CHAPTER 11

COMPLEX VARIABLE THEORY

The imaginary numbers are a wonderful flight of God's spirit; they are almost an amphibian between being and not being.

GOTTFRIED WILHELM VON LEIBNIZ, 1702

We turn now to a study of complex variable theory. In this area we develop some of the most powerful and widely useful tools in all of analysis. To indicate, at least partly, why complex variables are important, we mention briefly several areas of application.

1. In two dimensions, the electric potential, viewed as a solution of Laplace's equation, can be written as the real (or the imaginary) part of a complex-valued function, and this identification enables the use of various features of complex variable theory (specifically, conformal mapping) to obtain formal solutions to a wide variety of electrostatics problems.
2. The time-dependent Schrödinger equation of quantum mechanics contains the imaginary unit i, and its solutions are complex.
3. In Chapter 9 we saw that the second-order differential equations of interest in physics may be solved by power series. The same power series may be used in the complex plane to replace x by the complex variable z. The dependence of the solution $f(z)$ at a given z_0 on the behavior of $f(z)$ elsewhere gives us greater insight into the behavior of our solution and a powerful tool (analytic continuation) for extending the region in which the solution is valid.
4. The change of a parameter k from real to imaginary, $k \to ik$, transforms the Helmholtz equation into the time-independent diffusion equation. The same change connects the spherical and hyperbolic trigonometric functions, transforms Bessel functions into their *modified* counterparts, and provides similar connections between other superficially dissimilar functions.

5. Integrals in the complex plane have a wide variety of useful applications:
   - Evaluating definite integrals and infinite series,
   - Inverting power series,
   - Forming infinite products,
   - Obtaining solutions of differential equations for large values of the variable (asymptotic solutions),
   - Investigating the stability of potentially oscillatory systems,
   - Inverting integral transforms.
6. Many physical quantities that were originally real become complex as a simple physical theory is made more general. The real index of refraction of light becomes a complex quantity when absorption is included. The real energy associated with an energy level becomes complex when the finite lifetime of the level is considered.

11.1 COMPLEX VARIABLES AND FUNCTIONS

We have already seen (in Chapter 1) the definition of complex numbers $z = x + iy$ as ordered pairs of two real numbers, x and y. We reviewed there the rules for their arithmetic operations, identified the **complex conjugate** z^* of the complex number z, and discussed both the Cartesian and polar representations of complex numbers, introducing for that purpose the **Argand diagram** (complex plane). In the polar representation $z = re^{i\theta}$, we noted that r (the magnitude of the complex number) is also called its **modulus**, and the angle θ is known as its **argument**. We proved that $e^{i\theta}$ satisfies the important equation

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{11.1}$$

This equation shows that for real θ, $e^{i\theta}$ is of unit magnitude and is therefore situated on the unit circle, at an angle θ from the real axis.

Our focus in the present chapter is on functions of a complex variable and on their analytical properties. We have already noted that by defining complex functions $f(z)$ to have the same power-series expansion (in z) as the expansion (in x) of the corresponding real function $f(x)$, the real and complex definitions coincide when z is real. We also showed that by use of the polar representation, $z = re^{i\theta}$, it becomes clear how to compute powers and roots of complex quantities. In particular, we noted that roots, viewed as fractional powers, become **multivalued** functions in the complex domain, due to the fact that $\exp(2n\pi i) = 1$ for all positive and negative integers n. We thus found $z^{1/2}$ to have two values (not a surprise, since for positive real x, we have $\pm\sqrt{x}$). But we also noted that $z^{1/m}$ will have m different complex values. We also noted that the logarithm becomes multivalued when extended to complex values, with

$$\ln z = \ln(re^{i\theta}) = \ln r + i(\theta + 2n\pi), \tag{11.2}$$

with n any positive or negative integer (including zero).

If necessary, the reader should review the topics mentioned above by rereading Section 1.8.

11.2 CAUCHY-RIEMANN CONDITIONS

Having established complex functions of a complex variable, we now proceed to differentiate them. The derivative of $f(z)$, like that of a real function, is defined by

$$\lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{(z+\delta z) - z} = \lim_{\delta z \to 0} \frac{\delta f(z)}{\delta z} = \frac{df}{dz} = f'(z), \tag{11.3}$$

provided that the limit is **independent** of the particular approach to the point z. For real variables we require that the right-hand limit ($x \to x_0$ from above) and the left-hand limit ($x \to x_0$ from below) be equal for the derivative $df(x)/dx$ to exist at $x = x_0$. Now, with z (or z_0) some point in a plane, our requirement that the limit be independent of the direction of approach is very restrictive.

Consider increments δx and δy of the variables x and y, respectively. Then

$$\delta z = \delta x + i\delta y. \tag{11.4}$$

Also, writing $f = u + iv$,

$$\delta f = \delta u + i\delta v, \tag{11.5}$$

so that

$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}. \tag{11.6}$$

Let us take the limit indicated by Eq. (11.3) by two different approaches, as shown in Fig. 11.1. First, with $\delta y = 0$, we let $\delta x \to 0$. Equation (11.3) yields

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x} \right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{11.7}$$

assuming that the partial derivatives exist. For a second approach, we set $\delta x = 0$ and then let $\delta y \to 0$. This leads to

$$\lim_{\delta z \to 0} \frac{\delta f}{\delta z} = \lim_{\delta y \to 0} \left(-i\frac{\delta u}{\delta y} + \frac{\delta v}{\delta y} \right) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \tag{11.8}$$

If we are to have a derivative df/dz, Eqs. (11.7) and (11.8) must be identical. Equating real parts to real parts and imaginary parts to imaginary parts (like components of vectors), we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{11.9}$$

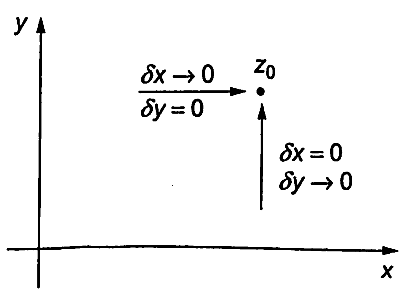

FIGURE 11.1 Alternate approaches to z_0.

These are the famous **Cauchy-Riemann** conditions. They were discovered by Cauchy and used extensively by Riemann in his development of complex variable theory. These Cauchy-Riemann conditions are necessary for the existence of a derivative of $f(z)$. That is, in order for df/dz to exist, the Cauchy-Riemann conditions must hold.

Conversely, if the Cauchy-Riemann conditions are satisfied and the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous, the derivative df/dz exists. To show this, we start by writing

$$\delta f = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y, \tag{11.10}$$

where the justification for this expression depends on the continuity of the partial derivatives of u and v. Using the Cauchy-Riemann equations, Eq. (11.9), we convert Eq. (11.10) to the form

$$\begin{aligned}\delta f &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y \\ &= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\delta y).\end{aligned} \tag{11.11}$$

Replacing $\delta x + i\delta y$ by δz and bringing it to the left-hand side of Eq. (11.11), we reach

$$\frac{\delta f}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}, \tag{11.12}$$

an equation whose right-hand side is independent of the direction of δz (i.e., the relative values of δx and δy). This independence of directionality meets the condition for the existence of the derivative, df/dz.

Analytic Functions

If $f(z)$ is differentiable and single-valued in a region of the complex plane, it is said to be an **analytic** function in that region.[1] Multivalued functions can also be analytic under certain restrictions that make them single-valued in specific regions; this case, which is of great importance, is taken up in detail in Section 11.6. If $f(z)$ is analytic everywhere in the (finite) complex plane, we call it an **entire** function. Our theory of complex variables here is one of analytic functions of a complex variable, which points up the crucial importance of the Cauchy-Riemann conditions. The concept of analyticity carried on in advanced theories of modern physics plays a crucial role in the dispersion theory (of elementary particles). If $f'(z)$ does not exist at $z = z_0$, then z_0 is labeled a **singular point**; singular points and their implications will be discussed shortly.

To illustrate the Cauchy-Riemann conditions, consider two very simple examples.

[1] Some writers use the term **holomorphic** or **regular**.

Example 11.2.1 z^2 IS ANALYTIC

Let $f(z) = z^2$. Multiplying out $(x - iy)(x - iy) = x^2 - y^2 + 2ixy$, we identify the real part of z^2 as $u(x, y) = x^2 - y^2$ and its imaginary part as $v(x, y) = 2xy$. Following Eq. (11.9),

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}.$$

We see that $f(z) = z^2$ satisfies the Cauchy-Riemann conditions throughout the complex plane. Since the partial derivatives are clearly continuous, we conclude that $f(z) = z^2$ is analytic, and is an entire function. ■

Example 11.2.2 z^* IS NOT ANALYTIC

Let $f(z) = z^*$, the complex conjugate of z. Now $u = x$ and $v = -y$. Applying the Cauchy-Riemann conditions, we obtain

$$\frac{\partial u}{\partial x} = 1 \neq \frac{\partial v}{\partial y} = -1.$$

The Cauchy-Riemann conditions are not satisfied for any values of x or y and $f(z) = z^*$ is nowhere an analytic function of z. It is interesting to note that $f(z) = z^*$ is continuous, thus providing an example of a function that is everywhere continuous but nowhere differentiable in the complex plane. ■

The derivative of a real function of a real variable is essentially a local characteristic, in that it provides information about the function only in a local neighborhood, for instance, as a truncated Taylor expansion. The existence of a derivative of a function of a complex variable has much more far-reaching implications, one of which is that the real and imaginary parts of our analytic function must separately satisfy Laplace's equation in two dimensions, namely

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

To verify the above statement, we differentiate the first Cauchy-Riemann equation in Eq. (11.9) with respect to x and the second with respect to y, obtaining

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

Combining these two equations, we easily reach

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \tag{11.13}$$

confirming that $u(x, y)$, the real part of a differentiable complex function, satisfies the Laplace equation. Either by recognizing that if $f(z)$ is differentiable, so is $-if(z) = v(x, y) - iu(x, y)$, or by steps similar to those leading to Eq. (11.13), we can confirm that $v(x, y)$ also satisfies the two-dimensional (2-D) Laplace equation. Sometimes u and v are referred to as **harmonic functions** (not to be confused with **spherical harmonics**, which we will later encounter as the angular solutions to central force problems).

The solutions $u(x,y)$ and $v(x,y)$ are complementary in that the curves of constant $u(x,y)$ make orthogonal intersections with the curves of constant $v(x,y)$. To confirm this, note that if (x_0, y_0) is on the curve $u(x,y)=c$, then x_0+dx, y_0+dy is also on that curve if

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0,$$

meaning that the slope of the curve of constant u at (x_0, y_0) is

$$\left(\frac{dy}{dx}\right)_u = \frac{-\partial u/\partial x}{\partial u/\partial y}, \tag{11.14}$$

where the derivatives are to be evaluated at (x_0, y_0). Similarly, we can find that the slope of the curve of constant v at (x_0, y_0) is

$$\left(\frac{dy}{dx}\right)_v = \frac{-\partial v/\partial x}{\partial v/\partial y} = \frac{\partial u/\partial y}{\partial u/\partial x}, \tag{11.15}$$

where the last member of Eq. (11.15) was reached using the Cauchy-Riemann equations. Comparing Eqs. (11.14) and (11.15), we note that at the same point, the slopes they describe are orthogonal (to check, verify that $dx_u dx_v + dy_u dy_v = 0$).

The properties we have just examined are important for the solution of 2-D electrostatics problems (governed by the Laplace equation). If we have identified (by methods outside the scope of the present text) an appropriate analytic function, its lines of constant u will describe electrostatic equipotentials, while those of constant v will be the stream lines of the electric field.

Finally, the global nature of our analytic function is also illustrated by the fact that it has not only a first derivative, but in addition, derivatives of all higher orders, a property which is not shared by functions of a real variable. This property will be demonstrated in Section 11.4.

Derivatives of Analytic Functions

Working with the real and imaginary parts of an analytic function $f(z)$ is one way to take its derivative; an example of that approach is to use Eq. (11.12). However, it is usually easier to use the fact that complex differentiation follows the same rules as those for real variables. As a first step in establishing this correspondence, note that, **if $f(z)$ is analytic,** then, from Eq. (11.12),

$$f'(z) = \frac{\partial f}{\partial x},$$

and that

$$\begin{aligned}\left[f(z)g(z)\right]' &= \left(\frac{d}{dz}\right)\left[f(z)g(z)\right] = \left(\frac{\partial}{\partial x}\right)\left[f(z)g(z)\right]\\ &= \left(\frac{\partial f}{\partial x}\right)g(z) + f(z)\left(\frac{\partial g}{\partial x}\right) = f'(z)g(z) + f(z)g'(z),\end{aligned}$$

the familiar rule for differentiating a product. Given also that

$$\frac{dz}{dz} = \frac{\partial z}{\partial x} = 1,$$

we can easily establish that

$$\frac{dz^2}{dz} = 2z, \quad \text{and, by induction,} \quad \frac{dz^n}{dz} = nz^{n-1}.$$

Functions defined by power series will then have differentiation rules identical to those for the real domain. Functions not ordinarily defined by power series also have the same differentiation rules as for the real domain, but that will need to be demonstrated case by case. Here is an example that illustrates the establishment of a derivative formula.

Example 11.2.3 DERIVATIVE OF LOGARITHM

We want to verify that $d \ln z/dz = 1/z$. Writing, as in Eq. (1.138),

$$\ln z = \ln r + i\theta + 2n\pi i,$$

we note that if we write $\ln z = u + iv$, we have $u = \ln r$, $v = \theta + 2n\pi$. To check whether $\ln z$ satisfies the Cauchy-Riemann equations, we evaluate

$$\frac{\partial u}{\partial x} = \frac{1}{r}\frac{\partial r}{\partial x} = \frac{x}{r^2}, \qquad \frac{\partial u}{\partial y} = \frac{1}{r}\frac{\partial r}{\partial y} = \frac{y}{r^2},$$

$$\frac{\partial v}{\partial x} = \frac{\partial \theta}{\partial x} = \frac{-y}{r^2}, \qquad \frac{\partial v}{\partial y} = \frac{\partial \theta}{\partial y} = \frac{x}{r^2}.$$

The derivatives of r and θ with respect to x and y are obtained from the equations connecting Cartesian and polar coordinates. Except at $r = 0$, where the derivatives are undefined, the Cauchy-Riemann equations can be confirmed.

Then, to obtain the derivative, we can simply apply Eq. (11.12),

$$\frac{d \ln z}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{x - iy}{r^2} = \frac{1}{x + iy} = \frac{1}{z}.$$

Because $\ln z$ is multivalued, it will not be analytic except under conditions restricting it to single-valuedness in a specific region. This topic will be taken up in Section 11.6. ■

Point at Infinity

In complex variable theory, infinity is regarded as a single point, and behavior in its neighborhood is discussed after making a change of variable from z to $w = 1/z$. This transformation has the effect that, for example, $z = -R$, with R large, lies in the w plane close to $z = +R$, thereby among other things influencing the values computed for derivatives. An elementary consequence is that entire functions, such as z or e^z, have singular points at $z = \infty$. As a trivial example, note that at infinity the behavior of z is identified as that of $1/w$ as $w \to 0$, leading to the conclusion that z is singular there.

Exercises

11.2.1 Show whether or not the function $f(z) = \Re(z) = x$ is analytic.

11.2.2 Having shown that the real part $u(x, y)$ and the imaginary part $v(x, y)$ of an analytic function $w(z)$ each satisfy Laplace's equation, show that neither $u(x, y)$ nor $v(x, y)$ **can have either a maximum or a minimum** in the interior of any region in which $w(z)$ is analytic. (They can have saddle points only.)

11.2.3 Find the analytic function

$$w(z) = u(x, y) + iv(x, y)$$

(a) if $u(x, y) = x^3 - 3xy^2$, (b) if $v(x, y) = e^{-y} \sin x$.

11.2.4 If there is some common region in which $w_1 = u(x, y) + iv(x, y)$ and $w_2 = w_1^* = u(x, y) - iv(x, y)$ are both analytic, prove that $u(x, y)$ and $v(x, y)$ are constants.

11.2.5 Starting from $f(z) = 1/(x + iy)$, show that $1/z$ is analytic in the entire finite z plane except at the point $z = 0$. This extends our discussion of the analyticity of z^n to negative integer powers n.

11.2.6 Show that given the Cauchy-Riemann equations, the derivative $f'(z)$ has the same value for $dz = a\,dx + ib\,dy$ (with neither a nor b zero) as it has for $dz = dx$.

11.2.7 Using $f(re^{i\theta}) = R(r, \theta)e^{i\Theta(r,\theta)}$, in which $R(r, \theta)$ and $\Theta(r, \theta)$ are differentiable real functions of r and θ, show that the Cauchy-Riemann conditions in polar coordinates become

(a) $\displaystyle \frac{\partial R}{\partial r} = \frac{R}{r}\frac{\partial \Theta}{\partial \theta}$, (b) $\displaystyle \frac{1}{r}\frac{\partial R}{\partial \theta} = -R\frac{\partial \Theta}{\partial r}$.

Hint. Set up the derivative first with δz radial and then with δz tangential.

11.2.8 As an extension of Exercise 11.2.7 show that $\Theta(r, \theta)$ satisfies the 2-D Laplace equation in polar coordinates,

$$\frac{\partial^2 \Theta}{\partial r^2} + \frac{1}{r}\frac{\partial \Theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} = 0.$$

11.2.9 For each of the following functions $f(z)$, find $f'(z)$ and identify the maximal region within which $f(z)$ is analytic.

(a) $\displaystyle f(z) = \frac{\sin z}{z}$, (d) $f(z) = e^{-1/z}$,

(b) $\displaystyle f(z) = \frac{1}{z^2 + 1}$, (e) $f(z) = z^2 - 3z + 2$,

(c) $\displaystyle f(z) = \frac{1}{z(z + 1)}$, (f) $f(z) = \tan(z)$,

(g) $f(z) = \tanh(z)$.

11.2.10 For what complex values do each of the following functions $f(z)$ have a derivative?

(a) $f(z) = z^{3/2}$,
(b) $f(z) = z^{-3/2}$,
(c) $f(z) = \tan^{-1}(z)$,
(d) $f(z) = \tanh^{-1}(z)$.

11.2.11 Two-dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x, v) + iv(x, y)$. We label the real part, $u(x, y)$, the velocity potential, and the imaginary part, $v(x, y)$, the stream function. The fluid velocity $\mathbf{V}$ is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic:

(a) Show that $df/dz = V_x - iV_y$.
(b) Show that $\nabla \cdot \mathbf{V} = 0$ (no sources or sinks).
(c) Show that $\nabla \times \mathbf{V} = 0$ (irrotational, nonturbulent flow).

11.2.12 The function $f(z)$ is analytic. Show that the derivative of $f(z)$ with respect to z^* does not exist unless $f(z)$ is a constant.

Hint. Use the chain rule and take $x = (z + z^*)/2$, $y = (z - z^*)/2i$.

Note. This result emphasizes that our analytic function $f(z)$ is not just a complex function of two real variables x and y. It is a function of the complex variable $x + iy$.

11.3 CAUCHY'S INTEGRAL THEOREM

Contour Integrals

With differentiation under control, we turn to integration. The integral of a complex variable over a path in the complex plane (known as a **contour**) may be defined in close analogy to the (Riemann) integral of a real function integrated along the real x-axis.

We divide the contour, from z_0 to z_0', designated C, into n intervals by picking $n - 1$ intermediate points $z_1, z_2, \ldots$ on the contour (Fig. 11.2). Consider the sum

$$S_n = \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}),$$

where ζ_j is a point on the curve between z_j and z_{j-1}. Now let $n \to \infty$ with

$$|z_j - z_{j-1}| \to 0$$

for all j. If $\lim_{n\to\infty} S_n$ exists, then

$$\lim_{n\to\infty} \sum_{j=1}^{n} f(\zeta_j)(z_j - z_{j-1}) = \int_{z_0}^{z_0'} f(z)\,dz = \int_C f(z)\,dz. \tag{11.16}$$

The right-hand side of Eq. (11.16) is called the contour integral of $f(z)$ (along the specified contour C from $z = z_0$ to $z = z_0'$).

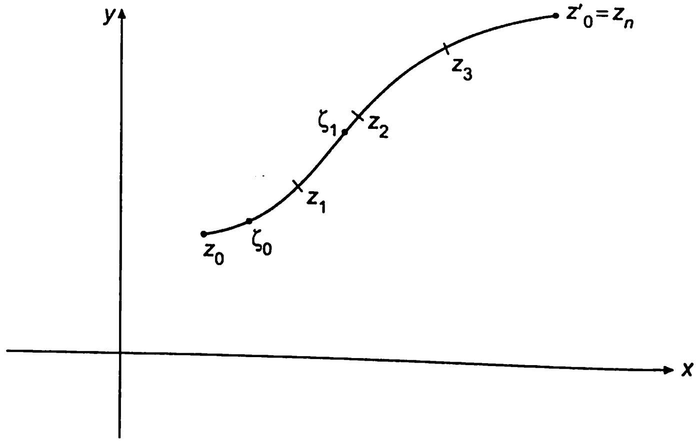

FIGURE 11.2 Integration path.

As an alternative to the above, the contour integral may be defined by

$$\int_{z_1}^{z_2} f(z)dz = \int_{x_1,y_1}^{x_2,y_2} [u(x,y) + iv(x,y)][dx + i\,dy]$$

$$= \int_{x_1,y_1}^{x_2,y_2} [u(x,y)dx - v(x,y)dy] + i \int_{x_1,y_1}^{x_2,y_2} [v(x,y)dx + u(x,y)dy], \qquad (11.17)$$

with the path joining (x_1, y_1) and (x_2, y_2) specified. This reduces the complex integral to the complex sum of real integrals. It is somewhat analogous to the replacement of a vector integral by the vector sum of scalar integrals.

Often we are interested in contours that are **closed**, meaning that the start and end of the contour are at the same point, so that the contour forms a closed loop. We normally define the region enclosed by a contour as that which lies to the left when the contour is traversed in the indicated direction; thus a contour intended to surround a finite area will normally be deemed to be traversed in the counterclockwise direction. If the origin of a polar coordinate system is within the contour, this convention will cause the normal direction of travel on the contour to be that in which the polar angle θ increases.

Statement of Theorem

Cauchy's integral theorem states that:

> *If f(z) is an analytic function at all points of a simply connected region in the complex plane and if C is a closed contour within that region, then*

$$\oint_C f(z)\,dz = 0. \qquad (11.18)$$

To clarify the above, we need the following definition:

- A region is **simply connected** if every closed curve within it can be shrunk continuously to a point that is within the region.

In everyday language, a simply connected region is one that has no holes. We also need to explain that the symbol $\oint$ will be used from now on to indicate an integral over a closed contour; a subscript (such as C) is attached when further specification of the contour is desired. Note also that for the theorem to apply, the contour must be "within" the region of analyticity. That means it cannot be on the boundary of the region.

Before proving Cauchy's integral theorem, we look at some examples that do (and do not) meet its conditions.

Example 11.3.1 z^n ON CIRCULAR CONTOUR

Let's examine the contour integral $\oint_C z^n dz$, where C is a circle of radius $r > 0$ around the origin $z = 0$ in the positive mathematical sense (counterclockwise). In polar coordinates, cf. Eq. (1.125), we parameterize the circle as $z = re^{i\theta}$ and $dz = ire^{i\theta}d\theta$. For $n \neq -1$, n an integer, we then obtain

$$\oint_C z^n dz = i\, r^{n+1} \int_0^{2\pi} \exp[i(n+1)\theta]\, d\theta$$

$$= i\, r^{n+1} \left[\frac{e^{i(n+1)\theta}}{i(n+1)} \right]_0^{2\pi} = 0 \tag{11.19}$$

because 2π is a period of $e^{i(n+1)\theta}$. However, for $n = -1$

$$\oint_C \frac{dz}{z} = i \int_0^{2\pi} d\theta = 2\pi i, \tag{11.20}$$

independent of r but nonzero.

