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A direct approximation technique of log magnitude response for digital filters

15 April 2014

Outline

- Introduction
- Cepstrum representation of log magnitude responses
- The elemental digital filter
- Design of digital filters realizing desired log magnitude responses
- Design example
- A filter for speech synthesis
- Conclusions

Introduction

- The technique for magnitude approximation uses the facts that
  - The log magnitude response of digital filters can be expanded into Fourier series
  - A fairly accurate cosine type log magnitude response can be realized by the elemental digital filter presented in this paper
- The system functions obtained by this method provide the best mean-square approximation to an arbitrarily prescribed log magnitude response

Cepstrum representation of log magnitude responses

- A desired log magnitude response S(f) must be real and even in order to be realized by a digital filter with real coefficients

Cepstrum representation of log magnitude responses

- A de … $S(f)$ must be re … zed by a digit

$$\hat{c}_m = \frac{1}{2F}\int_{-F}^{F} \hat{S}(f)\exp(j2\pi m f \Delta T)\, df$$

$$\hat{S}(f) = \sum_{m=-\infty}^{\infty} \hat{c}_m \exp(-j2\pi m f \Delta T).$$

Sampled:

$$\hat{S}_k = \hat{S}(2kF/N) = \sum_{m=-\infty}^{\infty} \hat{c}_m \exp(-j2\pi mk/N)$$

$$= \sum_{m=0}^{N-1} \hat{c}_m^p \exp(-j2\pi mk/N)$$

$$\hat{c}_m^p = \sum_{r=-\infty}^{\infty} \hat{c}_{m+rN}.$$

$$\hat{c}_m^p = \frac{1}{N}\sum_{k=0}^{N-1} \hat{S}_k \exp(j2\pi mk/N).$$

Cepstrum representation of log magnitude responses

- For all practical applications, the cepstrum C_m is of infinite extent
- In general, the cepstrum C_m decays very fast, so that it is to be expected that the aliasing would be arbitrarily reduced by increasing N
- Since the function S(f) is real and even, the sample S_k is symmetric, that is, $S_{N-k} = S_k$

$$\hat{S}_k = \sum_{m=0}^{[N/2]} \hat{C}_m \cos(2\pi mk/N)$$

$$\hat{C}_m = \begin{cases} \hat{c}_m^p, & (m = 0) \\ 2\hat{c}_m^p, & (1 \leqq m < [N/2]) \\ (1 + N - 2[N/2])\,\hat{c}_m^p, & (m = [N/2]). \end{cases}$$

The best mean-square approximation of desired log magnitude responses

- The log magnitude response S(f) of the digital filter with system function H(z) is given by

$$\tilde{S}_k = \ln|\tilde{H}[\exp(j2\pi k/N)]|^2.$$

Mean Squared Error:

$$\epsilon_A = \frac{1}{N}\sum_{k=0}^{N-1}(\tilde{S}_k - \hat{S}_k)^2.$$

From a fundamental result of Fourier series theory the partial sum of degree M of the desired log magnitude response provides the best mean-square approximation for a given value M

$$\tilde{S}_k = \sum_{m=0}^{M}\hat{C}_m\cos(2\pi mk/N), \quad (M \leq [N/2]).$$

$$\epsilon_A = \frac{1}{2}\sum_{m=M+1}^{[N/2]}\hat{C}_m^2.$$

The best mean-square approximation of desired log magnitude responses

- The ratio ε_R of the mean-squared error to the mean-squared value of the desired log magnitude response S_k is given by

$$\epsilon_R = \frac{\frac{1}{N}\sum_{k=0}^{N-1}(\tilde{S}_k - \hat{S}_k)^2}{\frac{1}{N}\sum_{k=0}^{N-1}\hat{S}_k^2} = \frac{\sum_{m=M+1}^{[N/2]}\hat{C}_m^2}{\sum_{m=0}^{[N/2]}\hat{C}_m^2}.$$

- The ratio ε_R would become very small for a sufficiently large value M since the cepstrum C_m generally decays very fast
- However, the amplitude error in the log magnitude response does not always decrease as M increase

The best mean-square approximation of desired log magnitude responses

- The squared magnitude response $H[\exp(j2\pi k/N)]^2$ of the optimal filter is given by

$$\begin{aligned}|\tilde{H}[\exp(j2\pi k/N)]|^2 &= \exp(\tilde{S}_k) \\ &= \exp\left(\sum_{m=0}^{M} \hat{C}_m \cos(2\pi mk/N)\right) \\ &= \prod_{m=0}^{M} \exp(\hat{C}_m \cos(2\pi mk/N)).\end{aligned}$$

