Project report: The VIX and Related Derivatives

Derivatives
FIN-404

Group O

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Spring Semester – 2025

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0 Documentation

0.1 The Market for Variance Derivatives: History and Recent Trends

The development and growth of financial markets over the last decades, especially since the Global Financial Crisis and the Covid 2019 turmoil, have significantly changed the role of volatility. In addition to being a crucial component of portfolio theory, asset volatilities are also major factors of the theoretical valuation of derivatives, referring for example to the standard option pricing model of Black and Scholes (1973). Moreover, volatility has become an asset class since the mid-1990's, when Variance Derivatives appeared. The value of these financial instruments is derived from the future volatility of an underlying asset, rather than its price level. Volatility has evolved from a statistical footnote to a tradable asset class. The change began in the early 1990s, when UBS transacted the first over-the-counter (OTC) variance swap (Carr & Lee, 2009). Demand for such hedges ballooned after the 1997–1998 Asia/LTCM turmoil, prompting exchanges to create visible benchmarks.

In 1993 the Chicago Board Options Exchange (Cboe) launched the VIX index, then based on Black-Scholes at-the-money volatility for S&P 100 options. A 2003 overhaul replaced that model with a strip-of-options formula, aligning VIX with variance-swap replication. Exchange-traded derivatives followed quickly: VIX futures debuted on the Cboe Futures Exchange in March 2004, and VIX options in February 2006.

Activity is now deep and liquid. Cboe processed 3.8 billion listed-option contracts in 2024 (Average Daily Volume (ADV) near 14.9 million). VIX products alone accounted for 59 million futures and 209 million options contracts(one third of S&P 500 Options) , underscoring investors' appetite to hedge, or harvest, market volatility.(Source: CBOE, volume statistics, 2024)

0.2 Variance Derivatives: Purpose and Practical Uses

Volatility can be measured using actual historical price changes (realized volatility) or it can be a measure of expected future volatility that is implied by option prices. In fact, as a measure of risk and uncertainty, volatility is usually expressed by the historical standard deviation of stock returns. When investors may have insights into the level of future volatility, they are interested in taking positions that profit if volatility changes.

According to Demeterfi et al. (1999), standard stock derivatives are impure because they provide exposure to both the direction of the stock price and its volatility. The purpose of variance derivatives is to provide pure volatility exposure by considering volatility as a tradable asset.

Because realised volatility tends to spike when equities fall, its payoff is negatively correlated with stock returns. Adding a long-variance position can therefore reduce portfolio draw-downs and smooth risk metrics. Today both retail traders and institutional desks can access liquid, centrally-cleared variance futures alongside the deeper OTC swap market to implement hedging, relative-value, or outright speculative views on volatility.

0.3 What is the VIX Volatility Index?

In 2003, to consider the leading indicator of the broad U.S. stock market, the new VIX is based on S&P 500 index options prices. Recognized as the first benchmark index of U.S. equity market volatility, "it measures the market's expectation of future volatility. It is used as a barometer for market uncertainty, providing market participants and observers with a measure of constant, 30-day expected volatility of the broad U.S. stock market"(CBOE). Although it cannot be traded directly, its construction—based on S&P 500 option prices—provides a framework for synthetically gaining exposure to volatility. This foundation enabled the development of derivative instruments such as VIX futures and options, now widely used to monitor and manage market risk.

According to CBOE White Paper (2024), the VIX index is often referred to as "the fear gauge" and is widely used by financial academics, risk managers and volatility traders. It offers a quantifiable measure of market risk and investor sentiment. It mostly rises when stocks fall, and decreases when stocks rise and vice versa. It is generally admitted that high VIX levels are associated with large volatility resulting from increased uncertainty, risk, and investor fear, whereas low VIX values correspond to stable and stress-free periods in the markets. Figure 1 (S&P 500 level in blue, VIX in red; data from Investing.com) visualises that inverse relationship over 2004-2025. During the 2008 Global Financial Crisis the S&P 500 collapsed by more than 50 percent and the VIX spiked to an all-time high near 80, signalling extreme counter-party and funding stress; a dozen years later, the COVID-19 shock produced the fastest equity drawdown on record and sent the VIX back above 80 as the market priced unprecedented economic shutdowns.

0.4 Key Variance Derivatives and Their Practical Uses

Table 1: *Main CBOE Variance Derivatives: Definition and Uses*

Instruments	Definition	Uses
Variance Futures (VA)	Standardized exchange-traded Futures contracts that provide exposure to the realized variance of the S&P 500 index over the life of the contract (1- to 3-month listed tenors).	- Replicate variance swaps - Hedging variance risk more directly than delta-hedging options - Volatility arbitrage - Tail risk strategies: long VA to profit from volatility spikes in market stress
VIX Futures	Exchange-traded derivatives contracts based on expectations of the future value of VIX index on various expiration dates in the future.	- Hedging tail volatility risk: protection against market sell-offs - Speculating on directional move of volatility - Diversifying investment portfolios with an asset that is typically negatively correlated to equities
VIX Options	European Style options based on VIX index.	- Cheap convex tail hedge: far-out-of-the-money calls jump in crashes - Speculating on directional move of volatility ith limited, upfront premium risk
VIX Futures Options	Cash-settled American Style options where the underling is specific VIX Futures Contracts rather than the VIX index itself.	- Hedging and speculating on positions in VIX Futures with more precise and more flexible strategies

Source: www.cboe.com/tradable_products

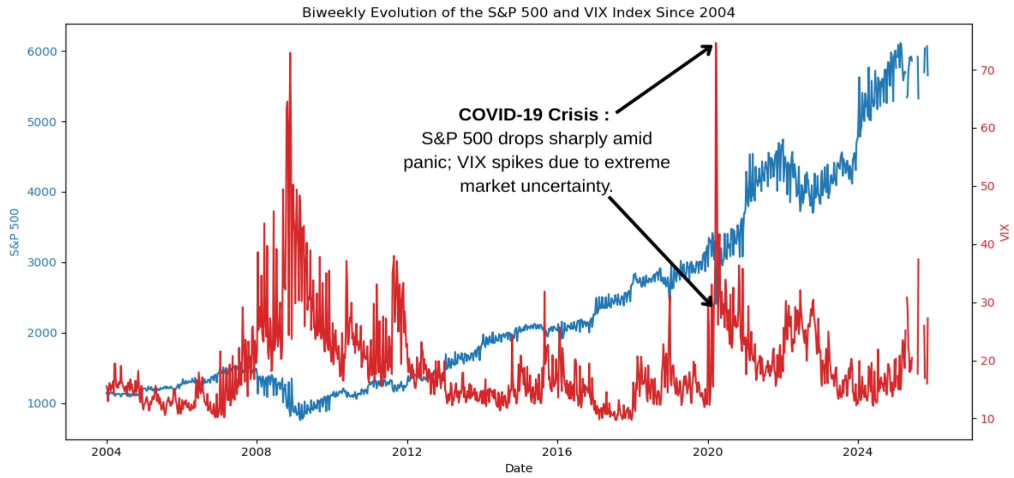

Figure 1: Biweekly evolution of the S&P 500 level and the VIX index from 2004 to 2025

0.5 What distinguishes the VIX from a Variance Swap?

The Variance Swap is a forward contract on future realized stock volatility. At maturity, the realized volatility is compared with a pre-agreed strike volatility level. For example, for the swap buyer, who anticipated that the markets will be more volatile than expected, if the spread is positive, he receives the difference times the contract value. The main uses of this financial instrument are speculating on future volatility levels, trading the spread between realized volatility and implied volatility, and hedging volatility exposure, for example for portfolio managers or hedge funds.