The fact that Eq. (11.19) is satisfied for all integers $n \geq 0$ is required by Cauchy's theorem, because for these n values z^n is analytic for all finite z, and certainly for all points within a circle of radius r. Cauchy's theorem does not apply for any negative integer n because, for these n, z^n is singular at $z = 0$. The theorem therefore does not prescribe any particular values for the integrals of negative n. We see that one such integral (that for $n = -1$) has a nonzero value, and that others (for integral $n \neq -1$) do vanish. ■

Example 11.3.2 z^n ON SQUARE CONTOUR

We next examine the integration of z^n for a different contour, a square with vertices at $\pm\frac{1}{2} \pm \frac{1}{2}i$. It is somewhat tedious to perform this integration for general integer n, so we illustrate only with $n = 2$ and $n = -1$.

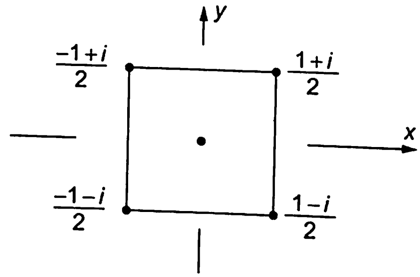

FIGURE 11.3 Square integration contour.

For $n = 2$, we have $z^2 = x^2 - y^2 + 2ixy$. Referring to Fig. 11.3, we identify the contour as consisting of four line segments. On Segment 1, $dz = dx$ ($y = -\frac{1}{2}$ and $dy = 0$); on Segment 2, $dz = i\,dy$, $x = \frac{1}{2}$, $dx = 0$; on Segment 3, $dz = dx$, $y = \frac{1}{2}$, $dy = 0$; and on Segment 4, $dz = i\,dy$, $x = -\frac{1}{2}$, $dx = 0$. Note that for Segments 3 and 4 the integration is in the direction of decreasing value of the integration variable. These segments therefore contribute as follows to the integral:

$$\text{Segment 1:} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} dx\left(x^2 - \tfrac{1}{4} - ix\right) = \frac{1}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] - \frac{1}{4} - \frac{i}{2}(0) = -\frac{1}{6},$$

$$\text{Segment 2:} \quad \int_{-\frac{1}{2}}^{\frac{1}{2}} i\,dy\left(\tfrac{1}{4} - y^2 + iy\right) = \frac{i}{4} - \frac{i}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] - \frac{1}{2}(0) = \frac{i}{6},$$

$$\text{Segment 3:} \quad \int_{\frac{1}{2}}^{-\frac{1}{2}} (dx)\left(x^2 - \tfrac{1}{4} + ix\right) = -\frac{1}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] + \frac{1}{4} - \frac{i}{2}(0) = \frac{1}{6},$$

$$\text{Segment 4:} \quad \int_{\frac{1}{2}}^{-\frac{1}{2}} (i\,dy)\left(\tfrac{1}{4} - y^2 - iy\right) = -\frac{i}{4} + \frac{i}{3}\left[\frac{1}{8} - \left(-\frac{1}{8}\right)\right] - \frac{1}{2}(0) = -\frac{i}{6}.$$

We find that the integral of z^2 over the square vanishes, just as it did over the circle. This is required by Cauchy's theorem.

For $n = -1$, we have, in Cartesian coordinates,

$$z^{-1} = \frac{x - iy}{x^2 + y^2},$$

and the integral over the four segments of the square contour takes the form

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{x+i/2}{x^2+\frac{1}{4}}\,dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\frac{1}{2}-iy}{y^2+\frac{1}{4}}(i\,dy) + \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{x-i/2}{x^2+\frac{1}{4}}\,dx + \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{\frac{1}{2}+iy}{y^2+\frac{1}{4}}(i\,dy).$$

Several of the terms vanish because they involve the integration of an odd integrand over an even interval, and others simply cancel. All that remains is

$$\int_{\square} z^{-1}dz = i\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{dx}{x^2+\frac{1}{4}} = 2i\int_{-1}^{1} \frac{du}{u^2+1} = 2i\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = 2\pi i,$$

the same result as was obtained for the integration of z^{-1} around a circle of any radius. Cauchy's theorem does not apply here, so the nonzero result is not problematic. ■

Cauchy's Theorem: Proof

We now proceed to a proof of Cauchy's integral theorem. The proof we offer is subject to a restriction originally accepted by Cauchy but later shown unnecessary by Goursat. What we need to show is that

$$\oint_C f(z)\,dz = 0,$$

subject to the requirement that C is a closed contour within a simply connected region R where $f(z)$ is analytic. See Fig. 11.4. The restriction needed for Cauchy's (and the present) proof is that if we write $f(z) = u(x, y) + iv(x, y)$, the partial derivatives of u and v are continuous.

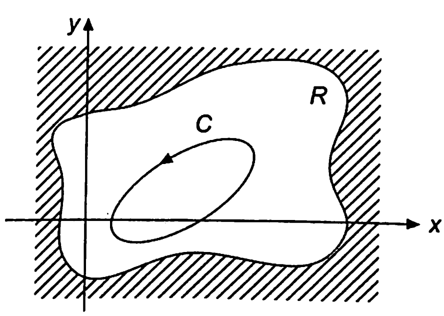

FIGURE 11.4 A closed-contour C within a simply connected region R.

We intend to prove the theorem by direct application of Stokes' theorem (Section 3.8). Writing $dz = dx + i\,dy$,

$$\oint_C f(z)\,dz = \oint_C (u + iv)(dx + i\,dy)$$

$$= \oint_C (u\,dx - v\,dy) + i \oint_C (v\,dx + u\,dy). \tag{11.21}$$

These two line integrals may be converted to surface integrals by Stokes' theorem, a procedure that is justified because we have assumed the partial derivatives to be continuous within the area enclosed by C. In applying Stokes' theorem, note that the final two integrals of Eq. (11.21) are real.

To proceed further, we note that all the integrals involved here can be identified as having integrands of the form $(V_x\hat{\mathbf{e}}_x + V_y\hat{\mathbf{e}}_y) \cdot d\mathbf{r}$, the integration is around a loop in the xy plane, and the value of the integral will be the surface integral, over the enclosed area, of the z component of $\nabla \times (V_x\hat{\mathbf{e}}_x + V_y\hat{\mathbf{e}}_y)$. Thus, Stokes' theorem says that

$$\oint_C (V_x\,dx + V_y\,dy) = \int_A \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx\,dy, \tag{11.22}$$

with A being the 2-D region enclosed by C.

For the first integral in the second line of Eq. (11.21), let $u = V_x$ and $v = -V_y$.[2] Then

$$\oint_C (u\,dx - v\,dy) = \oint_C (V_x\,dx + V_y\,dy)$$

$$= \int_A \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) dx\,dy = -\int_A \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy. \tag{11.23}$$

For the second integral on the right side of Eq. (11.21) we let $u = V_y$ and $v = V_x$. Using Stokes' theorem again, we obtain

$$\oint_C (v\,dx + u\,dy) = \int_A \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy. \tag{11.24}$$

Inserting Eqs. (11.23) and (11.24) into Eq. (11.21), we now have

$$\oint_C f(z)\,dz = -\int_A \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy + i\int_A \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy = 0. \tag{11.25}$$

Remembering that $f(z)$ has been assumed analytic, we find that both the surface integrals in Eq. (11.25) are zero because application of the Cauchy-Riemann equations causes their integrands to vanish. This establishes the theorem.

[2] For Stokes' theorem, V_x and V_y are any two functions with continuous partial derivatives, and they need not be connected by any relations stemming from complex variable theory.

Multiply Connected Regions

The original statement of Cauchy's integral theorem demanded a simply connected region of analyticity. This restriction may be relaxed by the creation of a barrier, a narrow region we choose to exclude from the region identified as analytic. The purpose of the barrier construction is to permit, within a multiply connected region, the identification of curves that can be shrunk to a point within the region, that is, the construction of a subregion that is simply connected.

Consider the multiply connected region of Fig. 11.5, in which $f(z)$ is only analytic in the unshaded area labeled R. Cauchy's integral theorem is not valid for the contour C, as shown, but we can construct a contour C' for which the theorem holds. We draw a barrier from the interior forbidden region, R', to the forbidden region exterior to R and then run a new contour, C', as shown in Fig. 11.6.

The new contour, C', through $ABDEFGA$, never crosses the barrier that converts R into a simply connected region. Incidentally, the three-dimensional analog of this technique was used in Section 3.9 to prove Gauss' law. Because $f(z)$ is in fact continuous across the barrier dividing DE from GA and the line segments DE and GA can be arbitrarily close together, we have

$$\int_G^A f(z)\,dz = -\int_E^D f(z)\,dz. \tag{11.26}$$

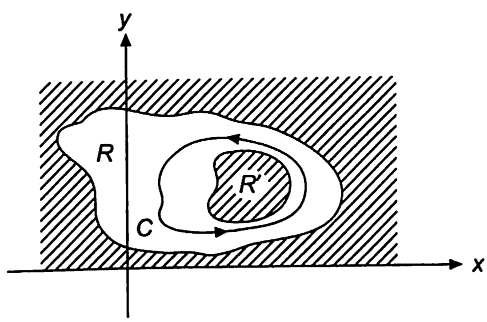

FIGURE 11.5 A closed contour C in a multiply connected region.

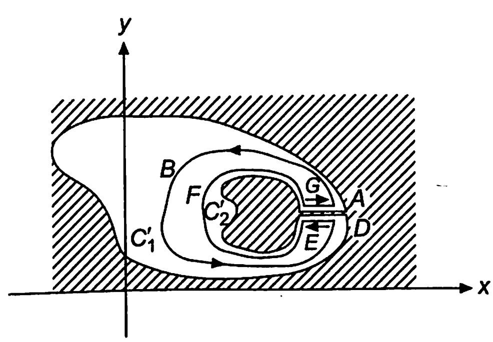

FIGURE 11.6 Conversion of a multiply connected region into a simply connected region.

Then, invoking Cauchy's integral theorem, because the contour is now within a simply connected region, and using Eq. (11.26) to cancel the contributions of the segments along the barrier,

$$\oint_{C'} f(z)\,dz = \int_{ABD} f(z)\,dz + \int_{EFG} f(z)\,dz = 0. \tag{11.27}$$

Now that we have established Eq. (11.27), we note that A and D are only infinitesimally separated and that $f(z)$ is actually continuous across the barrier. Hence, integration on the path ABD will yield the same result as a truly closed contour $ABDA$. Similar remarks apply to the path EFG, which can be replaced by $EFGE$. Renaming $ABDA$ as C_l' and $EFGE$ as $-C_2'$, we have the simple result

$$\oint_{C_1'} f(z)dz = \oint_{C_2'} f(z)dz, \tag{11.28}$$

in which C_1' and C_2' are both traversed in the same (counterclockwise, that is, positive) direction.

This result calls for some interpretation. What we have shown is that the integral of an analytic function over a closed contour surrounding an "island" of nonanalyticity can be subjected to any continuous deformation within the region of analyticity without changing the value of the integral. The notion of *continuous deformation* means that the change in contour must be able to be carried out via a series of small steps, which precludes processes whereby we "jump over" a point or region of nonanalyticity. Since we already know that the integral of an analytic function over a contour in a simply connected region of analyticity has the value zero, we can make the more general statement

The integral of an analytic function over a closed path has a value that remains unchanged over all possible continuous deformations of the contour within the region of analyticity.

Looking back at the two examples of this section, we see that the integrals of z^2 vanished for both the circular and square contours, as prescribed by Cauchy's integral theorem for an analytic function. The integrals of z^{-1} did not vanish, and vanishing was not required because there was a point of nonanalyticity within the contours. However, the integrals of z^{-1} for the two contours had the same value, as either contour can be reached by continuous deformation of the other.

We close this section with an extremely important observation. By a trivial extension to Example 11.3.1 plus the fact that closed contours in a region of analyticity can be deformed continuously without altering the value of the integral, we have the valuable and useful result:

The integral of $(z - z_0)^n$ around any counterclockwise closed path C that encloses z_0 has, for any integer n, the values

$$\oint_C (z - z_0)^n\,dz = \begin{cases} 0, & n \neq -1, \\ 2\pi i, & n = -1. \end{cases} \tag{11.29}$$

Exercises

11.3.1 Show that $\int\limits_{z_1}^{z_2} f(z)\,dz = -\int\limits_{z_2}^{z_1} f(z)\,dz.$

11.3.2 Prove that $\left|\int\limits_C f(z)dz\right| \le |f|_{\max} \cdot L,$

where $|f|_{\max}$ is the maximum value of $|f(z)|$ along the contour C and L is the length of the contour.

11.3.3 Show that the integral

$$\int\limits_{3+4i}^{4-3i} (4z^2 - 3iz)\,dz$$

has the same value on the two paths: (a) the straight line connecting the integration limits, and (b) an arc on the circle $|z| = 5$.

11.3.4 Let $F(z) = \int\limits_{\pi(1+i)}^{z} \cos 2\zeta\, d\zeta.$

Show that $F(z)$ is independent of the path connecting the limits of integration, and evaluate $F(\pi i)$.

11.3.5 Evaluate $\oint_C (x^2 - iy^2)\,dz$, where the integration is (a) clockwise around the unit circle, (b) on a square with vertices at $\pm 1 \pm i$. Explain why the results of parts (a) and (b) are or are not identical.

11.3.6 Verify that

$$\int\limits_0^{1+i} z^* dz$$

depends on the path by evaluating the integral for the two paths shown in Fig. 11.7. Recall that $f(z) = z^*$ is not an analytic function of z and that Cauchy's integral theorem therefore does not apply.

11.3.7 Show that

$$\oint\limits_C \frac{dz}{z^2 + z} = 0,$$

in which the contour C is a circle defined by $|z| = R > 1$.

Hint. Direct use of the Cauchy integral theorem is illegal. The integral may be evaluated by expanding into partial fractions and then treating the two terms individually. This yields 0 for $R > 1$ and $2\pi i$ for $R < 1$.

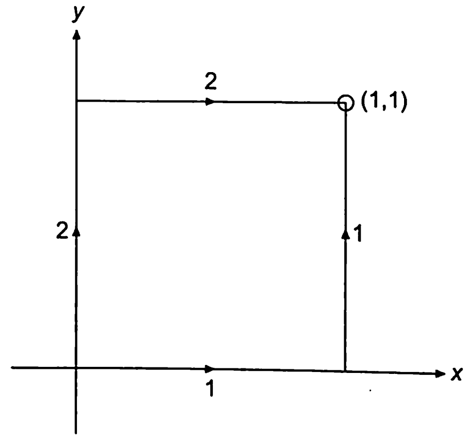

FIGURE 11.7 Contours for Exercise 11.3.6.

11.4 CAUCHY'S INTEGRAL FORMULA

As in the preceding section, we consider a function $f(z)$ that is analytic on a closed contour C and within the interior region bounded by C. This means that the contour C is to be traversed in the **counterclockwise** direction. We seek to prove the following result, known as **Cauchy's integral formula**:

$$\frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz = f(z_0), \tag{11.30}$$

in which z_0 is any point in the interior region bounded by C. Note that since z is on the contour C while z_0 is in the interior, $z - z_0 \neq 0$ and the integral Eq. (11.30) is well defined. Although $f(z)$ is assumed analytic, the integrand is $f(z)/(z-z_0)$ and is not analytic at $z = z_0$ unless $f(z_0) = 0$. We now deform the contour, to make it a circle of small radius r about $z = z_0$, traversed, like the original contour, in the counterclockwise direction. As shown in the preceding section, this does not change the value of the integral. We therefore write $z = z_0 + re^{i\theta}$, so $dz = ire^{i\theta}d\theta$, the integration is from $\theta = 0$ to $\theta = 2\pi$, and

$$\oint_C \frac{f(z)}{z-z_0}\,dz = \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\,ire^{i\theta}\,d\theta.$$

Taking the limit $r \to 0$, we obtain

$$\oint_C \frac{f(z)}{z-z_0}dz = if(z_0)\int_0^{2\pi} d\theta = 2\pi i f(z_0), \tag{11.31}$$

where we have replaced $f(z)$ by its limit $f(z_0)$ because it is analytic and therefore continuous at $z = z_0$. This proves the Cauchy integral formula.

Here is a remarkable result. The value of an analytic function $f(z)$ is given at an arbitrary interior point $z = z_0$ once the values on the boundary C are specified.

It has been emphasized that z_0 is an interior point. What happens if z_0 is exterior to C? In this case the entire integrand is analytic on and within C. Cauchy's integral theorem, Section 11.3, applies and the integral vanishes. Summarizing, we have

$$\frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z-z_0} = \begin{cases} f(z_0), & z_0 \text{ within the contour,} \\ 0, & z_0 \text{ exterior to the contour.} \end{cases}$$

Example 11.4.1 An Integral

Consider

$$I = \oint_C \frac{dz}{z(z+2)},$$

where the integration is counterclockwise over the unit circle. The factor $1/(z+2)$ is analytic within the region enclosed by the contour, so this is a case of Cauchy's integral formula, Eq. (11.30), with $f(z) = 1/(z+2)$ and $z_0 = 0$. The result is immediate:

$$I = 2\pi i \left[\frac{1}{z+2}\right]_{z=0} = \pi i.$$

■

Example 11.4.2 Integral with Two Singular Factors

Consider now

$$I = \oint_C \frac{dz}{4z^2-1},$$

also integrated counterclockwise over the unit circle. The denominator factors into $4\left(z-\frac{1}{2}\right)\left(z+\frac{1}{2}\right)$, and it is apparent that the region of integration contains two singular factors. However, we may still use Cauchy's integral formula if we make the partial fraction expansion

$$\frac{1}{4z^2-1} = \frac{1}{4}\left(\frac{1}{z-\frac{1}{2}} - \frac{1}{z+\frac{1}{2}}\right),$$

after which we integrate the two terms individually. We have

$$I = \frac{1}{4}\left[\oint_C \frac{dz}{z-\frac{1}{2}} - \oint_C \frac{dz}{z+\frac{1}{2}}\right].$$

Each integral is a case of Cauchy's formula with $f(z) = 1$, and for both integrals the point $z_0 = \pm\frac{1}{2}$ is within the contour, so each evaluates to $2\pi i$, and their sum is zero. So $I = 0$. ■

Derivatives

Cauchy's integral formula may be used to obtain an expression for the derivative of $f(z)$. Differentiating Eq. (11.30) with respect to z_0, and interchanging the differentiation and the z integration,[3]

$$f'(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^2}\, dz. \tag{11.32}$$

Differentiating again,

$$f''(z_0) = \frac{2}{2\pi i} \oint \frac{f(z)\, dz}{(z - z_0)^3}.$$

Continuing, we get[4]

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)\, dz}{(z - z_0)^{n+1}}; \tag{11.33}$$

that is, the requirement that $f(z)$ be analytic guarantees not only a first derivative but derivatives of **all** orders as well! The derivatives of $f(z)$ are automatically analytic. As indicated in a footnote, this statement assumes the Goursat version of the Cauchy integral theorem. This is a reason why Goursat's contribution is so significant in the development of the theory of complex variables.

Example 11.4.3 USE OF DERIVATIVE FORMULA

Consider

$$I = \oint_C \frac{\sin^2 z\, dz}{(z - a)^4},$$

where the integral is counterclockwise on a contour that encircles the point $z = a$. This is a case of Eq. (11.33) with $n = 3$ and $f(z) = \sin^2 z$. Therefore,

$$I = \frac{2\pi i}{3!} \left[\frac{d^3}{dz^3} \sin^2 z \right]_{z=a} = \frac{\pi i}{3} \Big[-8 \sin z \cos z \Big]_{z=a} = -\frac{8\pi i}{3} \sin a \cos a.$$

■

[3]The interchange can be proved legitimate, but the proof requires that Cauchy's integral theorem not be subject to the continuous derivative restriction in Cauchy's original proof. We are therefore now depending on Goursat's proof of the integral theorem.

[4]This expression is a starting point for defining derivatives of **fractional order**. See A. Erdelyi, ed., *Tables of Integral Transforms*, Vol. 2. New York: McGraw-Hill (1954). For more recent applications to mathematical analysis, see T. J. Osler, An integral analogue of Taylor's series and its use in computing Fourier transforms, *Math. Comput.* **26**: 449 (1972), and references therein.

Morera's Theorem

A further application of Cauchy's integral formula is in the proof of Morera's **theorem,** which is the converse of Cauchy's integral theorem. The theorem states the following:

> *If a function $f(z)$ is continuous in a simply connected region R and $\oint_C f(z)dz = 0$ for every closed contour C within R, then $f(z)$ is analytic throughout R.*

To prove the theorem, let us integrate $f(z)$ from z_1 to z_2. Since every closed-path integral of $f(z)$ vanishes, this integral is independent of path and depends only on its end-points. We may therefore write

$$F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(z)\,dz, \tag{11.34}$$

where $F(z)$, presently unknown, can be called the indefinite integral of $f(z)$. We then construct the identity

$$\frac{F(z_2) - F(z_1)}{z_2 - z_1} - f(z_1) = \frac{1}{z_2 - z_1}\int_{z_1}^{z_2}\Big[f(t) - f(z_1)\Big]\,dt, \tag{11.35}$$

where we have introduced another complex variable, t. Next, using the fact that $f(t)$ is continuous, we write, keeping only terms to first order in $t - z_1$,

$$f(t) - f(z_1) = f'(z_1)(t - z_1) + \cdots,$$

which implies that

$$\int_{z_1}^{z_2}\Big[f(t) - f(z_1)\Big]\,dt = \int_{z_1}^{z_2}\Big[f'(z_1)(t - z_1) + \cdots\Big]\,dt = \frac{f'(z_1)}{2}(z_2 - z_1)^2 + \cdots.$$

It is thus apparent that the right-hand side of Eq. (11.35) approaches zero in the limit $z_2 \to z_1$, so

$$f(z_1) \doteq \lim_{z_2 \to z_1}\frac{F(z_2) - F(z_1)}{z_2 - z_1} = F'(z_1). \tag{11.36}$$

Equation (11.36) shows that $F(z)$, which by construction is single-valued, has a derivative at all points within R and is therefore analytic in that region. Since $F(z)$ is analytic, then so also must be its derivative, $f(z)$, thereby proving Morera's theorem.

At this point, one comment might be in order. Morera's theorem, which establishes the analyticity of $F(z)$ in a simply connected region, cannot be extended to prove that $F(z)$, as well as $f(z)$, is analytic throughout a multiply connected region via the device of introducing a barrier. It is not possible to show that $F(z)$ will have the same value on both sides of the barrier, and in fact it does not always have that property. Thus, if extended to a multiply connected region, $F(z)$ may fail to have the single-valuedness that is one of the requirements for analyticity. Put another way, a function which is analytic in a multiply connected region will have analytic derivatives of all orders in that region, but its integral is not guaranteed to be analytic in the entire multiply connected region. This issue is elaborated in Section 11.6.

The proof of Morera's theorem has given us something additional, namely that the indefinite integral of $f(z)$ is its antiderivative, showing that:

The rules for integration of complex functions are the same as those for real functions.