- The resulting digital filter cannot be realized in a simple non-recursive form

The Elemental Digital Filter

- Let us assume that the digital filter with the desired log magnitude response is realized by connection in cascade M elemental filters
- Each with log magnitude response $C_m \cos(2\pi mk/N)$

The Elemental Digital Filter

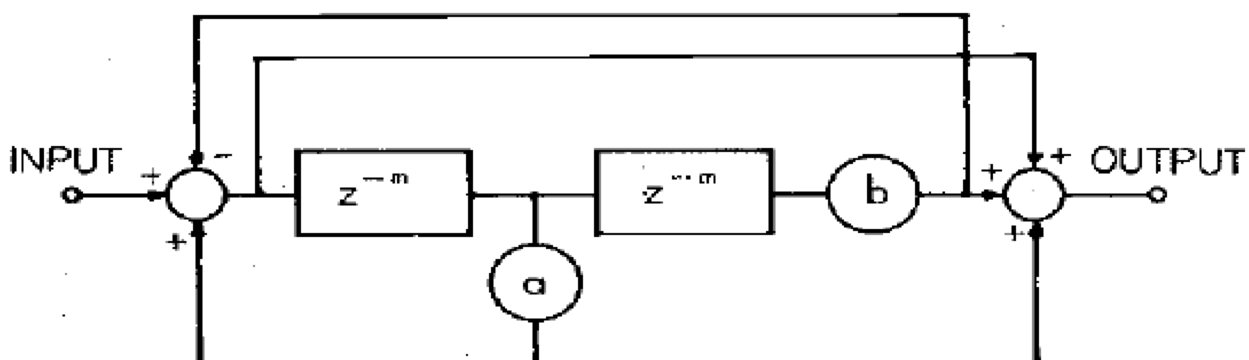
$$H_m(z) = \frac{1 + az^{-m} + bz^{-2m}}{1 - az^{-m} + bz^{-2m}}.$$

Squared Magnitude:

$$|H_m[\exp(j2\pi k/N)]|^2$$

$$= \frac{1 + \alpha\cos(2\pi mk/N) + \beta\cos^2(2\pi mk/N)}{1 - \alpha\cos(2\pi mk/N) + \beta\cos^2(2\pi mk/N)}$$

$$\alpha = \frac{2a(1+b)}{1 + a^2 + b^2 - 2b} \qquad \beta = \frac{4b}{1 + a^2 + b^2 - 2b}.$$



The Elemental Digital Filter

- The exponential function exp(x) can be approximated by the rational function

$$R'(x) = \frac{1 + A'x + B'x^2}{1 - A'x + B'x^2}, \qquad (|x| \leqq X)$$

$$E'(x) = (R'(x) - \exp(x))/\exp(x).$$

- For minimax approximation, choose the coefficients such that

The Elemental Digital Filter

$$A'(X) = A'|\min_{A',B'} \max_{|x| \leq X} |E'(x)| \qquad B'(X) = B'|\min_{A',B'} \max_{|x| \leq X} |E'(x)|.$$

$$\frac{dE'(x)}{dx} = \frac{[(2A'-1)+(A'^2-2B'-2A'B')x^2 - B'^2x^4]\exp(x)}{(1-A'x+B'x^2)^2}.$$

For minimizing the maximum of the absolute error $|E'(x)|$, all four roots $\pm x_1$, $\pm x_2 (0 < x_1 < x_2 < X)$ satisfying $dE'(x)/dx = 0$ must be real and they must satisfy the following equations:

$$E'(x_1) + E'(x_2) = 0 \qquad E'(x_2) + E'(X) = 0.$$

$$\bar{E}'(X) = \max_{|x| \leq X} |E'(x)|_{A'=A'(X),\, B'=B'(X)}.$$

TABLE I

COEFFICIENTS $A'(X)$ AND $B'(X)$ OF THE RATIONAL FUNCTION $R'(x)$ FOR MINIMAX APPROXIMATION AND THE MAXIMUM ABSOLUTE VALUE $\overline{E}'(x)$ OF THE APPROXIMATION ERROR

X	$A'(X)$	$B'(X)$	$\overline{E}'(X)$
0	0.500000	0.0833333	0
0.25	0.499999	0.0832249	$\approx 1 \times 10^{-7}$
0.50	0.499987	0.0828993	2.86×10^{-6}
0.75	0.499932	0.0823562	2.06×10^{-5}
1.00	0.499787	0.0815953	8.75×10^{-5}
1.25	0.499482	0.0806127	2.66×10^{-4}
1.50	0.498934	0.0794102	6.60×10^{-4}
1.75	0.498043	0.0779858	1.42×10^{-3}
2.00	0.496705	0.0763458	2.77×10^{-3}
2.25	0.494803	0.0744917	5.00×10^{-3}
2.50	0.492210	0.0724236	8.44×10^{-3}
2.75	0.488813	0.0701526	1.36×10^{-2}
3.00	0.484495	0.0676899	2.09×10^{-2}

