

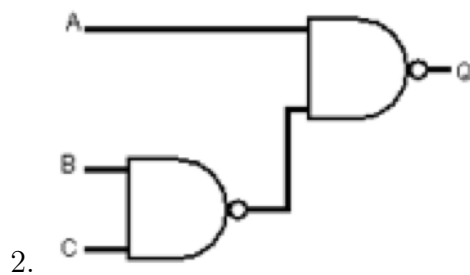
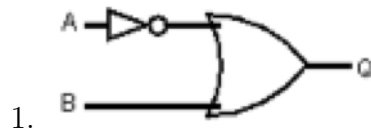
Discrete Mathematics

Tutorial sheet

Boolean Algebra

Question 1.

What is the output for each of the following logic circuits:

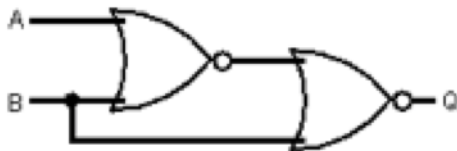


Solution:

1. $\bar{A} + B$
2. $A(\overline{BC})$

Question 2.

Write down the truth table for the output Q of the following circuit.



Solution:

A	B	$A + B$	$\overline{A + B}$	$\overline{A + B} + B$	$Q = \overline{A + B} + B$
0	0	0	1	1	0
0	1	1	0	1	0
1	0	1	0	0	1
1	1	1	0	1	0

Question 3.

Simplify each Boolean expression to one of the following expressions: $0, 1, A, B, AB, A + B, \overline{AB}, \overline{A + B}, \overline{AB}$ and $A\overline{B}$

1. $\overline{\overline{A + B}}$
2. $A(A + \overline{A}) + B$

$$3. (A + B)(\bar{A} + B)\bar{B}$$

Solution:

$$1. \overline{\bar{A} + \bar{B}} = A.B \text{ (De Morgan's law)}$$

$$2. A(A + \bar{A}) + B$$

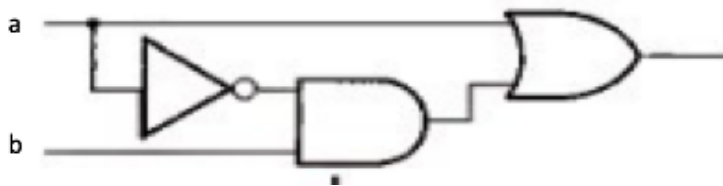
$$\begin{aligned} A(A + \bar{A}) + B &= A.A + A.\bar{A} + B && \text{Distributivity} \\ &= A + 0 + B && A.A = A \text{ and } A.\bar{A} = 0 \\ &= A + B && A + 0 = A \end{aligned}$$

$$3. (A + B)(\bar{A} + B)\bar{B}$$

$$\begin{aligned} (A + B)(\bar{A} + B)\bar{B} &= (A + B)(\bar{A}.\bar{B} + B.\bar{B}) && \text{distributivity} \\ &= (A + B)(\bar{A}.\bar{B} + 0) && B.\bar{B} = 0 \\ &= (A + B)(\bar{A}.\bar{B}) \\ &= (A.\bar{A}.\bar{B} + B.\bar{A}.\bar{B}) && \text{distributivity} \\ &= (A.\bar{A}.\bar{B} + \bar{A}.B.\bar{B}) && \text{commutativity} \\ &= (0.\bar{B} + \bar{A}.0) && A\bar{A} = B.\bar{B} = 0 \\ &= (0 + 0) && 0.\bar{B} = \bar{A}.0 = 0 \\ &= 0 \end{aligned}$$

Question 4.

1. Use the laws of boolean algebra to simplify the boolean expression:
 $a + \bar{a}b = a + b$.
2. Use the truth table prove that $a + \bar{a}b = a + b$.
3. Use the results from 1 and 2 to find a simplified circuit for the following logic circuit:



Solution:

$$\begin{aligned} a + \bar{a}b &= a.1 + \bar{a}b && \text{Identity law} \\ &= a.(1 + b) + \bar{a}b && \text{Identity law} \\ &= a.1 + ab + \bar{a}b && \text{Distributive law} \\ &= a.1 + b(a + \bar{a}) && \text{Distributive law} \\ &= a.1 + b.(1) && \text{Complement law} \\ &= a + b && \text{Identity law} \end{aligned}$$

2.