Further Applications

An important application of Cauchy's integral formula is the following **Cauchy inequality**. If $f(z) = \sum a_n z^n$ is analytic and bounded, $|f(z)| \le M$ on a circle of radius r about the origin, then

$$|a_n| r^n \le M \quad \text{(Cauchy's inequality)} \tag{11.37}$$

gives upper bounds for the coefficients of its Taylor expansion. To prove Eq. (11.37) let us define $M(r) = \max_{|z|=r} |f(z)|$ and use the Cauchy integral for $a_n = f^{(n)}(z)/n!$,

$$|a_n| = \frac{1}{2\pi} \left| \oint_{|z|=r} \frac{f(z)}{z^{n+1}} dz \right| \le M(r) \frac{2\pi r}{2\pi r^{n+1}}.$$

An immediate consequence of the inequality, Eq. (11.37), is **Liouville's theorem**: If $f(z)$ is analytic and bounded in the entire complex plane it is a constant. In fact, if $|f(z)| \le M$ for all z, then Cauchy's inequality Eq. (11.37), applied for $|z| = r$, gives $|a_n| \le M r^{-n}$. If now we choose to let r approach ∞, we may conclude that for all $n > 0$, $|a_n| = 0$. Hence $f(z) = a_0$.

Conversely, the slightest deviation of an analytic function from a constant value implies that there must be at least one singularity somewhere in the infinite complex plane. Apart from the trivial constant functions then, singularities are a fact of life, and we must learn to live with them. As pointed out when introducing the concept of the point at infinity, even innocuous functions such as $f(z) = z$ have singularities at infinity; we now know that this is a property of every entire function that is not simply a constant. But we shall do more than just tolerate the existence of singularities. In the next section, we show how to expand a function in a Laurent series at a singularity, and we go on to use singularities to develop the powerful and useful calculus of residues in a later section of this chapter.

A famous application of Liouville's theorem yields the **fundamental theorem of algebra** (due to C. F. Gauss), which says that any polynomial $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ with $n > 0$ and $a_n \neq 0$ has n roots. To prove this, suppose $P(z)$ has no zero. Then $1/P(z)$ is analytic and bounded as $|z| \to \infty$, and, because of Liouville's theorem, $P(z)$ would have to be a constant. To resolve this contradiction, it must be the case that $P(z)$ has at least one root λ that we can divide out, forming $P(z)/(z - \lambda)$, a polynomial of degree $n - 1$. We can repeat this process until the polynomial has been reduced to degree zero, thereby finding exactly n roots.

Exercises

Unless explicitly stated otherwise, closed contours occurring in these exercises are to be understood as traversed in the mathematically positive (counterclockwise) direction.

11.4.1 Show that

$$\frac{1}{2\pi i}\oint z^{m-n-1}dz, \quad m \text{ and } n \text{ integers}$$

(with the contour encircling the origin once), is a representation of the Kronecker δ_{mn}.

11.4.2 Evaluate

$$\oint_C \frac{dz}{z^2-1},$$

where C is the circle $|z-1|=1$.

11.4.3 Assuming that $f(z)$ is analytic on and within a closed contour C and that the point z_0 is within C, show that

$$\oint_C \frac{f'(z)}{z-z_0}dz = \oint_C \frac{f(z)}{(z-z_0)^2}dz.$$

11.4.4 You know that $f(z)$ is analytic on and within a closed contour C. You suspect that the nth derivative $f^{(n)}(z_0)$ is given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz.$$

Using mathematical induction (Section 1.4), prove that this expression is correct.

11.4.5 (a) A function $f(z)$ is analytic within a closed contour C (and continuous on C). If $f(z) \neq 0$ within C and $|f(z)| \le M$ on C, show that

$$|f(z)| \le M$$

for all points within C.

Hint. Consider $w(z) = 1/f(z)$.

(b) If $f(z) = 0$ within the contour C, show that the foregoing result does not hold and that it is possible to have $|f(z)| = 0$ at one or more points in the interior with $|f(z)| > 0$ over the entire bounding contour. Cite a specific example of an analytic function that behaves this way.

11.4.6 Evaluate

$$\oint_C \frac{e^{iz}}{z^3}dz,$$

for the contour a square with sides of length $a > 1$, centered at $z = 0$.

11.4.7 Evaluate

$$\oint_C \frac{\sin^2 z - z^2}{(z-a)^3}\,dz,$$

where the contour encircles the point $z = a$.

11.4.8 Evaluate

$$\oint_C \frac{dz}{z(2z+1)},$$

for the contour the unit circle.

11.4.9 Evaluate

$$\oint_C \frac{f(z)}{z(2z+1)^2}\,dz,$$

for the contour the unit circle.

Hint. Make a partial fraction expansion.

11.5 LAURENT EXPANSION

Taylor Expansion

The Cauchy integral formula of the preceding section opens up the way for another derivation of Taylor's series (Section 1.2), but this time for functions of a complex variable. Suppose we are trying to expand $f(z)$ about $z = z_0$ and we have $z = z_1$ as the nearest point on the Argand diagram for which $f(z)$ is not analytic. We construct a circle C centered at $z = z_0$ with radius less than $|z_1 - z_0|$ (Fig. 11.8). Since z_1 was assumed to be the nearest point at which $f(z)$ was not analytic, $f(z)$ is necessarily analytic on and within C.

From the Cauchy integral formula, Eq. (11.30),

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{z'-z} \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)} \\ &= \frac{1}{2\pi i}\oint_C \frac{f(z')\,dz'}{(z'-z_0)[1-(z-z_0)/(z'-z_0)]}. \end{aligned} \tag{11.38}$$

Here z' is a point on the contour C and z is any point interior to C. It is not legal yet to expand the denominator of the integrand in Eq. (11.38) by the binomial theorem, for

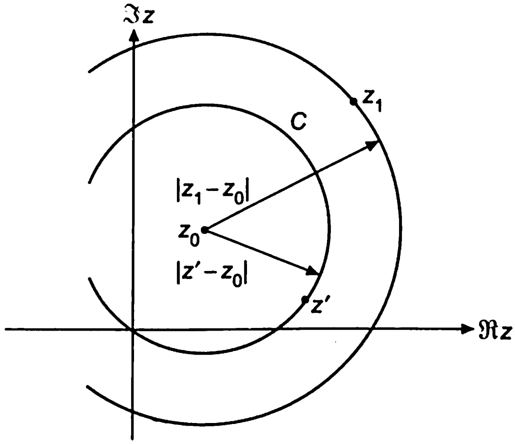

FIGURE 11.8 Circular domains for Taylor expansion.

we have not yet proved the binomial theorem for complex variables. Instead, we note the identity

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{n=0}^{\infty} t^n, \tag{11.39}$$

which may easily be verified by multiplying both sides by $1 - t$. The infinite series, following the methods of Section 1.2, is convergent for $|t| < 1$.

Now, for a point z interior to C, $|z - z_0| < |z' - z_0|$, and, using Eq. (11.39), Eq. (11.38) becomes

$$f(z) = \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \frac{(z - z_0)^n f(z')\, dz'}{(z' - z_0)^{n+1}}. \tag{11.40}$$

Interchanging the order of integration and summation, which is valid because Eq. (11.39) is uniformly convergent for $|t| < 1 - \varepsilon$, with $0 < \varepsilon < 1$, we obtain

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_C \frac{f(z')\, dz'}{(z' - z_0)^{n+1}}. \tag{11.41}$$

Referring to Eq. (11.33), we get

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n, \tag{11.42}$$

which is our desired Taylor expansion.

It is important to note that our derivation not only produces the expansion given in Eq. (11.41); it also shows that this expansion converges when $|z - z_0| < |z_1 - z_0|$. For this reason the circle defined by $|z - z_0| = |z_1 - z_0|$ is called the **circle of convergence** of our

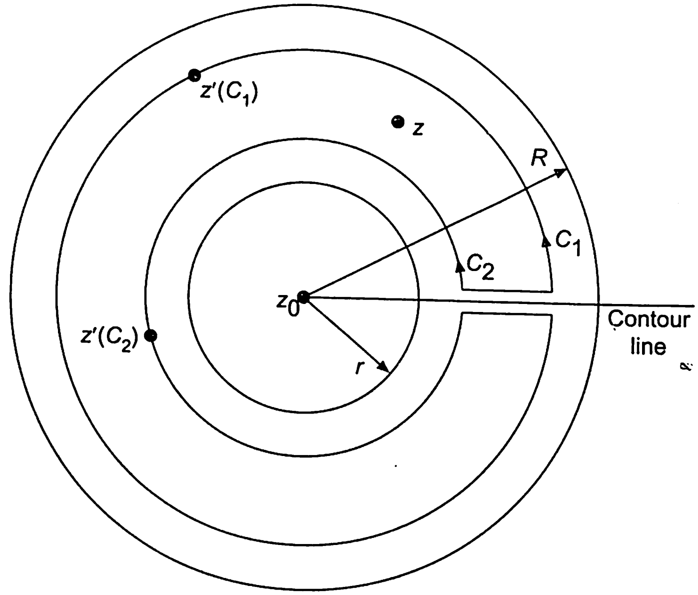

FIGURE 11.9 Annular region for Laurent series. $|z' - z_0|_{C_1} > |z - z_0|$; $|z' - z_0|_{C_2} < |z - z_0|$.

Taylor series. Alternatively, the distance $|z_1 - z_0|$ is sometimes referred to as the **radius of convergence** of the Taylor series. In view of the earlier definition of z_1, we can say that:

> *The Taylor series of a function $f(z)$ about any interior point z_0 of a region in which $f(z)$ is analytic is a* **unique expansion** *that will have a radius of convergence equal to the distance from z_0 to the singularity of $f(z)$ closest to z_0, meaning that the Taylor series will converge* **within** *this circle of convergence. The Taylor series may or may not converge at individual points* **on** *the circle of convergence.*

From the Taylor expansion for $f(z)$ a binomial theorem may be derived. That task is left to Exercise 11.5.2.

Laurent Series

We frequently encounter functions that are analytic in an annular region, say, between circles of inner radius r and outer radius R about a point z_0, as shown in Fig. 11.9. We assume $f(z)$ to be such a function, with z a typical point in the annular region. Drawing an imaginary barrier to convert our region into a simply connected region, we apply Cauchy's integral formula to evaluate $f(z)$, using the contour shown in the figure. Note that the contour consists of the two circles centered at z_0, labeled C_1 and C_2 (which can be considered closed since the barrier is fictitious), plus segments on either side of the barrier whose contributions will cancel. We assign C_2 and C_1 the radii r_2 and r_1, respectively, where $r < r_2 < r_1 < R$. Then, from Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')\,dz'}{z' - z} - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z')\,dz'}{z' - z}. \tag{11.43}$$

Note that in Eq. (11.43)) an explicit minus sign has been introduced so that the contour C_2 (like C_1) is to be traversed in the positive (counterclockwise) sense. The treatment of

Eq. (11.43) now proceeds exactly like that of Eq. (11.38) in the development of the Taylor series. Each denominator is written as $(z' - z_0) - (z - z_0)$ and expanded by the binomial theorem, which is now regarded as proven (see Exercise 11.5.2).

Noting that for C_1, $|z' - z_0| > |z - z_0|$, while for C_2, $|z' - z_0| < |z - z_0|$, we find

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z')\,dz'}{(z' - z_0)^{n+1}} + \frac{1}{2\pi i} \sum_{n=1}^{\infty} (z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z')\,dz'. \tag{11.44}$$

The minus sign of Eq. (11.43) has been absorbed by the binomial expansion. Labeling the first series S_1 and the second S_2 we have

$$S_1 = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z')\,dz'}{(z' - z_0)^{n+1}}, \tag{11.45}$$

which has the same form as the regular Taylor expansion, convergent for $|z - z_0| < |z' - z_0| = r_1$, that is, for all z **interior** to the larger circle, C_1. For the second series in Eq. (6.65) we have

$$S_2 = \frac{1}{2\pi i} \sum_{n=1}^{\infty} (z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z')\,dz', \tag{11.46}$$

convergent for $|z - z_0| > |z' - z_0| = r_2$, that is, for all z **exterior** to the smaller circle, C_2. Remember, C_2 now goes counterclockwise.

These two series are combined into one series,[5] known as a **Laurent series**, of the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \tag{11.47}$$

where

$$a_n = \frac{1}{2\pi i} \oint_{C} \frac{f(z')\,dz'}{(z' - z_0)^{n+1}}. \tag{11.48}$$

Since convergence of a binomial expansion is not relevant to the evaluation of Eq. (11.48), C in that equation may be any contour within the annular region $r < |z - z_0| < R$ that encircles z_0 once in a counterclockwise sense. If such an annular region of analyticity does exist, then Eq. (11.47) is the Laurent series, or Laurent expansion, of $f(z)$.

The Laurent series differs from the Taylor series by the obvious feature of negative powers of $(z - z_0)$. For this reason the Laurent series will always diverge at least at $z = z_0$ and perhaps as far out as some distance r. In addition, note that Laurent series coefficients need not come from evaluation of contour integrals (which may be very intractable). Other techniques, such as ordinary series expansions, may provide the coefficients.

Numerous examples of Laurent series appear later in this book. We limit ourselves here to one simple example to illustrate the application of Eq. (11.47).

[5] Replace n by $-n$ in S_2 and add.

Example 11.5.1 LAURENT EXPANSION

Let $f(z) = [z(z-1)]^{-1}$. If we choose to make the Laurent expansion about $z_0 = 0$, then $r > 0$ and $R < 1$. These limitations arise because $f(z)$ diverges both at $z = 0$ and $z = 1$. A partial fraction expansion, followed by the binomial expansion of $(1-z)^{-1}$, yields the Laurent series

$$\frac{1}{z(z-1)} = -\frac{1}{1-z} - \frac{1}{z} = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots = -\sum_{n=-1}^{\infty} z^n. \tag{11.49}$$

From Eqs. (11.49), (11.47), and (11.48), we then have

$$a_n = \frac{1}{2\pi i}\oint \frac{dz'}{(z')^{n+2}(z'-1)} = \begin{cases} -1 & \text{for } n \ge -1, \\ 0 & \text{for } n < -1, \end{cases} \tag{11.50}$$

where the contour for Eq. (11.50) is counterclockwise in the annular region between $z' = 0$ and $|z'| = 1$.

The integrals in Eq. (11.50) can also be directly evaluated by insertion of the geometric-series expansion of $(1-z')^{-1}$:

$$a_n = \frac{-1}{2\pi i}\oint \sum_{m=0}^{\infty} (z')^m \frac{dz'}{(z')^{n+2}}. \tag{11.51}$$

Upon interchanging the order of summation and integration (permitted because the series is uniformly convergent), we have

$$a_n = -\frac{1}{2\pi i}\sum_{m=0}^{\infty}\oint (z')^{m-n-2}\, dz'. \tag{11.52}$$

The integral in Eq. (11.52) (including the initial factor $1/2\pi i$, but not the minus sign) was shown in Exercise 11.4.1 to be an integral representation of the Kronecker delta, and is therefore equal to $\delta_{m,n+1}$. The expression for a_n then reduces to

$$a_n = -\sum_{m=0}^{\infty} \delta_{m,n+1} = \begin{cases} -1, & n \ge -1, \\ 0, & n < -1, \end{cases}$$

in agreement with Eq. (11.50). ■

Exercises

11.5.1 Develop the Taylor expansion of $\ln(1+z)$.

ANS. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$.

11.5.2 Derive the binomial expansion

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1 \cdot 2} z^2 + \cdots = \sum_{n=0}^{\infty} \binom{m}{n} z^n$$

for m, any real number. The expansion is convergent for $|z| < 1$. Why?

11.5.3 A function $f(z)$ is analytic on and within the unit circle. Also, $|f(z)| < 1$ for $|z| \le 1$ and $f(0) = 0$. Show that $|f(z)| < |z|$ for $|z| \le 1$.

Hint. One approach is to show that $f(z)/z$ is analytic and then to express $[f(z_0)/z_0]^n$ by the Cauchy integral formula. Finally, consider absolute magnitudes and take the nth root. This exercise is sometimes called Schwarz's theorem.

11.5.4 If $f(z)$ is a real function of the complex variable $z = x + iy$, that is, $f(x) = f^*(x)$, and the Laurent expansion about the origin, $f(z) = \sum a_n z^n$, has $a_n = 0$ for $n < -N$, show that all of the coefficients a_n are real.

Hint. Show that $z^N f(z)$ is analytic (via Morera's theorem, Section 11.4).

11.5.5 Prove that the Laurent expansion of a given function about a given point is unique; that is, if

$$f(z) = \sum_{n=-N}^{\infty} a_n (z-z_0)^n = \sum_{n=-N}^{\infty} b_n (z-z_0)^n,$$

show that $a_n = b_n$ for all n.

Hint. Use the Cauchy integral formula.

11.5.6 Obtain the Laurent expansion of e^z/z^2 about $z = 0$.

11.5.7 Obtain the Laurent expansion of $ze^z/(z-1)$ about $z = 1$.

11.5.8 Obtain the Laurent expansion of $(z-1)\,e^{1/z}$ about $z = 0$.

11.6 SINGULARITIES

Poles

We define a point z_0 as an **isolated** singular point of the function $f(z)$ if $f(z)$ is not analytic at $z = z_0$ but is analytic at all neighboring points. There will therefore be a Laurent expansion about an isolated singular point, and one of the following statements will be true:

1. The most negative power of $z - z_0$ in the Laurent expansion of $f(z)$ about $z = z_0$ will be some finite power, $(z-z_0)^{-n}$, where n is an integer, or
2. The Laurent expansion of $f(z)$ about $z - z_0$ will continue to negatively infinite powers of $z - z_0$.

In the first case, the singularity is called a **pole**, and is more specifically identified as a pole of **order** n. A pole of order 1 is also called a **simple pole**. The second case is not referred to as a "pole of infinite order," but is called an **essential singularity**.

One way to identify a pole of $f(z)$ without having available its Laurent expansion is to examine

$$\lim_{z \to z_0} (z - z_0)^n f(z_0)$$

for various integers n. The smallest integer n for which this limit exists (i.e., is finite) gives the order of the pole at $z = z_0$. This rule follows directly from the form of the Laurent expansion.

Essential singularities are often identified directly from their Laurent expansions. For example,

$$\begin{aligned} e^{1/z} &= 1 + \frac{1}{z} + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n \end{aligned}$$

clearly has an essential singularity at $z = 0$. Essential singularities have many pathological features. For instance, we can show that in any small neighborhood of an essential singularity of $f(z)$ the function $f(z)$ comes arbitrarily close to any (and therefore every) preselected complex quantity w_0.[6] Here, the entire w-plane is mapped by f into the neighborhood of the point z_0.

The behavior of $f(z)$ as $z \to \infty$ is defined in terms of the behavior of $f(1/t)$ as $t \to 0$. Consider the function

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}. \tag{11.53}$$

As $z \to \infty$, we replace the z by $1/t$ to obtain

$$\sin\left(\frac{1}{t}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,t^{2n+1}}. \tag{11.54}$$

It is clear that $\sin(1/t)$ has an essential singularity at $t = 0$, from which we conclude that $\sin z$ has an essential singularity at $z = \infty$. Note that although the absolute value of $\sin x$ for all real x is equal to or less than unity, the absolute value of $\sin iy = i \sinh y$ increases exponentially without limit as y increases.

A function that is analytic throughout the finite complex plane except for isolated poles is called **meromorphic**. Examples are ratios of two polynomials, also $\tan z$ and $\cot z$. As previously mentioned, functions that have no singularities in the finite complex plane are called **entire** functions. Examples are $\exp z$, $\sin z$, and $\cos z$.

[6]This theorem is due to Picard. A proof is given by E. C. Titchmarsh, *The Theory of Functions*, 2nd ed. New York: Oxford University Press (1939).

Branch Points

In addition to the isolated singularities identified as poles or essential singularities, there are singularities uniquely associated with multivalued functions. It is useful to work with these functions in ways that to the maximum possible extent remove ambiguity as to the function values. Thus, if at a point z_0 (at which $f(z)$ has a derivative) we have chosen a specific value of the multivalued function $f(z)$, then we can assign to $f(z)$ values at nearby points in a way that causes continuity in $f(z)$. If we think of a succession of closely spaced points as in the limit of zero spacing defining a path, our current observation is that a given value of $f(z_0)$ then leads to a unique definition of the value of $f(z)$ to be assigned to each point on the path. This scheme creates no ambiguity so long as the path is entirely **open**, meaning that the path does not return to any point previously passed. But if the path returns to z_0, thereby forming a **closed loop**, our prescription might lead, upon the return, to a different one of the multiple values of $f(z_0)$.

Example 11.6.1 VALUE OF $z^{1/2}$ ON A CLOSED LOOP

We consider $f(z) = z^{1/2}$ on the path consisting of counterclockwise passage around the unit circle, starting and ending at $z = +1$. At the start point, where $z^{1/2}$ has the multiple values $+1$ and -1, let us choose $f(z) = +1$. See Fig. 11.10. Writing $f(z) = e^{i\theta/2}$, we note that this form (with $\theta = 0$) is consistent with the desired starting value of $f(z)$, $+1$. In the figure, the start point is labeled A. Next, we note that passage counterclockwise on the unit circle corresponds to an increase in θ, so that at the points marked B, C, and D in the figure, the respective values of θ are $\pi/2$, π, and $3\pi/2$. Note that because of the path we have decided to take, we cannot assign to point C the θ value $-\pi$ or to point D the θ value $-\pi/2$. Continuing further along the path, when we return to point A the value of θ has become 2π (not zero).

Now that we have identified the behavior of θ, let's examine what happens to $f(z)$. At the points B, C, and D, we have

$$f(z_B) = e^{i\theta_B/2} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}},$$

$$f(z_C) = e^{i\pi/2} = +i,$$

$$f(z_D) = e^{3i\pi/4} = \frac{-1+i}{\sqrt{2}}.$$

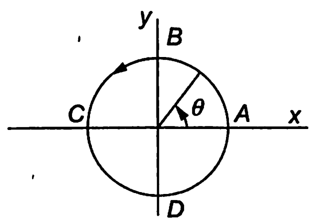

FIGURE 11.10 Path encircling $z = 0$ for evaluation of $z^{1/2}$.

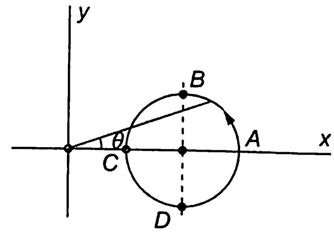

FIGURE 11.11 Path not encircling $z = 0$ for evaluation of $z^{1/2}$.

When we return to point A, we have $f(+1) = e^{i\pi} = -1$, which is the other value of the multivalued function $z^{1/2}$.

If we continue for a second counterclockwise circuit of the unit circle, the value of θ would continue to increase, from 2π to 4π (reached when we arrive at point A after the second loop). We now have $f(+1) = e^{(4\pi i)/2} = e^{2\pi i} = 1$, so a second circuit has brought us back to the original value. It should now be clear that we are only going to be able to obtain two different values of $z^{1/2}$ for the same point z. ■

Example 11.6.2 ANOTHER CLOSED LOOP

Let's now see what happens to the function $z^{1/2}$ as we pass counterclockwise around a circle of unit radius centered at $z = +2$, starting and ending at $z = +3$. See Fig. 11.11. At $z = 3$, the values of $f(z)$ are $+\sqrt{3}$ and $-\sqrt{3}$; let's start with $f(z_A) = +\sqrt{3}$. As we move from point A through point B to point C, note from the figure that the value of θ first increases (actually, to 30°) and then decreases again to zero; further passage from C to D and back to A causes θ first to decrease (to $-30°$) and then to return to zero at A. So in this example the closed loop does not bring us to a different value of the multivalued function $z^{1/2}$. ■

The essential difference between these two examples is that in the first, the path encircled $z = 0$; in the second it did not. What is special about $z = 0$ is that (from a complex-variable viewpoint) it is singular; the function $z^{1/2}$ does not have a derivative there. The lack of a well-defined derivative means that ambiguity in the function value will result from paths that circle such a singular point, which we call a **branch point**. The **order** of a branch point is defined as the number of paths around it that must be taken before the function involved returns to its original value; in the case of $z^{1/2}$, we saw that the branch point at $z = 0$ is of order 2.