The Elemental Digital Filter

- The coefficients A'(X) and B'(X) shown in table can be approximated respectively by the rational functions

$$A(X) = \frac{1 + X^2/48}{2(1 + X^2/48 + (X^2/48)^2)}, \qquad (X \leqq \bar{X})$$

$$B(X) = \frac{1}{12(1 + X^2/48 + (X^2/48)^2)}, \qquad (X \leqq \bar{X})$$

exp(x):

$$R(x) = R'(x)|_{A'=A(X),B'=B(X)}$$

$$= \frac{1 + A(X)x + B(X)x^2}{1 - A(X)x + B(X)x^2}, \qquad (|x| \leqq X)$$

Approximation Error

$$E(x) = E'(x)|_{R'(x)=R(x)} = (R(x) - \exp(x))/\exp(x).$$

Max absolute value:

$$\bar{E}(X) = (0.87X^5 + 0.01X^7) \times 10^{-4}, \qquad (X \lessapprox 3.1).$$

The Elemental Digital Filter

$$x = \hat{C}_m \cos(2\pi mk/N)$$

$$X = \max|x| = \hat{C}_m \quad (|\hat{C}_m| \leqq \bar{X})$$

$$R(\hat{C}_m \cos(2\pi mk/N)) = \frac{1 + A(|\hat{C}_m|)\hat{C}_m \cos(2\pi mk/N) + B(|\hat{C}_m|)\hat{C}_m^2 \cos(2\pi mk/N)}{1 - A(|\hat{C}_m|)\hat{C}_m \cos(2\pi mk/N) + B(|\hat{C}_m|)\hat{C}_m^2 \cos(2\pi mk/N)}$$

$$A(|\hat{C}_m|)\hat{C}_m = \alpha, \quad (|C_m| \leqq \bar{X}) \qquad B(|\hat{C}_m|)\hat{C}_m^2 = \beta, \quad (|C_m| \leqq \bar{X})$$

$$R(\hat{C}_m \cos(2\pi mk/N)) = |H_m[\exp(j2\pi k/N)]|^2$$

$$E(x) = (R(x) - \exp(x))/\exp(x).$$

$$\ln R(x) = x + \ln(1 + E(x)) \simeq x + E(x), \quad (|x| \leqq X \leqq \bar{X}).$$

The Log magnitude response:

$$S_{mk} = \ln|H_m[\exp(j2\pi k/N)]|^2 = \ln R(\hat{C}_m \cos(2\pi mk/N))$$

$$\simeq \hat{C}_m \cos(2\pi mk/N) + E(\hat{C}_m \cos(2\pi mk/N))$$

$$(|\hat{C}_m| \leqq \bar{X}).$$

The Elemental Digital Filter

- From the above equations we get the following equations

$$\frac{a(1+b)}{1+a^2+b^2-2b} = \frac{\hat{C}_m(1+\hat{C}_m^2/48)}{4(1+\hat{C}_m^2/48+(\hat{C}_m^2/48)^2)} \qquad \frac{b}{1+a^2+b^2-2b} = \frac{\hat{C}_m^2}{48(1+\hat{C}_m^2/48+(\hat{C}_m^2/48)^2)}.$$

- By inspection we get

$$a = \hat{C}_m/4, \quad (|\hat{C}_m| \leqq \bar{X}) \qquad b = \hat{C}_m^2/48, \quad (|\hat{C}_m| \leqq \bar{X})$$

- The elemental digital filter is stable if the coefficients a and b in its system function satisfy either the condition

$$1)\ |b| < 1 \quad \text{for} \quad a^2 < 4b \quad \textbf{Or} \quad 2)\ \left|\frac{a}{2} \pm \left[\left(\frac{a}{2}\right)^2 - b\right]^{1/2}\right| \quad \text{for} \quad a^2 \geqq 4b$$

The Elemental Digital Filter

If $|\hat{C}_m| \leqq \bar{X}$ and $\bar{X} \leqq \sqrt{48}$, the coefficients a and b satisfy the above condition 1).

