Discrete Mathematics

Tutorial sheet

Predicate Logic

Question 1.
Let $P(x)$ be the predicate "$x^2 > x$" with the domain the set $\mathbb{R}$ of all real numbers. Write $P(2)$, $P(\frac{1}{2})$, and $P(-\frac{1}{2})$ and indicate which of these statements are true and which are false.
Solution:

- P(2): $2^2 > 2$ or $4 > 2$ is true
- $P(\frac{1}{2}):\ (\frac{1}{2})^2 > \frac{1}{2}$ or $\frac{1}{4} > \frac{1}{2}$ is false
- $P(-\frac{1}{2}):\ (-\frac{1}{2})^2 > -\frac{1}{2}$ or $\frac{1}{4} > -\frac{1}{2}$ is true

Question 2.
Let $P(x)$ be the predicate "$x^2 > x$" with the domain the set $\mathbb{R}$ of all real numbers. What are the values $P(2) \wedge P(\frac{1}{2})$, and $P(2) \vee P(\frac{1}{2})$?
Solution:

- $P(2) \wedge P(\frac{1}{2}):\ (2^2 > 2) \wedge\ ((\frac{1}{2})^2 > \frac{1}{2}) = (4 > 2) \wedge (\frac{1}{4} > \frac{1}{2}) = T \wedge F = F$
- $P(2) \vee P(\frac{1}{2}):\ (2^2 > 2) \vee\ ((\frac{1}{2})^2 > \frac{1}{2}) = (4 > 2) \vee (\frac{1}{4} > \frac{1}{2}) = T \vee F = T$

Question 3.

1. Let $D = \{1, 2, 3, 4\}$, and consider the following statement:

$$\forall x \in D, x^2 \geq x.$$

   Write one way to read this statement, and show that is it true.

2. Show that the following statement is false.

$$\forall x \in \mathbb{R}, x^2 \geq x$$

Solution:

1. "For every x in the set D, x^2 is greater than or equal to x". The inequalities below show that that " $x^2 \geq x$" is true for each individual x in D.

$$1^2 \geq 1, 2^2 \geq 2, 3^2 \geq 3, 4^2 \geq 4$$

   Hence, "$\forall x \in D, x^2 \geq x$" is true.

2. the statement claims that $x^2 \geq x$ for every real number x, however, this is not true as for $x = \frac{1}{2}$ for example $(\frac{1}{2})^2 = \frac{1}{4} \ngeq \frac{1}{2}$.
   Hence, "$\forall x \in \mathbb{R}, x^2 \geq x$" is false.

Question 4.

1. Consider the following statement:

   $$\exists n \in \mathbb{Z}^+ \text{ such that } n^2 = n$$

   Write one way to read this statement, and show that is it true.

2. Let $E = \{5, 6, 7, 8\}$, and consider the following statement:

   $$\exists n \in E, n^2 = n.$$

   Show that this statement is false.

Solution:

1. There exists at least one positive integer n such that $n^2 = n$. 1 is positive integer and $1^=1$. Thus $n^2 = n$ is true for a positive integer. Hence, "$\exists n \in E, n^2 = n$" is true.
2. $5^2 = 25 \neq 5, 6^2 = 36 \neq 6, 7^2 = 49 \neq 7, 8^2 = 64 \neq 8$. Thus "$\exists n \in E, n^2 = n$" is false.

Question 5.
Rewrite each of the following statements formally, Use quantifiers and variables.

1. All triangles have three sides.
2. No dogs have wings.
3. Some programs are structured.

Solution:

1. All triangles have three sides: $\forall$ triangle t, t has three sides.
   Or, $\forall t \in T$, t has three sides (where T is set of all triangles)
2. No dogs have wings: $\forall$ dog d, d does not have wings.
   Or, $\forall d \in D$, d does not have wings (where D is set of all dogs).
3. Some programs are structured: $\exists$ a program p such such that p is structured
   Or: $\exists p \in P, p$ is structured (where P is the set of all programs).

Question 6.
Rewrite the following statements in form of $\forall$_______ if ______ then ______

1. If a real number is an integer, then it is a rational number
2. All bytes have eight bits
3. No fire trucks are green

Solution:

1. If a real number is an integer, then it is a rational number: $\forall$ real number x, if x is an integer, then x is a rational number
   Or: $\forall x \in \mathbb{R}$, if $x \in \mathbb{Z}$ then $x \in \mathbb{Q}$.
2. All bytes have eight bits: $\forall x$, if x is a byte, then x has eight bits.
3. No fire trucks are green: $\forall x$, if x is a fire truck, then x is not green.
   it is common for (1) and (2) above, to omit explicit identification of the domain of the predicate variables in universal conditional statements.