Then, it appears that the VIX and the Variance Swaps are fundamentally different in what they measure, how they are constructed and how they are used. In fact, the VIX is a volatility index, based on implied volatility, not tradable but is used as the underlying for tradable instruments. While, Variance Swap is an OTC instrument which measures the spread between realized and implied variance of a given asset. To replicate the Variance Swap, the CBOE introduced Variance Futures as a tradable variance derivative on listed markets.

0.6 What distinguishes a Variance Swap from a Variance Futures?

A variance future is the listed sibling of an OTC variance swap—same exposure (payoff tied to the realised variance of the S&P500), but very different legal and operational plumbing.

- **Market venue and clearing.** Centrally cleared by the Options Clearing Corporation, variance futures pool counter-party risk across all members. Variance swaps are private ISDA deals, leaving each side exposed to the other's default unless they opt to clear.

- **Contract design and settlement.** Variance futures are straightforward: each point equals one unit of realised S&P 500 variance, so expiry cash-settles at 10 000 $\times$ the annualised figure. Swaps are fully customisable—sampling, calendar, start/finish—perfect for bespoke hedges but far harder to draft and price.

- **Execution dynamics and liquidity.** Futures trade on a public order book with market-makers, block trades, and ECRPs, giving continuous two-way quotes and easy exits. Swap prices are private, and closing a position requires a fresh bilateral negotiation with a dealer.

- **Accessibility and regulatory burden :** Variance futures are open to anyone with a standard futures margin account, whereas swaps are institution-only: they require an ISDA, bilateral margin under UMR, and extensive Dodd-Frank/EMIR reporting.

1 The Carr-Madan formula

1.1 Question 1

Since $H \in C^2([x_0, x])$, in particular H' is continuous on $[x_0, x]$, so we can apply the Fundamental Theorem of Calculus:

$$H(x) - H(x_0) = \int_{x_0}^{x} H'(k)\, dk \tag{1}$$

Now the goal is to integrate by part :

$$\int_{x_0}^{x} H'(k)\, dk$$

By taking : u(k) = $H'(k)$ and v(k) = $(k - x)$. Also knowing that $H \in C^2([x_0, x])$, we have that :

$$\begin{aligned}
\int_{x_0}^{x} H'(k)\, dk &= [H'(k)(k-x)]_{x_0}^{x} - \int_{x_0}^{x} H''(k)(k-x)\, dk \\
&= -H'(x_0)(x_0 - x) - \int_{x_0}^{x} H''(k)(k-x)\, dk \\
&= (x - x_0)H'(x_0) + \int_{x_0}^{x} H''(k)(x-k)\, dk
\end{aligned}$$

Now by substituing what we found for the integration by part back to the FTC Equation (1). We have that :

$$H(x) = H(x_0) + (x - x_0)H'(x_0) + \int_{x_0}^{x} H''(k)(x-k)\, dk \tag{2}$$

for all $(x, x_0) \in \mathbb{R}_+^2$ such that H' is continuous at x_0

1.2 Question 2

From previous question we have the Equation (2). The goal is now to derive this equation using the formula given in instructions :

$$(x - k) = (x - k)^+ - (k - x)^+$$

Hence, we can write the integral :

$$\begin{aligned}
\int_{x_0}^{x} H''(k)(x-k)\, dk &= \int_{x_0}^{x} H''(k)((x-k)^+ - (k-x)^+) \\
&\overset{(1)}{=} \int_{x_0}^{x} (\mathbf{1}_{x \geq x_0} + \mathbf{1}_{x < x_0}) H''(k)(x-k)^+\, dk - \int_{x_0}^{x} (\mathbf{1}_{x \geq x_0} + \mathbf{1}_{x < x_0}) H''(k)(k-x)^+\, dk \\
&\overset{(2)}{=} \int_{x_0}^{x} \mathbf{1}_{x \geq x_0} H''(k)(x-k)^+\, dk + \int_{x}^{x_0} \mathbf{1}_{x < x_0} H''(k)(k-x)^+\, dk
\end{aligned}$$

$$
\begin{aligned}
&\overset{(3)}{=} \int_{x_0}^{\infty} (\mathbf{1} - \mathbf{1}_{x<x_0}) H''(k)(x-k)^+ \, dk + \int_0^{x_0} (\mathbf{1} - \mathbf{1}_{x\geq x_0}) H''(k)(k-x)^+ \, dk \\
&= \int_{x_0}^{\infty} H''(k)(x-k)^+ \, dk - \underbrace{\int_{x_0}^{\infty} \mathbf{1}_{x<x_0} H''(k)(x-k)^+ \, dk}_{=0} \\
&\quad + \int_0^{x_0} H''(k)(k-x)^+ \, dk - \underbrace{\int_0^{x_0} \mathbf{1}_{x\geq x_0} H''(k)(k-x)^+ \, dk}_{=0} \\
&= \int_{x_0}^{\infty} H''(k)(x-k)^+ \, dk + \int_0^{x_0} H''(k)(k-x)^+ \, dk
\end{aligned}
$$

By replacing this new formula directly to Equation (2). We finally have :

$$
H(x) = H(x_0) + (x-x_0)H'(x_0) + \int_0^{x_0} H''(k)(k-x)^+ \, dk + \int_{x_0}^{\infty} H''(k)(x-k)^+ \, dk \tag{3}
$$

