Introduction to Abstract and Motivic Homotopy Theory

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Summer 2019

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Lecture 1

1.1 Introduction

Abstract and motivic homotopy theory. You get a special gift if you're the first to ask a mathematical question, so ask questions!

Totally agree with this email about the spirit of this course. Intervene, stop him, don't let him get carried away. Steer this course in any direction which suits you best. Not too determined about what has to happen in here. Quick or slower as you like.

Most of today (all of today) will be the abstract homotopy theory part. That's maybe a bit dry if you have zero motivation, so that's what we'll start.

1.2 A Picture of Homotopy Theory

A hexagon which will haunt you for the second part of the course.

Notations:

1. Top will denote the category of CW-complexes.

2. We have a pointed version Top_* of pointed CW-complexes, which is the comma category $*/Top$. The objects here are maps $* \to X$ and morphisms are triangles

$$\begin{tikzcd} & * \arrow[dl] \arrow[dr] & \\ X \arrow[rr] & & Y \end{tikzcd}$$

3. We'll let $I := [0,1] \subseteq \mathbb{R}$.

The best kind of invariant are cohomology theories.

Definition 1.2.1. A *reduced cohomology theory* is a functor

$$E : {Top_*}^{op} \to \mathbf{Ab}^{\mathbb{Z}}$$

(sequences of abelian groups), and we often write $X \mapsto E^*(X) = \bigoplus_{n\in\mathbb{Z}} E^n(X)$ such that

1. (*Homotopy Invariance*) $X \times I \xrightarrow{\text{pr}} X$ induces $E^*(X) \xrightarrow{\sim} E^*(X \times I)$

2. (*Suspension Axiom*) We have an isomorphism natural in X $E^n(X) \cong E^{n+1}(\Sigma X)$. This makes sense: the nth group is made out of the n-cells, n-dimensional spheres, and if you shift up by a suspension you get $n+1$-spheres.

   Remark 1.2.2. The *suspension* is defined as follows: if $X \ni p$ is a pointed space,
   $$\Sigma X := X \times [0,1]/(X \times \{0\} \cup X \times \{1\} \times \{p\} \times [0,1]).$$
   ◁

3. (*Wedge Axiom*)

   Remark 1.2.3. The wedge is defined as follows: If $X_i \ni p_i$ are pointed topological spaces, $i \in J$, then
   $$\bigvee_{i \in J} X_i := \coprod X_i/(p_i \sim p_j \,|\, i, j \in J)$$
   This is the coproduct in the category of pointed spaces. ◁

   The wedge axiom says that this coproduct should turn into a product. This one is induced by canonical maps:
   $$E^*(\bigvee_{i \in J} X_i) \cong \prod_{i \in J} E^*(X_i)$$

4. A 'good enough' inclusion $A \hookrightarrow X \to X/A$ should induce an exact sequence
   $$E^*(X/A) \to E^*(X) \to E^*(A).$$

◁

We don't include the cup product because there are cohomology theories without a ring structure. But it's a meaningful structure you can put on it.

Definition 1.2.4. A cohomology theory is a functor of the form $Top \to Top_* \xrightarrow{E} \mathbf{Ab}^{\mathbb{Z}}$ where E is a reduced cohomology theory (the functor $Top \to Top_*$ given by $X \mapsto X \sqcup \{*\}$). ◁

Example 1.2.5.

1. Singular cohomology H_{sing};
2. KU, KO complex/real topological K-theory (which measures vector bundles)
3. MU complex cobordism.

◁

Such cohomology theories factorize through a bunch of functors and categories along the way. We'll follow this way a bit.

The axioms tell us that singular cohomology factorizes as follows: the first axiom tells us that cohomology factors through localization.

It's a pointed category: there's an object which points to everything universally, namely the zero abelian group. Because it goes into a pointed category, cohomology must go through that too, so it factors through $\mathbf{Spaces}_*$.