Since the elemental filter is utilized in the variable range $\bar{X} < \sqrt{48}$ such that the maximum absolute error $\bar{E}(|\hat{C}_m|)$ is still very small

the resulting elemental digital filter is stable

Log Magnitude Response S_{mk}

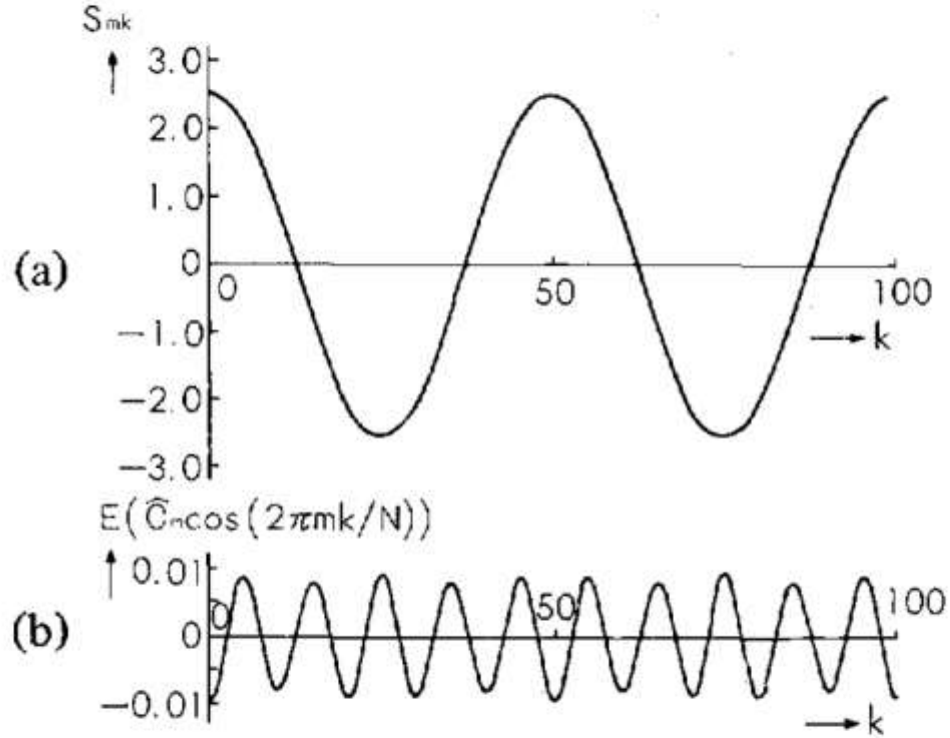


Fig. 2. (a) Log magnitude response S_{mk} of the elemental digital filter. (b) The corresponding approximation error $E(\hat{C}_m \cos(2\pi mk/N))$ ($m = 2$, $\hat{C}_m = 2.5$, $N = 100$).

Coefficients Sensitivities

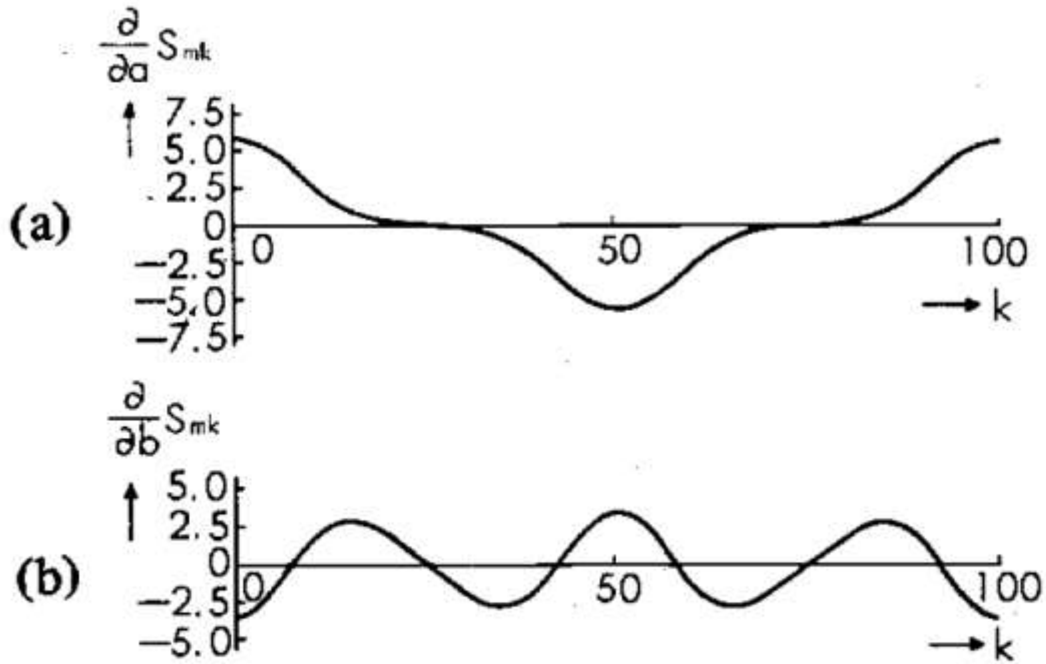


Fig. 3. Coefficient sensitivities of the log magnitude response of the elemental digital filter, $m = 1$, $\hat{C}_m = 3.0$, $N = 100$. (a) $\partial S_{mk}/\partial a$. (b) $\partial S_{mk}/\partial b$.