The different steps to come to this final expression are, first, in (1), by linearity of the integral, we separate the two terms.
Then in (2) we also used the linearity of the integrals to separate into 4 terms, however we remark that $\int_{x_0}^{x} \mathbf{1}_{x<x_0} H''(k)(x-k)^+ \, dk = 0$ since $(x-k)^+ = 0$ when $k \in [x, x_0]$. Respectively, using the same thinking $\int_{x_0}^{x} \mathbf{1}_{x\geq x_0} H''(k)(k-x)^+ \, dk = 0$ when $k \in [x_0, x]$. Then we used the identity : $\int_{x_0}^{x} \mathbf{1}_{x<x_0} H''(k)(k-x)^+ \, dk = -\int_{x}^{x_0} \mathbf{1}_{x<x_0} H''(k)(k-x)^+ \, dk$
For (4), we used the fact that $\int_x^{\infty} \mathbf{1}_{x\geq x_0} H''(k)(x-k)^+ \, dk = 0$ such that we have
$\int_{x_0}^{x} \mathbf{1}_{x\geq x_0} H''(k)(x-k)^+ \, dk = \int_{x_0}^{\infty} \mathbf{1}_{x\geq x_0} H''(k)(x-k)^+ \, dk$. Respectively, using the same method, we have that $\int_{x}^{x_0} \mathbf{1}_{x<x_0} H''(k)(k-x)^+ \, dk = \int_0^{x_0} \mathbf{1}_{x<x_0} H''(k)(k-x)^+ \, dk$

1.3 Question 3

By using the previous Equation (3), the terminal payoff of the European derivative can be expressed as :

$$
\begin{aligned}
H(S_T) &= H(x_0) + H'(x_0)(S_T - x_0) + \int_0^{x_0} H''(k) \underbrace{(k-x)^+}_{P_T(T,k)} \, dk + \int_{x_0}^{\infty} H''(k) \underbrace{(x-k)^+}_{C_T(T,k)} \, dk \\
&= H(x_0) + H'(x_0)(S_T - x_0) + \int_0^{x_0} H''(k) P_T(T,k) \, dk + \int_{x_0}^{\infty} H''(k) C_T(T,k) \, dk
\end{aligned}
$$

Now, we want to apply the law of one price in its neutral form such that with a constant interest rate r, the price at time 0 of the European Derivative's payoff can be expressed as :

$$
V_0 = e^{-rT} \mathbb{E}^{\mathbb{Q}}[H(S_T)]
$$

Which means, we have :

$$
\begin{aligned}
V_0 &= e^{-rT} \mathbb{E}^{\mathbb{Q}}[H(x_0) + H'(x_0)(S_T - x_0) + \int_0^{x_0} H''(k) P_T(T,k) \, dk + \int_{x_0}^{\infty} H''(k) C_T(T,k) \, dk] \\
&= e^{-rT} H(x_0) + e^{-rT} H'(x_0)(E^Q[S_T] - x_0) \\
&\quad + \int_0^{x_0} H''(k) \underbrace{e^{-rT} E^Q[P_T(T,k)]}_{P_0(T,k)} \, dk + \int_{x_0}^{\infty} H''(k) \underbrace{e^{-rT} E^Q[C_T(T,k)]}_{C_0(T,k)} \, dk \\
&= e^{-rT} H(x_0) + e^{-rT} H'(x_0)(E^Q[S_T] - x_0) + \int_0^{x_0} H''(k) P_0(T,k) \, dk + \int_{x_0}^{\infty} H''(k) C_0(T,k) \, dk
\end{aligned}
$$

Denote respectively $P_0(T,k)$ and $C_0(T,k)$ as the price today of an European put and call option with terminal date T and strike price k.

Moreover, from the lessons we know that :

$$e^{-rT}E^Q[S_T] = e^{-rT}e^{(r-\delta)T}S_0 = e^{-\delta T}S_0$$

Consequently, the final price's formula at time 0 of the European Derivative's payoff is :

$$V_0 = e^{-rT}(H(x_0) - x_0H'(x_0)) + H'(x_0)e^{-\delta T}S_0 + \int_0^{x_0} H''(k)P_0(T,k)\,dk + \int_{x_0}^{\infty} H''(k)C_0(T,k)\,dk \tag{4}$$

Hence, we have that an European derivative with terminal payoff $H(S_T)$ can be replicated by a static portfolio that holds

- $n_0 = H'(x_0)e^{-\delta T}$ units of the underlying asset
- An amount $a_0 = e^{-rT}(H(x_0) - x_0H'(x_0))$ invested at the risk free rate
- $w(k) = H''(k)$ units of a put with strike k for all $k \leq x_0$
- $w(k) = H''(k)$ units of a call with strike k for all $k > x_0$

For the special case, when $x_0 = F_0(T)$ is the forward price, we have $F_0(T) = e^{(r-\delta)T}S_0$, so $S_0 = e^{-(r-\delta)T}F_0(T)$. Now, by replacing S_0 and x_0 with their new values into Equation (4).

$$\begin{aligned} V_0 &= e^{-rT}(H(F_0(T) - F_0H'(F_0(T)) + H'(F_0(T)e^{-\delta T}e^{-(r-\delta)T}F_0(T) \\ &\quad + \int_0^{F_0(T)} H''(k)P_0(T,k)\,dk + \int_{F_0(T)}^{\infty} H''(k)C_0(T,k)\,dk \\ &= e^{-rT}H(F_0(T)) + \int_0^{F_0(T)} H''(k)P_0(T,k)\,dk + \int_{F_0(T)}^{\infty} H''(k)C_0(T,k)\,dk \end{aligned}$$

Which means, for the special case, when $x_0 = F_0(T)$ an European derivative with terminal payoff $H(S_T)$ can be replicated by a static portfolio that holds

- $n_0 = 0$ units of the underlying asset
- An amount $a_0 = e^{-rT}H(F_0(T))$ invested at the risk free rate
- $w(k) = H''(k)$ units of a put with strike k for all $k \leq F_0(T)$
- $w(k) = H''(k)$ units of a call with strike k for all $k > F_0(T)$

1.4 Question 4

In the previous question, we computed the general Carr-Madam formula. Moreover, since the function $H(x) = x^p$ is a piecewise twice continuously differentiable function on the positive real line, we can directly apply the Carr-Madan formula to it to have the replicated static portfolio as :

- $n_0 = H'(x_0)e^{-\delta T} = px_0^{p-1}e^{-\delta T}$ units of the underlying asset
- An amount $a_0 = e^{-rT}(H(x_0) - x_0H'(x_0)) = e^{-rT}(1-p)x_0^p$ invested at the risk free rate
- $w(k) = H''(k) = p(p-1)k^{p-2}$ units of a put with strike k for all $k \leq x_0$
- $w(k) = H''(k) = p(p-1)k^{p-2}$ units of a call with strike k for all $k > x_0$

1.5 Question 5

The most fundamental limitation comes from the gap between theoretical assumptions and market reality. In practice, only a finite number of option strikes are available in the market, requiring the continuous Carr-Madan formula to be discretized accordingly. This discretization leads to imperfect hedging and introduces approximation errors that can significantly impact pricing accuracy. Beyond that, deep-out-of-the-money puts and far-out calls are often illiquid or absent, forcing traders to cut off the integral and extrapolate the tails. Also, Carr-Madan needs simultaneous prices across all strikes, yet live feeds update at different times; stale or flickering quotes inject micro-timing noise that propagates into the integral.