Shifting (suspension) on the category of abelian sequences is an invertible endofunctor. It will factor through this spectra category because Σ is sent to an invertible endofunctor.

Axiom 3 tells us that E^* satisfies some properties which are satisfied by representable functors. Brown representability theory tells us that this is sort of enough. Namely, 3 and 4 tell us that cohomology theories are representable in the category of spectra.

Multiplication in our cohomology theory is often represented by multiplication on spectra. Some spectra have a monoidal structure, namely smash product $\wedge$. So we have $H\mathbb{Z} \wedge H\mathbb{Z} \to H\mathbb{Z}$ on the Eilenberg-MacLane spectrum, which gives cup product.

We can form, say, the category $H\mathbb{Z}$-Mod of modules: $H\mathbb{Z} \wedge M \to M$ where M is another spectra, with what you'd expect from a module.

Motivic homotopy theory mimics this picture here. We want to do the same for schemes. So we put algebraic varieties/schemes in the upper corner, because we want consider cohomology theories on algebraic varieties.

1.3 Localization of Categories

Let's get more detailed... and more boring.

Definition 1.3.1. A *relative category* is a pair $(\mathcal{C}, W$ where $\mathcal{C}$ is a category and $W \subseteq \operatorname{Mor} \mathcal{C}$. We want to understand the morphism in W are equivalences. $\lhd$

Example 1.3.2.

1. $\mathcal{C} = Top$, $W =$ homotopy equivalences (recall that Top consists of CW-complexes.)

2. $\mathcal{C} = \{\text{All topological spaces}\} = \mathbb{T}op$, $W =$ homotopy equivalences.

3. $\mathcal{C} = \mathbb{T}op$, $W =$ weak equivalences $= \{X \xrightarrow{f} Y \mid \pi_n(f) \text{ iso } \forall n \in \mathbb{N}_0\}$. Recall that
$$\pi_n(X) := [S^n, X]$$
(homotopy classes of base-point preserving continuous maps from the n-sphere to X. We've implicitly chosen a base-point.)

   A homotopy is a deformation of a map $f : X \to Y$ to another $g : X \to Y$. From another perspective, if $x \in X$, then there is a path between $f(x)$ and $g(x)$. Modding out $\mathbb{T}op(S^n, X)$ by homotopy equivalence, you get $[S^n, X]$.

   By definition $\pi_n(X)$ is corepresentable, and it turns out it's group-valued, but whatever it's functor. It measures n-dimensional holes, what kinds of holes are there and how many.

4. $\mathcal{C} = \mathbb{T}op$, $W = \{X \xrightarrow{f} Y \mid H_n(f) \text{ iso } \forall n \in \mathbb{N}_0\}$.

   (3) and (4) are of the form $\mathbb{C} \xrightarrow{F} \mathcal{D}$, $W := F^{-1}(\operatorname{Iso}(\mathcal{D})$.

5. R a ring, then $\mathbb{C} = \mathrm{Ch}_R =$ chain complexes of R-mod, and $W =$ quasi-isomorphisms (morphisms inducing isos on all H_n).

6. $\mathcal{C}$ any category, $W = \mathrm{Mor}\,\mathcal{C}$ (when we localize, it will be the groupoidification of $\mathcal{C}$).

7. $\mathcal{C} = \mathbf{Cat}$, $W =$ category equivalences.

8. $\mathcal{C} = \mathbf{Rings}$, $W = \{R \to S \mid - \otimes_R S \text{ induces equiv.} \operatorname{Mor}(R) \to \operatorname{Mor}(S)\}$ (induces an equivalence of module categories). Faithfully flat means this functor is fully faithful.

