CM1015 Computational Mathematics Mid-term coursework

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Question1

(a) A given number in base x can be converted to any other base y.
According to the expansion method, if **abc.de** is any given number in base x, write its value in base 10.

Solution:

The number **abc.de** in base x is expanded using positional values:
$ax^2+bx+c+dx^{-1}+ex^{-2}$ (base 10)

(b)
(i) Convert 10.00112 to decimal

Solution:

Integer part: $10_2=1*2^1+0*2^0=2$
Fractional part: $0.0011_2=1*2^{-3}+1*2^{-4}=1/8+1/16=3/16=0.1875$

Therefore, $10.0011_2=2.1875$.

(ii) Place values of the digits 1 in (0.0011_2)

Solution:

Binary fractional place values:

- 1st place: 2^{-1} = $1/2$
- 2nd place: 2^{-2} = $1/4$
- 3rd place: 2^{-3} = $1/8$
- 4th place: 2^{-4} = $1/16$

Thus the place values of the 1s are: $1/8, 1/16$

(iii) What is the sum of (1+1+1+1) in the binary system

Solution:

$1+1+1+1=4_{10}=100_2$

(iv) Calculate 101 divided by 10 using long division in base 2.

Solution:

$101_2 \div 10_2=10.1_2$

long division is following.

```
        10.1
     ---------
10 ) 101
      10
      ---
        01
        00
        ---
        01
        00
```

10
10

0

(c) Find the value of x from the following equation, if all numbers are in base 2. Express your final answer in base 2.

$$\frac{11_2}{x}=\frac{1000_2}{x+101_2}$$

<u>Solution:</u>
$11_2=3_{10}, 1000_2=8_{10}, 101_2=5_{10}$

$$\frac{11_2}{x}=\frac{1000_2}{x+101_2}$$
$$\frac{3}{x}=\frac{8}{x+5}$$
$$3(x+5)=8x$$
$$3x+15=8x$$
$$15=5x$$
$$x=3$$
$$x=11_2$$

(d) In some base b, the following equation holds: $AB_b+BA_b=121_{10}$
Where A and B are digits (A ≠ B). Find the base b, and the values of A and B.

<u>Solution:</u>
$AB_b=A*b+B$ and $BA_b=B*b+A$

Adding them gives:
$b*(A+B)+(A+B)$.

Therefore, the equation becomes:
$(A+B)*(b+1)=121$.

Case 1: A+B=1, b=120
Here A and B must be digits in base 120.
If we restrict digits to 0≤A, B≤9 (as is usual in this course),
the only solutions with A =B and A+B=1 are:

or (A, B)=(0,1) or (1,0).

Indeed,

$01_{120}=1, 10_{120}=120, 1+120=121.$

So (b, A, B)=(120, 0, 1) or (120, 1, 0) are valid solutions in a very large base.

Case 2: A+B=11, b=10

Now we are in ordinary decimal (base 10).
Digits satisfy $0 \le A, B \le 9$ and $A \neq B$.
We need

$A+B=11$.

The distinct digit pairs (A, B) with sum 11 are:

(2,9), (3,8), (4,7), (5,6)

and their reverses

(9,2), (8,3), (7,4), (6,5).

For example, with (A, B)=(2,9):

$AB_{10}=29, BA_{10}=92, 29+92=121$.

Thus, in base 10 we have:

- $b=10$,
- A, B are any distinct decimal digits such that $A+B=11$,
  i.e. $(A, B) \in \{(2, 9), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3), (9, 2)\}$.

The final answer is the combination of Case 1 and Case 2.

Question2

(a) The sum culture of bacteria doubles every 3 hours. Initially, there are 200 bacteria. After 12 hours, a sterilising agent is applied that instantly kills 70% of the bacteria. The remaining bacteria continue to grow at the same rate (doubling every 3 hours). After an additional 36 hours, a second identical sterilising agent is applied, again killing 70% of the bacteria instantly.
How many bacteria are present immediately after the second sterilisation?