Response with Coefficients Error

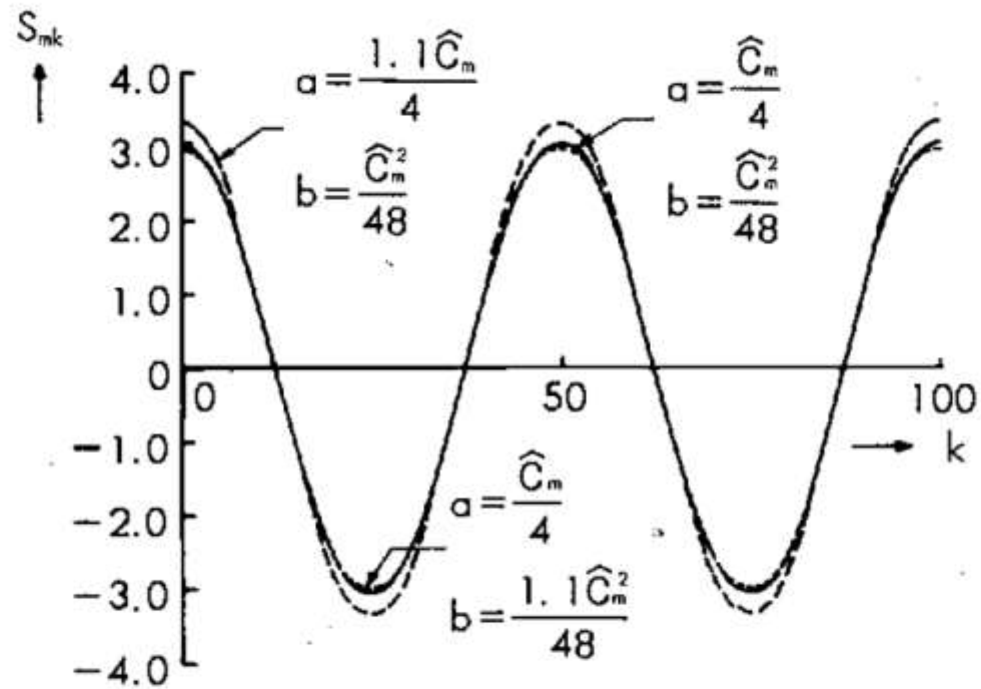


Fig. 4. Log magnitude responses of the elemental filter for $\hat{C}_m = 3.0$ when an error of ten percent occurs in the coefficient a or b, respectively.