$\lhd$

Definition 1.3.3. A (/the) *localization* of the relative category $(\mathcal{C}, W)$ (or the localization of $\mathcal{C}$ w.r.t. W) is a functor

$$L : \mathcal{C} \to \mathcal{C}[W^{-1}]$$

such that

1. $L(W) \subseteq \mathrm{Iso}(\mathcal{C}[W^{-1}])$

2. It should be initial, the most economical way of doing this. Formally, for any $F : \mathcal{C} \to \mathcal{D}$ such that $F(W) \subseteq \mathrm{Iso}(\mathcal{D})$ there should exist up to natural isomorphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, "L"] \arrow[dr, "F"'] & \mathcal{C}[W^{-1}] \arrow[d, dotted, "\exists! \text{ up to na. iso}"] \\ & \mathcal{D}. \end{tikzcd}$$

A *strict* localization is the same without the natural isos in (b). It's an "evil" notion, in the sense that it's not invariant under equivalence of categories. $\lhd$

Exercise 1.3.4. Any category $\widetilde{\mathcal{C}[W^{-1}]}$ equivalent to a strict localization will be a localization but not in general strict (*Hint*: consider $\mathbb{C} = (a \to b$. Not much to localize. Can find a strict localization, and can find equivalent categories to this strict localization which are not strict.) $\lhd$

Remark 1.3.5. $\mathcal{C} \xrightarrow{L} \mathcal{C}[W^{-1}]$ is a strict localization iff the following is a pushout

$$\begin{tikzcd} \coprod_W (* \to *) \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "L"] \\ \coprod_W (* \xrightarrow{\sim} *) \arrow[r] & \mathcal{C}[W^{-1}]. \end{tikzcd}$$

See video for explanation. $\lhd$

1.3.1 Construction

$\mathrm{Ob}(\mathcal{C}[W^{-1}]) = \mathrm{Ob}\,\mathcal{C}$.

$\mathcal{C}[W^{-1}](A,B) := \{A \xleftarrow{\sim} X_1 \to X_2 \xleftarrow{\sim} X_3 \to \dots X_k \xleftarrow{\sim} B\}/\sim$ where the backwards arrows are only from W. We call this a zig-zag.

The $\sim$ should be the equivalence relation generated by the following relations:

1. We want to see the backwards morphisms as the inverses of the corresponding morphisms:

$$\begin{gathered} \cdots \rightleftarrows X_i \xleftarrow{f} X_{i+1} \xrightarrow{f} X_i \rightleftarrows \cdots \\ \wr \\ \cdots \rightleftarrows X_i = X_i \rightleftarrows \cdots \end{gathered}$$

2. The same for arrows going into X_{i+1}.

3. We should be able to compose:

$$\begin{gathered} \cdots \rightleftarrows X_i \xrightarrow{h} X_{i+1} \xrightarrow{k} X_{i+2} \rightleftarrows \cdots \\ \wr \\ \cdots \rightleftarrows X_i \xrightarrow{k \circ h} X_{i+2} \rightleftarrows \cdots \end{gathered}$$

   and same for arrows going backwards - for this we demand that W is closed under composition.

Composition is concatenation of zig-zags.

Exercise 1.3.6. This defines a category (possibly with $\mathcal{C}[W^{-1}](A,B)$ a class, not a set). We have a set-theoretic problem here. We usually demand a class of objects and a set of morphisms. Certainly before dividing by equivalence the collection of zig-zags is a proper class in general. Sometimes when you mod out by the equivalence relation you identify enough to get a set. $\lhd$

Proposition 1.3.7. *It is a strict localization.* $\lhd$

Proof. Define the morphism $\mathcal{C} \to \mathcal{C}[W^{-1}]$ by

$$X \xrightarrow{f} Y \mapsto \{X \xrightarrow{f} Y\}$$

where the latter is the zig-zag of length 1. It's pretty easy to show that it's universal (see the video). $\square$

It's an unwieldly and strange category we've constructed, a priori at least. The outcome of the construction is hard to control, in a way we'll make more precise. For example, this functor L needn't be full or faithful:

Say $\mathcal{C} = (\mathbb{N}, +)$, i.e. $\mathrm{Ob}\,\mathcal{C} = \{*\}$ and $(\mathcal{C}(*,*) = \mathbb{N})$. Composition is addition, and zero will be the identity. Let $W = \mathbb{N}$ (or equivalently $W = \{1\}$, since everything else is composition of that.) Then $\mathcal{C}[W^{-1}] = \mathbb{Z}$, so the morphism is not full.