Solution:

Growth factor: doubling every 3 hours $\rightarrow$ in time t hours, factor is $2^{t/3}$.

1. First 12 hours

Number of doublings: 12/3 = 4.

$N_1 = 200 * 2^4 = 200 * 16 = 32000.$

2. First sterilisation (kills 70%)

30% remain: $N_2 = 0.3 * 32000 = 960.$

3. Next 36 hours

Number of doublings: $36/3 = 12.$

$N_3 = 960 * 2^{12} = 960 * 4096 = 3{,}932{,}160.$

4. Second sterilisation (kills 70%)

30% remain: $N_4 = 0.3 * 3{,}932{,}160 = 1{,}179{,}648.$

Answer (a): There are **1,179,648 bacteria** immediately after the second sterilisation.

(b) A geometric sequence has a first term of 4 and a common ratio of 3. The sum of the first n terms are 364.

(i) Find the number of terms n.

Solution:

First term a=4, common ratio r=3, sum $S_n = 364$.

The sum of the first terms of a geometric sequence is

$$S_n = a\frac{(r^n - 1)}{r-1}$$
$$364 = 4\frac{(3^n - 1)}{3-1}$$
$$364 = 2(3^n - 1)$$
$$3^n = 183$$

$3^4 = 81 < 183 < 243 = 3^5$

Therefore3^n can never equal 183 for any integer n.

Hence no such geometric sequence exists.

(ii) If possible, find the last term of the sequence.

Solution:

Since there is no valid integer n, **the last term does not exist / cannot be found.**

Note: If the sum were 484 instead of 364, then n = 5 and the last term would be 4 ×

3^4**= 324.**

(c) The 2nd, 4th, and 6th terms of a geometric progression form an arithmetic progression. If the first term is positive and the common ratio is not 1, find the possible values of the common ratio.

Solution:

Let the geometric progression have first term a>0 and common ratio $r \neq 1$.

Terms:

- 2nd term: $a * r$
- 4th term: $a * r^3$
- 6th term: $a * r^5$

These form an arithmetic progression, so the middle term is the average of the other two:

$$2ar^3 = ar + ar^5$$
$$2r^3 = r + r^5$$
$$r^5 - 2r^3 + r = 0$$
$$r(r^2 - 1)^2 = 0$$
$$r = 0 \vee r = 1 \vee r = -1$$

Thus the possible solutions are r=0, r=1, r=−1.

The question states that the common ratio is **not 1**, so we discard r=1.

Depending on the convention, some courses also exclude r=0 for a geometric progression, but mathematically both r=0 and r=−1 satisfy the condition.

For this midterm, the intended answer is most likely:

r=−1 (and possibly r=0 if 0 is allowed).

(d) Ben buys 10 books from an online store. The price of the first book is £3, and the price of each subsequent book doubles. However, after the 6th book, he gets a 15% discount on the price of each additional book. How much does Ben pay for all 10 books?

Solution:

First 6 books (no discount)

These are 6 terms of a geometric sequence:

$$S_6 = 3\frac{(2^6 - 1)}{2 - 1}$$
$$S_6 = 3 \cdot 63 = 189$$

Last 4 books (with 15% discount)

Base prices for books 7–10 form another geometric sequence:

- First of this block:$a_7 = 3 * 26 = 192$,
- Ratio r = 2,
- Number of terms: 4.

$$S_4 = 192\frac{(2^4 - 1)}{2 - 1}$$
$$S_4 = 192 \cdot 15 = 2880$$

Each of these books gets a **15% discount**, so Ben pays only **85%** of this amount:

$0.85 * 2880 = 2448$.

Total cost: 189+2448=2637.

Answer (d): Ben pays **£2,637** in total.

Question3

(a) Find the smallest positive integer x that satisfies the following system:

$x \equiv 5 \left(mod\, 6\right)$

$x \equiv 3 \left(mod\, 7\right)$

Solution:

From the first congruence: $x = 5 + 6k$.