We are now ready to see what must be done to cause a multivalued function to be restricted to single-valuedness on a portion of the complex plane. We simply need to prevent its evaluation on paths that encircle a branch point. We do so by drawing a line (known as a **branch line**, or more commonly, a **branch cut**) that the evaluation path cannot cross; the branch cut must start from our branch point and continue to infinity (or if consistent with maintaining single-valuedness) to another finite branch point. The precise path of a branch cut can be chosen freely; what must be chosen appropriately are its endpoints.

Once appropriate branch cut(s) have been drawn, the originally multivalued function has been restricted to being single-valued in the region bounded by the branch cut(s); we call the function as made single-valued in this way a **branch** of our original function. Since we could construct such a branch starting from any one of the values of the original function at a single arbitrary point in our region, we identify our multivalued function as having multiple branches. In the case of $z^{1/2}$, which is double-valued, the number of branches is two.

Note that a function with a branch point and a corresponding branch cut will not be continuous across the cut line. Hence line integrals in opposite directions on the two sides of the branch cut will not generally cancel each other. Branch cuts, therefore, are real boundaries to a region of analyticity, in contrast to the artificial barriers we introduced in extending Cauchy's integral theorem to multiply connected regions.

While from a fundamental viewpoint all branches of a multivalued function $f(z)$ are equally legitimate, it is often convenient to agree on the branch to be used, and such a branch is sometimes called the **principal branch**, with the value of $f(z)$ on that branch called its **principal value**. It is common to take the branch of $z^{1/2}$ which is positive for real, positive z as its principal branch.

An observation that is important for complex analysis is that by drawing appropriate branch cut(s), we have restricted a multivalued function to single-valuedness, so that it can be an analytic function within the region bounded by the branch cut(s), and we can therefore apply Cauchy's two theorems to contour integrals within the region of analyticity.

Example 11.6.3 $\ln z$ Has an Infinite Number of Branches

Here we examine the singularity structure of $\ln z$. As we already saw in Eq. (1.138), the logarithm is multivalued, with the polar representation

$$\ln z = \ln\left(re^{i(\theta+2n\pi)}\right) = \ln r + i(\theta + 2n\pi), \tag{11.55}$$

where n can have **any** positive or negative integer value.

Noting that $\ln z$ is singular at $z = 0$ (it has no derivative there), we now identify $z = 0$ as a branch point. Let's consider what happens if we encircle it by a counterclockwise path on a circle of radius r, starting from the initial value $\ln r$, at $z = r = re^{i\theta}$ with $\theta = 0$. Every passage around the circle will add 2π to θ, and after n complete circuits the value we have for $\ln z$ will be $\ln r + 2n\pi i$. The branch point of $\ln z$ at $z = 0$ is of infinite order, corresponding to the infinite number of its multiple values. (By encircling $z = 0$ repeatedly in the *clockwise* direction, we can also reach all negative integer values of n.)

We can make $\ln z$ single-valued by drawing a branch cut from $z = 0$ to $z = \infty$ in **any** way (though there is ordinarily no reason to use cuts that are not straight lines). It is typical to identify the branch with $n = 0$ as the principal branch of the logarithm. Incidentally, we note that the inverse trigonometric functions, which can be written in terms of logarithms, as in Eq. (1.137), will also be infinitely multivalued, with principal values that are usually chosen on a branch that will yield real values for real z. Compare with the usual choices of the values assigned the real-variable forms of $\sin^{-1} x = \arcsin x$, etc. ■

Using the logarithm, we are now in a position to look at the singularity structures of expressions of the form z^p, where both z and p may be complex. To do so, we write

$$z = e^{\ln z}, \quad \text{so } z^p = e^{p \ln z}, \tag{11.56}$$

which is single-valued if p is an integer, t-valued if p is a real rational fraction (in lowest terms) of the form s/t, and infinitely multivalued otherwise.

Example 11.6.4 MULTIPLE BRANCH POINTS

Consider the function

$$f(z) = (z^2 - 1)^{1/2} = (z+1)^{1/2}(z-1)^{1/2}.$$

The first factor on the right-hand side, $(z+1)^{1/2}$, has a branch point at $z = -1$. The second factor has a branch point at $z = +1$. At infinity $f(z)$ has a simple pole. This is best seen by substituting $z = 1/t$ and making a binomial expansion at $t = 0$:

$$(z^2 - 1)^{1/2} = \frac{1}{t}(1 - t^2)^{1/2} = \frac{1}{t}\sum_{n=0}^{\infty}\binom{1/2}{n}(-1)^n t^{2n} = \frac{1}{t} - \frac{1}{2}t - \frac{1}{8}t^3 + \cdots .$$

We want to make $f(z)$ single-valued by making appropriate branch cut(s). There are many ways to accomplish this, but one we wish to investigate is the possibility of making a branch cut from $z = -1$ to $z = +1$, as shown in Fig. 11.12.

To determine whether this branch cut makes our $f(z)$ single-valued, we need to see what happens to each of the multivalent factors in $f(z)$ as we move around on its Argand diagram. Figure 11.12 also identifies the quantities that are relevant for this purpose, namely those that relate a point P to the branch points. In particular, we have written the position relative to the branch point at $z = 1$ as $z - 1 = \rho e^{i\varphi}$, with the position relative to $z = -1$ denoted $z + 1 = re^{i\theta}$. With these definitions, we have

$$f(z) = r^{1/2}\rho^{1/2}e^{(\theta+\varphi)/2}.$$

Our mission is to note how φ and θ change as we move along the path, so that we can use the correct value of each for evaluating $f(z)$.

We consider a closed path starting at point A in Fig. 11.13, proceeding via points B through F, then back to A. At the start point, we choose $\theta = \varphi = 0$, thereby causing the multivalued $f(z_A)$ to have the specific value $+\sqrt{3}$. As we pass **above** $z = +1$ on the way to point B, θ remains essentially zero, but φ increases from zero to π. These angles do not change as we pass from B to C, but on going to point D, θ increases to π, and then, passing **below** $z = -1$ on the way to point E, it further increases to 2π (not zero!). Meanwhile, φ remains essentially at π. Finally, returning to point A **below** $z = +1$, φ increases to 2π, so that upon the return to point A both φ and θ have become 2π. The behavior of these angles and the values of $(\theta + \varphi)/2$ (the argument of $f(z)$) are tabulated in Table 11.1.

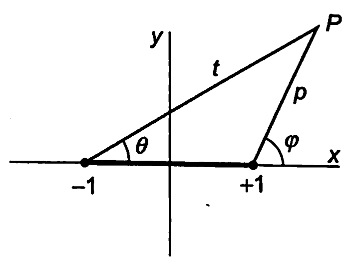

FIGURE 11.12 Possible branch cut for Example 11.6.4 and the quantities relating a point P to the branch points.

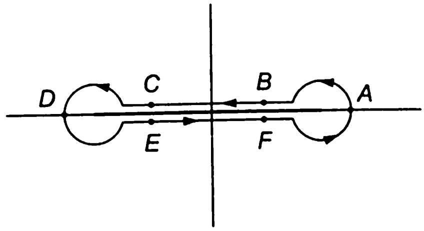

FIGURE 11.13 Path around the branch cut in Example 11.6.4.

Table 11.1 Phase Angles, Path in Fig. 11.13

Point	θ	φ	$(\theta+\varphi)/2$
A	0	0	0
B	0	π	$\pi/2$
C	0	π	$\pi/2$
D	π	π	π
E	2π	π	$\pi/2$
F	2π	π	$3\pi/2$
A	2π	2π	2π

Two features emerge from this analysis:

1. The phase of $f(z)$ at points B and C is not the same as that at points E and F. This behavior can be expected at a branch cut.
2. The phase of $f(z)$ at point A' (the return to A) exceeds that at point A by 2π, meaning that the function $f(z)=(z^2-1)^{1/2}$ is **single-valued** for the contour shown, encircling **both** branch points.

What actually happened is that each of the two multivalued factors contributed a sign change upon passage around the closed loop, so the two factors together restored the original sign of $f(z)$.

Another way we could have made $f(z)$ single-valued would have been to make a separate branch cut from each branch point to infinity; a reasonable way to do this would be to make cuts on the real axis for all $x > 1$ and for all $x < -1$. This alternative is explored in Exercises 11.6.2 and 11.6.4. ■

Analytic Continuation

We saw in Section 11.5 that a function $f(z)$ which is analytic within a region can be uniquely expanded in a Taylor series about any interior point z_0 of the region of analyticity, and that the resulting expansion will be convergent within a circle of convergence extending to the singularity of $f(z)$ closest to z_0. Since

- The coefficients in the Taylor series are proportional to the derivatives of $f(z)$,
- An analytic function has derivatives of all orders that are independent of direction, and therefore
- The values of $f(z)$ on a single finite line segment with z_0 as a interior point will suffice to determine all derivatives of $f(z)$ at $z = z_0$,

we conclude that if two apparently different analytic functions (e.g., a closed expression vs. an integral representation or a power series) have values that coincide on a range as restricted as a single finite line segment, then they are actually **the same function** within the region where both functional forms are defined.

The above conclusion will provide us with a technique for extending the definition of an analytic function beyond the range of any particular functional form initially used to define it. All we will need to do is to find another functional form whose range of definition is not entirely included in that of the initial form and which yields the same function values on at least a finite line segment within the area where both functional forms are defined.

To make the approach more concrete, consider the situation illustrated in Fig. 11.14, where a function $f(z)$ is defined by its Taylor expansion about a point z_0 with a circle of convergence C_0 defined by the singularity nearest to z_0, labeled z_s. If we now make a Taylor expansion about some point z_1 within C_0 (which we can do because $f(z)$ has known values in the neighborhood of z_1), this new expansion may have a circle of convergence C_1 that is not entirely within C_0, thereby defining a function that is analytic in the region that is the union of C_1 and C_2. Note that if we need to obtain actual values of $f(z)$ for z within the intersection of C_0 and C_1 we may use either Taylor expansion, but in the region within only one circle we must use the expansion that is valid there (the other expansion will not converge). A generalization of the above analysis leads to the beautiful and valuable result that if two analytic functions coincide in any region, or even on any finite line segment, they are the same function, and therefore defined over the entire range of both function definitions.

After Weierstrass this process of enlarging the region in which we have the specification of an analytic function is called **analytic continuation**, and the process may be carried out repeatedly to maximize the region in which the function is defined. Consider the situation pictured in Fig. 11.15, where the only singularity of $f(z)$ is at z_s and $f(z)$ is originally defined by its Taylor expansion about z_0, with circle of convergence C_0. By making analytic continuations as shown by the series of circles $C_1, \ldots$, we can cover the entire annular region of analyticity shown in the figure, and can use the original Taylor series to generate new expansions that apply to regions within the other circles.

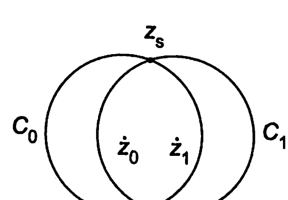

FIGURE 11.14 Analytic continuation. One step.

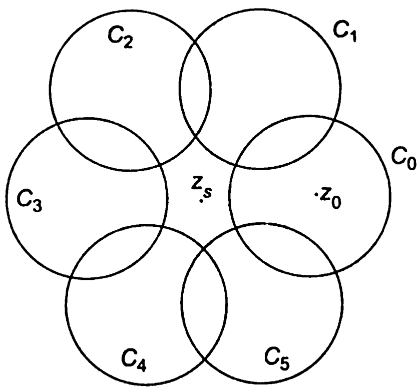

FIGURE 11.15 Analytic continuation. Many steps.

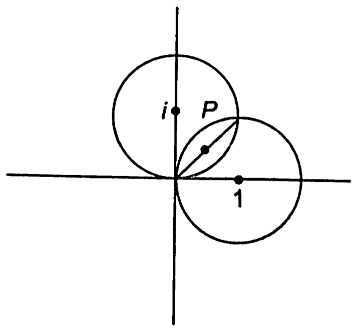

FIGURE 11.16 Radii of convergence of power-series expansions for Example 11.6.5.

Example 11.6.5 ANALYTIC CONTINUATION

Consider these two power-series expansions:

$$f_1(z) = \sum_{n=0}^{\infty} (-1)^n (z-1)^n, \tag{11.57}$$

$$f_2(z) = \sum_{n=0}^{\infty} i^{n-1} (z-i)^n. \tag{11.58}$$

Each has a unit radius of convergence; the circles of convergence overlap, as can be seen from Fig. 11.16.

To determine whether these expansions represent the same analytic function in overlapping domains, we can check to see if $f_1(z) = f_2(z)$ for at least a line segment in the region of overlap. A suitable line is the diagonal that connects the origin with $1+i$, passing through the intermediate point $(1+i)/2$. Setting $z = (\alpha + \frac{1}{2})(1+i)$ (chosen to make $\alpha = 0$ an interior point of the overlap region), we expand f_1 and f_2 about $\alpha = 0$ to find out whether their power series coincide. Initially we have (as functions of α)

$$f_1 = \sum_{n=0}^{\infty} (-1)^n \left[(1+i)\alpha - \frac{1-i}{2} \right]^n,$$

$$f_2 = \sum_{n=0}^{\infty} i^{n-1} \left[(1+i)\alpha + \frac{1-i}{2} \right]^n.$$

Applying the binomial theorem to obtain power series in α, and interchanging the order of the two sums,

$$f_1 = \sum_{j=0}^{\infty} (-1)^j (1+i)^j \alpha^j \sum_{n=j}^{\infty} \binom{n}{j} \left(\frac{1-i}{2} \right)^{n-j},$$

$$f_2 = \sum_{j=0}^{\infty} i^{j-1} (1+i)^j \alpha^j \sum_{n=j}^{\infty} i^{n-j} \binom{n}{j} \left(\frac{1-i}{2} \right)^{n-j}$$

$$= \sum_{j=0}^{\infty} \frac{1}{i} (-1)^j (1-i)^j \alpha^j \sum_{n=j}^{\infty} \binom{n}{j} \left(\frac{1+i}{2} \right)^{n-j}.$$

To proceed further we need to evaluate the summations over n. Referring to Exercise 1.3.5, where it was shown that

$$\sum_{n=j}^{\infty} \binom{n}{j} x^{n-j} = \frac{1}{(1-x)^{j+1}},$$

we get

$$f_1 = \sum_{j=0}^{\infty} (-1)^j (1+i)^j \alpha^j \left(\frac{2}{1+i} \right)^{j+1} = \sum_{j=0}^{\infty} \frac{(-1)^j 2^{j+1} \alpha^j}{1+i},$$

$$f_2 = \sum_{j=0}^{\infty} \frac{1}{i} (-1)^j (1-i)^j \alpha^j \left(\frac{2}{1-i} \right)^{j+1} = \sum_{j=0}^{\infty} \frac{(-1)^j 2^{j+1} \alpha^j}{i(1-i)} = f_1,$$

confirming that f_1 and f_2 are the same analytic function, now defined over the union of the two circles in Fig. 11.16.

Incidentally, both f_1 and f_2 are expansions of $1/z$ (about the respective points 1 and i), so $1/z$ could also be regarded as an analytic continuation of f_1, f_2, or both to the entire complex plane except the singular point at $z = 0$. The expansion in powers of α is also a representation of $1/z$, but its range of validity is only a circle of radius $1/\sqrt{2}$ about $(1+i)/2$ and it does not analytically continue $f(z)$ outside the union of C_1 and C_2. ■

The use of power series is not the only mechanism for carrying out analytic continuations; an alternative and powerful method is the use of **functional relations**, which are formulas that relate values of the same analytic function $f(z)$ at different z. As an example of a functional relation, the integral representation of the gamma function, given in

Table 1.2, can be manipulated (see Chapter 13) to show that $\Gamma(z+1) = z\Gamma(z)$, consistent with the elementary result that $n! = n(n-1)!$. This functional relation can be used to analytically continue $\Gamma(z)$ to values of z for which the integral representation does not converge.

Exercises

11.6.1 As an example of an essential singularity consider $e^{1/z}$ as z approaches zero. For any complex number z_0, $z_0 \neq 0$, show that

$$e^{1/z} = z_0$$

has an infinite number of solutions.

11.6.2 Show that the function

$$w(z) = (z^2 - 1)^{1/2}$$

is single-valued if we make branch cuts on the real axis for $x > 1$ and for $x < -1$.

11.6.3 A function $f(z)$ can be represented by

$$f(z) = \frac{f_1(z)}{f_2(z)},$$

in which $f_1(z)$ and $f_2(z)$ are analytic. The denominator, $f_2(z)$, vanishes at $z = z_0$, showing that $f(z)$ has a pole at $z = z_0$. However, $f_1(z_0) \neq 0$, $f_2'(z_0) \neq 0$. Show that a_{-1}, the coefficient of $(z - z_0)^{-1}$ in a Laurent expansion of $f(z)$ at $z = z_0$, is given by

$$a_{-1} = \frac{f_1(z_0)}{f_2'(z_0)}.$$

11.6.4 Determine a unique branch for the function of Exercise 11.6.2 that will cause the value it yields for $f(i)$ to be the same as that found for $f(i)$ in Example 11.6.4. Although Exercise 11.6.2 and Example 11.6.4 describe the same multivalued function, the specific values assigned for various z will not agree everywhere, due to the difference in the location of the branch cuts. Identify the portions of the complex plane where both these descriptions do and do not agree, and characterize the differences.

11.6.5 Find all singularities of

$$z^{-1/3} + \frac{z^{-1/4}}{(z-3)^3} + (z-2)^{1/2},$$

and identify their types (e.g., second-order branch point, fifth-order pole, ...). Include any singularities at the point at infinity.

Note. A branch point is of nth order if it requires n, but no fewer, circuits around the point to restore the original value.

11.6.6 The function $F(z) = \ln(z^2 + 1)$ is made single-valued by straight-line branch cuts from $(x, y) = (0, -1)$ to $(-\infty, -1)$ and from $(0, +1)$ to $(0, +\infty)$. See Fig. 11.17. If $F(0) = -2\pi i$, find the value of $F(i - 2)$.

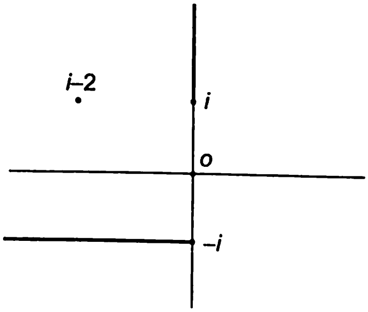

FIGURE 11.17 Branch cuts for Exercise 11.6.6.

11.6.7 Show that negative numbers have logarithms in the complex plane. In particular, find $\ln(-1)$.

ANS. $\ln(-1) = i\pi$.

11.6.8 For noninteger m, show that the binomial expansion of Exercise 11.5.2 holds only for a suitably defined branch of the function $(1+z)^m$. Show how the z-plane is cut. Explain why $|z| < 1$ may be taken as the circle of convergence for the expansion of this branch, in light of the cut you have chosen.

11.6.9 The Taylor expansion of Exercises 11.5.2 and 11.6.8 is **not** suitable for branches other than the one suitably defined branch of the function $(1+z)^m$ for noninteger m. (Note that other branches cannot have the same Taylor expansion since they must be distinguishable.) Using the same branch cut of the earlier exercises for all other branches, find the corresponding Taylor expansions, detailing the phase assignments and Taylor coefficients.

11.6.10 (a) Develop a Laurent expansion of $f(z) = [z(z-1)]^{-1}$ about the point $z = 1$ valid for small values of $|z-1|$. Specify the exact range over which your expansion holds. This is an analytic continuation of the infinite series in Eq. (11.49).

(b) Determine the Laurent expansion of $f(z)$ about $z = 1$ but for $|z-1|$ large.

Hint. Make a partial fraction decomposition of this function and use the geometric series.

11.6.11 (a) Given $f_1(z) = \int_0^\infty e^{-zt}\,dt$ (with t real), show that the domain in which $f_1(z)$ exists (and is analytic) is $\Re e(z) > 0$.

(b) Show that $f_2(z) = 1/z$ equals $f_1(z)$ over $\Re e(z) > 0$ and is therefore an analytic continuation of $f_1(z)$ over the entire z-plane except for $z = 0$.

(c) Expand $1/z$ about the point $z = -i$. You will have

$$f_3(z) = \sum_{n=0}^{\infty} a_n (z+i)^n.$$

What is the domain of this formula for $f_3(z)$?

ANS. $\dfrac{1}{z} = i\sum_{n=0}^{\infty} i^{-n}(z+i)^n, \quad |z+i| < 1.$

11.7 CALCULUS OF RESIDUES

Residue Theorem

If the Laurent expansion of a function,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

is integrated term by term by using a closed contour that encircles one isolated singular point z_0 once in a counterclockwise sense, we obtain, applying Eq. (11.29),

$$a_n \oint (z - z_0)^n dz = 0, \quad n \neq -1. \tag{11.59}$$

However, for $n = -1$, Eq. (11.29) yields

$$a_{-1} \oint (z - z_0)^{-1} dz = 2\pi i a_{-1}. \tag{11.60}$$

Summarizing Eqs. (11.59) and (11.60), we have

$$\oint f(z)\, dz = 2\pi i a_{-1}. \tag{11.61}$$

The constant a_{-1}, the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion, is called the **residue** of $f(z)$ at $z = z_0$.

Now consider the evaluation of the integral, over a closed contour C, of a function that has isolated singularities at points $z_1, z_2, \ldots$. We can handle this integral by deforming our contour as shown in Fig. 11.18. Cauchy's integral theorem (Section 11.3) then leads to

$$\oint_C f(z)\, dz + \oint_{C_1} f(z)\, dz + \oint_{C_2} f(z)\, dz + \cdots = 0, \tag{11.62}$$

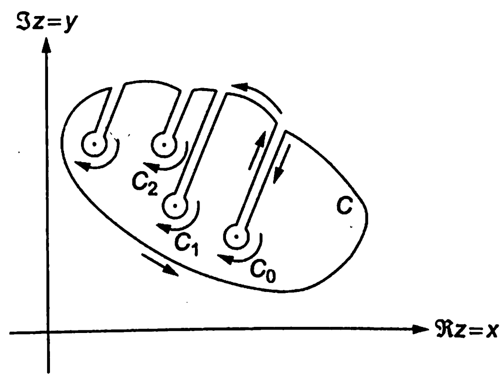

FIGURE 11.18 Excluding isolated singularities.

where C is in the positive, counterclockwise direction, but the contours $C_1, C_2, \ldots$, that, respectively, encircle $z_1, z_2, \ldots$ are all clockwise. Thus, referring to Eq. (11.61), the integrals C_i about the individual isolated singularities have the values

$$\oint_{C_i} f(z)\,dz = -2\pi i a_{-1,i}, \tag{11.63}$$

where $a_{-1,i}$ is the residue obtained from the Laurent expansion about the singular point $z = z_i$. The negative sign comes from the clockwise integration. Combining Eqs. (11.62) and (11.63), we have

$$\begin{aligned}\oint_C f(z)\,dz &= 2\pi i(a_{-1,1} + a_{-1,2} + \cdots)\\ &= 2\pi i\,(\text{sum of the enclosed residues}).\end{aligned} \tag{11.64}$$

This is the **residue theorem**. The problem of evaluating a set of contour integrals is replaced by the algebraic problem of computing residues at the enclosed singular points.