2 The VIX index

2.1 Question 1 – Realized variance formula

Under the risk-neutral measure Q, the S&P 500 index S_t follows the SDE :

$$\frac{dS_t}{S_t} = (r-\delta)\,dt + \sqrt{V_t}\,dB_t^Q \ ,$$

where r is the constant risk-free rate, δ the dividend yield, and V_t is the instantaneous squared volatility (i.e. variance rate) at time t. To relate the realized variance to the log returns of S_t, we apply Itô's formula to the log of the price. For $X_t = \ln S_t$, Itô's lemma gives:

$$dX_t = d(\ln S_t) = \frac{dS_t}{S_t} - \tfrac{1}{2}(\text{volatility})^2\,dt = \frac{dS_t}{S_t} - \tfrac{1}{2}V_t\,dt \ ,$$

since $(\text{volatility})^2 = V_t$. Now integrate this expression from time t to T:

$$\ln S_T - \ln S_t \ = \ \int_t^T \frac{dS_u}{S_u} \ - \ \frac{1}{2}\int_t^T V_u\,du \ .$$

Rearranging terms, we obtain an identity relating the cumulative variance to the stock's logarithmic return and gross return:

$$\int_t^T V_u\,du \ = \ 2\int_t^T \frac{dS_u}{S_u} \ - \ 2\ln\frac{S_T}{S_t} \ .$$

By definition, the realized variance over $[t,T]$ is $V_{t,T} := \int_t^T V_s\,ds$. Thus we can rewrite the above result as:

$$V_{t,T} \ = \ 2\int_t^T \frac{dS_u}{S_u} \ - \ 2\ln\frac{S_T}{S_t} \ ,$$

or equivalently:

$$\frac{1}{2}V_{t,T} \ = \ \int_t^T \frac{dS_u}{S_u} \ - \ \ln\frac{S_T}{S_t} \ .$$

Comparing with the required form $x\,V_{t,T} = \int_t^T \frac{dS_u}{S_u} - \ln(S_T/S_t)$, we identify the constant $x = \frac{1}{2}$.

2.2 Question 2 – Expected continuous return integral

Taking expectations of the SDE, we note that under the risk-neutral measure Q the drift of S_t is $r-\delta$. The stochastic integral has zero mean, so:

$$\mathbb{E}_t^Q\left[\int_t^T \frac{dS_u}{S_u}\right] \ = \ \int_t^T \mathbb{E}_t^Q\left[\frac{dS_u}{S_u}\right] \ = \ \int_t^T (r-\delta)\,du \ = \ (r-\delta)\,(T-t) \ .$$

Thus $\alpha = r - \delta$. In particular,

$$\mathbb{E}_t^Q\Big[\int_t^T \frac{dS_u}{S_u}\Big] = \alpha(T-t),$$

a linear function of the remaining time $T-t$.

2.3 Question 3 – Log return in terms of OTM options

Recall that for any twice-differentiable payoff $H(S_T)$, one convenient representation (from Part 1) was:

$$H(S_T) = H(K_0) + H'(K_0)\,(S_T - K_0) \;+\; \int_0^{K_0} H''(k)\,(k - S_T)^+ dk \;+\; \int_{K_0}^{\infty} H''(k)\,(S_T - k)^+ dk\ ,$$

with K_0 an arbitrary reference strike (often chosen near the forward price). We apply this formula to the function $H(x) = \ln x$ (the log payoff), and choose the reference strike $x_0 = K_0$ to follow the CBOE convention (i.e. K_0 is the strike at or just below the forward $F_t(T)$). We first compute the derivatives of $H(x) = \ln x$:

$$H'(x) = \frac{1}{x}, \qquad H''(x) = -\frac{1}{x^2}.$$

Plugging these into the formula gives the log payoff in terms of option payoffs:

$$\ln S_T \;=\; \ln K_0 \;+\; \frac{1}{K_0}(S_T - K_0) \;-\; \int_0^{K_0} \frac{1}{k^2}(k - S_T)^+ dk \;-\; \int_{K_0}^{\infty} \frac{1}{k^2}(S_T - k)^+ dk\ .$$

Here $(k - S_T)^+$ is the payoff of a put with strike k, and $(S_T - k)^+$ is the payoff of a call with strike k. We can recognize these as European option payoffs. Now, taking the risk-neutral expectation $\mathbb{E}_t^Q[\cdot]$ of both sides and using the fact that under Q the discounted expected payoff equals the current option price, we replace each expected option payoff with the present value of the option:

- $\mathbb{E}_t^Q[(k - S_T)^+] = e^{r(T-t)}\,P_t(T,k)$, where $P_t(T,k)$ is the time-t price of an OTM put with strike k maturing at T.
- $\mathbb{E}_t^Q[(S_T - k)^+] = e^{r(T-t)}\,C_t(T,k)$, where $C_t(T,k)$ is the price of an OTM call with strike k (for $k \geq K_0$).

Also, $\mathbb{E}_t^Q[S_T] = F_t(T)$, the forward price for delivery at T (because $F_t(T) = S_t e^{(r-\delta)(T-t)}$ under no-arbitrage). Substituting these into the above decomposition yields:

$$\mathbb{E}_t^Q[\ln S_T] \;=\; \ln K_0 + \frac{1}{K_0}\big(F_t(T) - K_0\big) \;-\; e^{r(T-t)}\int_0^{K_0} \frac{1}{k^2} P_t(T,k)\,dk \;-\; e^{r(T-t)}\int_{K_0}^{\infty} \frac{1}{k^2} C_t(T,k)\,dk\ .$$

We define

$$\mathcal{P}_t(T,K_0) \;:=\; e^{r(T-t)}\int_0^{K_0} \frac{P_t(T,k)}{k^2}\,dk \;+\; e^{r(T-t)}\int_{K_0}^{\infty} \frac{C_t(T,k)}{k^2}\,dk\ ,$$

which represents the time-t price of a log payoff (with strike K_0) expressed as an integral over out-of-the-money puts and calls.