Substitute into the second congruence:

$5 + 6k \equiv 3 \left(mod\, 7\right)$

$6k \equiv -2 \equiv 5 \left(mod\, 7\right)$

Since $6 \equiv -1 \left(mod\, 7\right)$, $-k \equiv 5 \left(mod\, 7\right) \Rightarrow k \equiv -5 \equiv 2 \left(mod\, 7\right)$.

Use the smallest k=2: $x = 5 + 6 \cdot 2 = 17$.

Answer (a): x=17.

(b) A 24-hour clock shows 14:00 now. What time will it show after 224 hours?

Solution:

We compute the remainder modulo 24:

$224\, mod\, 24 = 8$.

So 224 hours later is the same as **8 hours later**.

14:00+8h=22:00.

Answer (b): 22:00.

(c) A number leaves a remainder 1 when divided by 2, 2 when divided by 3, and 3 when divided by 4 … and 9 when divided by 10. What is the smallest positive integer that satisfies all these?

Solution:

The conditions are:

$$x \equiv 1\, mod\, 2$$
$$x \equiv 2\, mod\, 3$$
$$\cdots$$
$$x \equiv 9\, mod\, 10$$

Notice that:

$1 \equiv -1 \left(mod\, 2\right), 2 \equiv -1 \left(mod\, 3\right), 3 \equiv -1 \left(mod\, 4\right), \ldots, 9 \equiv -1 \left(mod\, 10\right)$.

Thus **all conditions are the same** as:

$x \equiv -1 \left(mod\, m\right) for\, m = 2,3, \ldots, 10$.

Equivalently:

$x+1 \equiv 0 (mod\, m)$.

So **x+1 must be divisible by every number from 2 to 10**.

Compute the least common multiple:

$lcm(2,3,4,5,6,7,8,9,10)=2520$.

Therefore:

$x+1=2520 \Rightarrow x=2519$.

Answer (c): x=2519.

(d) Find all integers x satisfying $6x \equiv 15 (mod\, 21)$.

Solution:

First compute:

$gcd(6,21)=3$.

Because 3 divides 15, **solutions exist**.

Divide the whole congruence by 3: $2x \equiv 5 (mod\, 7)$.

Find the inverse of 2 modulo 7: $2 \cdot 4=8 \equiv 1 (mod\, 7)$,

so $2^{-1}=4$.

Multiply both sides by 4: $x \equiv 4 \cdot 5 \equiv 20 \equiv 6 (mod\, 7)$.

Thus: $x=6+7k (k \in Z)$.

If we want distinct solutions modulo 21, plug in k=0, 1, 2:

x=6, 13, 20

These are all distinct modulo 21.

Answer (d): x=6+7k, i.e., x=6, 13, 20(mod21).

Question4

(a) A triangle ABC has side lengths a=7, b=8, and angle C=x.

We follow the standard convention:
a, b are the sides adjacent to angle C, and c is opposite angle C.

(i) Write an equation involving x using the Law of Cosines.

Solution:

The Law of Cosines says: $c^2=a^2+b^2-2*a*b*cosC$.

Here a=7, b=8, C=x. So: $c^2=7^2+8^2-2*7*8*cosx$.

(ii) Solve for x if c=9.

Solution:

Substitute c=9: $9^2=7^2+8^2-2*7*8*cosx$.

Compute: $81=49+64-112*cosx=113-112*cosx$.

Rearrange: $112*cosx=113-81=32 \Rightarrow cosx=32/112=2/7$.

(iii) Find the angle C in degrees to 2 decimal places.

Solution:

$x=\cos^{-1}(2/7)\approx 73.40°$

(b) In triangle ABC, you are given AB=10, AC=13, Area=30.
Find angle A using the formula: $Area=1/2*a*b*sinC$

Solution:

Here angle A is between sides AB and AC, so we can use:

$Area=1/2*AB*AC*sinA$

Substitute: $30=1/2 \cdot 10 \cdot 13 \cdot sinA=65 sinA$.

