$$\lim_{R\to\infty} \int_{\text{semicircle}} f(z)\, e^{iaz}\, dz = 0$$

This result is called Jordan's lemma.

Example: $I = \int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx$

consider the integral $\int \frac{e^{iz}}{z}\,dz$ on the below contour.



we have

$$\oint \frac{e^{iz}}{z}\,dz = 0.$$

$$\Rightarrow \int_{C_2} \frac{e^{iz}}{z}\,dz + \int_{-R}^{-\delta} \frac{e^{ix}}{x}\,dx + \int_{C_1} \frac{e^{iz}}{z}\,dz$$

$$+ \int_{\delta}^{R} \frac{e^{ix}}{x}\,dx = 0$$

Using Jordan's lemma, ~~in the~~ with $R \to \infty$,

$$\int_{C_2} \frac{e^{iz}}{z}\,dz = 0.$$

with $R \to \infty$ and $\delta \to 0$,

$$\int_{-R}^{-\delta} \frac{e^{ix}}{x}\,dx + \int_{\delta}^{R} \frac{e^{ix}}{x}\,dx = P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx$$

$$\int_{C_1} \frac{e^{iz}}{z}\,dz = \int_{\pi}^{0} \frac{\left[1 + iz + \frac{(iz)^2}{2!} + \cdots\right]}{z}\,dz$$

on C_1, $z = \delta e^{i\theta}$. in the limit $\delta \to 0$,

$$\int_{C_1} \frac{e^{iz}}{z}\,dz = \int_{\pi} \frac{dz}{z} = -\pi i$$

$\therefore$ we get

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\, dx = \pi i$$

Equating imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi \quad \text{or} \quad \int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

Example: In Quantum Mechanics, we have integrals like

$$I(\sigma) = \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 - \sigma^2}\, dx, \quad \sigma \text{ is real}$$

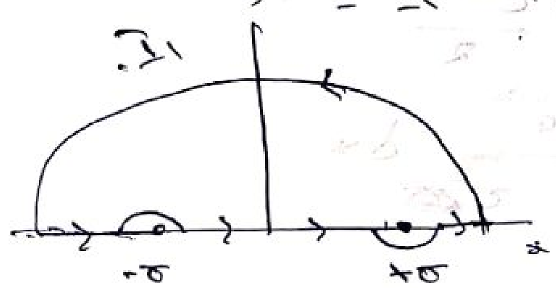
Consider,

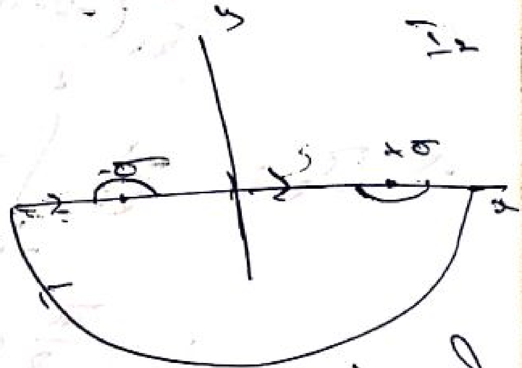
$$I = \int \frac{z \sin z}{z^2 - \sigma^2}\, dz$$

$$\sin z = \frac{1}{2i} e^{iz} - \frac{1}{2i} e^{-iz}$$

$$\therefore I = \frac{1}{2i} \int \frac{z e^{iz}}{z^2 - \sigma^2}\, dz - \frac{1}{2i} \int \frac{z e^{-iz}}{z^2 - \sigma^2}\, dz$$

$$= I_1 + I_2$$





We can perform I_1 and I_2 integrals in a different limiting process.

Take $\sigma \to \sigma + i\delta, \quad \delta \to 0$

$-\sigma \to -\sigma - i\delta$

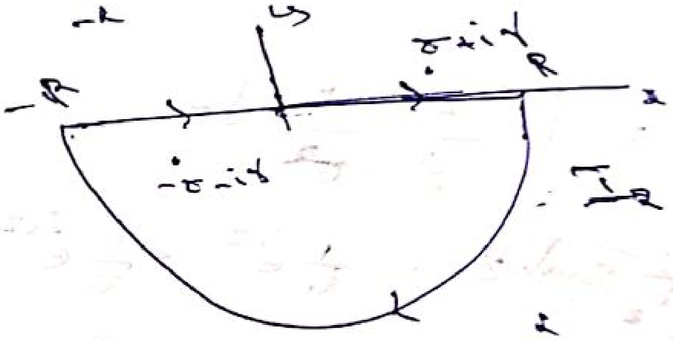
$$\frac{1}{2i}\int_{\text{inf. semicircle}} \frac{z e^{iz}}{z^2-(\sigma+i\delta)^2}\,dz + \frac{1}{2i}\int_{-\alpha}^{\alpha} \frac{x e^{ix}}{x^2-(\sigma+i\delta)^2}\,dx$$