Let $\mathcal{C}$ be the monoid $\{1, e\}$ with $e^2 = e$ and $W = \{e\}$. Then in $\mathcal{C}[W^{-1}]$ we have an inverse to e call it $[e^{-1}]$. Then $\mathrm{id} = [e] \circ [e^{-1}] = [e] \circ [e] \circ [e^{-1}] = [e]$, so the morphism is not faithful.

Other counterexample to faithfulness: If in $\mathcal{C}$ we have $s : b \to a$ and $f, g : a \to b$ such that $f \circ s = g \circ s$, then if $s \in W$ we have in $\mathcal{C}[W^{-1}]$:

$$[f] = [f] \circ [s] \circ [s]^{-1} = [g] \circ [s] \circ [s]^{-1} = [g]$$

So we can start forcing relations we didn't expect before.

Exercise 1.3.8. Let $\mathrm{Ho}\,Top$ denote the category with objects topological spaces and $\mathrm{Ho}\,Top(A, B) = \mathrm{Top}(A, B)/\text{homotopy}$. Show that $\mathrm{Ho}(Top) \cong \mathrm{Top}[(X \times I \xrightarrow{\mathrm{pr}} X)^{-1}]$. Use the counterexample above.

This category has a universal property: there's a functor $\mathrm{Top} \to \mathrm{Ho}\,Top$, and it's an intial functor among all functors which identify homotopical maps. Then you have to use the universal property to get maps both ways, and show that they compose to the identity. ◁

Exercise 1.3.9. Let $\mathcal{C}$ be a reflective subcategory of $\mathcal{D}$, i.e., a subcategory for which the inclusion $i : \mathcal{C} \to \mathcal{D}$ has a left-adjoint r.

Show that

$$\mathcal{C} \cong \mathcal{D}[r^{-1}(\mathrm{Iso}(\mathcal{C})^{-1}] \cong \mathcal{D}[(\text{units } d \to ir(d))^{-1}].$$

◁

Lecture 2

2.1 Interlude: Simplicial Sets

Definition 2.1.1. Δ is the category of (nonempty) finite linearly ordered sets with order-preserving morphisms. So $[n] = \{1 < 2 < \cdots < n\}$. $\lhd$

Example 2.1.2. Some particular morphisms: $\delta_i : [n] \to [n+1]$ which omits i. The other way around, $\sigma_i : [n+1] \to [n]$, $\sigma_i(i) = \sigma_i(i+1)$ and $\sigma_i(j) = j$ otherwise. So you merge two of them and leave everything else the same. $\lhd$

Every morphism in Δ is the composition of such morphisms.

Definition 2.1.3. A *simplicial set* is a functor $X : \Delta^{op} \to \mathbf{Set}$, say $[n] \mapsto X_n$.
s $\lhd$

We can specify such a functor by the images of its objects and the δ_i and σ_i. This can be drawn as a diagram $X_0, X_1, X_2, X_3, \ldots\ldots$ with the morphisms between them. The usual picture (see video).

[The category Δ has some relations between the face and boundary maps. Any category which has arrows which satisfy (the opposite of) these relations is a simplicial set.]

Definition 2.1.4. The *n-simplex* Δ^n is defined to be

$$\Delta^n := \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \,|\, 0 \leq x_i \leq 1, \sum x_i = 1\}.$$

[The equation defines a place, and the inequalities bound the plane.] $\lhd$

We think of the maps in the simplicial set as gluing data. It's a blueprint for gluing a space.