Subtracting $\ln S_t$ from both sides, the expectation of the log return becomes:

$$\mathbb{E}_t^Q\Big[\ln\frac{S_T}{S_t}\Big] = \ln K_0 - \ln S_t \;+\; \frac{F_t(T) - K_0}{K_0} \;-\; \mathcal{P}_t(T,K_0)\ .$$

Finally, we notice that $\ln K_0 - \ln S_t = \ln(K_0/S_t)$. Also, $\frac{F_t(T)-K_0}{K_0} = \frac{F_t(T)}{K_0} - 1$. Thus,

$$\mathbb{E}_t^Q\Big[\ln\frac{S_T}{S_t}\Big] \;=\; \ln\frac{K_0}{S_t} \;-\; \Big(1 - \frac{F_t(T)}{K_0}\Big) \;-\; \mathcal{P}_t(T,K_0)\ ,$$

which is the desired formula.

2.4 Question 4 – Expected variance versus option prices

Taking the expectation $\mathbb{E}_t^Q[\cdot]$ of both sides of the identity from **Question 1** yields:

$$\frac{1}{2}\,\mathbb{E}_t^Q[V_{t,T}] \;=\; \mathbb{E}_t^Q\Big[\int_t^T \frac{dS_u}{S_u}\Big] \;-\; \mathbb{E}_t^Q\Big[\ln\frac{S_T}{S_t}\Big]\;.$$

Substituting the results of **Question 2** and **Question 3** into this equation, we get:

$$\frac{1}{2}\,\mathbb{E}_t^Q[V_{t,T}] \;=\; (r-\delta)(T-t) \;-\; \Big[\ln\frac{K_0}{S_t} - \Big(1-\frac{F_t(T)}{K_0}\Big) - P_t(T,K_0)\Big]\;.$$

Notice that $(r-\delta)(T-t) - \ln\frac{K_0}{S_t} \;=\; \ln\Big(S_t e^{(r-\delta)(T-t)}\Big) - \ln K_0 \;=\; \ln\frac{F_t(T)}{K_0}$, because $F_t(T) = S_t e^{(r-\delta)(T-t)}$. Thus we have:

$$\frac{1}{2}\,\mathbb{E}_t^Q[V_{t,T}] \;=\; \ln\frac{F_t(T)}{K_0} \;+\; \Big(1-\frac{F_t(T)}{K_0}\Big) \;+\; P_t(T,K_0)\;.$$

Now, since K_0 is chosen close to $F_t(T)$ then $K_0 \approx F_t(T)$, so the term $\frac{F_t(T)}{K_0}$ is close to 1. Let $\Delta := \frac{F_t(T)}{K_0} - 1$, which is a small number if K_0 is the at-the-money strike. Using a second-order Taylor expansion for $\ln(1+\Delta)$:

$$\ln\frac{F_t(T)}{K_0} \;=\; \ln(1+\Delta) \;\approx\; \Delta - \frac{1}{2}\Delta^2\,, \qquad \text{for small } \Delta.$$

Substituting $\ln(F_t/K_0) \approx \Delta - \frac{1}{2}\Delta^2$ and $1 - F_t/K_0 = -\Delta$ into the expression above gives:

$$\frac{1}{2}\,\mathbb{E}_t^Q[V_{t,T}] \;\approx\; P_t(T,K_0) \;+\; (\Delta - \tfrac{1}{2}\Delta^2) \;-\; \Delta \;=\; P_t(T,K_0) \;-\; \tfrac{1}{2}\Delta^2\;.$$

Finally, we have $\frac{1}{2}\Delta^2 = \frac{1}{2}(1-\frac{F_t(T)}{K_0})^2$. Hence we obtain the desired relation:

$$x\,\mathbb{E}_t^Q[V_{t,T}] \;\approx\; P_t(T,K_0) \;-\; \frac{1}{2}\Big(1-\frac{F_t(T)}{K_0}\Big)^2\;.$$

Interpretation

- **VIX as the market expectation of future volatility.** The above result explains the CBOE's claim that the VIX index measures the market's expectation of near-term volatility. $(\mathrm{VIX}t/100)^2$ equals the risk-neutral expected 30-day variance of the S&P500. In other words, VIX (squared) is proxied by $\frac{1}{T-t}\mathbb{E}_t^Q[Vt,T]$, the market's forecast of variance over the next 30 days (annualized). Taking the square root (and multiplying by 100) yields a volatility percentage. It uses current option prices rather than past price data. The small adjustment term $\frac{1}{2}\big(1-F_t(T)/K_0\big)^2$ is minor if K_0 is close to the forward, so essentially VIX is determined by the weighted sum of OTM option prices $P_t(T,K_0)$. Thus, the VIX formula provides a model-free estimate of expected volatility, derived purely from market prices under the no-arbitrage Q-measure.

- **Advantage over 30 days realized variance.** Historical realized volatility is computed from past index fluctuations, so it may not capture sudden changes in outlook. The VIX, by contrast, will jump immediately if market participants expect future volatility to increase, since option premiums rise. Furthermore, realized variance is just one path outcome, whereas VIX aggregates the market's view of the distribution of future outcomes. This makes VIX a more relevant metric for forward-looking risk management.

- **Advantage over Black-Scholes implied volatility.** A Black–Scholes implied volatility usually refers to the volatility parameter that, when plugged into the Black–Scholes formula, matches the price of a single option (with a specific strike and maturity). VIX, on the other hand, is a strike-aggregated, model-free measure: it uses a broad range of strikes to capture the entire risk-neutral distribution of future returns. This means: VIX accounts for skewness and tail risk by incorporating OTM put prices. Also VIX is model-independent in that it does not assume constant volatility or any particular diffusion model; In contrast, quoting an implied vol inherently assumes the Black-Scholes model holds for that option. Another practical advantage is that VIX provides a standardized volatility index that is easier to communicate and trade, whereas a single option's implied vol is tied to that strike and maturity.

- **Key assumptions.**
  - No arbitrage, market completeness, and a frictionless market where a continuum of strikes is available for trading for the Carr–Madan replication to hold.
  - Continuous-time stochastic process for the asset price S_t: $dS_t/S_t = (r - \delta)dt + \sqrt{V_t}, dB_t^Q$ with $V_t \geq 0$ being the instantaneous variance.
  - Constant r and δ: We took interest rates and dividend yields to be constant.
  - Risk-Neutral Expectation: Interpreting VIX as an expected volatility assumes that the risk-neutral measure Q is being used to represent the market's consensus.

