Temperaturas em Manhattan

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MÉTODOS DE CÁLCULO NUMÉRICO I

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Sumário

Modelagem do problema

$$\frac{\partial T(x,t)}{\partial t} = \alpha \frac{\partial^2 T(x,t)}{\partial x^2}$$

$$T(x+1) = T(x) + T'(x) + \frac{T''(x)}{2} + \frac{T^{(3)}(x)}{6} + \frac{T^{(4)}(x)}{24} + \cdots$$

$$T(x-1) = T(x) - T'(x) + \frac{T''(x)}{2} - \frac{T^{(3)}(x)}{6} + \frac{T^{(4)}(x)}{24} - \cdots$$

$$T(x+1) + T(x-1) = 2T(x) + T''(x) + \frac{T^{(4)}(x)}{12} + \cdots$$

$$T''(x) = T(x+1) - 2T(x) + T(x-1) - \mathcal{O}(1)$$

$$\frac{dT_i(t)}{dt} = \alpha(T_{i+1}(t) - 2T_i(t) + T_{i-1}(t))$$

Modelagem do problema

$$LT = \left(\begin{bmatrix} G_1 & 0 & \cdots & 0 \\ 0 & G_2 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & G_n \end{bmatrix} - \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \right)_{n \times n} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ \vdots \\ T_n \end{bmatrix}_{n \times 1} = \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ \vdots \\ C_n \end{bmatrix}_{n \times 1}$$

$$C_i = G_i T_i - \sum_{j \to i} T_j$$

$$\frac{dT_i(t)}{dt} = -\alpha C_i = -\alpha (LT)_i$$

$$\frac{dT}{dt} = -\alpha LT$$

$$0 = \alpha LT \iff \alpha LT = b \iff (L + P)T = Pb$$

Métodos Diretos

$$\text{Cholesky:} \quad M = L \cdot L^T \longrightarrow \mathcal{O}\left(\frac{1}{3}n^3\right)$$

$$M \cdot T = c$$

$$(L \cdot L^T) \cdot T = c \begin{cases} y = L^T \cdot T \\ L \cdot y = c \end{cases}$$

$$\text{LU:} \quad P \cdot M = L \cdot U \longrightarrow \mathcal{O}\left(\frac{2}{3}n^3\right)$$

$$M \cdot T = c$$

$$P \cdot M \cdot T = P \cdot c$$

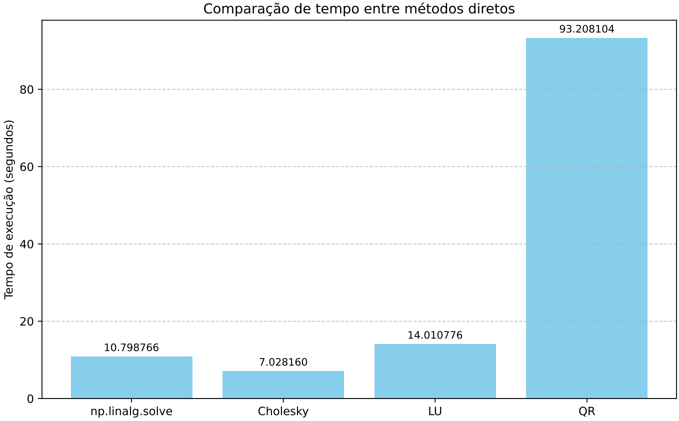
$$(L \cdot U) \cdot T = P \cdot c \begin{cases} y = U \cdot T \\ L \cdot y = P \cdot c \end{cases}$$

Métodos Diretos

$$
\begin{aligned}
\text{QR:}\quad & M = Q \cdot R \longrightarrow \mathcal{O}\left(\frac{4}{3}n^3\right) \\
& M \cdot T = c \\
& (Q \cdot R) \cdot T = c \\
& Q^T \cdot (Q \cdot R \cdot T) = Q^T \cdot c \\
& (Q^T \cdot Q) \cdot R \cdot T = Q^T \cdot c \\
& R \cdot T = Q^T \cdot c
\end{aligned}
$$