More formally:

Definition 2.1.5. Given a simplicial set $X_\bullet$ its *geometric realization* is defined to be

$$|X_\bullet| := \left(\coprod_{n \in \mathbb{N}_0} X_n \times \Delta^n \right) / \sim = \int^n X_n \times \Delta^n.$$

where $\sim$ identifies simplicies as prescribed by the "face maps". (It's a special kind of colimit.) $\lhd$

It's a combinatorial blueprint for making topological spaces. It's okay not to think of the degeneracy maps if you're imagining it as a way of constructing topological spaces.

Definition 2.1.6. Let Y be a topological space. Then we get a simplicial set

$$\mathrm{Sing}(Y)_n := \{\Delta^n \to Y \text{ continuous}\}$$

where the face maps $\mathrm{Sing}(Y)_n \to \mathrm{Sing}(Y)_{n-1} = \{\Delta^{n-1} \to Y\}$ are given by the restriction map to faces.

From this you construct a (co)complex

$$\mathrm{Sing}(Y)_\bullet = \mathbb{Z}\,\mathrm{Sing}_0(Y) \xleftarrow{d_0 - d_1} \mathbb{Z}\,\mathrm{Sing}_1 \xleftarrow{d_0 - d_1 + d_2} \mathbb{Z}\,\mathrm{Sing}_2(Y) \ldots$$

which level-wise is the free abelian group generated by $\mathrm{Sing}(Y)_n$. Your differentials are the alternating sum of these degeneracy maps. Then singular homology is the homology of this complex.

To get cohomology, you apply the contravariant functor $\mathrm{Hom}(-, \mathbb{Z})$ to the complex $\mathrm{Sing}(Y)_\bullet$ then take cohomology. $\lhd$

Fact: The geometric realization functor $|-|$ is left-adjoint to $\mathrm{Sing}(-)$. The counit $|\mathrm{Sing}(Y)_\bullet| \to Y$ is a weak-equivalence, i.e., it induces an isomorphism on the higher homotopy groups.

Explicitly

$$|\mathrm{Sing}(Y)_\bullet| = \left(\coprod_n \{\Delta^n \xrightarrow{f} Y\} \times \Delta^n \right) / \sim \to Y$$

is given by $\overline{(f, x)} \mapsto f(x)$. So every topological space is weak-equivalent to a CW-complex.

Fact: If $X_\bullet \in \mathbf{Set}^{\Delta^{op}}$ is a simplicial set, then the unit $X_\bullet \to \mathrm{Sing}(|X_\bullet|)$ is also a weak equivalence, if one defines a weak equivalence $X_\bullet \xrightarrow{f} X'_\bullet$ to be a weak equivalence $|X_\bullet| \xrightarrow{|f|} |X'_\bullet|$.

This is a shadow of the fact that these two things define the same homotopy theories. There's a more refined statement about these that we'll come to. Actually: $\mathrm{Top}[\text{weak eq.s}^{-1}] \cong \mathbf{Set}^{\Delta^{op}}[\text{weak eq.s}^{-1}]$ is an equivalence of categories. We can deduce this now, but maybe we can do it better after this lecture.

This is an example of how different relative categories can give the same categories after inversion, i.e., the "same" localizations.

That's why we can always use simplicial sets of instead of topological spaces - when we're working with weak homotopy, at least, we can use either category. The category of simplicial sets is much nicer, and in the category of topological spaces you have to do more work.

2.2 Derived Functors

Previously we had the general situation of a relative category $(\mathcal{C}, W)$, and we now know how to associate it a localization $\mathcal{C} \xrightarrow{L} \mathcal{C}[W^{-1}]$. In this category the notion of equivalence became an isomorphism, in the best way possible.

How to work in $\mathcal{C}[W^{-1}]$? The disappointment is that you can't work in there alone. For example, limits and colimits don't really work here, and there's a result that if you can take limits and colimits you were in the situation where it was a reflexive subcategory.

But usually we come from a category which is not so bad.