3 Futures pricing

3.1 Question 1

We know that :

$$\left(\frac{VIX_t}{100}\right)^2 = \frac{1}{\eta}\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\eta} V_u \, du\right]$$

with $\eta = \frac{30}{252}$ expressed in years and $V_t \geq 0$ a process that models the squared volatility of the index. Therefore, using Tonelli's Theorem, we can rewrite :

$$\left(\frac{VIX_t}{100}\right)^2 = \frac{1}{\eta}\int_t^{t+\eta} \mathbb{E}_t^{\mathbb{Q}}\left[V_u\right] \, du \tag{5}$$

Moreover, we assume that :

$$dV_t = \lambda(\theta - V_t)dt + \xi\rho\sqrt{V_t}dB_t^{\mathbb{Q}} + \xi\sqrt{1-\rho^2}\sqrt{V_t}dZ_t^{\mathbb{Q}} = \lambda(\theta - V_t)dt + \xi\sqrt{V_t}dW_t^{\mathbb{Q}} \tag{6}$$

where λ, θ, $\xi > 0$ and $\rho \in$ [-1, 1] are constants and $W_t^{\mathbb{Q}}$ ($dW_t^{\mathbb{Q}} = \rho dB_t^{\mathbb{Q}} + \sqrt{1-\rho^2}dZ_t^{\mathbb{Q}}$) is a $\mathbb{Q}-$Brownian motion.

Now, let's compute the expectation of both sides at time T, applying (6), together with the fact that a Brownian motion satisfies $W_t \sim \mathcal{N}(0, t)$:

$$\mathbb{E}_T^{\mathbb{Q}}\left[dV_t\right] = \mathbb{E}_T^{\mathbb{Q}}\left[\lambda(\theta - V_t)dt + +\xi\sqrt{V_t}dW_t^{\mathbb{Q}}\right] \tag{7}$$

$$\mathbb{E}_T^{\mathbb{Q}}\left[dV_t\right] = \lambda(\theta - \mathbb{E}_T^{\mathbb{Q}}\left[V_t\right])dt \tag{8}$$

$$\frac{d}{dt}\mathbb{E}_T^{\mathbb{Q}}\left[V_t\right] = \lambda(\theta - \mathbb{E}_T^{\mathbb{Q}}\left[V_t\right]) \tag{9}$$

The equation (9) is a linear first-order ODE, with initial condition $\mathbb{E}_T^{\mathbb{Q}}[V_T] = V_T$; its solution is:

$$\mathbb{E}_T^{\mathbb{Q}}[V_t] = \theta + (V_T - \theta)e^{-\lambda(t-T)} \tag{10}$$

Finally, we can show that :

$$\begin{aligned}
\left(\frac{VIX_T}{100}\right)^2 = \frac{1}{\eta}\int_T^{T+\eta} \mathbb{E}_T^{\mathbb{Q}}[V_u]\, du &= \frac{1}{\eta}\int_T^{T+\eta} \theta + (V_T - \theta)e^{-\lambda(u-T)}\, du \\
&= \frac{1}{\eta}\left(\eta\theta + (V_T - \theta)\left[\frac{-1}{\lambda}e^{-\lambda(u-T)}\right]_T^{T+\eta}\right) \\
&= \frac{1}{\eta}\left(\eta\theta + (V_T - \theta)\frac{1-e^{-\lambda\eta}}{\lambda}\right) \\
&= \frac{1}{\eta}\left(\theta\left(\eta - \frac{1-e^{-\lambda\eta}}{\lambda}\right) + V_T\frac{1-e^{-\lambda\eta}}{\lambda}\right)
\end{aligned}$$

Therefore, we showed that :

$$\left(\frac{VIX_T}{100}\right)^2 = \frac{1}{\eta}(a + bV_T) \tag{11}$$

with $a = \theta\left(\eta - \frac{1-e^{-\lambda\eta}}{\lambda}\right)$ and $b = \frac{1-e^{-\lambda\eta}}{\lambda}$

3.2 Question 2

Let's rewrite the equation to isolate $\mathbb{E}_t^{\mathbb{Q}}[e^{-sV_T}]$:

$$\begin{aligned}
-\log \mathbb{E}_t^{\mathbb{Q}}[e^{-sV_T}] &= c(T-t;s) + d(T-t;s)V_t \\
\log \mathbb{E}_t^{\mathbb{Q}}[e^{-sV_T}] &= -c(T-t;s) - d(T-t;s)V_t \\
\mathbb{E}_t^{\mathbb{Q}}[e^{-sV_T}] &= e^{-c(T-t;s)-d(T-t;s)V_t}
\end{aligned}$$

We define $\mathbb{E}_t^{\mathbb{Q}}[e^{-sV_T}] = f(t, V_t)$, which is a martingale under $\mathbb{Q}$. Since $f(t, V_t)$ is a martingale, its drift term must vanish when applying itô's lemma, using the assumption (6) :

$$df(t, V_t) = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial V}dV_t + \frac{1}{2}\frac{\partial^2 f}{\partial V^2}(dV_t)^2$$

The variation term $(dV_t)^2 = \xi^2 V_t (dW_t^{\mathbb{Q}})^2 = \xi^2 V_t dt$, since $(dW_t^{\mathbb{Q}})^2 = dt$ and the cross-term $dB_t^{\mathbb{Q}} dZ_t^{\mathbb{Q}} = 0$ (independent Brownian motions). This allows us to continue with the following equation:

$$df(t, V_t) = (\frac{\partial f}{\partial t} + \lambda(\theta - V_t)\frac{\partial f}{\partial V} + \frac{1}{2}\xi^2 V_t \frac{\partial^2 f}{\partial V^2})dt + \xi\sqrt{V_t}\frac{\partial f}{\partial V}dW_t^{\mathbb{Q}}$$

Therefore, using the drift term we establish that:

$$\begin{aligned}
0 &= \frac{\partial f}{\partial t} + \lambda(\theta - V_t)\frac{\partial f}{\partial V} + \frac{1}{2}\xi^2 V_t \frac{\partial^2 f}{\partial V^2} \\
0 &= (\frac{\partial c(T-t;s)}{\partial t} + \frac{\partial d(T-t;s)}{\partial t}V_t)f(t, V_t) - \lambda(\theta - V_t)d(T-t;s)f(t, V_t) + \frac{1}{2}\xi^2 V_t (d(T-t;s))^2 f(t, V_t) \\
0 &= (\frac{\partial c(T-t;s)}{\partial t} + \frac{\partial d(T-t;s)}{\partial t}V_t) - \lambda(\theta - V_t)d(T-t;s) + \frac{1}{2}\xi^2 V_t (d(T-t;s))^2 \\
0 &= \frac{\partial c(T-t;s)}{\partial t} - \lambda\theta d(T-t;s) + (\lambda\theta d(T-t;s) + \frac{\partial d(T-t;s)}{\partial t} + \frac{1}{2}\xi^2 (d(T-t;s))^2)V_t
\end{aligned}$$

Therefore, to solve for $c(T-t;s)$ and $d(T-t;s)$, we need to solve the two following ODEs:

$$\frac{\partial c(T-t;s)}{\partial t} = \lambda\theta d(T-t;s) \qquad \frac{\partial d(T-t;s)}{\partial t} = -\lambda d(T-t;s) - \frac{1}{2}\xi^2(d(T-t;s))^2$$