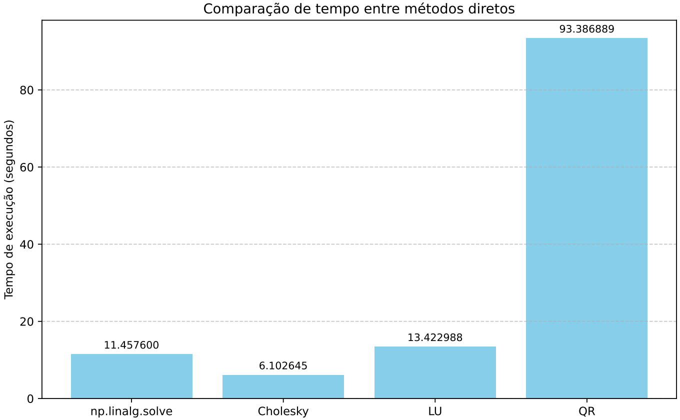
Tempos

Quando fixamos 9 pontos dentre os 8708, obtemos



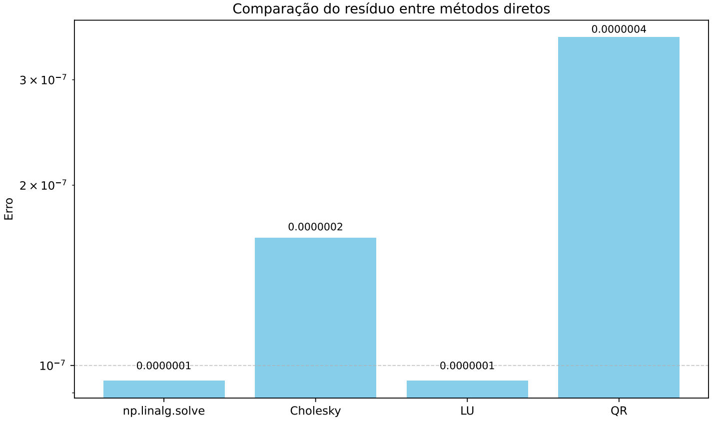
Tempos

Quando fixamos 435 pontos dentre os 8708, obtemos



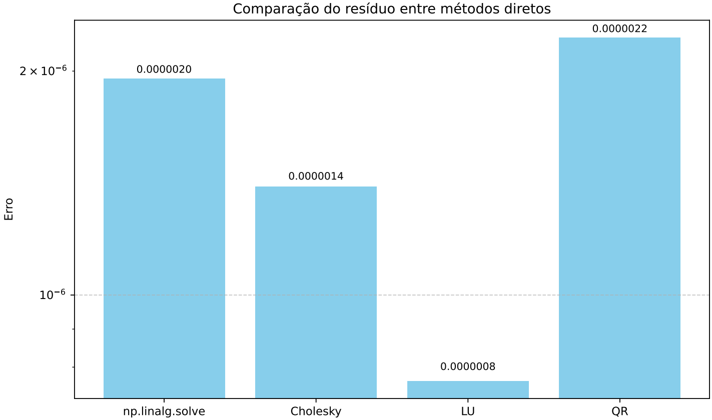
Erros

Quando fixamos 9 pontos dentre os 8708, obtemos



Erros

Quando fixamos 435 pontos dentre os 8708, obtemos



Se fizermos D a matriz diagonal de M, temos

$$
\text{Gauss-Jacobi:}\quad
\begin{aligned}
&M \cdot T = c \\
&(M - D + D) \cdot T = c \\
&(M - D) \cdot T + D \cdot T = c \\
&(M - D) \cdot T^{(k)} + D \cdot T^{(k+1)} = c \\
&D \cdot T^{(k+1)} = (D - M) \cdot T^{(k)} + c \\
&T^{(k+1)} = D^{-1} \cdot (D - M) \cdot T^{(k)} + D^{-1} \cdot c \\
&T^{(k+1)} = (I - D^{-1} \cdot M) \cdot T^{(k)} + D^{-1} \cdot c \\
&T^{(k+1)} = C \cdot T^{(k)} + g
\end{aligned}
$$

Métodos iterativos

Se fizermos L a matriz diagonal inferior de M e R, a triangular superior sem a diagonal, temos

$$
\text{Gauss-Seidel:} \quad \begin{aligned}
& M \cdot T = c \\
& (L + R) \cdot T = c \\
& L \cdot T + R \cdot T = c \\
& L \cdot T^{(k+1)} + R \cdot T^{(k)} = c \\
& L \cdot T^{(k+1)} = -R \cdot T^{(k)} + c \\
& T^{(k+1)} = (-L^{-1} \cdot R) \cdot T^{(k)} + L^{-1} \cdot c \\
& T^{(k+1)} = C \cdot T^{(k)} + g
\end{aligned}
$$