Computing Residues

It is, of course, not necessary to obtain an entire Laurent expansion of $f(z)$ about $z = z_0$ to identify a_{-1}, the coefficient of $(z - z_0)^{-1}$ in the expansion. If $f(z)$ has a simple pole at $z - z_0$, then, with a_n the coefficients in the expansion of $f(z)$,

$$(z - z_0)f(z) = a_{-1} + a_0(z - z_0) + a_1(z - z_0)^2 + \cdots, \tag{11.65}$$

and, recognizing that $(z - z_0)f(z)$ may not have a form permitting an obvious cancellation of the factor $z - z_0$, we take the limit of Eq. (11.65) as $z \to z_0$:

$$a_{-1} = \lim_{z \to z_0} \Big((z - z_0)f(z)\Big). \tag{11.66}$$

If there is a pole of order $n > 1$ at $z - z_0$, then $(z - z_0)^n f(z)$ must have the expansion

$$(z - z_0)^n f(z) = a_{-n} + \cdots + a_{-1}(z - z_0)^{n-1} + a_0(z - z_0)^n + \cdots. \tag{11.67}$$

We see that a_{-1} is the coefficient of $(z - z_0)^{n-1}$ in the Taylor expansion of $(z - z_0)^n f(z)$, and therefore we can identify it as satisfying

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z \to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \Big((z - z_0)^n f(z)\Big)\right], \tag{11.68}$$

where a limit is indicated to take account of the fact that the expression involved may be indeterminate. Sometimes the general formula, Eq. (11.68), is found to be more complicated than the judicious use of power-series expansions. See items 4 and 5 in Example 11.7.1 below.

Essential singularities will also have well-defined residues, but finding them may be more difficult. In principle, one can use Eq. (11.48) with $n = -1$, but the integral involved may seem intractable. Sometimes the easiest route to the residue is by first finding the Laurent expansion.

Example 11.7.1 COMPUTING RESIDUES

Here are some examples:

1. Residue of $\frac{1}{4z+1}$ at $z=-\frac{1}{4}$ is $\lim_{z=-\frac{1}{4}}\left(\frac{z+\frac{1}{4}}{4z+1}\right)=\frac{1}{4}$,

2. Residue of $\frac{1}{\sin z}$ at $z=0$ is $\lim_{z\to 0}\left(\frac{z}{\sin z}\right)=1$,

3. Residue of $\frac{\ln z}{z^2+4}$ at $z=2e^{\pi i}$ is

$$\lim_{z\to 2e^{\pi i}}\left(\frac{(z-2e^{\pi i})\ln z}{z^2+4}\right)=\frac{(\ln 2+\pi i)}{4i}=\frac{\pi}{4}-\frac{i\ln 2}{4},$$

4. Residue of $\frac{z}{\sin^2 z}$ at $z=\pi$; the pole is second order, and the residue is given by

$$\frac{1}{1!}\lim_{z\to\pi}\left(\frac{d}{dz}\frac{z(z-\pi)}{\sin^2 z}\right).$$

However, it may be easier to make the substitution $w=z-\pi$, to note that $\sin^2 z=\sin^2 w$, and to identify the residue as the coefficient of $1/w$ in the expansion of $(w+\pi)/\sin^2 w$ about $w=0$. This expansion can be written

$$\frac{w+\pi}{\left(w-\frac{w^3}{3!}+\cdots\right)^2}=\frac{w+\pi}{w^2-\frac{w^4}{3}+\cdots}.$$

The denominator expands entirely into even powers of w, so the π in the numerator cannot contribute to the residue. Then, from the w in the numerator and the leading term of the denominator, we find the residue to be 1.

5. Residue of $f(z)=\frac{\cot\pi z}{z(z+2)}$ at $z=0$.

The pole at $z=0$ is second-order, and direct application of Eq. (11.48) leads to a complicated indeterminate expression requiring multiple applications of l'Hôpital's rule. Perhaps easier is to introduce the initial terms of the expansions about $z=0$: $\cot\pi z=(\pi z)^{-1}+O(z)$, $1/(z+2)=\frac{1}{2}[1-(z/2)+O(z^2)]$, reaching

$$f(z)=\frac{1}{z}\left[\frac{1}{\pi z}+O(z)\right]\left(\frac{1}{2}\right)\left[1-\frac{z}{2}+O(z^2)\right],$$

from which we can read out the residue as the coefficient of z^{-1}, namely $-1/4\pi$.

6. Residue of $e^{-1/z}$ at $z=0$. This is at an essential singularity; from the Taylor series of e^w with $w=-1/z$, we have

$$e^{-1/z}=1-\frac{1}{z}+\frac{1}{2!}\left(-\frac{1}{z}\right)^2+\cdots,$$

from which we read out the value of the residue, -1. ■

Cauchy Principal Value

Occasionally an isolated pole will be directly on the contour of an integration, causing the integral to diverge. A simple example is provided by an attempt to evaluate the real integral

$$\int_{-a}^{b} \frac{dx}{x}, \tag{11.69}$$

which is divergent because of the logarithmic singularity at $x = 0$; note that the indefinite integral of x^{-1} is $\ln x$. However, the integral in Eq. (11.69) can be given a meaning if we obtain a convergent form when replaced by a limit of the form

$$\lim_{\delta \to 0+} \int_{-a}^{-\delta} \frac{dx}{x} + \int_{\delta}^{b} \frac{dx}{x}. \tag{11.70}$$

To avoid issues with the logarithm of negative values of x, we change the variable in the first integral to $y = -x$, and the two integrals are then seen to have the respective values $\ln\delta - \ln a$ and $\ln b - \ln\delta$, with sum $\ln b - \ln a$. What has happened is that the increase toward $+\infty$ as $1/x$ approaches zero from positive values of x is compensated by a decrease toward $-\infty$ as $1/x$ approaches zero from negative x. This situation is illustrated graphically in Fig. 11.19.

Note that the procedure we have described does **not** make the original integral of Eq. (11.69) convergent. In order for that integral to be convergent, it would be necessary

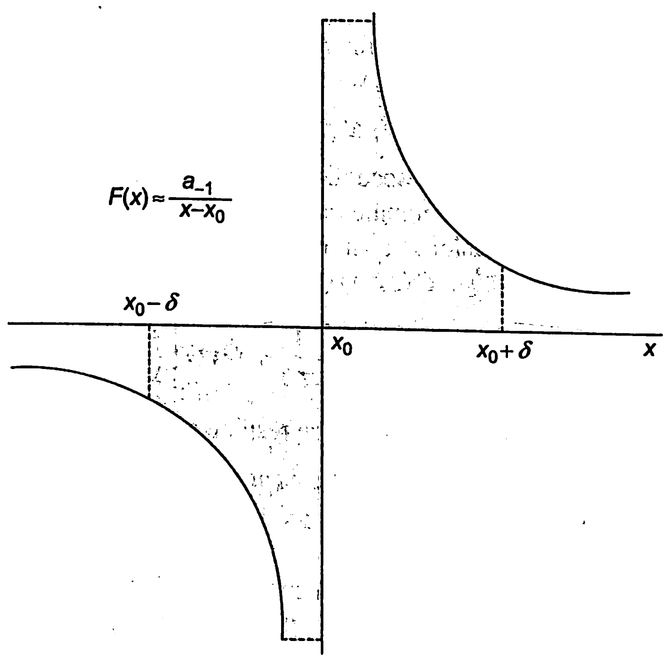

FIGURE 11.19 Cauchy principal value cancellation, integral of $1/z$.

that

$$\lim_{\delta_1,\delta_2\to 0^+}\left[\int_{-a}^{-\delta_1}\frac{dx}{x}+\int_{\delta_2}^{b}\frac{dx}{x}\right]$$

exist (meaning that the limit has a unique value) when δ_1 and δ_2 approach zero **independently**. However, different rates of approach to zero by δ_1 and δ_2 will cause a change in value of the integral. For example, if $\delta_2 = 2\delta_1$, then an evaluation like that of Eq. (11.70) would yield the result $(\ln\delta_1 - \ln a) + (\ln b - \ln\delta_2) = \ln b - \ln a - \ln 2$. The limit then has no definite value, confirming our original statement that the integral diverges.

Generalizing from the above example, we define the **Cauchy principal value** of the real integral of a function $f(x)$ with an isolated singularity on the integration path at the point x_0 as the limit

$$\lim_{\delta\to 0^+}\int^{x_0-\delta} f(x)\,dx + \int_{x_0+\delta} f(x)\,dx. \tag{11.71}$$

The Cauchy principal value is sometimes indicated by preceding the integral sign by P or by drawing a horizontal line through the integration sign, as in

$$P\int f(x)\,dx \quad \text{or} \quad ⨍ f(x)\,dx.$$

This notation, of course, presumes that the location of the singularity is known.

Example 11.7.2 A Cauchy Principal Value

Consider the integral

$$I = \int_0^\infty \frac{\sin x}{x}\,dx. \tag{11.72}$$

If we substitute for $\sin x$ the equivalent formula

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i},$$

we then have

$$I = \int_0^\infty \frac{e^{ix}-e^{-ix}}{2ix}\,dx. \tag{11.73}$$

We would like to separate this expression for I into two terms, but if we do so, each will become a logarithmically divergent integral. However, if we change the integration range in Eq. (11.72), originally $(0,\infty)$, to (δ,∞), that integral remains unchanged in the limit of small δ, and the integrals in Eq. (11.73) remain convergent so long as δ is not precisely zero. Then, rewriting the second of the two integrals in Eq. (11.73), to reach

$$\int_{\delta}^{\infty} \frac{e^{-ix}}{2ix}\,dx = \int_{-\infty}^{-\delta} \frac{e^{ix}}{2ix}\,dx,$$

we see that the two integrals which together form I can be written (in the limit $\delta \to 0^+$) as the Cauchy principal value integral

$$I = \fint_{-\infty}^{\infty} \frac{e^{ix}}{2ix}\,dx. \tag{11.74}$$

■

The Cauchy principal value has implications for complex variable theory. Suppose now that, instead of having a break in the integration path from $x_0 - \delta$ to $x_0 + \delta$, we connect the two parts of the path by a circular arc passing, in the complex plane, either above or below the singularity at x_0. Let's continue the discussion in conventional complex-variable notation, denoting the singular point as z_0, so our arc will be a half circle (of radius δ) passing either counterclockwise **below** the singularity at z_0 or clockwise **above** z_0. We restrict further analysis to singularities no stronger than $1/(z - z_0)$, so we are dealing with a simple pole. Looking at the Laurent expansion of the function $f(z)$ to be integrated, it will have initial terms

$$\frac{a_{-1}}{z - z_0} + a_0 + \cdots,$$

and the integration over a semicircle of radius δ will take (in the limit $\delta \to 0^+$) one of the two forms (in the polar representation $z - z_0 = re^{i\theta}$, with $dz = ire^{i\theta}d\theta$ and $r = \delta$):

$$I_{\text{over}} = \int_{\pi}^{0} d\theta\, i\delta e^{i\theta}\left[\frac{a_{-1}}{\delta e^{i\theta}} + a_0 + \cdots\right] = \int_{\pi}^{0}\Big(ia_{-1} + i\delta e^{i\theta}a_0 + \cdots\Big)\,d\theta \to -i\pi a_{-1}, \tag{11.75}$$

$$I_{\text{under}} = \int_{\pi}^{2\pi} d\theta\, i\delta e^{i\theta}\left[\frac{a_{-1}}{\delta e^{i\theta}} + a_0 + \cdots\right] = \int_{\pi}^{2\pi}\Big(ia_{-1} + i\delta e^{i\theta}a_0 + \cdots\Big)\,d\theta \to i\pi a_{-1}. \tag{11.76}$$

Note that all but the first term of each of Eqs. (11.75) and (11.76) vanishes in the limit $\delta \to 0^+$, and that each of these equations yields a result that is in magnitude half the value that would have been obtained by a full circuit around the pole. The signs associated with the semicircles correspond as expected to the direction of travel, and the two semicircular integrals average to zero.

We occasionally will want to evaluate a contour integral of a function $f(z)$ on a closed path that includes the two pieces of a Cauchy principal value integral $\fint f(z)\,dz$ with a simple pole at z_0, a semicircular arc connecting them at the singularity, and whatever other curve C is needed to close the contour (see Fig. 11.20).

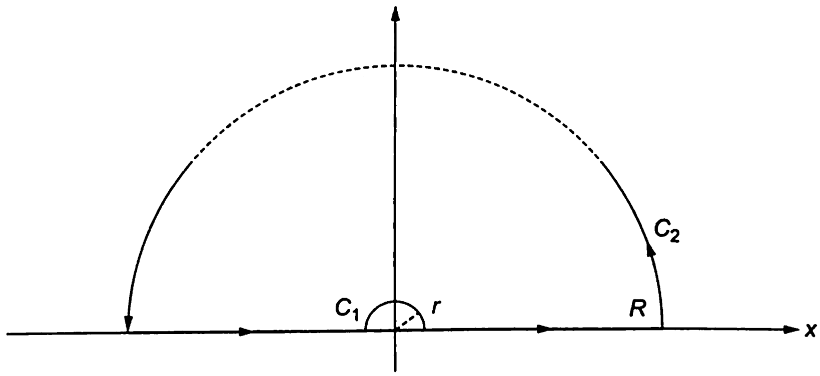

FIGURE 11.20 A contour including a Cauchy principal value integral.

These contributions combine as follows, noting that in the figure the contour passes **over** the point z_0:

$$\oint f(z)\,dz + I_{\text{over}} + \int_{C_2} f(z)\,dz = 2\pi i \sum \text{residues (other than at } z_0\text{)},$$

which rearranges to give

$$\oint f(z)\,dz = -I_{\text{over}} - \int_{C_2} f(z)\,dz + 2\pi i \sum \text{residues (other than at } z_0\text{)}. \qquad (11.77)$$

On the other hand, we could have chosen the contour to pass **under** z_0, in which case, instead of Eq. (11.77) we would get

$$\oint f(z)\,dz = -I_{\text{under}} - \int_{C_2} f(z)\,dz + 2\pi i \sum \text{residues (other than at } z_0\text{)} + 2\pi i a_{-1}, \qquad (11.78)$$

where the residue denoted a_{-1} is from the pole at z_0. Equations (11.77) and (11.78) are in agreement because $2\pi i a_{-1} - I_{\text{under}} = -I_{\text{over}}$, so for the purpose of evaluating the Cauchy principal value integral, it makes no difference whether we go below or above the singularity on the original integration path.

Pole Expansion of Meromorphic Functions

Analytic functions $f(z)$ that have only isolated poles as singularities are called **meromorphic**. Mittag-Leffler showed that, instead of making an expansion about a single regular point (a Taylor expansion) or about an isolated singular point (a Laurent expansion), it was also possible to make an expansion each of whose terms arises from a different pole of $f(z)$. Mittag-Leffler's theorem assumes that $f(z)$ is analytic at $z=0$ and at all other points (excluding infinity) with the exception of discrete simple poles at points z_1, $z_2, \ldots$, with respective residues $b_1, b_2, \ldots$. We choose to order the poles in a way such that $0 < |z_1| \le |z_2| \le \cdots$, and we assume that in the limit of large z, $|f(z)/z| \to 0$. Then, Mittag-Leffler's theorem states that

$$f(z) = f(0) + \sum_{n=1}^{\infty} b_n \left(\frac{1}{z - z_n} + \frac{1}{z_n} \right). \tag{11.79}$$

To prove the theorem, we make the preliminary observation that the quantity being summed in Eq. (11.79) can be written

$$\frac{z\, b_n}{z_n(z_n - z)},$$

suggesting that it might be useful to consider a contour integral of the form

$$I_N = \oint_{C_N} \frac{f(w)\, dw}{w(w - z)},$$

where w is another complex variable and C_N is a circle enclosing the first N poles of $f(z)$. Since C_N, which has a radius we denote R_N, has total arc length $2\pi R_N$, and the absolute value of the integrand asymptotically approaches $|f(R_N)|/R_N^2$, the large-z behavior of $f(z)$ guarantees that $\lim_{R_N \to \infty} I_N = 0$.

We now obtain an alternate expression for I_N using the residue theorem. Recognizing that C_N encircles simple poles at $w = 0$, $w = z$, and $w = z_n$, $n = 1 \ldots N$, that $f(w)$ is nonsingular at $w = 0$ and $w = z$, and that the residue of $f(z)/w(w - z)$ at z_n is just $b_n/z_n(z_n - z)$, we have

$$I_N = 2\pi i \frac{f(0)}{-z} + 2\pi i \frac{f(z)}{z} + \sum_{n=1}^{N} \frac{2\pi i b_n}{z_n(z_n - z)}.$$

Taking the large-N limit, in which $I_N = 0$, we recover Mittag-Leffler's theorem, Eq. (11.79). The pole expansion converges when the condition $\lim_{z\to\infty} |f(z)/z| = 0$ is satisfied.

Mittag-Leffler's theorem leads to a number of interesting pole expansions. Consider the following examples.

Example 11.7.3 POLE EXPANSION OF tan z

Writing

$$\tan z = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})},$$

we easily see that the only singularities of $\tan z$ are for real values of z, and they occur at the zeros of $\cos x$, namely at $\pm\pi/2, \pm 3\pi/2, \ldots$, or in general at $z_n = \pm(2n + 1)\pi/2$.

To obtain the residues at these points, we take the limit (using l'Hôpital's rule)

$$b_n = \lim_{z \to \frac{(2n+1)\pi}{2}} \frac{(z-(2n+1)\pi/2)\sin z}{\cos z}$$

$$= \left. \frac{\sin z + (z-(2n+1)\pi/2)\cos z}{-\sin z} \right|_{z=\frac{(2n+1)\pi}{2}} = -1,$$

the same value for every pole.

Noting that $\tan(0) = 0$, and that the poles within a circle of radius $(N+1)\pi$ will be those (of both signs) referred to here by n values 0 through N, Eq. (11.79) for the current case (but only through N) yields

$$\tan z = \sum_{n=0}^{N}(-1)\left(\frac{1}{z-(2n+1)\pi/2} + \frac{1}{(2n+1)\pi/2}\right)$$

$$+ \sum_{n=0}^{N}(-1)\left(\frac{1}{z+(2n+1)\pi/2} + \frac{1}{-(2n+1)\pi/2}\right)$$

$$= \sum_{n=0}^{N}(-1)\left(\frac{1}{z-(2n+1)\pi/2} + \frac{1}{z+(2n+1)\pi/2}\right).$$

Combining terms over a common denominator, and taking the limit $N \to \infty$, we reach the usual form of the expansion:

$$\tan z = 2z\left(\frac{1}{(\pi/2)^2 - z^2} + \frac{1}{(3\pi/2)^2 - z^2} + \frac{1}{(5\pi/2)^2 - z^2} + \cdots\right). \tag{11.80}$$

■

Example 11.7.4 POLE EXPANSION OF cot z

This example proceeds much as the preceding one, except that $\cot z$ has a simple pole at $z = 0$, with residue $+1$. We therefore consider instead $\cot z - 1/z$, thereby removing the singularity. The singular points are now simple poles at $\pm n\pi$ ($n \neq 0$), with residues (again obtained via l'Hôpital's rule)

$$b_n = \lim_{z \to n\pi}(z-n\pi)\cot z = \lim_{z \to n\pi} \frac{(z-n\pi)(z\cos z - \sin z)}{z \sin z}$$

$$= \left. \frac{z\cos z - \sin z + (z-n\pi)(-z\sin z)}{\sin z + z\cos z} \right|_{z=n\pi} = +1.$$

Noting that $\cot z - 1/z$ is zero at $z = 0$ (the second term in the expansion of $\cot z$ is $-z/3$), we have

$$\cot z - \frac{1}{z} = \sum_{n=1}^{N}\left(\frac{1}{z-n\pi} + \frac{1}{n\pi} + \frac{1}{z+n\pi} + \frac{1}{-n\pi}\right),$$

which rearranges to

$$\cot z = \frac{1}{z} + 2z\left(\frac{1}{z^2-\pi^2} + \frac{1}{z^2-(2\pi)^2} + \frac{1}{z^2-(3\pi)^2} + \cdots\right). \tag{11.81}$$

■

In addition to Eqs. (11.80) and (11.81), two other pole expansions of importance are

$$\sec z = \pi\left(\frac{1}{(\pi/2)^2-z^2} - \frac{3}{(3\pi/2)^2-z^2} + \frac{5}{(5\pi/2)^2-z^2} - \right), \tag{11.82}$$

$$\csc z = \frac{1}{z} - 2z\left(\frac{1}{z^2-\pi^2} - \frac{1}{z^2-(2\pi)^2} + \frac{1}{z^2-(3\pi)^2} + \cdots\right). \tag{11.83}$$

Counting Poles and Zeros

It is possible to obtain information about the numbers of poles and zeros of a function $f(z)$ that is otherwise analytic within a closed region by consideration of its logarithmic derivative, namely $f'(z)/f(z)$. The starting point for this analysis is to write an expression for $f(z)$ relative to a point z_0 where there is either a zero or a pole in the form

$$f(z) = (z-z_0)^{\mu} g(z),$$

with $g(z)$ finite and nonzero at $z = z_0$. That requirement identifies the limiting behavior of $f(z)$ near z_0 as proportional to $(z-z_0)^{\mu}$, and also causes f'/f to assume near $z = z_0$ the form

$$\frac{f'(z)}{f(z)} = \frac{\mu(z-z_0)^{\mu-1}g(z) + (z-z_0)^{\mu}g'(z)}{(z-z_0)^{\mu}g(z)} = \frac{\mu}{z-z_0} + \frac{g'(z)}{g(z)}. \tag{11.84}$$

Equation (11.84) shows that, for all nonzero μ (i.e., if z_0 is either a zero or a pole), f'/f has a simple pole at $z = z_0$ with residue μ. Note that because $g(z)$ is required to be nonzero and finite, the second term of Eq. (11.84) cannot be singular.

Applying now the residue theorem to Eq. (11.84) for a closed region within which $f(z)$ is analytic except possibly at poles, we see that the integral of f'/f around a closed contour yields the result

$$\oint_C \frac{f'(z)}{f(z)}dz = 2\pi i\Big(N_f - P_f\Big), \tag{11.85}$$

where P_f is the number of poles of $f(z)$ within the region enclosed by C, each multiplied by its order, and N is the number of zeros of $f(z)$ enclosed by C, each multiplied by its multiplicity.

The counting of zeros is often facilitated by using **Rouché's theorem**, which states

> *If $f(z)$ and $g(z)$ are analytic in the region bounded by a curve C and $|f(z)| > |g(z)|$ on C, then $f(z)$ and $f(z)+g(z)$ have the same number of zeros in the region bounded by C.*

To prove Rouché's theorem, we first write, from Eq. (11.85),

$$\oint_C \frac{f'(z)}{f(z)}dz = 2\pi i N_f \quad \text{and} \quad \oint_C \frac{f'(z)+g'(z)}{f(z)+g(z)}dz = 2\pi i N_{f+g},$$

where N_f designates the number of zeros of f within C. Then we observe that because the indefinite integral of f'/f is $\ln f$, N_f is the number of times the argument of f cycles through 2π when C is traversed once in the counterclockwise direction. Similarly, we note that N_{f+g} is the number of times the argument of $f+g$ cycles through 2π on traversal of the contour C.

We next write

$$f+g = f\left(1+\frac{g}{f}\right) \quad \text{and} \quad \arg(f+g) = \arg(f) + \arg\left(1+\frac{g}{f}\right), \tag{11.86}$$

using the fact that the argument of a product is the sum of the arguments of its factors. It is then clear that the number of cycles through 2π of $\arg(f+g)$ is equal to the number of cycles of $\arg(f)$ **plus** the number of cycles of $\arg(1+g/f)$. But because $|g/f| < 1$, the real part of $1+g/f$ never becomes negative, and its argument is therefore restricted to the range $-\pi/2 < \arg(1+g/f) < \pi/2$. Therefore $\arg(1+g/f)$ cannot cycle through 2π, the number of cycles of $\arg(f+g)$ must be equal to the number of cycles of $\arg f$, and $f+g$ and f must have the same number of zeros within C. This completes the proof of Rouché's theorem.