So, $sinA=30/65=6/13$

Take inverse sine: $A=\sin^{-1}(6/13)\approx 27.49°$

(c) Solve the equation $\sin 3x=sinx$ in the interval 0◦≤x≤360◦.

Solution:

Start from: $\sin 3x=sinx \Rightarrow \sin 3x-sinx=0$

Use the identity: $\sin 3x-\sin x=2\cos\frac{(3x+x)}{2}\sin\frac{(3x-x)}{2}=2\cos(2x)\sin x$

So, $2\cos(2x)sinx=0 \Rightarrow sinx=0 \vee \cos 2x=0$

Case 1: $sinx=0$

ln $0° \leqq x \leqq 360°$: $x=0°, 180°, 360°$

Case 2: $\cos 2x=0$

$\cos 2x=0 \Rightarrow 2x=90°+180°*k \Rightarrow x=45°+90°*k$

For integer k giving $0° \leqq x \leqq 360°$:

k=0,1,2,3: $x=45°, 135°, 225°, 315°$

Combine all distinct solutions: $x=0°, 45°, 135°, 180°, 225°, 315°, 360°$

(d) Solve for all values of $x \in [0^\circ, 360^\circ]$: $sinx + \sin 2x + \sin 3x = 0$.

Solution:

We first group $sinx$ and $\sin 3x$: $sinx + \sin 3x = 2\sin 2x * cosx$

(using the sum formula $\sin A + \sin B = 2\sin A + \frac{B}{2}\cos A - \frac{B}{2}$)

So $sinx + \sin 2x + \sin 3x = 2\sin 2x * cosx + \sin 2x = \sin 2x * (2cosx + 1)$

Therefore the equation becomes: $\sin 2x * (2cosx + 1) = 0$

So either

1. $\sin 2x = 0$

2. $2\cos x + 1 = 0 (i.e. \cos x = \frac{-1}{2})$

Case 1: $\sin 2x = 0$

$\sin 2x = 0 \Rightarrow 2x = 180°k \Rightarrow x = 90°k$

For $0^\circ \le x \le 360^\circ$, k=0,1,2,3,4: $x = 0°, 90°, 180°, 270°, 360°$

Case 2: $cosx = -1/2$

On $0^\circ \le x \le 360^\circ$, we know $cosx = -1/2 \Rightarrow x = 120°, 240°$

Combine all distinct solutions: $x = 0°, 90°, 120°, 180°, 240°, 270°, 360°$

Question5

(a)

i. An object is moving at constant speed. Which statement MUST be true? Show your work.

Solution:

A constant **speed** does **not** imply constant **velocity**, because velocity includes direction.
For example, an object moving in uniform circular motion has constant speed but its direction changes, and its acceleration is non-zero.

Therefore, **none** of the statements 1–3 must always be true.

Correct answer: 5 (None of the above).

ii. A tennis player tosses a tennis ball straight up in the air. If *a* is the acceleration of the ball, and *v* is its velocity, which statement is true when the ball reaches the highest point of its trajectory? Show your work.

Solution:

At the highest point:

- The velocity is **zero** (instantaneously).
- The acceleration is **not zero** (gravity acts downward at $a = -9.8\, m/s^2$).

Correct answer: 2 (Only v is zero and a is not).

iii. When a stone is thrown directly upwards with an initial velocity of $30.0\, m/s$, what will be the maximum height it will reach, and when will it be? Acceleration due to gravity is $10\, m/s^2$. Show your work

Solution:

$v = u + at \Rightarrow 0 = 30 - 10t \Rightarrow t = 3[s]$

$v2 = u2 + 2as \Rightarrow 0 = 30^2 - 20h \Rightarrow h = 45[m]$

Correct answer: 1 (45 m in 3s)

(b) Consider the function given by $f(x) = y = \frac{x-3}{x^2+9x-22}$

i. State the domain of the function

Solution:

Denominator zero when: $x^2 + 9x - 22 = (x-2)(x+11)$

So x≠2, x≠−11.

Domain:ℝ \ {−11, 2}

ii. State the range of the function

Solution:

Treat $y = \frac{x-3}{x^2+9x-22}$ as quadratic in x.

