

Discrete Mathematics

Tutorial sheet

Topic-9

Relations

Question 1.

Let \mathcal{S} be a set and \mathcal{R} be a relation on \mathcal{S} . Explain what it means (you are expected to give mathematical definitions.) to say that \mathcal{R} is:

1. reflexive;
2. symmetric;
3. anti-symmetric;
4. transitive;
5. an equivalence relation;
6. a partial order.

In each case give an example of a relation which has the given property and another relation which does not have it.

Question 2.

Let $S = \{a, b, c\}$ and $A = \{(c, c), (a, b), (b, b), (b, c), (c, b)\}$.

Define a relation R on S by “ x is related to y whenever $(x, y) \in A$ ”.

1. Draw the relationship digraph.
2. The relation R is not reflexive. What pair (x, y) should be added to A to make R reflexive?
3. The relation R is not symmetric. What pair (x, y) should be added to A to make R symmetric?
4. The relation R is not anti-symmetric. What pair (x, y) should be removed to make R anti-symmetric?
5. The relation R is not transitive. What pair (x, y) should be added to A to make R transitive?

Question 3.

The following relations are defined on a set $S = \{a, b, c\}$.

R_1 is the relation given by $\{(a,a), (a,b), (a,c), (b,a), (b,b), (c,a), (c,c)\}$

R_2 is given by $\{(a,a), (a,b), (b,a), (b,b), ((c,c))\}$

R_3 is given by $\{(a,b), (a,c), (b,a), (b,c), (c,a), (c,b)\}$

R_4 is given by $\{(a,a), (a,b), (a,c), ((b,b), (b,c), (c,c))\}$

Complete the table below. If the relation is an equivalence relation give the equivalence classes. Also state whether any of the relations is a partial order, justifying your answer.

	reflexive	symmetric	antisymmetric	transitive	equivalence rel.	
\mathcal{R}_1						
\mathcal{R}_2						
\mathcal{R}_3						
\mathcal{R}_4						

Question 4.

Let $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and let \mathcal{P} be the partition on \mathcal{S} given by

$$\{\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}\}.$$

Define \mathcal{R} to be the equivalence relation associated to \mathcal{P} .

1. Give two conditions for \mathcal{P} to be a partition.
2. Draw the relationship digraph.
3. Write down the equivalence class $[5]$ as a set.

Question 5.

Let $S = \mathbb{Z} \times \mathbb{N}^+$ and Let \mathbb{R} be relation on S defined as follows:

$$(a, b) \mathcal{R} (c, d) \text{ whenever } ad = bc$$

1. Show that \mathcal{R} is an equivalence relation
2. Define the equivalence class generated by (a, b) , for $a \in \mathbb{Z}$ and $b \in \mathbb{N}^+$

Question 6.

Let A and B be two sets where:

$A = \{\text{France, Germany, Switzerland, England, Morocco}\}$ and

$B = \{\text{French, German, English, Arabic}\}$. Let \mathcal{R} be relation defined from A to B , given by $a\mathcal{R}b$ when b is a national language of a . The national language

of each of these countries is as follows: French for France, German for Germany, English for England, Arabic for Morocco, whereas, Switzerland has two national languages, French and German. Find the logical matrix for the relation \mathcal{R} .

Question 7.

For each of the following relations on the set of all people, state if it is an equivalence relation. Explain your answer.

1. $\mathcal{R}_1 = \{(x, y) | x \text{ and } y \text{ are the same height}\}$.
2. $\mathcal{R}_2 = \{(x, y) | x \text{ and } y \text{ have, at some time, lived in the same country}\}$.
3. $\mathcal{R}_3 = \{(x, y) | x \text{ and } y \text{ have the same first name}\}$.
4. $\mathcal{R}_4 = \{(x, y) | x \text{ is taller than } y\}$.
5. $\mathcal{R}_5 = \{(x, y) | x \text{ and } y \text{ have the same colour hair}\}$.

Question 8.

Let $\mathcal{S} = \{\{1\}, \{1, 2\}, \{1, 2, 3\}, \{1, 2, 3, 4\}, \{1, 2, 3, 4, 5\}\}$. Define a relation \mathcal{R} between the elements of \mathcal{S} by

X is related to Y whenever $X \subseteq Y$.

1. Draw the relationship digraph.
2. Determine whether or not \mathcal{R} is reflexive, symmetric, antisymmetric or transitive. Give a brief justification for each of your answers.
3. State, with reasons, whether or not \mathcal{R} is an equivalence relation, whether or not it is a partial order and whether or not it is a total order.

Question 9.

Let $\mathcal{S} = \{a, b, c, d\}$ and let $A \subseteq \mathcal{S} \times \mathcal{S}$ be given by

$$\{(a, a), (a, c), (b, b), (b, d), (c, a), (c, c), (d, b), (d, d)\}.$$

A relation \mathcal{R} on \mathcal{S} is defined by

x is related to y whenever $(x, y) \in A$.