The Max Absolute Coeff Sensitivities

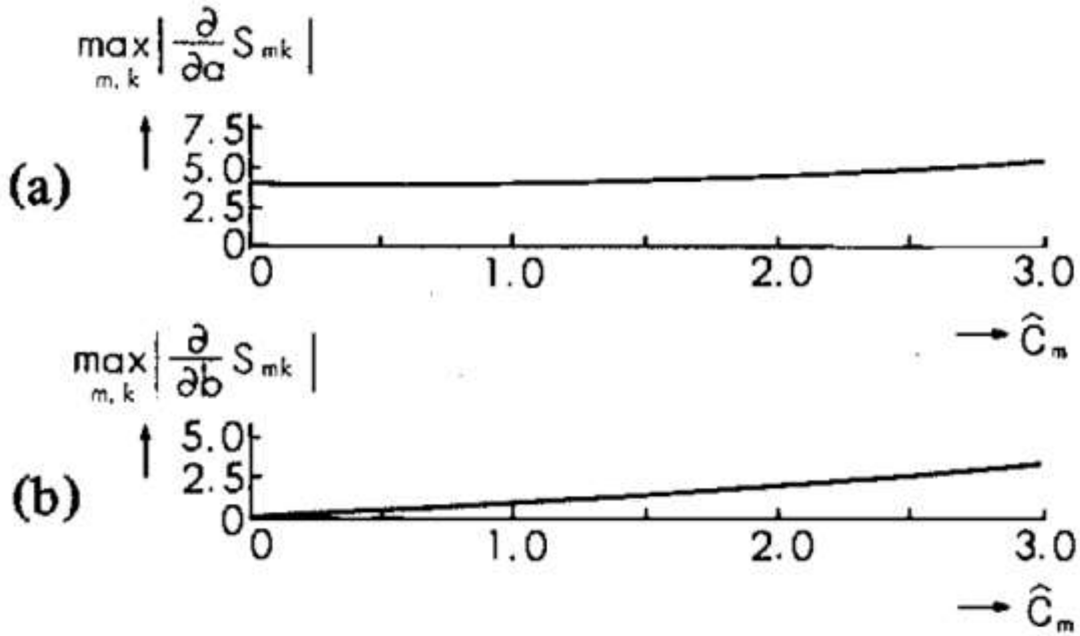


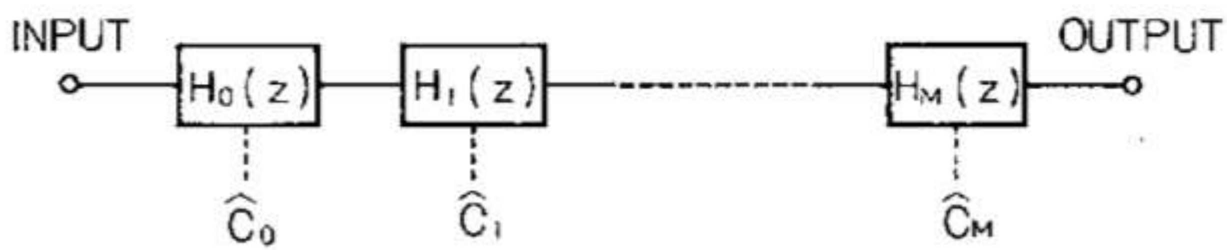
Fig. 5. The maximum absolute values of the coefficient sensitivities. (a) $\max_{m,k}|\partial S_{mk}/\partial a|$. (b) $\max_{m,k}|\partial S_{mk}/\partial b|$.

The Elemental Digital Filter

- The elemental digital filter is stable
- The numerator of its system function $H_m(z)$ has the same form as the denominator with the only difference in the sign of the coefficient a
- $H_m(Z)$ has all its poles and zeros inside the unit circle in the Z-plane
- The system function $H_m(z)$ is minimum phase
- The elemental digital filter has an accurate sine type phase response since it has an accurate cosine type log magnitude

Design of digital filters realizing desired log magnitude responses

- Desired log magnitude responses can be realized by the cascade connection of elemental digital filters with the coefficients expressed in terms of the cepstrum



Cascade digital filter realizing the desired log magnitude response.

Design of digital filters realizing desired log magnitude responses

$$H(z) = \prod_{m=0}^{M} H_m(z).$$

Log Magnitude Response:

$$S_k = \sum_{m=0}^{M} \ln |H_m[\exp(j2\pi k/N)]|^2 = \sum_{m=0}^{M} S_{mk}$$

$$= \sum_{m=0}^{M} \hat{C}_m \cos(2\pi mk/N) + \sum_{m=0}^{M} E(\hat{C}_m \cos(2\pi mk/N))$$

$$(|\hat{C}_m| \leqq \bar{X}).$$

Design of digital filters realizing desired log magnitude responses

- If the absolute value of cepstrum C_m is not too large, the error caused by the elemental digital filter for the approximation of cosine type log magnitude response is very small
- Therefore the approximation error is caused mainly by the Fourier series expansion of the desired log magnitude response
- As C_m increases, the approximation error increases because of the deviation of the filter from an ideal cosine type log magnitude response

Design of digital filters realizing desired log magnitude responses

- For large Cm we utilize the cascade form of digital filter with the coefficients corresponding to almost equally distributed values

$$\gamma_l \hat{C}_m (l = 1, 2, \cdots, L; \gamma_l \simeq 1/L, \sum_{l=1}^{L} \gamma_l = 1)$$

$$S'_{mk} = \sum_{l=1}^{L} \gamma_l \hat{C}_m \cos(2\pi mk/N) + \sum_{l=1}^{L} E(\gamma_l \hat{C}_m \cos(2\pi mk/N))$$