Example 11.7.5 COUNTING ZEROS

Our problem is to determine the number of zeros of $F(z) = z^3 - 2z + 11$ with moduli between 1 and 3. Since $F(z)$ is analytic for all finite z, we could in principle simply apply Eq. (11.85) for the contour consisting of the circles $|z| = 1$ (clockwise) and $|z| = 3$ (counterclockwise), setting $P_F = 0$ and solving for N_F. However, that approach will in practice prove difficult. Instead, we simplify the problem by using Rouché's theorem.

We first compute the number of zeros within $|z| = 1$, writing $F(z) = f(z) + g(z)$, with $f(z) = 11$ and $g(z) = z^3 - 2z$. It is clear that $|f(z)| > |g(z)|$ when $|z| = 1$, so, by Rouché's theorem, f and $f+g$ have the same number of zeros within this circle. Since $f(z) = 11$ has no zeros, we conclude that all the zeros of $F(z)$ are outside $|z| = 1$.

Next we compute the number of zeros within $|z| = 3$, taking for this purpose $f(z) = z^3$, $g(z) = 11 - 2z$. When $|z| = 3$, we have $|f(z)| = 27 > |g(z)|$, so F and f have the same number of zeros, namely three (the three-fold zero of f at $z = 0$). Thus, the answer to our problem is that F has three zeros, all with moduli between 1 and 3. ■

Product Expansion of Entire Functions

We remind the reader that a function $f(z)$ that is analytic for all finite z is called an **entire** function. Referring to Eq. (11.84), we see that if $f(z)$ is an entire function, then $f'(z)/f(z)$ will be meromorphic, with all its poles simple. Assuming for simplicity that the zeros of f

are simple and at points z_n, so that μ in Eq. (11.84) is 1, we can invoke the Mittag-Leffler theorem to write f'/f as the pole expansion

$$\frac{f'(z)}{f(z)} = \frac{f'(0)}{f(0)} + \sum_{n=1}^{\infty}\left[\frac{1}{z-z_n} + \frac{1}{z_n}\right]. \tag{11.87}$$

Integrating Eq. (11.87) yields

$$\int_0^z \frac{f'(z)}{f(z)}dz = \ln f(z) - \ln f(0)$$

$$= \frac{zf'(0)}{f(0)} + \sum_{n=1}^{\infty}\left[\ln(z-z_n) - \ln(-z_n) + \frac{z}{z_n}\right].$$

Exponentiating, we obtain the product expansion

$$f(z) = f(0)\exp\left(\frac{zf'(0)}{f(0)}\right)\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)e^{z/z_n}. \tag{11.88}$$

Examples are the product expansions for

$$\sin z = z\prod_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left(1-\frac{z}{n\pi}\right)e^{z/n\pi} = z\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2\pi^2}\right), \tag{11.89}$$

$$\cos z = \prod_{n=1}^{\infty}\left(1-\frac{z^2}{(n-1/2)^2\pi^2}\right). \tag{11.90}$$

The expansion of $\sin z$ cannot be obtained directly from Eq. (11.88), but its derivation is the subject of Exercise 11.7.5. We also point out here that the gamma function has a product expansion, discussed in Chapter 13.

Exercises

11.7.1 Determine the nature of the singularities of each of the following functions and evaluate the residues ($a > 0$).

(a) $\dfrac{1}{z^2+a^2}$. (b) $\dfrac{1}{(z^2+a^2)^2}$.

(c) $\dfrac{z^2}{(z^2+a^2)^2}$. (d) $\dfrac{\sin 1/z}{z^2+a^2}$.

(e) $\dfrac{ze^{+iz}}{z^2+a^2}$. (f) $\dfrac{ze^{+iz}}{z^2-a^2}$.

(g) $\dfrac{e^{+iz}}{z^2-a^2}$. (h) $\dfrac{z^{-k}}{z+1}$, $0 < k < 1$.

Hint. For the point at infinity, use the transformation $w = 1/z$ for $|z| \to 0$. For the residue, transform $f(z)dz$ into $g(w)dw$ and look at the behavior of $g(w)$.

11.7.2 Evaluate the residues at $z = 0$ and $z = -1$ of $\pi \cot \pi z / z(z+1)$.

11.7.3 The classical definition of the exponential integral $\mathrm{Ei}(x)$ for $x > 0$ is the Cauchy principal value integral

$$\mathrm{Ei}(x) = \fint_{-\infty}^{x} \frac{e^t}{t}\, dt,$$

where the integration range is cut at $x = 0$. Show that this definition yields a convergent result for positive x.

11.7.4 Writing a Cauchy principal value integral to deal with the singularity at $x = 1$, show that, if $0 < p < 1$,

$$\fint_0^{\infty} \frac{x^{-p}}{x-1}\, dx = -\pi \cot p\pi.$$

11.7.5 Explain why Eq. (11.88) is not directly applicable to the product expansion of $\sin z$. Show how the expansion, Eq. (11.89), can be obtained by expanding instead $\sin z / z$.

11.7.6 Starting from the observations

1. $f(z) = a_n z^n$ has n zeros, and
2. for sufficiently large $|R|$, $|\sum_{m=0}^{n-1} a_m R^m| < |a_n R^n|$,

use Rouché's theorem to prove the fundamental theorem of algebra (namely that every polynomial of degree n has n roots).

11.7.7 Using Rouché's theorem, show that all the zeros of $F(z) = z^6 - 4z^3 + 10$ lie between the circles $|z| = 1$ and $|z| = 2$.

11.7.8 Derive the pole expansions of $\sec z$ and $\csc z$ given in Eqs. (11.82) and (11.83).

11.7.9 Given that $f(z) = (z^2 - 3z + 2)/z$, apply a partial fraction decomposition to f'/f and show directly that $\oint_C f'(z)/f(z)\, dz = 2\pi i(N_f - P_f)$, where N_f and P_f are, respectively, the numbers of zeros and poles encircled by C (including their multiplicities).

11.7.10 The statement that the integral halfway around a singular point is equal to one-half the integral all the way around was limited to simple poles. Show, by a specific example, that

$$\int_{\text{Semicircle}} f(z)\, dz = \frac{1}{2} \oint_{\text{Circle}} f(z)\, dz$$

does not necessarily hold if the integral encircles a pole of higher order.

Hint. Try $f(z) = z^{-2}$.

11.7.11 A function $f(z)$ is analytic along the real axis except for a third-order pole at $z = x_0$. The Laurent expansion about $z = x_0$ has the form

$$f(z) = \frac{a_{-3}}{(z - x_0)^3} + \frac{a_{-1}}{z - x_0} + g(z),$$

with $g(z)$ analytic at $z = x_0$. Show that the Cauchy principal value technique is applicable, in the sense that

(a) $\lim_{\delta\to 0}\left\{\int_{-\infty}^{x_0-\delta} f(x)\,dx + \int_{x_0+\delta}^{\infty} f(x)\,dx\right\}$ is finite.

(b) $\int_{C_{x_0}} f(z)\,dz = \pm i\pi a_{-1}$,

where C_{x_0} denotes a **small semicircle** about $z = x_0$.

11.7.12 The unit step function is defined as (compare Exercise 1.15.13)

$$u(s-a) = \begin{cases} 0, & s < a \\ 1, & s > a. \end{cases}$$

Show that $u(s)$ has the integral representations

(a) $u(s) = \lim_{\varepsilon\to 0^+} \dfrac{1}{2\pi i}\displaystyle\int_{-\infty}^{\infty} \frac{e^{ixs}}{x - i\varepsilon}\,dx.$

(b) $u(s) = \dfrac{1}{2} + \dfrac{1}{2\pi i}\displaystyle\fint_{-\infty}^{\infty} \frac{e^{ixs}}{x}\,dx.$

Note. The parameter s is real.

11.8 Evaluation of Definite Integrals

Definite integrals appear repeatedly in problems of mathematical physics as well as in pure mathematics. In Chapter 1 we reviewed several methods for integral evaluation, there noting that contour integration methods were powerful and deserved detailed study. We have now reached a point where we can explore these methods, which are applicable to a wide variety of definite integrals with physically relevant integration limits. We start with applications to integrals containing trigonometric functions, which we can often convert to forms in which the variable of integration (originally an angle) is converted into a complex variable z, with the integration integral becoming a contour integral over the unit circle.

Trigonometric Integrals, Range $(0,2\pi)$

We consider here integrals of the form

$$I = \int_0^{2\pi} f(\sin\theta, \cos\theta)\,d\theta, \tag{11.91}$$

where f is finite for all values of θ. We also require f to be a rational function of $\sin\theta$ and $\cos\theta$ so that it will be single-valued. We make a change of variable to

$$z = e^{i\theta}, \quad dz = ie^{i\theta}d\theta,$$

with the range in θ, namely $(0, 2\pi)$, corresponding to $e^{i\theta}$ moving counterclockwise around the unit circle to form a closed contour. Then we make the substitutions

$$d\theta = -i\frac{dz}{z}, \quad \sin\theta = \frac{z - z^{-1}}{2i}, \quad \cos\theta = \frac{z + z^{-1}}{2}, \tag{11.92}$$

where we have used Eq. (1.133) to represent $\sin\theta$ and $\cos\theta$. Our integral then becomes

$$I = -i\oint f\left(\frac{z - z^{-1}}{2i}, \frac{z + z^{-1}}{2}\right)\frac{dz}{z}, \tag{11.93}$$

with the path of integration the unit circle. By the residue theorem, Eq. (11.64),

$$I = (-i)\,2\pi i\sum \text{residues within the unit circle.} \tag{11.94}$$

Note that we must use the residues of f/z. Here are two preliminary examples.

Example 11.8.1 Integral of cos in Denominator

Our problem is to evaluate the definite integral

$$I = \int_0^{2\pi}\frac{d\theta}{1 + a\cos\theta}, \qquad |a| < 1.$$

By Eq. (11.93) this becomes

$$\begin{aligned} I &= -i \oint_{\text{unit circle}} \frac{dz}{z[1 + (a/2)(z + z^{-1})]} \\ &= -i\frac{2}{a}\oint\frac{dz}{z^2 + (2/a)z + 1}. \end{aligned}$$

The denominator has roots

$$z_1 = -\frac{1 + \sqrt{1 - a^2}}{a} \quad \text{and} \quad z_2 = -\frac{1 - \sqrt{1 - a^2}}{a}.$$

Noting that $z_1 z_2 = 1$, it is easy to see that z_2 is within the unit circle and z_1 is outside. Writing the integral in the form

$$\oint\frac{dz}{(z - z_1)(z - z_2)},$$

we see that the residue of the integrand at $z = z_2$ is $1/(z_2 - z_1)$, so application of the residue theorem yields

$$I = -i\frac{2}{a}\cdot 2\pi i\frac{1}{z_2 - z_1}.$$

Inserting the values of z_1 and z_2, we obtain the final result

$$\int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta} = \frac{2\pi}{\sqrt{1-a^2}}, \qquad |a| < 1.$$

■

Example 11.8.2 ANOTHER TRIGONOMETRIC INTEGRAL

Consider

$$I = \int_0^{2\pi} \frac{\cos 2\theta \, d\theta}{5 - 4\cos\theta}.$$

Making the substitutions identified in Eqs. (11.92) and (11.93), the integral I assumes the form

$$\begin{aligned} I &= \oint \frac{\frac{1}{2}(z^2 + z^{-2})}{5 - 2(z + z^{-1})} \left(\frac{-i\,dz}{z}\right) \\ &= \frac{i}{4} \oint \frac{(z^4 + 1)\,dz}{z^2\left(z - \frac{1}{2}\right)(z - 2)}, \end{aligned}$$

where the integration is around the unit circle. Note that we identified $\cos 2\theta$ as $(z^2 + z^{-2})/2$, which is simpler than reducing it first to its equivalent in terms of $\sin z$ and $\cos z$. We see that the integrand has poles at $z = 0$ (of order 2), and simple poles at $z = 1/2$ and $z = 2$. Only the poles at $z = 0$ and $z = 1/2$ are within the contour.

At $z = 0$ the residue of the integrand is

$$\frac{d}{dz}\left[\frac{z^4 + 1}{\left(z - \frac{1}{2}\right)(z - 2)}\right]_{z=0} = \frac{5}{2},$$

while its residue at $z = 1/2$ is

$$\left[\frac{z^4 + 1}{z^2(z - 2)}\right]_{z=1/2} = -\frac{17}{6}.$$

Applying the residue theorem, we have

$$I = \frac{i}{4}(2\pi i)\left[\frac{5}{2} - \frac{17}{6}\right] = \frac{\pi}{6}.$$

■

We stress that integrals of the type now under consideration are evaluated after transforming them so that they can be identified as exactly equivalent to contour integrals to which we can apply the residue theorem. Further examples are in the exercises.

Integrals, Range $-\infty$ to ∞

Consider now definite integrals of the form

$$I = \int_{-\infty}^{\infty} f(x)\,dx, \tag{11.95}$$

where it is assumed that

- $f(z)$ is analytic in the upper half-plane except for a finite number of poles. For the moment will be assumed that there are no poles on the real axis. Cases not satisfying this condition will be considered later.
- In the limit $|z| \to \infty$ in the upper half-plane ($0 \le \arg z \le \pi$), $f(z)$ vanishes more strongly than $1/z$.

Note that there is nothing unique about the upper half-plane. The method described here can be applied, with obvious modifications, if $f(z)$ vanishes sufficiently strongly on the lower half-plane.

The second assumption stated above makes it useful to evaluate the contour integral $\oint f(z)\,dz$ on the contour shown in Fig. 11.21, because the integral I is given by the integration along the real axis, while the arc, of radius R, with $R \to \infty$, gives a negligible contribution to the contour integral. Thus,

$$I = \oint f(z)\,dz,$$

and the contour integral can be evaluated by applying the residue theorem.

Situations of this sort are of frequent occurrence, and we therefore formalize the conditions under which the integral over a large arc becomes negligible:

If $\lim_{R\to\infty} zf(z) = 0$ *for all* $z = Re^{i\theta}$ *with* θ *in the range* $\theta_1 \le \theta \le \theta_2$, *then*

$$\lim_{R\to\infty} \int_C f(z)\,dz = 0, \tag{11.96}$$

where C *is the arc over the angular range* θ_1 *to* θ_2 *on a circle of radius* R *with center at the origin.*

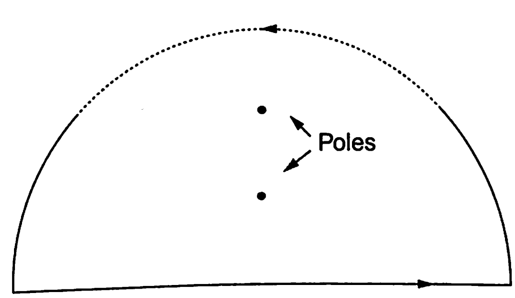

FIGURE 11.21 A contour closed by a large semicircle in the upper half-plane.

To prove Eq. (11.96), simply write the integral over C in polar form:

$$\lim_{R\to\infty}\left|\int_C f(z)\,dz\right| \le \int_{\theta_1}^{\theta_2} \lim_{R\to\infty}\left|f(Re^{i\theta})iRe^{i\theta}\right|\,d\theta$$

$$\le (\theta_2-\theta_1)\lim_{R\to\infty}\left|f(Re^{i\theta})Re^{i\theta}\right| = 0.$$

Now, using the contour of Fig. 11.21, letting C denote the semicircular arc from $\theta = 0$ to $\theta = \pi$,

$$\oint f(z)\,dz = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx + \lim_{R\to\infty}\int_C f(z)\,dz$$

$$= 2\pi i \sum \text{residues (upper half-plane)}, \tag{11.97}$$

where our second assumption has caused the vanishing of the integral over C.

Example 11.8.3 Integral of Meromorphic Function

Evaluate

$$I = \int_0^\infty \frac{dx}{1+x^2}.$$

This is not in the form we require, but it can be made so by noting that the integrand is even and we can write

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^2}. \tag{11.98}$$

We note that $f(z) = 1/(1+z^2)$ is meromorphic; all its singularities for finite z are poles, and it also has the property that $zf(z)$ vanishes in the limit of large $|z|$. Therefore, we may apply Eq. (11.97), so

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \frac{1}{2}(2\pi i)\sum \text{residues of } \frac{1}{1+z^2} \text{ (upper half-plane)}.$$

Here and in every other similar problem we have the question: Where are the poles? Rewriting the integrand as

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)},$$

we see that there are simple poles (order 1) at $z = i$ and $z = -i$. The residues are

$$\text{at } z = i\!: \quad \left.\frac{1}{z+i}\right|_{z=i} = \frac{1}{2i}, \quad \text{and} \quad \text{at } z = -i\!: \quad \left.\frac{1}{z-i}\right|_{z=-i} = -\frac{1}{2i}.$$

However, only the pole at $z = +i$ is enclosed by the contour, so our result is

$$\int_0^\infty \frac{dx}{1+x^2} = \frac{1}{2}(2\pi i)\frac{1}{2i} = \frac{\pi}{2}. \tag{11.99}$$

This result is hardly a surprise, as we presumably already know that

$$\int_0^\infty \frac{dx}{1+x^2} = \tan^{-1} x\Big|_0^\infty = \arctan x\Big|_0^\infty = \frac{\pi}{2},$$

but, as shown in later examples, the techniques illustrated here are also easy to apply when more elementary methods are difficult or impossible.

Before leaving this example, note that we could equally well have closed the contour with a semicircle in the lower half-plane, as $zf(z)$ vanishes on that arc as well as that in the upper half-plane. Then, taking the contour so the real axis is traversed from $-\infty$ to $+\infty$, the path would be **clockwise** (see Fig. 11.22), so we would need to take $-2\pi i$ times the residue of the pole that is now encircled (at $z = -i$). Thus, we have $I = -\frac{1}{2}(2\pi i)(-1/2i)$, which (as it must) evaluates to the same result we obtained previously, namely $\pi/2$. ■

Integrals with Complex Exponentials

Consider the definite integral

$$I = \int_{-\infty}^{\infty} f(x)e^{iax}\,dx, \tag{11.100}$$

with a real and positive. (This is a Fourier transform; see Chapter 19.) We assume the following two conditions:

- $f(z)$ is analytic in the upper half-plane except for a finite number of poles.
- $\lim_{|z|\to\infty} f(z) = 0, \quad 0 \le \arg\ z \le \pi.$

Note that this is a less restrictive condition than the second condition imposed on $f(z)$ for our previous integration of $\int_{-\infty}^{\infty} f(x)\,dx$.

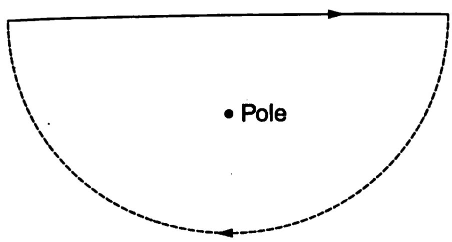

FIGURE 11.22 A contour closed by a large semicircle in the lower half-plane.

We again employ the half-circle contour shown in Fig. 11.21. The application of the calculus of residues is the same as the example just considered, but here we have to work harder to show that the integral over the (infinite) semicircle goes to zero. This integral becomes, for a semicircle of radius R,

$$I_R = \int_0^\pi f(Re^{i\theta})e^{iaR\cos\theta - aR\sin\theta}\, iRe^{i\theta}\, d\theta,$$

where the θ integration is over the upper half-plane, $0 \le \theta \le \pi$. Let R be sufficiently large that $|f(z)| = |f(Re^{i\theta})| < \varepsilon$ for all θ within the integration range. Our second assumption on $f(z)$ tells us that as $R \to \infty$, $\varepsilon \to 0$. Then

$$|I_R| \le \varepsilon R \int_0^\pi e^{-aR\sin\theta}\, d\theta = 2\varepsilon R \int_0^{\pi/2} e^{-aR\sin\theta}\, d\theta. \tag{11.101}$$

We now note that in the range $[0, \pi/2]$,

$$\frac{2}{\pi}\theta \le \sin\theta,$$

as is easily seen from Fig. 11.23. Substituting this inequality into Eq. (11.101), we have

$$|I_R| \le 2\varepsilon R \int_0^{\pi/2} e^{-2aR\theta/\pi}\, d\theta = 2\varepsilon R\, \frac{1 - e^{-aR}}{2aR/\pi} < \frac{\pi}{a}\varepsilon,$$

showing that

$$\lim_{R\to\infty} I_R = 0.$$

This result is also important enough to commemorate; it is sometimes known as **Jordan's lemma**. Its formal statement is

If $\lim_{R=\infty} f(z) = 0$ *for all* $z = Re^{i\theta}$ *in the range* $0 \le \theta \le \pi$*, then*

$$\lim_{R\to\infty} \int_C e^{iaz} f(z)\, dz = 0, \tag{11.102}$$

where $a > 0$ *and* C *is a semicircle of radius* R *in the upper half-plane with center at the origin.*

Note that for Jordan's lemma the upper and lower half-planes are not equivalent, because the condition $a > 0$ causes the exponent $-aR\sin\theta$ only to be negative and yield a negligible result in the upper half-plane. In the lower half-plane, the exponential is positive and the integral on a large semicircle there would diverge. Of course, we could extend the theorem by considering the case $a < 0$, in which event the contour to be used would then be a semicircle in the lower half-plane.

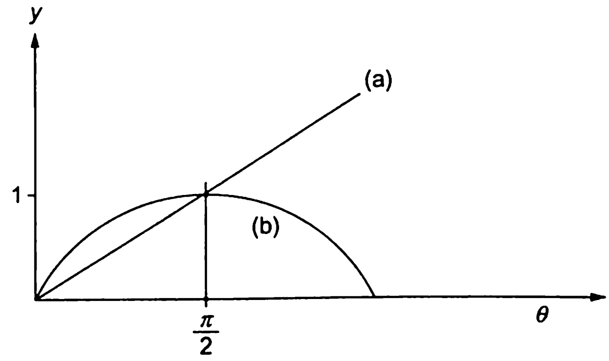

FIGURE 11.23 (a) $y = (2/\pi)\theta$, (b) $y = \sin\theta$.

Returning now to integrals of the type represented by Eq. (11.100), and using the contour shown in Fig. 11.21, application of the residue theorem yields the general result (for $a > 0$),

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i \sum \text{residues of } e^{iaz}f(z) \text{ (upper half-plane)}, \tag{11.103}$$

where we have used Jordan's lemma to set to zero the contribution to the contour integral from the large semicircle.

Example 11.8.4 OSCILLATORY INTEGRAL

Consider

$$I = \int_0^{\infty} \frac{\cos x}{x^2+1}dx,$$

which we initially manipulate, introducing $\cos x = (e^{ix} + e^{-ix})/2$, as follows:

$$I = \frac{1}{2}\int_0^{\infty} \frac{e^{ix}\,dx}{x^2+1} + \frac{1}{2}\int_0^{\infty} \frac{e^{-ix}\,dx}{x^2+1}$$

$$= \frac{1}{2}\int_0^{\infty} \frac{e^{ix}\,dx}{x^2+1} + \frac{1}{2}\int_0^{-\infty} \frac{e^{ix}\,d(-x)}{(-x)^2+1} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x^2+1},$$

thereby bringing I to the form presently under discussion.

We now note that in this problem $f(z) = 1/(z^2+1)$, which certainly approaches zero for large $|z|$, and the exponential factor is of the form e^{iaz}, with $a = +1$. We may therefore evaluate the integral using Eq. (11.103), with the contour shown in Fig. 11.21.

The quantity whose residues are needed is

$$\frac{e^{iz}}{z^2+1} = \frac{e^{iz}}{(z+i)(z-i)},$$

and we note that the exponential, an entire function, contributes no singularities. So our singularities are simple poles at $z = \pm i$. Only the pole at $z = +i$ is within the contour, and its residue is $e^{i^2}/2i$, which reduces to $1/2ie$. Our integral therefore has the value

$$I = \frac{1}{2}(2\pi i)\frac{1}{2ie} = \frac{\pi}{2e}.$$

■

Our next example is an important integral, the evaluation of which involves the principal-value concept and a contour that apparently needs to go through a pole.