At time T, we have $f(T, V_T) = \mathbb{E}_T^{\mathbb{Q}}[e^{-sV_T}] = e^{-sV_T}$, which implies:

$$c(0;s) = 0 \qquad d(0;s) = s$$

Using Wolfram α, we conclude:

$$c(T-t;s) = \lambda\theta\frac{2}{\xi^2}\log(2\lambda - s\xi^2(e^{\lambda(T-t)}-1)) - \lambda\theta\frac{2}{\xi^2}\log(2\lambda) \qquad d(T-t;s) = \frac{2s\lambda e^{\lambda(T-t)}}{2\lambda - s\xi^2(e^{\lambda(T-t)}-1)}$$

3.3 Question 3

Variance Futures

We know that the terminal payoff of a variance futures contract is:

$$f_T^{\text{VA}}(T) = \frac{10,000}{T-t_0}\int_{t_0}^{T} V_u\,du \tag{12}$$

Moreover, the futures price process of asset i for delivery at date T is uniquely given by:

$$f_t^{\text{VA}}(T) = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{10,000}{T-t_0}\int_{t_0}^{T} V_u\,du\right] \tag{13}$$

at time $t_0 \leq t \leq T$. Hence, we can write:

$$\begin{aligned}
f_t^{\text{VA}}(T) &= \frac{10,000}{T-t_0}\left(\int_{t_0}^{t} V_u\,du + \mathbb{E}_t^{\mathbb{Q}}\left[\int_{t}^{T} V_u\,du\right]\right) && (14)\\
&= \frac{10,000}{T-t_0}\left(\int_{t_0}^{t} V_u\,du + \int_{t}^{T} \mathbb{E}_t^{\mathbb{Q}}\left[V_u\right]\,du\right) && (15)\\
&= \frac{10,000}{T-t_0}\left(\int_{t_0}^{t} V_u\,du + \int_{t}^{T} \theta + (V_t-\theta)e^{-\lambda(u-t)}\,du\right) && (16)\\
&= \frac{10,000}{T-t_0}\left(\int_{t_0}^{t} V_u\,du + \theta\left(T-t-\frac{1-e^{-\lambda(T-t)}}{\lambda}\right) + V_t\frac{1-e^{-\lambda(T-t)}}{\lambda}\right) && (17)
\end{aligned}$$

where we used Tonelli's theorem in (15) and the expression of $\mathbb{E}_T^{\mathbb{Q}}[V_t]$ from equation (10) of Question 1 in line (16). Thus:

$$a^*(T-t) = \theta\left(T-t-\frac{1-e^{-\lambda(T-t)}}{\lambda}\right) \qquad b^*(T-t) = \frac{1-e^{-\lambda(T-t)}}{\lambda}$$

VIX Futures

In Question 1, we proved that:

$$\left(\frac{VIX_T}{100}\right)^2 = \frac{1}{\eta}(a+bV_T) \tag{18}$$

Rewriting this expression, we have:

$$VIX_T = \frac{100}{\sqrt{\eta}}\sqrt{(a+bV_T)} \tag{19}$$

which is the terminal value of the VIX futures price with maturity date T. We use one more time that the futures price process of asset i for delivery at date T is uniquely given by:

$$\begin{aligned}
f_t^{\text{VIX}}(T) &= \mathbb{E}_t^{\mathbb{Q}}\left[VIX_T\right] \\
&= \frac{100}{\sqrt{\eta}}\mathbb{E}_t^{\mathbb{Q}}\left[\sqrt{a+bV_T}\right]
\end{aligned}$$

Therefore, $a' = a$ and $b' = b$.

3.4 Question 4

The Laplace transform identity states that:

$$\begin{aligned}
\sqrt{\frac{\pi}{z}} = \int_0^{\infty} e^{-sz}\frac{ds}{\sqrt{s}} &\leftrightarrow \int_0^x \sqrt{\frac{\pi}{z}}\,dx = \int_0^x \left(\int_0^{\infty} e^{-sz}\frac{ds}{\sqrt{s}}\right) dx \\
&\leftrightarrow \sqrt{\pi}\int_0^x \frac{1}{\sqrt{z}}\,dx = \int_0^{\infty}\left(\int_0^x e^{-sz}\,dx\right)\frac{ds}{\sqrt{s}} \\
&\leftrightarrow \sqrt{\pi}2\sqrt{x} = \int_0^{\infty}\frac{1}{s}\left(1-e^{-sx}\right)\frac{ds}{\sqrt{s}} \\
&\leftrightarrow 2\sqrt{\pi}\sqrt{x} = \int_0^{\infty}\left(1-e^{-sx}\right)\frac{ds}{\sqrt{s^3}}
\end{aligned}$$

Therefore, we can use the identity:

$$\sqrt{x} = \frac{1}{2\sqrt{\pi}}\int_0^{\infty}(1-e^{-sx})\frac{ds}{\sqrt{s^3}}$$

And apply it to the VIX futures price:

$$\begin{aligned}
f_t^{\text{VIX}}(T) &= \frac{100}{\sqrt{\eta}}\mathbb{E}_t^{\mathbb{Q}}[\sqrt{a'+b'V_T}] \\
&= \frac{100}{\sqrt{\eta}}\mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{2\sqrt{\pi}}\int_0^{\infty}\left(1-e^{-s(a'+b'V_T)}\right)\frac{ds}{\sqrt{s^3}}\right] \\
&= \frac{100}{\sqrt{\eta}}\frac{1}{2\sqrt{\pi}}\int_0^{\infty}\left(1-e^{-sa'}\mathbb{E}_t^{\mathbb{Q}}\left[e^{-sb'V_T}\right]\right)\frac{ds}{\sqrt{s^3}}
\end{aligned}$$

We know from Question 2 that:

$$\mathbb{E}_t^{\mathbb{Q}}[e^{-sb'V_T}] = e^{-c(T-t;sb')-d(T-t;sb')V_t}$$

Thus, the VIX futures price is:

$$\begin{aligned}
f_t^{\text{VIX}}(T) &= \frac{50}{\sqrt{\eta\pi}}\int_0^{\infty}\left(1-e^{-sa'}e^{-c(T-t;sb')-d(T-t;sb')V_t}\right)\frac{ds}{\sqrt{s^3}} \\
&= \frac{50}{\sqrt{\eta\pi}}\int_0^{\infty}\left(1-e^{-sa'-c(T-t;sb')-d(T-t;sb')V_t}\right)\frac{ds}{\sqrt{s^3}} \\
&= \frac{50}{\sqrt{\eta\pi}}\int_0^{\infty}\left(1-e^{-\ell(s,T-t,V_t)}\right)\frac{ds}{\sqrt{s^3}}
\end{aligned}$$

where $\ell(s,T-t,V_t) = sa' + c(T-t;sb') + d(T-t;sb')V_t$

3.5 Question 5

We know that the VIX index satisfies:

$$\left(\frac{\text{VIX}_t}{100}\right)^2 = \frac{1}{\eta}\,\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\eta} V_u\,du\right] \tag{20}$$