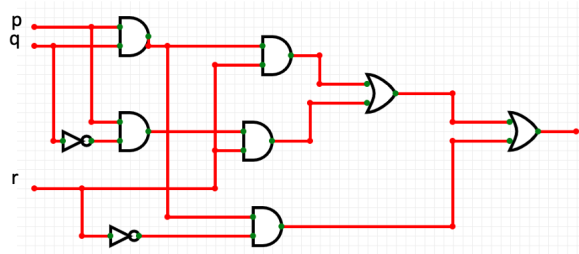
a	b	\bar{a}	$a + b$	$\bar{a}b$	$a + \bar{a}b$
0	0	1	0	0	0
0	1	1	1	1	1
1	0	0	1	0	1
1	1	0	1	0	1

The column for $\overline{(a + b)}$ is the same as the column $\bar{a}\bar{b}$, hence, $\overline{(a + b)} = \bar{a}\bar{b}$.



Question 5.

1. What is the output of the following logical circuit:



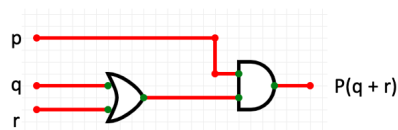
Solution:

$$pqr + p\bar{q}r + pq\bar{r}$$

2. Simplify the output form the circuit above and find a simpler circuit which has the same effect.

Solution:

1. $pqr + p\bar{q}r + pq\bar{r} = p(q + r)$. Hence, this circuit can be simplified to



2.

Question 6.

Use the truth table prove De Morgan's laws: $\overline{ab} = \overline{a} + \overline{b}$ and $\overline{a + b} = \overline{a}.\overline{b}$

Solution:

a	b	\overline{a}	\overline{b}	$a + b$	$\overline{a + b}$	$\overline{a}.\overline{b}$	ab	\overline{ab}	$\overline{a} + \overline{b}$
0	0	1	1	0	1	1	0	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	0	0	1	0	0

The column for $\overline{a + b}$ is the same as the column $\overline{a}.\overline{b}$, hence, $\overline{a + b} = \overline{a}.\overline{b}$.

The column for \overline{ab} is the same as the column $\overline{a} + \overline{b}$, hence, $\overline{ab} = \overline{a} + \overline{b}$.

Question 7.

Use the laws of boolean algebra to simplify the boolean expression:

$$\overline{ab}(\overline{a} + b)(\overline{b} + b)$$

Solution:

$$\begin{aligned}
 \overline{ab}(\overline{a} + b)(\overline{b} + b) &= \overline{ab}(\overline{a} + b).1 && \text{Complement law} \\
 &= \overline{ab}(\overline{a} + b) && \text{idempotent law} \\
 &= \overline{ab}\overline{a} + \overline{ab}b && \text{Distributive law} \\
 &= (\overline{a} + \overline{b})\overline{a} + (\overline{a} + \overline{b})b && \text{De Morgan's law} \\
 &= \overline{a}\overline{a} + \overline{b}\overline{a} + \overline{a}b + \overline{b}b && \text{Distributive law} \\
 &= \overline{a} + \overline{a}(\overline{b} + b) + 0 && \text{Idempotent law, Complement laws and distributive} \\
 &= \overline{a} + \overline{a}.1 + 0 && \text{complement law} \\
 &= \overline{a} + \overline{a} && \text{identity law} \\
 &= \overline{a} && \text{idempotent law}
 \end{aligned}$$

Question 8.

Use the laws of boolean algebra to simplify the boolean expression:

$$\overline{a}(a + b) + (b + aa)(a + \overline{b})$$

Solution:

$$\begin{aligned}
 \overline{a}(a + b) + (b + aa)(a + \overline{b}) &= \overline{a}(a + b) + (b + a)(a + \overline{b}) && \text{idempotent law} \\
 &= \overline{a}(a + b) + (a + b)(a + \overline{b}) && \text{Commutative law} \\
 &= (a + b).(\overline{a} + (a + \overline{b})) && \text{Distributive law} \\
 &= (a + b).((\overline{a} + a) + \overline{b}) && \text{Associative law} \\
 &= (a + b).(1 + \overline{b}) && \text{Complement law} \\
 &= (a + b).1 && \text{Annulment law} \\
 &= (a + b). && \text{Identity law}
 \end{aligned}$$

Question 9.