Example 11.8.5 SINGULARITY ON CONTOUR OF INTEGRATION

We now consider the evaluation of

$$I = \int_0^\infty \frac{\sin x}{x}\,dx. \tag{11.104}$$

Writing the integrand as $(e^{iz} - e^{-iz})/2iz$, an attempt to do as we did in Example 11.8.4 leads to the problem that each of the two integrals into which I can be separated is individually divergent. This is a problem we have already encountered in discussing the Cauchy principal value of this integral. Referring to (11.74), we write I as

$$I = \fint_{-\infty}^{\infty} \frac{e^{ix}\,dx}{2ix}, \tag{11.105}$$

suggesting that we consider the integral of $e^{iz}/2iz$ over a suitable closed contour.

We now note that although the gap at $x = 0$ is infinitesimal, that point is a pole of $e^{iz}/2iz$, and we must draw a contour which avoids it, using a small semicircle to connect the points at $-\delta$ and $+\delta$. Compare with the discussion at Eqs. (11.75) and (11.76). Choosing the small semicircle **above** the pole, as in Fig. 11.20, we then have a contour that encloses **no** singularities.

The integral around this contour can now be identified as consisting of (1) the two semi-infinite segments constituting the principal value integral in Eq. (11.105), (2) the large semicircle C_R of radius R ($R \to \infty$), and (3) a semicircle C_r of radius r ($r \to 0$), traversed **clockwise**, so

$$\oint \frac{e^{iz}}{2iz}dz = I + \int_{C_r} \frac{e^{iz}}{2iz}dz + \int_{C_R} \frac{e^{iz}}{2iz}dz = 0. \tag{11.106}$$

By Jordan's lemma, the integral over C_R vanishes. As discussed at Eq. (11.75), the clockwise path C_r half-way around the pole at $z = 0$ contributes half the value of a full circuit, namely (allowing for the clockwise direction of travel) $-\pi i$ times the residue of $e^{iz}/2iz$ at $z = 0$. This residue has value $1/2i$, so $\int_{C_r} = -\pi i(1/2i) = -\pi/2$, and, solving Eq. (11.106) for I, we then obtain

$$I = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}. \qquad (11.107)$$

Note that it was necessary to close the contour in the upper half-plane. On a large circle in the lower half-plane, e^{iz} becomes infinite and Jordan's lemma cannot be applied. ■

Another Integration Technique

Sometimes we have an integral on the real range $(0, \infty)$ that lacks the symmetry needed to extend the integration range to $(-\infty, \infty)$. However, it may be possible to identify a direction in the complex plane on which the integrand has a value identical to or conveniently related to that of the original integral, thereby permitting construction of a contour facilitating the evaluation.

Example 11.8.6 EVALUATION ON A CIRCULAR SECTOR

Our problem is to evaluate the integral

$$I = \int_0^\infty \frac{dx}{x^3+1},$$

which we cannot convert easily into an integral on the range $(-\infty, \infty)$. However, we note that along a line with argument $\theta = 2\pi/3$, z^3 will have the same values as at corresponding points on the real line; note that $(re^{2\pi i/3})^3 = r^3 e^{2\pi i} = r^3$. We therefore consider

$$\oint \frac{dz}{z^3+1}$$

on the contour shown in Fig. 11.24. The part of the contour along the positive real axis, labeled A, simply yields our integral I. The integrand approaches zero sufficiently rapidly for large $|z|$ that the integral on the large circular arc, labeled C in the figure, vanishes. On

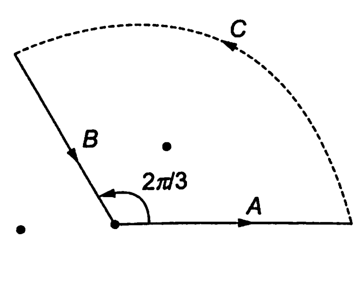

FIGURE 11.24 Contour for Example 11.8.6.

the remaining segment of the contour, labeled B, we note that $dz = e^{2\pi i/3}dr$, $z^3 = r^3$, and

$$\int_B \frac{dz}{z^3+1} = \int_\infty^0 \frac{e^{2\pi i/3}dr}{r^3+1} = -e^{2\pi i/3}\int_0^\infty \frac{dr}{r^3+1} = -e^{2\pi i/3} I.$$

Therefore,

$$\oint \frac{dz}{z^3+1} = \left(1 - e^{2\pi i/3}\right) I. \tag{11.108}$$

We now need to evaluate our complete contour integral using the residue theorem. The integrand has simple poles at the three roots of z^3+1, which are at $z_1 = e^{\pi i/3}$, $z_2 = e^{\pi i}$, and $z_3 = e^{5\pi i/3}$, as marked in Fig. 11.24. Only the pole at z_1 is enclosed by our contour. The residue at $z = z_1$ is

$$\lim_{z=z_1} \frac{z-z_1}{z^3+1} = \left.\frac{1}{3z^2}\right|_{z=z_1} = \frac{1}{3e^{2\pi i/3}}.$$

Equating $2\pi i$ times this result to the value of the contour integral as given in Eq. (11.108), we have

$$\left(1 - e^{2\pi i/3}\right) I = 2\pi i\left(\frac{1}{3e^{2\pi i/3}}\right).$$

Solution for I is facilitated if we multiply through by $e^{-\pi i/3}$, obtaining initially

$$\left(e^{-\pi i/3} - e^{\pi i/3}\right) I = 2\pi i\left(-\frac{1}{3}\right),$$

which is easily rearranged to

$$I = \frac{\pi}{3\sin\pi/3} = \frac{\pi}{3\sqrt{3}/2} = \frac{2\pi}{3\sqrt{3}}.$$

■

Avoidance of Branch Points

Sometimes we must deal with integrals whose integrands have branch points. In order to use contour integration methods for such integrals we must choose contours that avoid the branch points, enclosing only point singularities.

Example 11.8.7 INTEGRAL CONTAINING LOGARITHM

We now look at

$$I = \int_0^\infty \frac{\ln x\, dx}{x^3+1}. \tag{11.109}$$

The integrand in Eq. (11.109) is singular at $x = 0$, but the integration converges (the indefinite integral of $\ln x$ is $x\ln x - x$). However, in the complex plane this singularity manifests

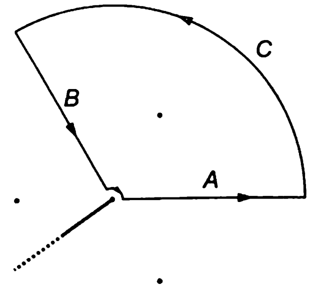

FIGURE 11.25 Contour for Example 11.8.7.

itself as a branch point, so if we are to recast this problem in a way involving a contour integral, we must avoid $z = 0$ and a branch cut from that point to $z = \infty$. It turns out to be convenient to use a contour similar to that for Example 11.8.6, except that we must make a small circular detour about $z = 0$ and then draw the branch cut in a direction that remains outside our chosen contour. Noting also that the integrand has poles at the same points as those of Example 11.8.6, we consider a contour integral

$$\oint \frac{\ln z\, dz}{z^3+1},$$

where the contour and the locations of the singularities of the integrand are as illustrated in Fig. 11.25.

The integral over the large circular arc, labeled C, vanishes, as the factor z^3 in the denominator dominates over the weakly divergent factor $\ln z$ in the numerator (which diverges more weakly than any positive power of z). We also get no contribution to the contour integral from the arc at small r, since we have there

$$\lim_{r\to 0} \int_0^{2\pi/3} \frac{\ln(re^{i\theta})}{1+r^3e^{3i\theta}}\, ire^{i\theta}\, d\theta,$$

which vanishes because $r\ln r \to 0$.

The integrals over the segments labeled A and B do not vanish. To evaluate the integral over these segments, we need to make an appropriate choice of the branch of the multi-valued function $\ln z$. It is natural to choose the branch so that on the real axis we have $\ln z = \ln x$ (and not $\ln x + 2n\pi i$ with some nonzero n). Then the integral over the segment labeled A will have the value I.[7]

To compute the integral over B, we note that on this segment $z^3 = r^3$ and $dz = e^{2\pi i/3}dr$ (as in Example 11.8.6), also but note that $\ln z = \ln r + 2\pi i/3$. There is little temptation here to use a different one of the multiple values of the logarithm, but for future reference note that we **must** use the value that is reached continuously from the value we already chose on the positive real axis, moving in a way that does not cross the branch cut. Thus, we cannot reach segment A by clockwise travel from the positive real axis (thereby getting

[7]Because the integration converges at $x = 0$, the value is not affected by the fact that this segment terminates infinitesimally before reaching that point.

$\ln z = \ln r - 4\pi i/3$) or any other value that would require multiple circuits around the branch point $z = 0$.

Based on the foregoing, we have

$$\int_B \frac{\ln z\, dz}{z^3+1} = \int_\infty^0 \frac{\ln r + 2\pi i/3}{r^3+1} e^{2\pi i/3}\, dr = -e^{2\pi i/3} I - \frac{2\pi i}{3} e^{2\pi i/3} \int_0^\infty \frac{dr}{r^3+1}. \tag{11.110}$$

Referring to Example 11.8.6 for the value of the integral in the final term of Eq. (11.110), and combining the contributions to the overall contour integral,

$$\oint \frac{\ln z\, dz}{z^3+1} = \left(1 - e^{2\pi i/3}\right) I - \frac{2\pi i}{3} e^{2\pi i/3} \left(\frac{2\pi}{3\sqrt{3}}\right). \tag{11.111}$$

Our next step is to use the residue theorem to evaluate the contour integral. Only the pole at $z = z_1$ lies within the contour. The residue we must compute is

$$\lim_{z=z_1} \frac{(z-z_1)\ln z}{z^3+1} = \left.\frac{\ln z}{3z^2}\right|_{z=z_1} = \frac{\pi i/3}{3e^{2\pi i/3}} = \frac{\pi i}{9} e^{-2\pi i/3},$$

and application of the residue theorem to Eq. (11.111) yields

$$\left(1 - e^{2\pi i/3}\right) I - \frac{2\pi i}{3} e^{2\pi i/3} \left(\frac{2\pi}{3\sqrt{3}}\right) = (2\pi i)\left(\frac{\pi i}{9}\right) e^{-2\pi i/3}. \tag{11.112}$$

Solving for I, we get

$$I = -\frac{2\pi^2}{27}. \tag{11.113}$$

Verification of the passage from Eq. (11.112) to (11.113) is left to Exercise 11.8.6. ∎

Exploiting Branch Cuts

Sometimes, rather than being an annoyance, a branch cut provides an opportunity for a creative way of evaluating difficult integrals.

Example 11.8.8 Using a Branch Cut

Let's evaluate

$$I = \int_0^\infty \frac{x^p\, dx}{x^2+1}, \qquad 0 < p < 1.$$

Consider the contour integral

$$\oint \frac{z^p\, dz}{z^2+1},$$

where the contour is that shown in Fig. 11.26. Note that $z = 0$ is a branch point, and we have taken the cut along the positive real axis. We assign z^p its usual principal value

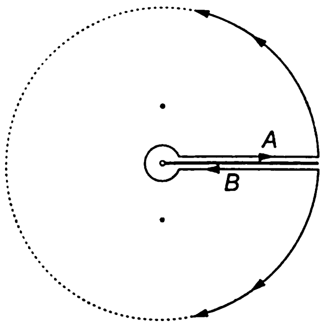

FIGURE 11.26 Contour for Example 11.8.8.

(which is x^p) just above the cut, so that the segment of the contour labeled A, which actually extends from ε to ∞, converges in the limit of small ε to the integral I. Neither the circle of radius ε nor that at $R \to \infty$ contributes to the value of the contour integral. On the remaining segment of the contour, labeled B, we have $z = re^{2\pi i}$, written this way so we can see that $z^p = r^p e^{2p\pi i}$. We use this value for z^p on segment B because we must get to B by encircling $z = 0$ in the counterclockwise, mathematically positive direction. The contribution of segment B to the contour integral is then seen to be

$$\int_{\infty}^{0} \frac{r^p e^{2p\pi i}\,dr}{r^2+1} = -e^{2p\pi i} I,$$

so

$$\oint \frac{z^p\,dz}{z^2+1} = \left(1 - e^{2p\pi i}\right) I. \tag{11.114}$$

To apply the residue theorem, we note that there are simple poles at $z_1 = i$ and $z_2 = -i$; to use these for evaluation of z^p we need to identify these as $z_1 = e^{\pi i/2}$ and $z_2 = e^{3\pi i/2}$. It would be a serious mistake to use $z_2 = e^{-\pi i/2}$ when evaluating z_2^p. We now find the residues to be:

$$\text{Residue at } z_1\text{: } \frac{e^{p\pi i/2}}{2i}, \quad \text{Residue at } z_2\text{: } \frac{e^{3p\pi i/2}}{-2i},$$

and we have, referring to Eq. (11.114),

$$\left(1 - e^{2p\pi i}\right) I = (2\pi i)\frac{1}{2i}\left(e^{p\pi i/2} - e^{3p\pi i/2}\right). \tag{11.115}$$

This equation simplifies to

$$I = \frac{\pi \sin(p\pi/2)}{\sin p\pi} = \frac{\pi}{2\cos(p\pi/2)}. \tag{11.116}$$

The details of the evaluation are left to Exercise 11.8.7. ■

The use of a branch cut, as illustrated in Example 11.8.8, is so helpful that sometimes it is advisable to insert a factor into a contour integral to create one that would not otherwise exist. To illustrate this, we return to an integral we evaluated earlier by another method.

Example 11.8.9 Introducing a Branch Point

Let's evaluate once again the integral

$$I = \int_0^\infty \frac{dx}{x^3+1},$$

which we previously considered in Example 11.8.6. This time, we proceed by setting up the contour integral

$$\oint \frac{\ln z\, dz}{z^3+1},$$

taking the contour to be that depicted in Fig. 11.26. Note that in the present problem the poles of the integrand are not those shown in Fig. 11.26, which was originally drawn to illustrate a different problem; for the locations of the poles of the present integrand, see Fig. 11.24.

The virtue of the introduction of the factor $\ln z$ is that its presence causes the integral segments above and below the positive real axis not to cancel completely, but to yield a net contribution corresponding to an integral of interest. In the present problem (using the labeling in Fig. 11.26), we again have vanishing contributions from the small and large circles, and (taking the usual principal value for the logarithm on segment A), that segment contributes to the contour integral the expected value

$$\int_A \frac{\ln z\, dz}{z^3+1} = \int_0^\infty \frac{\ln x\, dx}{x^3+1}. \tag{11.117}$$

However, segment B make the contribution

$$\int_B \frac{\ln z\, dz}{z^3+1} = \int_\infty^0 \frac{(\ln x + 2\pi i)\, dx}{x^3+1}, \tag{11.118}$$

and when Eqs. (11.117) and (11.118) are combined, the logarithmic terms cancel, and we are left with

$$\oint \frac{\ln z\, dz}{z^3+1} = \int_{A+B} \frac{\ln z\, dz}{z^3+1} = -2\pi i \int_0^\infty \frac{dx}{x^3+1} = -2\pi i\, I. \tag{11.119}$$

Note that what has happened is that the logarithm has disappeared (its contributions canceled), but its presence caused the integral of current interest to be proportional to the value of the contour integral we introduced.

To complete the evaluation, we need to evaluate the contour integral using the residue theorem. Note that the residues are those of the integrand, including the logarithmic factor, and this factor must be computed taking account of the branch cut. In the present problem, we identify poles at $z_1 = e^{\pi i/3}$, $z_2 = e^{\pi i}$, and $z_3 = e^{5\pi i/3}$ (**not** $e^{-\pi i/3}$). The contour now in use encircles all three poles. Their respective residues (denoted R_i) are

$$R_1 = \left(\frac{\pi i}{3}\right)\frac{1}{3\,e^{2\pi i/3}}, \quad R_2 = (\pi i)\,\frac{1}{3\,e^{6\pi i/3}}, \quad \text{and} \quad R_3 = \left(\frac{5\pi i}{3}\right)\frac{1}{3\,e^{10\pi i/3}},$$

where the first parenthesized factor of each residue comes from the logarithm.

Continuing, we have, referring to Eq. (11.119),

$$-2\pi i\, I = 2\pi i\,(R_1 + R_2 + R_3);$$

$$I = -(R_1 + R_2 + R_3) = -\frac{\pi i}{9}\left[e^{-2\pi i/3} + 3 + 5\,e^{2\pi i/3}\right] = \frac{2\pi}{3\sqrt{3}}.$$

More robust examples involving the introduction of $\ln z$ appear in the exercises. ■

Exploiting Periodicity

The periodicity of the trigonometric functions (and that, in the complex plane, of the hyperbolic functions) creates opportunities to devise contours in which multiple contributions corresponding to an integral of interest can be used to encircle singularities and enable use of the residue theorem. We illustrate with one example.

Example 11.8.10 INTEGRAND PERIODIC ON IMAGINARY AXIS

We wish to evaluate

$$I = \int_0^\infty \frac{x\,dx}{\sinh x}.$$

Taking account of the sinusoidal behavior of the hyperbolic sine in the imaginary direction, we consider

$$\oint \frac{z\,dz}{\sinh z} \tag{11.120}$$

on the contour shown in Fig. 11.27. In drawing the contour we needed to be mindful of the singularities of the integrand, which are poles associated with the zeros of $\sinh z$. Recognizing that

$$\sinh(x+iy) = \sinh x \cosh iy + \cosh x \sinh iy = \sinh x \cos y + i \cosh x \sin y, \tag{11.121}$$

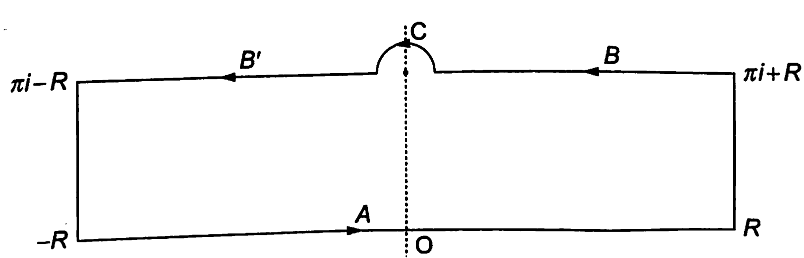

FIGURE 11.27 Contour for Example 11.8.10.

and that for all x, $\cosh x \geq 1$, we see that $\sinh z$ is zero only for $z = n\pi i$, with n an integer. Moreover, because $\lim_{z\to 0} z/\sinh z = 1$, the integrand of our present contour integral will not have a pole at $z = 0$, but will have poles at $z = n\pi i$ for all nonzero integral n. For that reason, the lower horizontal line of the contour in Fig. 11.27, marked A, continues through $z = 0$ as a straight line on the real axis, but the upper horizontal line (for which $y = \pi$), marked B and B', has an infinitesimal semicircular detour, marked C, around the pole at $z = \pi i$.

Because the integrand in Eq. (11.120) is an even function of z, the integral on segment A, which extends from $-\infty$ to $+\infty$, has the value $2I$. To evaluate the integral on segments B and B', we first note, using Eq. (11.121), that $\sinh(x + i\pi) = -\sinh x$, and that the integral on these segments is in the direction of negative x. Recognizing the integral on these segments as a Cauchy principal value, we write

$$\int_{B+B'} \frac{z\,dz}{\sinh z} = \fint_{-\infty}^{\infty} \frac{x + i\pi}{\sinh x} dx.$$

Because $x/\sinh x$ is even and nonsingular at $z = 0$, while $i\pi/\sinh x$ is odd, this integral reduces to

$$\fint_{-\infty}^{\infty} \frac{x + i\pi}{\sinh x} dx = 2I.$$

Combining what we have up to this point, invoking the residue theorem, and noting that the integrand is negligible on the vertical connections at $x = \pm\infty$. We have

$$\oint \frac{z\,dz}{\sinh z} = 4I + \int_C \frac{z\,dz}{\sinh z} = 2\pi i \text{ (residue of } z/\sinh z \text{ at } z = \pi i). \tag{11.122}$$

To complete the evaluation, we now note that the residue we need is

$$\lim_{z\to\pi i} \frac{z(z - \pi i)}{\sinh z} = \frac{\pi i}{\cosh \pi i} = -\pi i,$$

and, cf. Eqs. (11.75) and (11.76), the counterclockwise semicircle C evaluates to πi times this residue. We have then

$$4I + (\pi i)(-\pi i) = (2\pi i)(-\pi i), \quad \text{so } I = \frac{\pi^2}{4}.$$

■

Exercises

11.8.1 Generalizing Example 11.8.1, show that

$$\int_0^{2\pi} \frac{d\theta}{a \pm b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a \pm b\sin\theta} = \frac{2\pi}{(a^2 - b^2)^{1/2}}, \quad \text{for } a > |b|.$$

What happens if $|b| > |a|$?

11.8.2 Show that $\displaystyle\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{\pi a}{(a^2-1)^{3/2}}, \quad a > 1.$

11.8.3 Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{1-2t\cos\theta+t^2} = \frac{2\pi}{1-t^2}, \quad \text{for } |t| < 1.$

What happens if $|t| > 1$? What happens if $|t| = 1$?

11.8.4 Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 3\theta\, d\theta}{5-4\cos\theta}.$

ANS. $\pi/12$.

11.8.5 With the calculus of residues, show that

$$\int_0^{\pi} \cos^{2n}\theta\, d\theta = \pi \frac{(2n)!}{2^{2n}(n!)^2} = \pi \frac{(2n-1)!!}{(2n)!!}, \quad n = 0, 1, 2, \dots.$$

The double factorial notation is defined in Eq. (1.76).

Hint. $\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) = \frac{1}{2}(z + z^{-1}), \quad |z| = 1.$

11.8.6 Verify that simplification of the expression in Eq. (11.112) yields the result given in Eq. (11.113).

11.8.7 Complete the details of Example 11.8.8 by verifying that there is no contribution to the contour integral from either the small or the large circles of the contour, and that Eq. (11.115) simplifies to the result given as (11.116).

11.8.8 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\cos bx - \cos ax}{x^2}\, dx, \quad a > b > 0.$

ANS. $\pi(a-b)$.

11.8.9 Prove that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \frac{\pi}{2}.$

Hint. $\sin^2 x = \frac{1}{2}(1 - \cos 2x).$

11.8.10 Show that $\displaystyle\int_0^{\infty} \frac{x\sin x}{x^2+1}\, dx = \frac{\pi}{2e}.$

11.8.11 A quantum mechanical calculation of a transition probability leads to the function $f(t,\omega) = 2(1 - \cos\omega t)/\omega^2$. Show that

$$\int_{-\infty}^{\infty} f(t,\omega)\, d\omega = 2\pi t.$$

11.8.12 Show that $(a > 0)$:

(a) $$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2}\, dx = \frac{\pi}{a} e^{-a}.$$

How is the right side modified if $\cos x$ is replaced by $\cos kx$?

(b) $$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx = \pi\, e^{-a}.$$

How is the right side modified if $\sin x$ is replaced by $\sin kx$?