Question 7.
A **prime number** is an integer greater than 1 whose only positive integer factors are itself and 1. Consider the following predicate **Prime**(n): "n is is prime " and **Even**(n): "n is even". Use the notation **Prime**(n) and **Even**(n) to rewrite the following statement:

"There is an integer that is both prime and even "

Solution:
The statement "There is an integer that is both prime and even " can be written in two ways
$\exists n$ such that $Prime(n) \wedge Even(n)$
or
$\exists$ an even number n such that $Prime(n)$

Question 8.
Determine the truth value each of the following where $P(x, y) : y < x^2$, where x and y are real numbers:

1. $(\forall x)(\forall y)P(x, y)$
2. $(\exists x)(\exists y)P(x, y)$
3. $(\forall y)(\exists x)P(x, y)$
4. $(\exists x)(\forall y)P(x, y)$

Solution:

1. $(\forall x)(\forall y)P(x,y)$: this is false as there exists, $x, y \in \mathbb{R}$ where $x = 2$, and $y = 5$ such that $P(2,5)$ is false.
2. $(\exists x)(\exists y)P(x,y)$: this true as there exists x=2 and y = 3 for example such that $P(2,3) = 3 < 2^2$ is true.
3. $(\forall y)(\exists x)P(x,y)$ for all $y \in\ mathbbR$ there is exists $x = 2|\sqrt{|y|}|$ with $x^2 = 4|y| > y$. this is true.
4. $(\exists x)(\forall y)P(x,y)$ this is false as there exists $x, y \in \mathbb{R}$ where $x = 1$ and $y = 5$ such that $P(1,5)$ is false as $5 > 1^2$

Question 9.
Let $P(x)$ denote the statement x is taking discrete mathematics course. The domain of discourse is the set of all students. Write each of the following statements in words.

$$\forall x P(x), \quad \forall x \neg P(x), \quad \neg(\forall x P(x)), \quad \exists x P(x), \quad \exists x \neg P(x), \quad \neg(\exists x P(x)).$$

Solution:
$\forall x P(x)$: every students is taking discrete mathematics course.
$\forall x \neg P(x)$ every student is not taking discrete mathematics course.
$\neg(\forall x P(x))$ some student is not taking discrete mathematics course.
$\exists x P(x)$ some student is taking discrete mathematics course.
$\exists x \neg P(x)$ some student is not taking discrete mathematics course.
$\neg(\exists x P(x))$ every student is not taking mathematics course.

Question 10.
Let $P(x)$ denote the statement 'x is a professional athlete', and let $Q(x)$ denote the statement 'x plays football'. The domain of discourse is the set of all people. Write each of the following in words.

1. $\forall x(P(x) \to Q(x))$
2. $\exists x(Q(x) \to P(x))$
3. $\forall x(P(x) \wedge Q(x))$

Solution:

1. $\forall x(P(x) \to Q(x))$: every professional athlete plays football.
2. $\exists x(Q(x) \to P(x))$: either someone does not play football or some football player is a professional athlete.
3. $\forall x(P(x) \wedge Q(x))$: everyone is a professional athlete and plays football.

Question 11.
Let $P(x)$ denote the statement 'x is a professional athlete', and let $Q(x)$ denote the statement 'x plays football'. The domain of dis- course is the set of all people. Write the negation of each proposition symbolically and in words.

1. $\forall x(P(x) \to Q(x))$
2. $\exists x(Q(x) \to P(x))$
3. $\forall x(P(x) \wedge Q(x))$

Solution:

1. $\forall x(P(x) \to Q(x))$: its negation is $\exists x \neg(P(x) \to Q(x)) = \exists x(P(x) \wedge \neg Q(x))$:
2. $\exists x(Q(x) \to P(x))$: its negation is $\forall x \neg(Q(x) \to P(x)) = \forall x(Q(x) \wedge \neg P(x))$:
3. $\forall x(P(x) \wedge Q(x))$: its negation is $\exists x \neg(P(x) \wedge Q(x)) = \exists x(\neg P(x) \vee \neg Q(x))$:

Question 12.
Let P and Q denote the following propositional functions:

- $P(x)$: "x is greater than 2"
- $Q(x)$: "x^2 is greater than 4 "

where, the universe of discourse for both $P(x)$ and $Q(x)$ is the set of real number, $\mathbb{R}$.