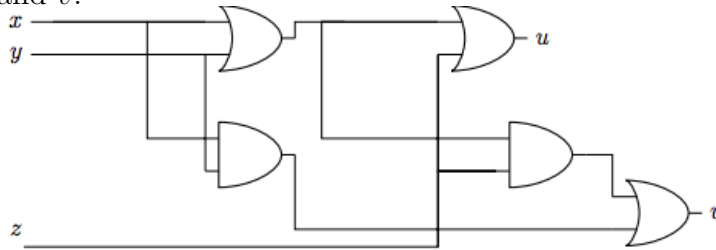
Prove that in a boolean algebra $a^2 = a$. You are required to explain your answer by making a reference to a boolean algebra axioms (laws).

Solution:

$$\begin{aligned}
 a &= a.1 && (a.1 = a) \\
 &= a.(a + \bar{a}) && (\bar{a} + a = 1) \\
 &= a.a + a.\bar{a} && (\text{distributivity of } . \text{ over } +) \\
 &= a^2 + 0 && (a.\bar{a} = 0) \\
 &= a^2 && a + 0 = a
 \end{aligned}$$

Question 10.

The following diagram shows a circuit with three inputs and two outputs, u and v .



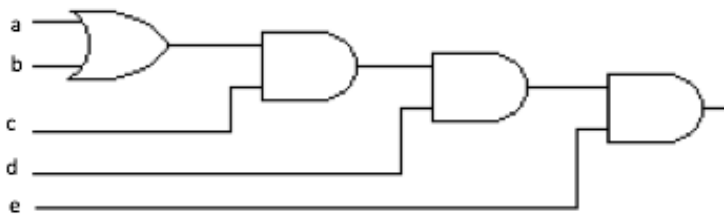
1. List the logic gates used in this circuit.
2. Describe each output u and v as a Boolean expression in terms of x , y and z .

Solution:

1. 2 and-gates and 3 or-gates
2. $u = x + y + z$ where as $v = ((x + y).z) + (x.y) = x.y + x.z + y.z$

Question 11.

Derive the Boolean expression for the following logic circuit shown below



Solution:

$$(a+b).c.d.e$$

Question 12.

1. Write down a boolean expression for the following input/output behaviour.

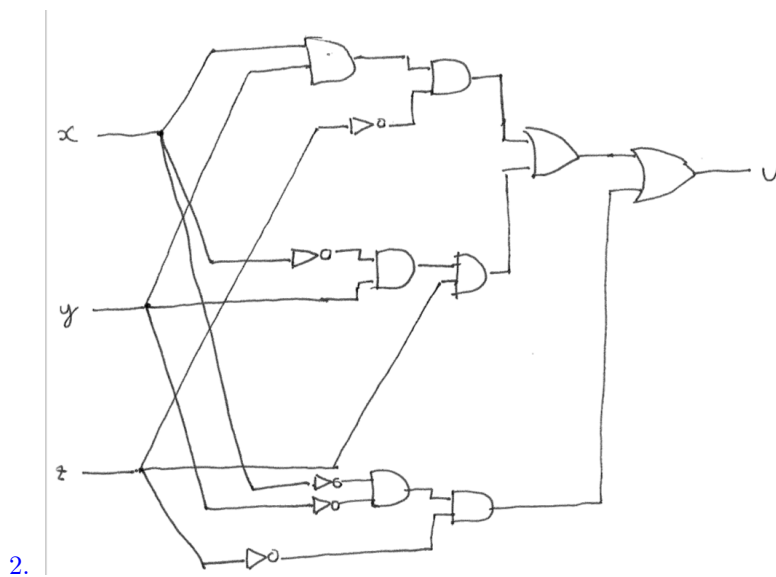
x	y	z	u
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

- Construct the corresponding circuit of the above expression using not-gates, and-gates and or-gates only.

Solution:

- In order to answer this questions we need to check all combination that makes the output 1 the and-gates then use or-gates to link all the possible true outputs. by doing this we will get:

$$u = (x.y.\bar{z}) + (\bar{x}.y.z) + (\bar{x}.\bar{y}.\bar{z}).$$

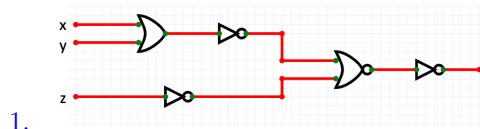


Question 13.