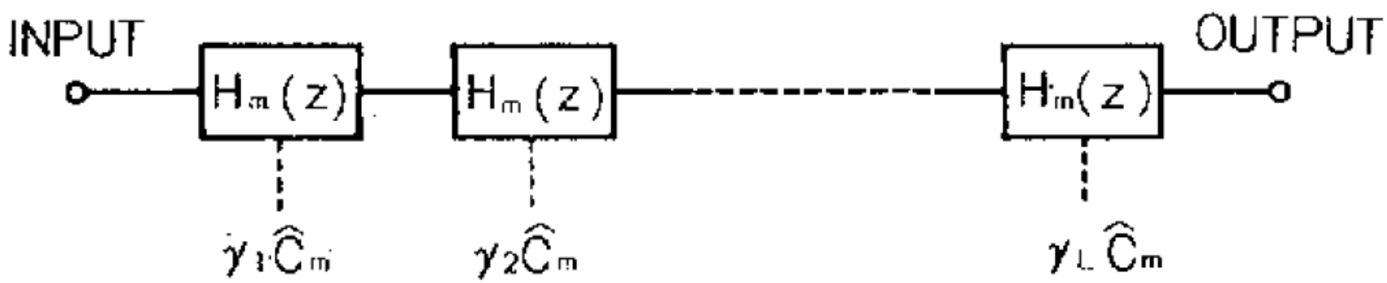
$$= \hat{C}_m \cos(2\pi mk/N) + \sum_{l=1}^{L} E(\gamma_l \hat{C}_m \cos(2\pi mk/N)),$$

$$(\gamma_l |\hat{C}_m| \leqq \bar{X}). \qquad (53)$$

Design of digital filters realizing desired log magnitude responses

- For large Cm we utilize the cascade form of digital filter with the coefficients corresponding to almost equally distributed values

$$\gamma_l \hat{C}_m (l = 1, 2, \cdots, L; \gamma_l \simeq 1/L, \sum^{L} \gamma_l = 1)$$



Cascade form elemental digital filter used for large cepstrum value $\hat{C}_m$.

$$(\gamma_l \, |C_m| \leqq X). \qquad (53)$$

Design Example

$$\hat{S}_k = \begin{cases} \ln 10^2 (0 \leqq k \leqq 63), \\ -\ln 10^2 (64 \leqq k \leqq 128), \end{cases} \qquad (N = 256).$$

CEPSTRUM $\hat{C}_m$ FOR A DESIRED LOG MAGNITUDE RESPONSE OF THE LOW-PASS FILTER

m	$\hat{C}_m$	m	$\hat{C}_m$
0	−0.0360	11	−0.5298
1	5.8632	12	−0.0720
2	0.0720	13	0.4472
3	−1.9536	14	0.0720
4	−0.0720	15	−0.3865
5	1.1712	16	−0.0720
6	0.0720	17	0.3399
7	−0.8356	18	0.0720
8	−0.0720	19	−0.3030
9	0.6488	20	−0.0720
10	0.0720	21	0.2730

Design Example

- 12 stages of elemental filters were cascaded
- For a large value C1 of the cepstrum, the cascade form elemental digital filter is used such that the equally distributed cepstrum values of the component filters are chosen