1. Use quantifiers and logical operators to write the following statement formally

   " if a real number is greater 2, then its square is greater than 4."

2. Write a formal and informal contrapositive, converse and inverse of the statement above in (1).

Solution:

1. " if a real number is greater 2, then its square is greater than 4" can be written formally as $\forall x(P(x)) \to Q(x))$.
2. The contrapositive of "if a real number is greater 2, then its square is greater than 4" is the statement "if the square of a real number is less or equal to 4 then the number is less or equal to 2" . this can be written using quantifiers as $\forall x(\neg Q(x)) \to \neg P(x))$.

3. The converse of " if a real number is greater 2, then its square is greater than 4" is the statement " if the square of a real number is greater than 4, then the number is greater than 2" is the statement $\forall x(Q(x(\rightarrow P(x))$.

   The inverse of " if a real number is greater 2, then its square is greater than 4" is the statement " if a real number is less or equal to 2 , then its square is less or equal 4" is the statement $\forall x(\neg(P) \rightarrow \neg Q(x))$.

Question 13.

1. Rewrite each of the following statements in English as simply as possible without using the symbols $\forall$ or $\exists$ or variables.
   (a) $\forall$ color c, $\exists$ an animal a such that a is colored c.
   (b) $\exists$ a book b such that $\forall$ person p, p has read b.
   (c) $\forall$ odd integer n, $\exists$ an integer k such that $n = 2k + 1$.
   (d) . $\forall$ $x \in \mathbb{R}$, $\exists$ a real number y such that $x + y = 0$.
2. Write a negation for each of the statements above.

Solution:

1. (a) $\forall$ color c, $\exists$ an animal a such that a is colored c. This can written as "For every color, there is an animal of that color".
   (b) $\exists$ a book b such that $\forall$ person p, p has read b. This can written as "There is a book that every person has read".
   (c) $\forall$ odd integer n, $\exists$ an integer k such that $n = 2k + 1$. This can be written as "For every odd number n, we can find an integer k with $n = 2k + 1$".
   (d) $\forall$ $x \in \mathbb{R}$, $\exists$ a real number y such that $x + y = 0$. This can be written as " Given any real, we can find another real number (possibly the same) such that the sum of both numbers is equal to 0".
2. (a) $\exists$ a color c, $\forall$ animal a, a is **NOT** colored c.
   (b) $\forall$ book b, $\forall$ a person p, p has **NOT** read b.
   (c) $\exists$ an odd integer n, such that $\forall$ integer k, $n \neq 2k + 1$.
   (d) . $\exists$ $x \in \mathbb{R}$, such that $\forall$ real number y, $x + y \neq 0$.

Question 14.

Rewrite the statement "No good cars are cheap " in the form "$\forall x$,if $P(x)$ then $\neg Q(x)$". Indicate whether each of the following arguments is valid or invalid, and justify your answers.

1. No good care are cheap
   A Ferrari is a good car
   $\therefore$ A Ferrari is not cheap

2. No good cars are cheap
   A BMW is not cheap
   $\therefore$ A BMW is no a good car

Solution:

$\forall x$, if x is a good car, then x is **NOT** cheap.

1. No good cars are cheap
   A Ferrari is a good car
   $\therefore$ A Ferrari is not cheap

   This is a valid argument, universal modus or universal instantiation.

2. No good cars are cheap
   A BMW is not cheap
   $\therefore$ A BMW is not a good car

   This is invalid, converse error.

Question 15.

Let x be any student and C(x), B(x) and P(x) be the following statements:
$C(x)$: "x is in this class".
$B(x)$: "x has read the book".
$P(x)$: "x has passed the first exam".
Rewrite the following symbolically and state whether it a valid argument.
A student in this class has not read the book
Everyone in this class passed the first exam
$\therefore$ Someone who passed the first exam has not read the book
Solution:

(1) $\exists x(C(x) \wedge \neg B(x))$
(2) $\forall x(C(x) \rightarrow P(x))$
$\therefore \exists x(C(x) \rightarrow \neg B(x))$
This a valid argument!.

End of questions