Given two localizations

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "L"] & \mathcal{D} \arrow[d, "L'"] \\
\mathcal{C}[W^{-1}] \arrow[r, dotted] & \mathcal{D}[V^{-1}].
\end{tikzcd}
$$

this dotted arrow exists if $L'(F(W)) \subseteq \operatorname{Iso}\mathcal{D}(V')$. But these categories don't know anything about their isomorphisms, so we can't expect this to happen in general.

Example 2.2.1. (Colimits) Let $\mathcal{C} = \text{Top}$. Pushout: glue along their boundary. We definitely want pushouts, gluing things along their boundaries, etc. Unfortunately the pushout functor does not respect equivalences - see video for a nice example. $\lhd$

Example 2.2.2. (Limits) See video. $\lhd$

Example 2.2.3. Let $\mathcal{C} = \mathbf{Cat}$. Consider a category as $s, t : \text{Mor} \to \text{Ob}$, which specifies the source and target of any morphism. It's cool but long. $\lhd$

Three examples of failure. What do we do about it? The remedy is derived functors.

Suppose we have a diagram

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "L"] & \mathcal{D} \arrow[d, "L'"] \\
\mathcal{C}[W^{-1}] & \mathcal{D}[V^{-1}].
\end{tikzcd}
$$

There's no functor making this commute. The answer is a Kan extensions. Left/right Kan extension of $L' \circ F$ along L. It's not obvious they exist, and they won't in general. [**See video**].

Suppose we have a full subcategory $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ on which F does preserve weak equivalences and such that $\widetilde{\mathcal{C}}[(W \cap \widetilde{\mathcal{C}})^{-1}] \cong \mathcal{C}[W^{-1}]$. So you add some isomorphisms and everything in $\mathcal{C}$ becomes isomorphic to one of these little objects. Then we can define

$$
\begin{tikzcd}
\mathcal{C} \arrow[r, "F|_{\widetilde{\mathcal{C}}}"] \arrow[d, "L"] & \mathcal{D} \arrow[d, "L'"] \\
\widetilde{\mathcal{C}}[W^{-1}] \cong \mathcal{C}[W^{-1}] \arrow[r, dotted, "\overline{F}"] & \mathcal{D}[V^{-1}].
\end{tikzcd}
$$

and the arrow *does* exist.

Theorem 2.2.4. *Suppose we have $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and $\widetilde{\mathcal{C}} \subseteq \mathcal{C}$ as above, a functor $Q : \mathcal{C} \to \widetilde{\mathcal{C}} \hookrightarrow \mathcal{C}$, and a natural transformation $\tau : Q \to \operatorname{Id}_{\mathcal{C}}$ which consists objectwise of weak equivalences. Also suppose W satisfies the 2-of-3 condition, i.e., if in a diagram*

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g \circ f"] & B \arrow[d, "g"] \\
& C
\end{tikzcd}
$$

two of the three are weak equivalences, then the third is a weak equivalence.

Then Q preserves equivalences by 2-of-3, so $F \circ Q$ preserves equivalences. So we have the following diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "\mathrm{Id}"] \arrow[r, bend right, "Q"'] \arrow[d, "L"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \arrow[d, "L'"] \\ \mathcal{C}[W^{-1}] \arrow[rr, dotted, "\mathrm{Ran}_L(L'\circ F)"] & & \mathcal{D}[V^{-1}] \end{tikzcd}$$

This is harmless for equivalences - if two of them are isomorphisms then the third should be too. We can enhance without changing the localization.

Proof. Exercise. Only do this if you're fond of abstract nonsense and categorical chasing. □

Example 2.2.5. Consider the colimit functor $\mathrm{Top}^{diag} \xrightarrow{\mathrm{colim}} \mathrm{Top}$. Fact: ths preserves weak equivalences of diagram of the form

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"] & B \\ C \end{tikzcd}$$

where A, B, C are CW-complexse and f and g are cofibrations, i.e., are inclusions of subspaces such that around the image there is a neighborhood which retracts onto it. (This is the "good inclusion" of the previous lecture.) ◁

Fact: Every map in Top factors as a cofibration followed by a weak equivalence. Namely, denote by Mf the mapping cylinder $Mf := (X \times I \cup Y)/(x, 0) \sim f(x)$. See paper.