A variance futures contract maturing at $T = t + \eta$ has terminal payoff (12) and its time-t futures price is (13). For a newly listed contract ($t_0 = t$), this simplifies to:

$$f_t^{VA}(t+\eta) = \frac{10,000}{\eta}\,\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\eta} V_u\,du\right] \tag{21}$$

Comparing (20) and (21):

$$\left(\frac{\text{VIX}_t}{100}\right)^2 = \frac{1}{\eta}\,\mathbb{E}_t^{\mathbb{Q}}\left[\int_t^{t+\eta} V_u\,du\right] = \frac{f_t^{VA}(t+\eta)}{10,000}$$

Thus, we obtain the relationship:

$$\text{VIX}_t^2 = f_t^{VA}(t+\eta) \tag{22}$$

The table 2 presents the strategy to implement based on the relative value of VIX_t^2 and the variance futures price $f_t^{VA}(t+\eta)$.

Table 2: Variance-VIX Relative Value Strategies

$f_t^{VA} < \text{VIX}_t^2$			$f_t^{VA} > \text{VIX}_t^2$		
Action	**Asset**	**Cash-Flow**	**Action**	**Asset**	**Cash-Flow**
Long	VA Futures	Borrow at rate r	Short	VA Futures	Difference invested at rate r
Short	VIX Futures	Invest difference at rate r	Long	VIX Futures	Fund purchase at r

3.6 Question 6 - 7 - 8

See Notebook ***(also accesible on the github of the project :*** *`https://github.com/najabba/derivatives_project_2025.git`* ***)***

3.7 Question 9

To estimate the correlation parameter ρ between increments of the S&P 500 index returns and its instantaneous variance under the risk-neutral measure $\mathbb{Q}$, we start with:

$$\frac{dS_t}{S_t} = (r-\delta)dt + \sqrt{V_t}\,dB_t^{\mathbb{Q}},$$

The natural discretized returns consistent with this model are the simple percentage changes

$$r_i = \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}.$$

These returns reflect the continuous-time dynamics and capture the relationship between price increments and volatility changes. Using a calibrated sequence of instantaneous variances $\{V_{t_i}\}$ derived from option prices, we compute increments $\Delta V_i = V_{t_i} - V_{t_{i-1}}$ and estimate the correlation ρ as the sample correlation coefficient:

$$\rho = \frac{\text{Cov}(r_i, \Delta V_i)}{\sqrt{\text{Var}(r_i)\text{Var}(\Delta V_i)}} = \frac{\sum_{i=1}^n r_i \Delta V_i}{\sqrt{\sum_{i=1}^n r_i^2}\sqrt{\sum_{i=1}^n (\Delta V_i)^2}}$$

Therefore, to perform this calibration, we would require historical SPX price data to calculate returns and the model-implied instantaneous variance series calibrated from option data.

3.8 Question 10

Let $C = C(S_t, t)$ denote the price at time t of a European call option on the S&P 500 index with strike $K = S_0$ and maturity $T = 1$. Under the Black-Scholes model, the option price is given by

$$C(S_t, t) = S_t e^{-\delta(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

where

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \delta + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_2 = d_1 - \sigma\sqrt{T-t},$$

and σ is the implied volatility:

$$\sigma_t(T, K) = \alpha(t, T, V_t) + \beta(t, T, V_t)\left(\log\frac{K}{S_t}\right) + \gamma(t, T, V_t)\left(\log\frac{K}{S_t}\right)^2$$

Our goal is to derive formulas for the initial portfolio weights w_S and w_{VA} in SPX futures and variance futures, respectively.
The sensitivity of the call price to the underlying price, or Delta, is

$$w_S = \Delta = \frac{\partial C}{\partial S}(S_0, 0) = \frac{\partial}{\partial S_0}(S_0 e^{-\delta}\Phi(d_1) - Ke^{-r}\Phi(d_2)) = e^{-\delta}\Phi(d_1).$$

Since the option is at-the-money with $K = S_0$, we have $\ln(S_0/K) = 0$, which simplifies $\sigma_0 = \alpha(0, T, V_0) = \alpha$ and $d_1 = \frac{r-\delta+\frac{1}{2}\alpha^2}{\alpha}$.

Then, because the variance process V_t is not directly observable or tradable, we express the option's sensitivity in terms of the variance futures price $f_0^{\mathrm{VA}}(T)$, which is a tradable asset. Using the chain rule, this leads to the portfolio weight $w_{\mathrm{VA}} = \frac{\partial C}{\partial f_0^{\mathrm{VA}}(T)} = \frac{\partial C}{\partial V_0} \cdot \left(\frac{df_0^{\mathrm{VA}}(T)}{dV_0}\right)^{-1}$.
We know :

$$\begin{aligned}
f_0^{\mathrm{VA}}(T) &= \frac{10,000}{T - t_0}\left(\int_{t_0}^{0} V_u\,du + \theta\left(T - \frac{1-e^{-\lambda T}}{\lambda}\right) + V_0\frac{1-e^{-\lambda T}}{\lambda}\right) \\
&= \frac{10,000}{T}\left(\theta\left(T - \frac{1-e^{-\lambda T}}{\lambda}\right) + V_0\frac{1-e^{-\lambda T}}{\lambda}\right) \\
&\to \frac{df_0^{\mathrm{VA}}(T)}{dV_0} = 10,000\frac{1-e^{-\lambda}}{\lambda}
\end{aligned}$$

Using the chain rule as before, with $\sigma = \alpha$, we have:

$$\frac{\partial C}{\partial V_0} = \frac{\partial C}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial V_0}$$

where we define the Vega of the option as :

$$\text{Vega} := \frac{\partial C}{\partial \sigma} = S_0 e^{-\delta}(\frac{1}{2} - \frac{r-\delta}{\alpha^2})\,\phi(d_1) - S_0 e^{-r}(-\frac{1}{2} - \frac{r-\delta}{\alpha^2})\,\phi(d_2) +$$

Therefore, the final expression for the weight in variance futures is:

$$w_{\mathrm{VA}} = \text{Vega} \cdot \frac{\partial \alpha}{\partial V_0} \cdot \left(\frac{df_0^{\mathrm{VA}}(T)}{dV_0}\right)^{-1} = \text{Vega} \cdot \frac{\partial \alpha}{\partial V_0} \cdot \frac{\lambda}{10,000(1-e^{-\lambda})}$$

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