11.8.13 Use the contour shown (Fig. 11.28) with $R \to \infty$ to prove that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \pi.$$

11.8.14 In the quantum theory of atomic collisions, we encounter the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin t}{t} e^{ipt}\, dt,$$

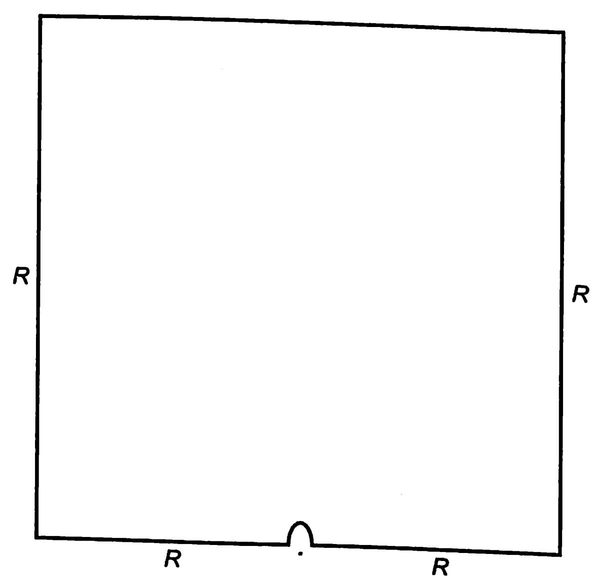

FIGURE 11.28 Contour for Exercise 11.8.13.

in which p is real. Show that

$$I = 0, \ |p| > 1$$
$$I = \pi, \ |p| < 1.$$

What happens if $p = \pm 1$?

11.8.15 Show that $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, \quad a > 0.$

11.8.16 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx.$

ANS. $\pi/\sqrt{2}$.

11.8.17 Evaluate $\displaystyle\int_0^\infty \frac{x^p \ln x}{x^2+1}\,dx, \quad 0 < p < 1.$

ANS. $\dfrac{\pi^2}{4}\dfrac{\sin(\pi p/2)}{\cos^2(\pi p/2)}$.

11.8.18 Evaluate $\displaystyle\int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx,$

(a) by appropriate series expansion of the integrand to obtain

$$4\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-3},$$

(b) and by contour integration to obtain $\dfrac{\pi^3}{8}$.

Hint. $x \to z = e^t$. Try the contour shown in Fig. 11.29, letting $R \to \infty$.

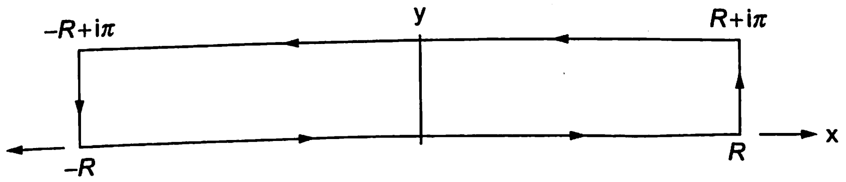

FIGURE 11.29 Contour for Exercise 11.8.18.

11.8.19 Prove that $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2}\,dx = \pi \ln 2.$

11.8.20 Show that

$$\int_0^\infty \frac{x^a}{(x+1)^2}\,dx = \frac{\pi a}{\sin \pi a},$$

where $-1 < a < 1$.

Hint. Use the contour shown in Fig. 11.26, noting that $z = 0$ is a branch point and the positive x-axis can be chosen to be a cut line.

11.8.21 Show that

$$\int_{-\infty}^\infty \frac{x^2 dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin \theta} = \frac{\pi}{2^{1/2}(1 - \cos 2\theta)^{1/2}}.$$

Exercise 11.8.16 is a special case of this result.

11.8.22 Show that

$$\int_0^\infty \frac{dx}{1 + x^n} = \frac{\pi/n}{\sin(\pi/n)}.$$

Hint. Try the contour shown in Fig. 11.30, with $\theta = 2\pi/n$.

11.8.23 (a) Show that

$$f(z) = z^4 - 2z^2 \cos 2\theta + 1$$

has zeros at $e^{i\theta}$, $e^{-i\theta}$, $-e^{i\theta}$, and $-e^{-i\theta}$.

(b) Show that

$$\int_{-\infty}^\infty \frac{dx}{x^4 - 2x^2 \cos 2\theta + 1} = \frac{\pi}{2 \sin \theta} = \frac{\pi}{2^{1/2}(1 - \cos 2\theta)^{1/2}}.$$

Exercise 11.8.22 ($n = 4$) is a special case of this result.

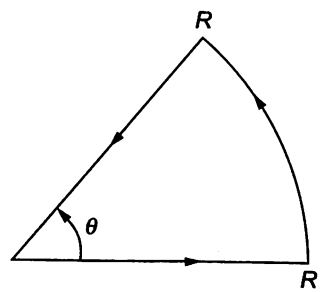

FIGURE 11.30 Sector contour.

11.8.24 Show that

$$\int_0^\infty \frac{x^{-a}}{x+1}\,dx = \frac{\pi}{\sin a\pi},$$

where $0 < a < 1$.

Hint. You have a branch point and you will need a cut line. Try the contour shown in Fig. 11.26.

11.8.25 Show that $\displaystyle\int_0^\infty \frac{\cosh bx}{\cosh x}\,dx = \frac{\pi}{2\cos(\pi b/2)}, \quad |b| < 1.$

Hint. Choose a contour that encloses one pole of $\cosh z$.

11.8.26 Show that

$$\int_0^\infty \cos(t^2)\,dt = \int_0^\infty \sin(t^2)\,dt = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$

Hint. Try the contour shown in Fig. 11.30, with $\theta = \pi/4$.

Note. These are the Fresnel integrals for the special case of infinity as the upper limit. For the general case of a varying upper limit, asymptotic expansions of the Fresnel integrals are the topic of Exercise 12.6.1.

11.8.27 Show that $\displaystyle\int_0^1 \frac{1}{(x^2 - x^3)^{1/3}}\,dx = 2\pi/\sqrt{3}.$

Hint. Try the contour shown in Fig. 11.31.

11.8.28 Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\tan^{-1} ax\,dx}{x(x^2+b^2)}$, for a and b positive, with $ab < 1$.

Explain why the integrand does not have a singularity at $x = 0$.

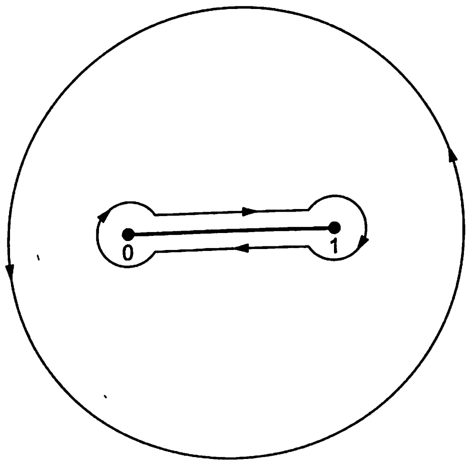

FIGURE 11.31 Contour for Exercise 11.8.27.

Hint. Try the contour shown in Fig. 11.32, and use Eq. (1.137) to represent $\tan^{-1} az$. After cancellation, the integrals on segments B and B' combine to give an elementary integral.

11.9 EVALUATION OF SUMS

The fact that the cotangent is a meromorphic function with regularly spaced poles, all with the same residue, enables us to use it to write a wide variety of infinite summations in terms of contour integrals. To start, note that $\pi \cot \pi z$ has simple poles at all integers on the real axis, each with residue

$$\lim_{z\to n} \frac{\pi \cos \pi z}{\sin \pi z} = 1.$$

Suppose that we now evaluate the integral

$$I_N = \oint_{C_N} f(z)\pi \cot \pi z \, dz,$$

where the contour is a circle about $z = 0$ of radius $N + \frac{1}{2}$ (thereby not passing close to the singularities of $\cot \pi z$). Assuming also that $f(z)$ has only isolated singularities, at points z_j other than real integers, we get by application of the residue theorem (see also Exercise 11.9.1),

$$I_N = 2\pi i \sum_{n=-N}^{N} f(n) + 2\pi i \sum_j (\text{residues of } f(z)\pi \cot \pi z \text{ at singularities } z_j \text{ of } f).$$

This integral over the circular contour C_N will be negligible for large $|z|$ if $zf(z) \to 0$ at large $|z|$.[8] When that condition is met, $\lim_{N\to\infty} I_N = 0$, and we have the useful result

$$\sum_{n=-\infty}^{\infty} f(n) = -\sum_j (\text{residues of } f(z)\pi \cot \pi z \text{ at singularities } z_j \text{ of } f). \qquad (11.123)$$

The condition required of $f(z)$ will usually be satisfied if the summation of Eq. (11.123) converges.

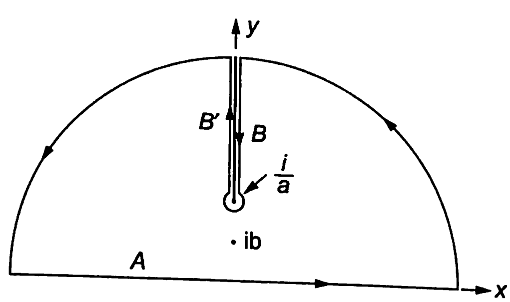

FIGURE 11.32 Contour for Exercise 11.8.28.

[8] See also Exercise 11.9.2.

Example 11.9.1 EVALUATING A SUM

Consider the summation

$$S=\sum_{n=1}^{\infty}\frac{1}{n^2+a^2},$$

where, for simplicity, we assume that a is nonintegral. To bring our problem to the form we know how to treat, we note that also

$$\sum_{n=-\infty}^{-1}\frac{1}{n^2+a^2}=S,$$

so that

$$\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2}=2S+\frac{1}{a^2}, \tag{11.124}$$

where we have added on the right-hand side the contribution from $n=0$ that was not included in S.

The summation is now identified as of the form of Eq. (11.123), with $f(z)=1/(z^2+a^2)$; $f(z)$ approaches zero at large z rapidly enough to make Eq. (11.123) applicable. We therefore proceed to the observation that the only singularities of $f(z)$ are simple poles at $z=\pm ia$. The residues we need are those of $\pi\cot(\pi z)/(z^2+a^2)$; they are

$$\frac{\pi\cot i\pi a}{2ia}=\frac{-\pi\coth\pi a}{2a}\quad\text{and}\quad\frac{\pi\cot(-i\pi a)}{-2ia}=\frac{-\pi\coth(-\pi a)}{-2a}.$$

These are equal, so from Eqs. (11.123) and (11.124),

$$2S+\frac{1}{a^2}=\frac{\pi\coth\pi a}{a},$$

which we easily solve to reach $S=\dfrac{\pi\coth\pi a}{2a}-\dfrac{1}{2a^2}$. ■

Additional types of summations can be performed if we replace $\cot\pi z$ by functions with other regularly repeating patterns of residues. For example, $\pi\csc\pi z$ has residues for integer z that alternate in sign between $+1$ and -1; $\pi\tan\pi z$ has residues that are all $+1$, but occur at the points $n+\frac{1}{2}$. And $\pi\sec\pi z$ has residues ±1 at the half-integers with a sign alternation. For convenience, we list in Table 11.2 the contour-integral formulas for the four types of summations we have just discussed.

We close this section with another example, this time illustrating what can be done if $f(z)$ has a pole at an integer value of z.

Example 11.9.2 ANOTHER SUM

Consider now the summation

$$S=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}.$$

Table 11.2 Contour-Integral-Based Formulas for Summations

Summation	Formula
$\sum_{n=-\infty}^{\infty} f(n)$	$-\sum$(residues of $f(z)\pi \cot \pi z$ at singularities of f).
$\sum_{n=-\infty}^{\infty} (-1)^n f(n)n$	$-\sum$(residues of $f(z)\pi \csc \pi z$ at singularities of f).
$\sum_{n=-\infty}^{\infty} f\left(n+\frac{1}{2}\right)$	$\sum$(residues of $f(z)\pi \tan \pi z$ at singularities of f).
$\sum_{n=-\infty}^{\infty} (-1)^n f\left(n+\frac{1}{2}\right)$	$\sum$(residues of $f(z)\pi \sec \pi z$ at singularities of f).

To extend the summation to $n = -\infty$, we note that $S = \sum_{n=-\infty}^{-2} \frac{1}{n(n+1)}$, so that

$$2S = {\sum_{n=-\infty}^{\infty}}' \frac{1}{n(n+1)}, \tag{11.125}$$

where the prime on the sum indicates that the terms for $n = 0$ and $n = -1$ are to be omitted. The derivation of Eq. (11.123) indicates that this equation will apply if we omit the (singular) $n = 0$ and $n = -1$ terms from the sum and include the points $z = 0$ and $z = -1$ as points where the residues of $f(z)\pi \cot \pi z$ are to be included.

Based on that insight, we find that in the present problem,

$$2S = -(\text{sum of residues of } \pi \cot \pi z / z(z+1) \text{ at } z = 0 \text{ and } z = -1).$$

The singularities at $z = 0$ and $z = -1$ are second-order poles, at which the residues are most easily computed by the method illustrated in item 5 of Example 11.7.1. In Exercise 11.7.2 it is shown that the residue at each pole has value -1. Completing the problem,

$$2S = -(-1-1) = 2, \quad \text{so } S = 1.$$

In this instance the result is easily verified by making the partial fraction expansion

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}.$$

When inserted in the summation S, all terms cancel except the initial term of the $1/n$ summation, yielding $S = 1$. ■

Exercises

11.9.1 Show that if $f(z)$ is analytic at $z = z_0$ and $g(z)$ has a simple pole at $z = z_0$ with residue b_0, then $f(z)g(z)$ also has a simple pole at $z = z_0$, with residue $f(z_0)b_0$.

11.9.2 Show that $\cot z$ has magnitude of order 1 for large $|z|$ when not extremely close to one of its poles and does not affect the limiting behavior of I_N.

11.9.3 Evaluate $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \cdots$.

11.9.4 Evaluate $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$.

11.9.5 Evaluate $\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(n+a)^2}$, where a is real and not an integer.

11.9.6 (a) Using a method based on contour integration, evaluate $\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.
(b) Check your work by relating your answer to an appropriate expression involving zeta functions.

11.9.7 Show that $\frac{1}{\cosh(\pi/2)} - \frac{1}{3\cosh(3\pi/2)} + \frac{1}{5\cosh(5\pi/2)} - \cdots = \frac{\pi}{8}$.

11.9.8 For $-\pi \le \varphi \le +\pi$, show that $\sum_{n=1}^{\infty} (-1)^n \frac{\sin n\varphi}{n^3} = \frac{\varphi}{12}(\varphi^2 - \pi^2)$.

11.10 MISCELLANEOUS TOPICS

Schwarz Reflection Principle

Our starting point for this topic is the observation that $g(z) = (z - x_0)^n$ for integral n and real x_0 satisfies

$$g^*(z) = [(z - x_0)^n]^* = (z^* - x_0)^n = g(z^*). \tag{11.126}$$

A generalization of the result in Eq. (11.126) is the Schwarz reflection principle:

If a function $f(z)$ is (1) analytic over some region including a portion of the real axis and (2) real when z is real, then

$$f^*(z) = f(z^*). \tag{11.127}$$

Expanding $f(z)$ about some point x_0 within the region of analyticity on the real axis,

$$f(z) = \sum_{n=0}^{\infty} (z - x_0)^n \frac{f^{(n)}(x_0)}{n!}.$$

Since $f(z)$ is analytic at $z = x_0$, this Taylor expansion exists. Since $f(z)$ is real when z is real, $f^{(n)}(x_0)$ must be real for all n. Then, invoking Eq. (11.126), the Schwarz reflection principle, Eq. (11.127), follows immediately. This completes the proof within a circle of convergence. Analytic continuation then permits the extension of this result to the entire region of analyticity.

Note that the reflection principle can also be derived by the consideration of Laurent expansions. See Exercise 11.10.2.

Mapping

An analytic function $w(z) = u(x, y) + iv(x, y)$ can be regarded as a **mapping** in which points or curves in an xy plane can be associated with the corresponding points or curves in a uv plane. As a relatively simple example, consider the transformation $w = 1/z$. From an examination of its polar form, with $z = re^{i\theta}$, $w = \rho e^{i\varphi}$, we see that $\rho = 1/r$ and $\varphi = -\theta$, leading to the conclusion that the interior of the unit circle maps into its exterior (see Fig. 11.33). Circles in other locations in the z plane are transformed by $w = 1/z$ into other circles (or straight lines, which can be thought of as circles of infinite radius). This statement is the subject of Exercise 11.10.6. The transformation of two such circles are shown in the four panels of Fig. 11.34. Compare the way in which the interiors of the circles transform in Figs. 11.33 and 11.34. Note that the transformation does not preserve lengths, as can be seen in the figure from the labeling of various points and their locations when mapped.

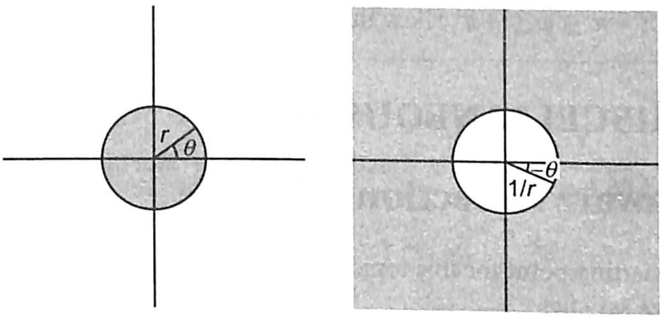

FIGURE 11.33 Mapping $w = 1/z$. The shaded areas transform into each other.

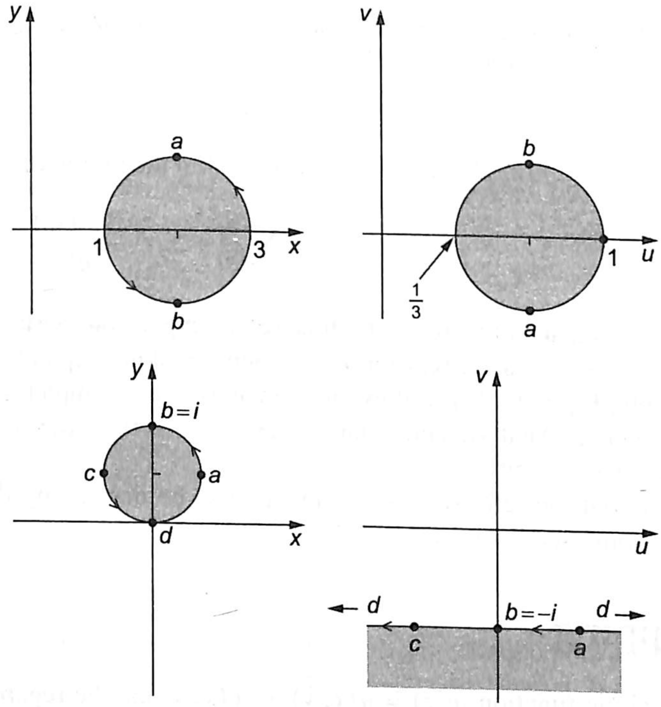

FIGURE 11.34 Left panels: circles in z plane. Right panels: their transformations in w plane under $w = 1/z$.

Historically, the notion of mapping was useful for identifying and carrying out transformations that would facilitate the solution of 2-D problems in electrostatics, fluid dynamics, and other areas of classical physics. An important aspect of such mappings is that they are **conformal**, meaning that (except at singularities of the transformation) the angles at which curves intersect remain unchanged when transformed. This feature preserves relations, e.g., between equipotentials and lines of force (stream lines). With the nearly universal use of high-speed computers, procedures based on conformal mapping are no longer central to the practical solution of most physics and engineering problems, and as a consequence will not be explored here in further detail. For problems where these techniques are still relevant, we refer the reader to earlier editions of this book and to sources identified under Additional Readings. In that connection, we call particular attention to the book by Spiegel, which contains (in chapter 8) descriptions of a large number of mappings and (in chapter 9) many applications to problems of fluid flow, electrostatics, and heat conduction.

Exercises

11.10.1 A function $f(z) = u(x, y) + iv(x, y)$ satisfies the conditions for the Schwarz reflection principle. Show that

(a) u is an even function of y. (b) v is an odd function of y.

11.10.2 A function $f(z)$ can be expanded in a Laurent series about the origin with the coefficients a_n real. Show that the complex conjugate of this function of z is the same function of the complex conjugate of z; that is,

$$f^*(z) = f(z^*).$$

Verify this explicitly for

(a) $f(z) = z^n$, n an integer. (b) $f(z) = \sin z$.

If $f(z) = iz (a_1 = i)$, show that the foregoing statement does not hold.

11.10.3 The function $f(z)$ is analytic in a domain that includes the real axis. When z is real $(z = x)$, $f(x)$ is pure imaginary.

(a) Show that

$$f(z^*) = -[f(z)]^*.$$

(b) For the specific case $f(z) = iz$, develop the Cartesian forms of $f(z)$, $f(z^*)$, and $f^*(z)$. Do not quote the general result of part (a).

11.10.4 How do circles centered on the origin in the z-plane transform for

(a) $w_1(z) = z + \dfrac{1}{z}$, (b) $w_2(z) = z - \dfrac{1}{z}$, for $z \neq 0$?

What happens when $|z| \to 1$?

11.10.5 What part of the z-plane corresponds to the interior of the unit circle in the w-plane if

$$\text{(a)}\quad w = \frac{z-1}{z+1}? \qquad \text{(b)}\quad w = \frac{z-i}{z+i}?$$

11.10.6 (a) Writing $z = x + iy$, $w = u + iv$, show that if $w = 1/z$, the circle in the xy plane defined by $(x-a)^2 + (y-b)^2 = r^2$ transforms into $(u-A)^2 + (v-B)^2 = R^2$.

(b) Does the center of the circle in the z plane transform into the center of the corresponding circle in the w plane?

11.10.7 Assume that a curve in the xy plane passes through point z_0 in the direction $dz = e^{i\theta} ds$, where s indicates arc length on the curve. Then, if $w = f(z)$, with $f(z)$ analytic at $z = z_0$, we have $dw = (dw/dz)dz = f'(z)e^{i\theta} ds$, where dw is in the direction the mapping of the xy curve passes through $w_0 = f(z_0)$ in the w plane. Use this observation to prove that if $f'(z_0) \neq 0$, the angle at which two curves intersect in the z plane is the same (both in magnitude and direction) as the angle of intersection of their mappings in the w plane.

Additional Readings

Ahlfors, L. V., *Complex Analysis*, 3rd ed. New York: McGraw-Hill (1979). This text is detailed, thorough, rigorous, and extensive.

Churchill, R. V., J. W. Brown, and R. F. Verkey, *Complex Variables and Applications*, 5th ed. New York: McGraw-Hill (1989). This is an excellent text for both the beginning and advanced student. It is readable and quite complete. A detailed proof of the Cauchy-Goursat theorem is given in Chapter 5.

Greenleaf, F. P., *Introduction to Complex Variables*. Philadelphia: Saunders (1972). This very readable book has detailed, careful explanations.

Kurala, A., *Applied Functions of a Complex Variable*. New York: Wiley (Interscience) (1972). An intermediate-level text designed for scientists and engineers. Includes many physical applications.

Levinson, N., and R. M. Redheffer, *Complex Variables*. San Francisco: Holden-Day (1970). This text is written for scientists and engineers who are interested in applications.

Morse, P. M., and H. Feshbach, *Methods of Theoretical Physics*. New York: McGraw-Hill (1953). Chapter 4 is a presentation of portions of the theory of functions of a complex variable of interest to theoretical physicists.

Remmert, R., *Theory of Complex Functions*. New York: Springer (1991).

Sokolnikoff, I. S., and R. M. Redheffer, *Mathematics of Physics and Modern Engineering*, 2nd ed. New York: McGraw-Hill (1966). Chapter 7 covers complex variables.

Spiegel, M. R., *Complex Variables*, in *Schaum's Outline Series*. New York: McGraw-Hill (original 1964, reprinted 1995). An excellent summary of the theory of complex variables for scientists.

Titchmarsh, E. C., *The Theory of Functions*, 2nd ed. New York: Oxford University Press (1958). A classic.

Watson, G. N., *Complex Integration and Cauchy's Theorem*. New York: Hafner (original 1917, reprinted 1960). A short work containing a rigorous development of the Cauchy integral theorem and integral formula. Applications to the calculus of residues are included. *Cambridge Tracts in Mathematics, and Mathematical Physics*, No. 15.