Given this other fact, we have a recipe for the derived functor of pushout, called homotopy pushout. (***)

Going back to the previous example (****)

What we did here was a right Kan extension for homotopy, which is a bit surprising.

Likewise for the derived limit we replace maps by *fibrations*, i.e. we have a homotopy lifting property (see video).

There are factorization systems on many categories. This has been done in abstract homotopy theory. For example, on **Cat** with W = category equivalences, cofibrations are functors which are injective on objects, and fibrations are functors which are $\mathcal{C} \to \mathcal{D}$ inducing surjections $\mathrm{Iso}(\mathcal{C}) \to \mathrm{Iso}(\mathcal{D})$ whose source lies in $\mathrm{Iso}(\mathcal{C})$.

Exercise 2.2.6. Find factorizations for these notions in the above style. Think again about the pushout

$$\begin{tikzcd} \coprod_W (* \to *) \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "L"] \\ \coprod_W (* \xrightarrow{\sim} *) \arrow[r] & \mathcal{C}[W^{-1}]. \end{tikzcd}$$

We can factor the upper arrow so that it's actually an inclusion on objects and... that should tell us something. $\triangleleft$

Example 2.2.7. (Homotopy Quotients) Let G be a topological group. Then we can form the category $G - \mathrm{Top}$ of topological spaces with a continuous G-action and G-equivariant continuous map. Let V be weak equivalences of the underlying spaces (ignoring the G-action). We have the quotient functor

$$G\text{-Top} \xrightarrow{-/G} \mathrm{Top}\,.$$

We know this can go bad morally in general. We can invert these G-equivalences:

$$\begin{tikzcd} G\text{-Top} \arrow[r, "-/G"] \arrow[d] & \mathrm{Top} \arrow[d] \\ G\text{-Top}[V^{-1}] & \mathrm{Top}[W^{-1}] \end{tikzcd}$$

As before, quotienting out the G-action does not respect weak equivalences. For example, let $S^\infty = \mathrm{colim}(S^0 \hookrightarrow S^1 \hookrightarrow S^2 \hookrightarrow \dots)$. (5*) The outcome will be a contractible space.

$\mathbb{Z}/2$ acts on both spaces; on S^∞ by multiplying coordinates by ± 1. Then quotient is left exact (?) so it commutes with colimits (??) so

$$\begin{aligned} S^\infty/(\mathbb{Z}/2) &= \mathrm{colim}(S^0/(\mathbb{Z}/2)) \hookrightarrow S^1/(\mathbb{Z}/2) \hookrightarrow \dots) \\ &= \mathrm{colim}(* \to \mathbb{RP}^1 \to \mathbb{RP}^2 \to \dots) \\ &= \mathbb{RP}^\infty \end{aligned}$$

But $\pi_1(\mathbb{RP}^\infty) = \mathbb{Z}/2 \neq 0 = \pi_1(*) = \pi_1(*/(\mathbb{Z}/2)$, so these spaces are not weakly equivalent.

So we have to derive the quotient. First we need a statement along the lines that any G-action can be replaced by a "good" G-action.

Fact: $-/G$ respects equivalences between spaces with *free* G-action (it really throws everything around - if an element has a fixed point it has to be the identity).

So given any G-space, need to replace it by an equivalent space with a free G-action, then take the quotient.

Consider X as the constant simplicial object (X at every level, all maps identity), and consider the simplicial space

$$\cdots \to G \times G \times G \times X \to G \times G \times X \to G \times X$$

(6*) Then hocolim $\widetilde{X_\bullet}$ is a good replacement. $\triangleleft$