Unary Labeling CNF

Notation

y_i is a vector representing the unary label of node i.

x_{ij} is a boolean representing the existence of an edge from node i to j.

We use $\iff$ to denote logical equivalence, and $\to$ and $\leftrightarrow$ to denote the conditional (implies) and the biconditional, respectively.

Intro

The acyclicity check given by the paper is defined as:

$$
\begin{aligned}
&\bigwedge(x_{ij} \to \text{less}(y_i, y_j)) \wedge \bigvee(\text{unary}(y_i)) \\
\iff &\bigwedge(x_{ij} \to \exists u \ \text{lessunr}(y_i, y_j, u)) \wedge \bigvee(\text{unary}(y_i))
\end{aligned}
$$

We can split circuit up at the conjunction (and) and convert each part to CNF. The conjunction of two CNFs is a CNF.

1. Converting the *lessunr* circuit

Let's first convert the lessunr circuit:

$$\bigwedge(x_{ij} \to \exists u \ \text{lessunr}(y_i, y_j, u))$$

We remove the existential by *existential instantiation*, replacing each existential with a new constant symbol.

Let U_{ij} be a vector of new auxiliary variables associated with y_i and y_j:

$$\bigwedge(x_{ij} \to \text{lessunr}(y_i, y_j, U_{ij}))$$

Replace the implies with the equivalent clause, then

$$
\begin{aligned}
&\bigwedge_{i,j\in[n]} (\neg x_{ij} \vee \text{lessunr}(y_i, y_j, U_{ij})) \\
\iff &\bigwedge_{i,j\in[n]} \left(\neg x_{ij} \vee \Big(\bigwedge_{k=1}^{n-1} \big((\neg y_{ik} \vee \neg U_{ijk}) \wedge (y_{jk} \vee \neg U_{ijk})\big) \wedge \bigvee_{k=1}^{n-1} (U_{ijk}) \Big) \right)
\end{aligned}
$$

Distribute $\neg x_{ij}$ disjunction (or) over conjunction (and), twice:

$$\bigwedge_{i,j\in[n]} \Bigg(\neg x_{ij} \vee \Big(\bigwedge_{k=1}^{n-1} \big((\neg y_{ik} \vee \neg U_{ijk}) \wedge (y_{jk} \vee \neg U_{ijk})\big) \wedge \bigvee_{k=1}^{n-1} (U_{ijk})\Big)\Bigg)$$

$$\iff \bigwedge_{i,j\in[n]} \Big(\bigwedge_{k=1}^{n-1} \big(\neg x_{ij} \vee ((\neg y_{ik} \vee \neg U_{ijk}) \wedge (y_{jk} \vee \neg U_{ijk}))\big) \wedge \bigvee_{k=1}^{n-1} (U_{ijk})\Big)$$

$$\iff \bigwedge_{i,j\in[n]} \Big(\bigwedge_{k=1}^{n-1} \big((\neg x_{ij} \vee \neg y_{ik} \vee \neg U_{ijk}) \wedge (\neg x_{ij} \vee y_{jk} \vee \neg U_{ijk})\big) \wedge \bigvee_{k=1}^{n-1} (U_{ijk})\Big)$$

Finally, rearrange the conjunctions (and) to make the CNF more clear:

$$\bigwedge_{i,j\in[n]} \Big(\bigwedge_{k=1}^{n-1} \big((\neg x_{ij} \vee \neg y_{ik} \vee \neg U_{ijk}) \wedge (\neg x_{ij} \vee y_{jk} \vee \neg U_{ijk})\big)\Big) \wedge \bigwedge_{i,j\in[n]} \big(\bigvee_{k=1}^{n-1} (U_{ijk})\big)$$

2. Converting the *unary* circuit

Now we convert the unary circut:

$$\bigwedge_{i\in[n]} \text{unary}(y_i)$$

$$\iff \bigwedge_{i\in[n]} \big(\bigwedge_{j=2}^{n-1} y_{i,j-1} \rightarrow y_{ij}\big)$$

$$\iff \bigwedge_{i\in[n]} \big(\bigwedge_{j=2}^{n-1} \neg y_{i,j-1} \vee y_{ij}\big)$$

And we're done!

The final CNF is given by: lessunrCNF $\wedge$ unaryCNF