$$= \frac{1}{2i}\,2\pi i \; \cancel{2i}\; \frac{e^{i(\sigma+i\delta)}}{2}$$

$$\Rightarrow I_1 = \frac{1}{2i}\int_{-\alpha}^{\alpha} \frac{x e^{ix}}{x^2-\sigma^2}\,dx = \frac{\pi}{2}e^{i\sigma}$$

$$\frac{z e^{iz}}{z+\sigma+i\delta}$$

$$\frac{-\sigma e^{-i\sigma}}{-2\sigma}$$



$$\frac{1}{2i}\int_{\text{inf. semicircle}} \frac{z e^{-iz}}{z^2-(\sigma+i\delta)^2}\,dz + \frac{1}{2i}\int_{-\alpha}^{\alpha} \frac{x e^{-ix}}{x^2-(\sigma+i\delta)^2}\,dx$$

$$= \frac{1}{2i}(-2\pi i)\; \cancel{2i}\; \frac{e^{+i(\sigma+i\delta)}}{2}$$

$$\Rightarrow I_2 = \frac{1}{2i}\int_{-\alpha}^{\alpha} \frac{x e^{-ix}}{x^2-\sigma^2}\,dx$$

$$= \frac{\pi}{2}e^{i\sigma}$$

$$\therefore\ I = I_1 + I_2 = \pi e^{i\sigma}$$

If you take the limiting process as $\sigma \to \sigma - i\delta$, $\delta \to 0$, then you would get

$I = I_1 + I_2 = \pi e^{i\pi}$

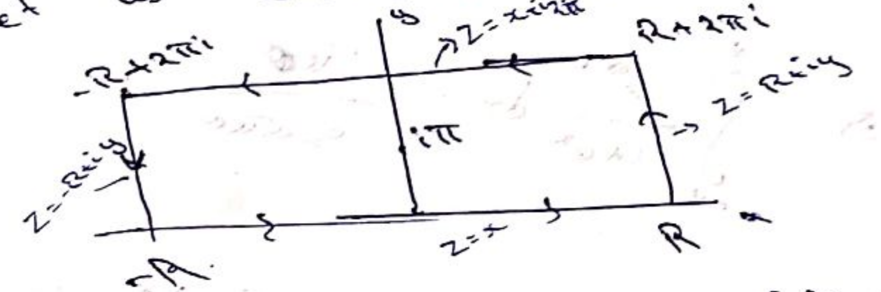
Example: consider $I = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}} dx, \; 0<a<1$

let $f(z) = \frac{e^{az}}{1+e^{z}}$

$e^{z} = -1 \Rightarrow z = \ln(-1) = \ln e^{i(2n+1)\pi}$ $n=0,1,\ldots$

$\therefore z = i(2n+1)\pi$

Let us choose the contour



Residue of $f(z)$ at $i\pi$ ~~can be found from~~

$$f(z) = \frac{e^{a(z-i\pi)}\, e^{ai\pi}}{1+e^{z-i\pi}\, e^{i\pi}}$$

$$= \frac{e^{ia\pi}\left[1 + a(z-i\pi) + \cdots\right]}{1 - \left[1 + (z-i\pi) + \frac{(z-i\pi)^2}{2!} + \cdots\right]}$$

$$= \frac{-e^{ia\pi}}{z-i\pi} + \cdots$$

From residue theorem,

$$\oint f(z)\,dz = 2\pi i\left(-e^{ia\pi}\right)$$

$$\Rightarrow \lim_{R\to\infty}\left[\int_{-R}^{R} \frac{e^{ax}}{1+e^{x}} dx + \int_{R}^{-R} \frac{e^{a(x+i2\pi)}}{1+e^{x+i2\pi}} dx\right.$$

$$\int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{(R+iy)}}\, i\,dy + \int_{2\pi}^{0} \frac{e^{a(-R+iy)}}{1+e^{-R+iy}}\, i\,dy$$

$R \to \infty$

$$= 2\pi i\left(-e^{ia\pi}\right)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}} - e^{ia2\pi} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = -2\pi i\, e^{ia\pi}$$

$$\Rightarrow \left(1 - e^{i2\pi a}\right) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = -2\pi i\, e^{ia\pi}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{x}}\,dx = \frac{\pi}{\sin \pi a}$$

$$\frac{e^{a(R+iy)}}{1+e^{R+iy}} = \frac{e^{aR}\, e^{aiy}}{e^{R}\left[e^{-R} + e^{iy}\right]} = \frac{e^{(a-1)R}\, e^{aiy}}{\left[e^{-R} + iy\right]}$$

$a-1<0$

$$\frac{e^{a(-R+iy)}}{1+e^{-R+iy}} = \frac{e^{-aR}\, e^{aiy}}{1+e^{-R}\, e^{iy}}$$

$0<a<1$