1. Draw the relationship digraph.

2. Determine whether or not \mathcal{R} is reflexive, symmetric, antisymmetric or transitive, giving a brief justification for your answer.
3. State, with reasons, whether or not \mathcal{R} is an equivalence relation, whether or not it is a partial order and whether or not it is a total order.

Question 10.

Let \mathcal{R} be a relation from a set A to a set B . The inverse of \mathcal{R} , denoted \mathcal{R}^{-1} , is the relation from B to A defined by $\mathcal{R}^{-1} = \{(y, x) : (x, y) \in \mathcal{R}\}$.

Given a relation \mathcal{R} from $A = \{2, 3, 4\}$ to $B = \{3, 4, 5, 6, 7\}$ defined by $(x, y) \in \mathcal{R}$ if x divides y .

1. List the elements of \mathcal{R} and write down the matrix, $M_{\mathcal{R}}$, of \mathcal{R} .
2. List the elements of \mathcal{R}^{-1} and write down the matrix, $M_{\mathcal{R}^{-1}}$, of \mathcal{R} .

Question 11.

Let \mathcal{R}_1 and \mathcal{R}_2 be the relations on a set $S = \{1, 2, 3, 4\}$ given by:

$\mathcal{R}_1 = \{(1, 1), (1, 2), (3, 4), (4, 2), (2, 4)\}$ $\mathcal{R}_2 = \{(1, 1), (3, 2), (4, 4), (2, 2), (4, 2)\}$.

1. Find the matrix representation \mathcal{R}_1 and that of \mathcal{R}_2 .
2. Find the matrix of the intersection of both matrices in (1).
3. Find the matrix of the union both matrices in (1).
4. list the element of $\mathcal{R}_1 \cap \mathcal{R}_2$.
5. list the element of $\mathcal{R}_1 \cup \mathcal{R}_2$.

Question 12.

Let \mathcal{R} be a relation on set A .

1. How can we quickly determine whether a relation \mathcal{R} is reflexive by examining the matrix of \mathcal{R} ?
2. How can we quickly determine whether a relation \mathcal{R} is symmetric by examining the matrix of \mathcal{R} ?
3. How can we quickly determine whether a relation \mathcal{R} is anti-symmetric by examining the matrix of \mathcal{R} ?

Question 13.

For each of following relations on a set $A = \{a, b, c\}$ defined by their corresponding Matrices, say whether it is reflexive, symmetric or anti-symmetric.

$$1. M_{\mathcal{R}_1} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

$$2. M_{\mathcal{R}_2} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

$$3. M_{\mathcal{R}_3} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{matrix}$$

$$4. M_{\mathcal{R}_4} = \begin{matrix} & \begin{matrix} a & b & c \end{matrix} \\ \begin{matrix} a \\ b \\ c \end{matrix} & \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \end{matrix}$$

Question 14.

Let $A = \{1, 2, \dots, 10\}$ and let \mathcal{R} be a relation on A defined by $x\mathcal{R}y$ if 3 divides $x - y$.

1. Show that the relation \mathcal{R} is an equivalence relation on A .
2. List all the equivalence classes of \mathcal{R} .

Question 15.

Let $A = \{1, 2, \dots, 10\}$ and let \mathcal{R} be a relation on A defined by $x\mathcal{R}y$ if $x \bmod 2 = y \bmod 2$.

1. Show that the relation \mathcal{R} is an equivalence relation on A .
2. List all the equivalence classes of \mathcal{R} .
3. is \mathcal{R} a partial or a total order?

Question 16.

Let $A = \{1, 2, \dots, 10\}$ and let \mathcal{R} be a relation on A defined by $x\mathcal{R}y$ if $x + y \bmod 2 = 0$.

1. Show that the relation \mathcal{R} is an equivalence relation on A .

2. List all the equivalence classes of \mathcal{R} .

3. is \mathcal{R} a partial or a total order?

Question 17.

Let \mathcal{R} be a relation on the set $A = \{1, 2, 3, 4, 5\}$ defined by the rule $x\mathcal{R}y$ if $x = y - 1$. Is this relation reflexive, symmetric, antisymmetric, transitive, equivalence, and/or a partial order?

Question 18.

Let $A = \{1, 2, \dots, 10\}$ and let \mathcal{R} be a relation on $A \times A$ defined by $(a, b)\mathcal{R}(c, d)$ if $a + d = b + c$. Show that \mathcal{R} is an equivalence relation on $A \times A$.

Question 19.

Let $X = \{1, 2, 3, 4\}$, $Y = \{3, 4\}$, and $C = \{1, 3\}$ and let \mathcal{R} be a relation on $\mathcal{P}(X)$, the set of all subsets of X , defined as

$$\forall A, B \in \mathcal{P}(X), \quad A\mathcal{R}B \text{ if } A \cup Y = B \cup Y$$

1. Show that \mathcal{R} is an equivalence relation.

2. List the elements of $[C]$, the equivalence class containing C .