Multiply both sides by the denominator: $y(x^2 + 9x - 22) = x - 3$.

Expand: $yx^2 + 9yx - 22y = x - 3$.

Bring all terms to one side: $yx^2 + (9y-1)x + (3-22y) = 0$.

This is a **quadratic equation in x**.
A real x exists exactly when the discriminant is ≥ 0.

The discriminant: $D(y) = (9y-1)^2 - 4y(3-22y)$.

Compute:

$(9y-1)^2 = 81y^2 - 18y + 1$

So, $D(y)=169y^2-30y+1$

We need $D(y)\geq 0$

The roots are: $y=\frac{15\pm 2\sqrt{14}}{169}$

Thus the range is: Range = { $y\in\mathbb{R}$ | $y\leq\frac{15-2\sqrt{14}}{169}$ or $y\geq\frac{15+2\sqrt{14}}{169}$}

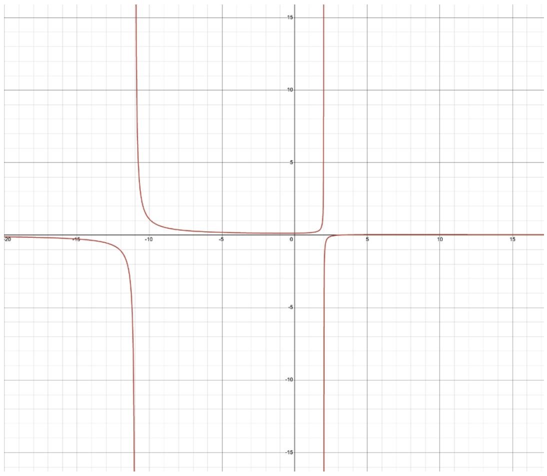
iii. Plot the graph of the function

Solution:

Important features:

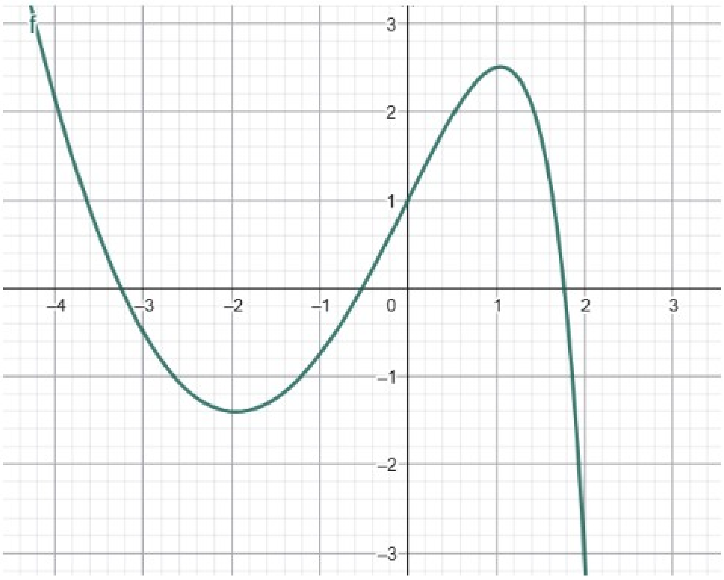
- Vertical asymptotes: x=2, −11 ($x^2+9x-22=(x-2)(x+11)$)
- Horizontal asymptote: y=0
- x-intercept: (3, 0)
- y-intercept: (0, 3/22)

Based on the above, plotting the graph yields the following result.



plotted by: https://www.desmos.com/calculator

(c) From the following plot of the function f(x), say whether it is one-to-one or not. Is it onto or not? Find the domain and the range of the function.



Solution:

One-to-one?
No. The graph fails the horizontal line test: several y-levels intersect the graph more than once.

Onto?
Not onto R. The function covers only a bounded set of y-values.

Domain (approx.):

[−4.5, 2.2]

Range (approx.):

[−1.4, 2.4]

(Any close approximation is acceptable since the graph is hand-read.)