Design Example

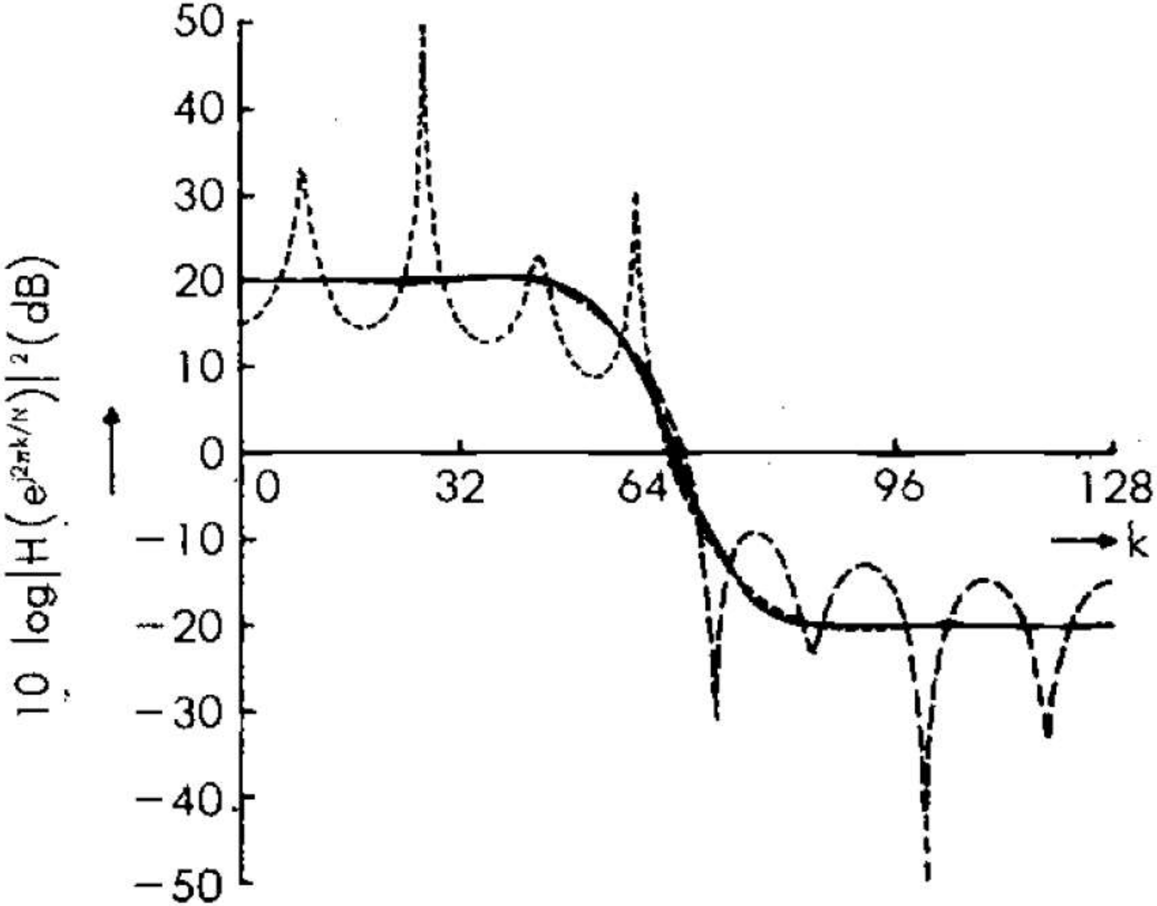


Fig. 8. Log magnitude responses of the low-pass filters obtained by this method (solid curve) and by Johnson's method (dashed curve: non-recursive, dotted curve: purely recursive).

A Filter for Speech Synthesis

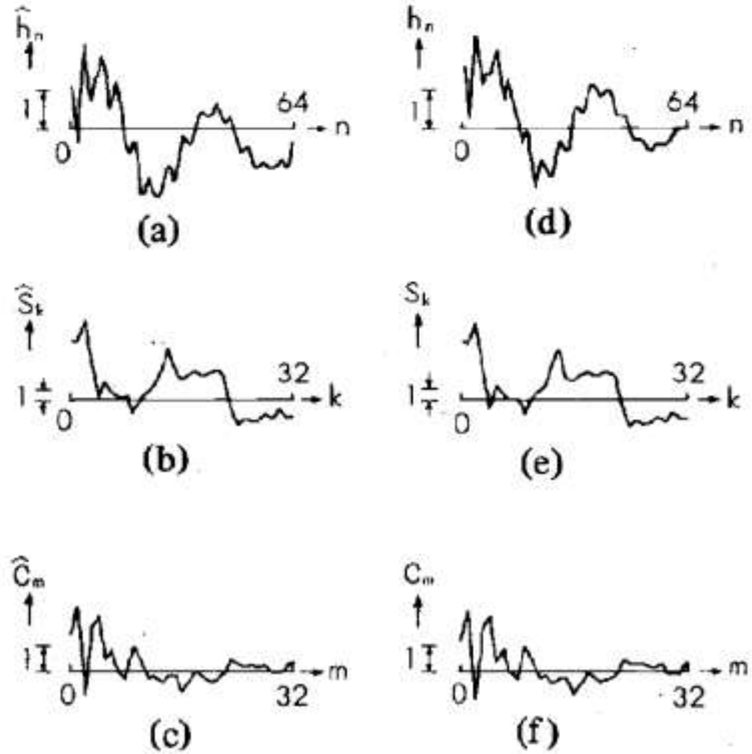


Fig. 9. Characteristics of the digital filter realizing the log magnitude response equal to the log spectrum of Japanese vowel /i/, pronounced like "i" in "ink". (a) Waveform $\hat{h}_n$ of /i/. (b) Log spectrum $\hat{S}_k$ of /i/. (c) Cepstrum $\hat{C}_m$ of /i/. (d) Impulse response h_n of the digital filter. (e) Log magnitude response S_k of the digital filter. (f) Cepstrum C_m of the impulse response.

Conclusion

- A direct approximation technique of an arbitrarily prescribed log magnitude response was presented
- The system functions of the resulting digital filters provide the best mean-square approximation to the arbitrarily prescribed log magnitude response
- The digital filters are realized by connecting elemental digital filters in cascade
- Its coefficients are easily obtained by the cepstrum of the impulse response which is the Fourier transform of the desired log magnitude response