Métodos iterativos

Gradientes Conjugados:

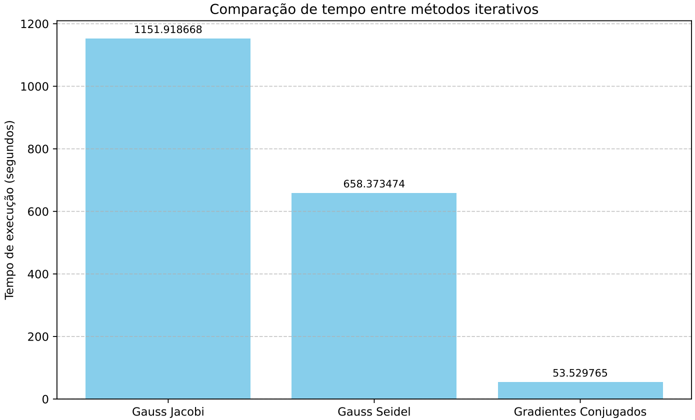
$$f(x) = \frac{1}{2}x^T M x - c^T x$$

$$\nabla f(x) = Mx - c$$

1. Chute inicial x_0;
2. $p_0 = r_0 = c - Mx_0$;
3. $\alpha = \dfrac{r_k^T \cdot r_k}{p_k^T \cdot M \cdot p_k}$;
4. $x_{k+1} = x_k + \alpha p_k$;
5. $r_{k+1} = r_k - \alpha(M \cdot p_k)$;
6. $\beta = \dfrac{r_{k+1}^T \cdot r_{k+1}}{r_k^T \cdot r_k}$;
7. $p_{k+1} = r_{k+1} + \beta p_k$;
8. Volta para o passo 3, enquanto $\|r_{k+1}\| > \text{tol}$.

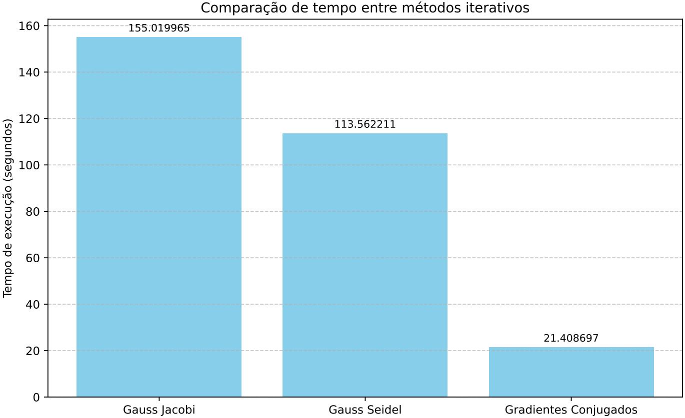
Tempos

Quando fixamos 9 pontos dentre os 8708 e uma tolerância de 10^{-1}, obtemos



Tempos

Quando fixamos 435 pontos dentre os 8708 e uma tolerância de 10^{-3}, obtemos



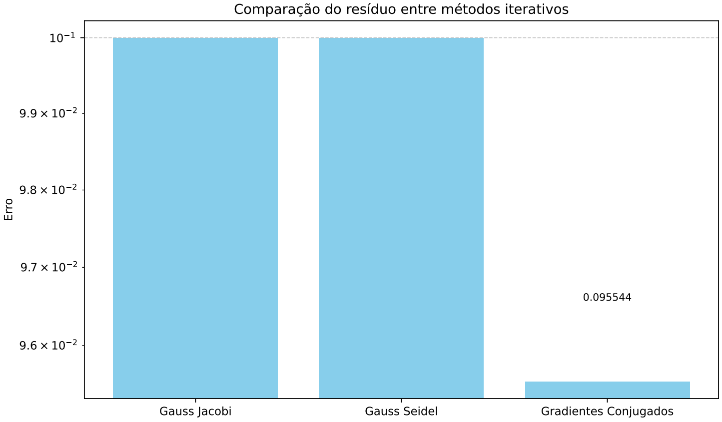
Tempos

Obtivemos também a quantidade de passos necessários para convergir

Pontos	Tolerância		Gauss-Jacobi	Gauss-Seidel	Gradientes Conjugados
9	10^{-1}	k	13000	7129	630
		t	19 min	11 min	1 min
435	10^{-3}	k	1261	647	251
		t	3 min	2 min	30 s

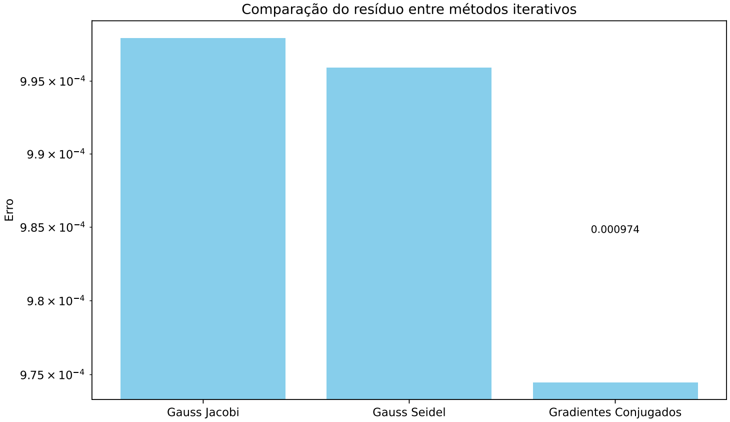
Erros

Quando fixamos 9 pontos dentre os 8708 e uma tolerância de 10^{-1}, obtemos

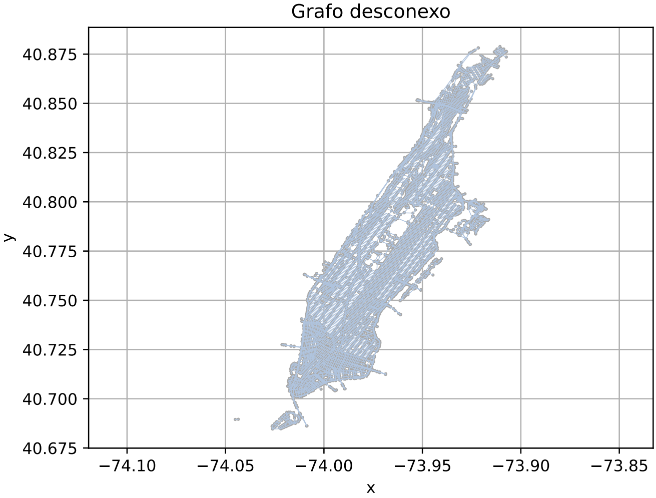


Erros

Quando fixamos 435 pontos dentre os 8708 e uma tolerância de 10^{-3}, obtemos

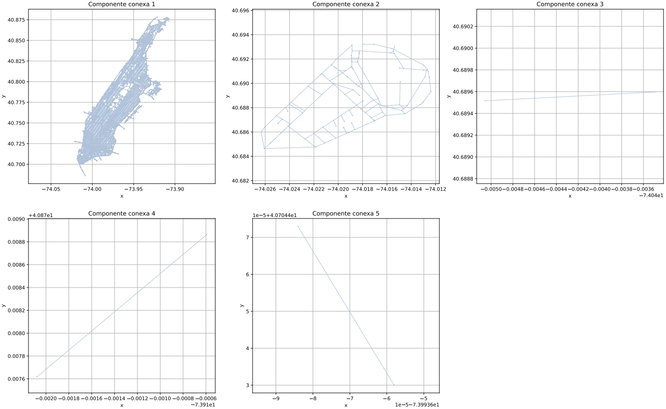


Resultados

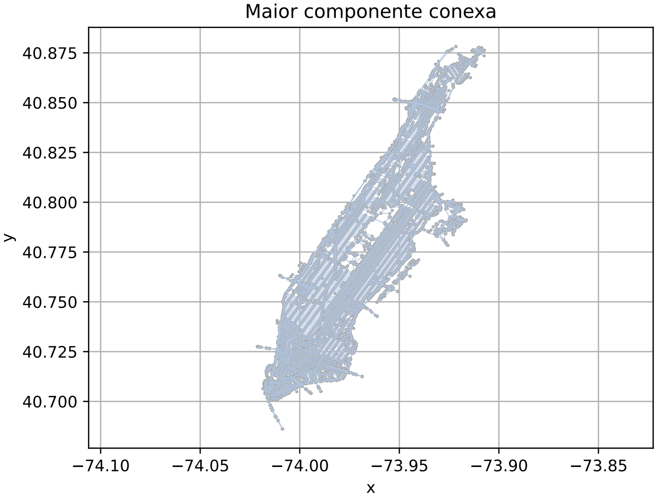


Grafo desconexo
40.875
40.850
40.825
40.800
40.775
40.750
40.725
40.700
40.675
y
−74.10
−74.05
−74.00
−73.95
−73.90
−73.85
x

Resultados

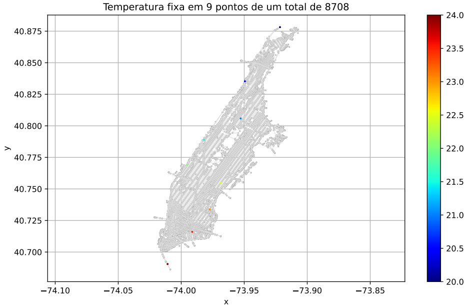

Componente conexa 1
40.875
40.850
40.825
40.800
40.775
40.750
40.725
40.700
y
−74.05
−74.00
−73.95
−73.90
x
Componente conexa 2
40.696
40.694
40.692
40.690
40.688
40.686
40.684
40.682
y
−74.026 −74.024 −74.022 −74.020 −74.018 −74.016 −74.014 −74.012
x
Componente conexa 3
40.6902
40.6900
40.6898
40.6896
40.6894
40.6892
40.6890
40.6888
y
−0.0050 −0.0048 −0.0046 −0.0044 −0.0042 −0.0040 −0.0038 −0.0036
x
−7.404e1
Componente conexa 4
+4.087e1
0.0090
0.0088
0.0086
0.0084
0.0082
0.0080
0.0078
0.0076
y
−0.0020 −0.0018 −0.0016 −0.0014 −0.0012 −0.0010 −0.0008 −0.0006
x
−7.391e1
Componente conexa 5
1e−5+4.07044e1
7
6
5
4
3
y
−9
−8
−7
−6
−5
x
1e−5−7.39936e1

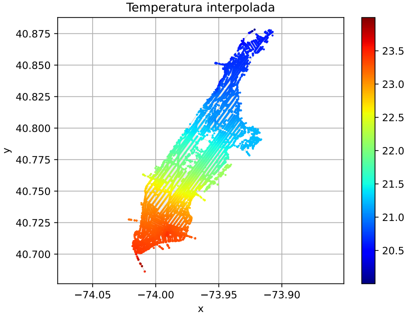
Resultados



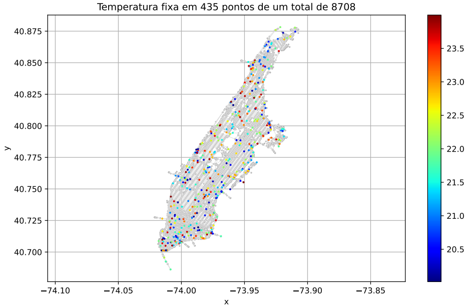
Maior componente conexa
40.875
40.850
40.825
40.800
40.775
40.750
40.725
40.700
y
−74.10
−74.05
−74.00
−73.95
−73.90
−73.85
x

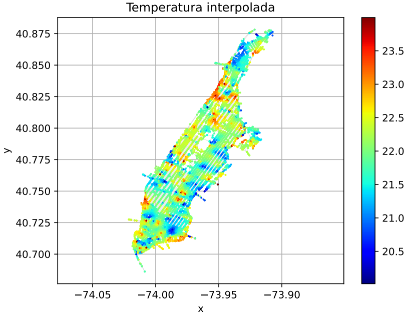
Resultados


Temperatura fixa em 9 pontos de um total de 8708
y
x
40.875
40.850
40.825
40.800
40.775
40.750
40.725
40.700
−74.10
−74.05
−74.00
−73.95
−73.90
−73.85
24.0
23.5
23.0
22.5
22.0
21.5
21.0
20.5
20.0


Temperatura interpolada
y
x
40.875
40.850
40.825
40.800
40.775
40.750
40.725
40.700
−74.05
−74.00
−73.95
−73.90
23.5
23.0
22.5
22.0
21.5
21.0
20.5

Resultados


Temperatura fixa em 435 pontos de um total de 8708
40.875
40.850
40.825
40.800
40.775
40.750
40.725
40.700
y
−74.10
−74.05
−74.00
−73.95
−73.90
−73.85
x
23.5
23.0
22.5
22.0
21.5
21.0
20.5


Temperatura interpolada
40.875
40.850
40.825
40.800
40.775
40.750
40.725
40.700
y
−74.05
−74.00
−73.95
−73.90
x
23.5
23.0
22.5
22.0
21.5
21.0
20.5

Dúvidas?