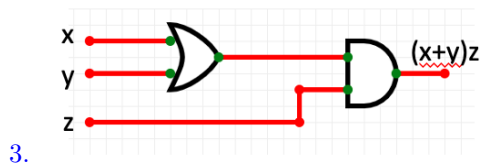
Given the following boolean expression $\overline{\overline{(x + y)} + \bar{z}}$.

- Construct its corresponding circuit.
- Use DeMorgan's laws to find a simpler form for this expression
- Construct the circuit the simplified expression.

Solution:



2. $(x + y)z$



Question 14.

Simplifying the following boolean expression using Karnaugh Map

$$\bar{a}\bar{b}\bar{c} + \bar{a}b\bar{c} + ab\bar{x}$$

Solution:

c \ ab	00	01	11	10
0	1	1	1	
1		1		

We can now group the 1's as follows:

c \ ab	00	01	11	10
0	1	1	1	
1		1		

Now for each grouping, we are not going to consider any variable that is changing.

For the first grouping in yellow, consists of two 1's which correspond to $a = 0, b = 0, c = 0$ and $a = 0, b = 1, c = 0$. In another way, the value in this first grouping is independent from the value of b . so when $a = 0$ and $c = 0$ the output is 1. hence, this can be reduced to just $\bar{a}\bar{c}$

For the green grouping which consists of two 1's corresponding to $a = 0, b = 1, c = 0$ and $a = 0, b = 1, c = 1$. In another way, the value in this first grouping is independent from the value of c . so when $a = 0$ and $b = 1$ the output is 1. hence this can be reduced to $\bar{a}b$

Finally, for the second grouping in orange consists of two 1's which correspond to $a = 0, b = 1, c = 0$ and $a = 1, b = 1, c = 0$. In another way, the value in this first grouping is independent from the value of a . so when $b = 1$ and $c = 0$ the output is 1. Hence this can be reduced to just $b\bar{c}$.

The final reduced expression is then $\bar{a}\bar{c} + \bar{a}b + b\bar{c}$

Question 15.

Given the following boolean function

$$f(a, b, c, d) = \bar{a}\bar{b}cd + \bar{a}bcd + abcd + \bar{a}bcd + ab\bar{c}\bar{d} + ab\bar{c}d + abc\bar{c}$$

1. Fill in the missing value in the following Karnaugh map of $f(a, b, c)$:

ab \ cd	00	01	11	10
00				
01				
11				
10				

2. Use K-map in (1) to find the minimum sum of products of $f(a, b, c)$.

Solution:

1.

ab \ cd	00	01	11	10
00	0	0	1	0
01	0	0	1	0
11	1	1	1	1
10	0	0	1	0

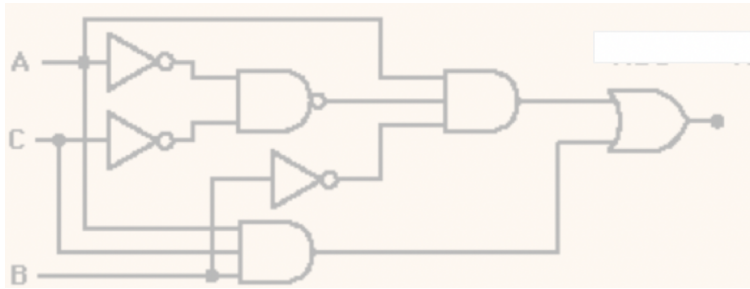
ab \ cd	00	01	11	10
00	0	0	1	0
01	0	0	1	0
11	1	1	1	1
10	0	0	1	0

- 2.

Hence, $f(a, b, c) = ab + cd$.

Question 16.

Given the following circuit:



1. Find the output of this circuit.
2. Use the laws of algebra to give a simpler expression for this output.
3. Use the result in 2 to draw a simpler circuit equivalent circuit.

Solution:

$$1. ABC + AB(\overline{AC})$$

$$\begin{aligned}
 ABC + AB(\overline{AC}) &= ABC + AB(\overline{A} + \overline{C}) && \text{De Morgan's law} \\
 &= ABC + AB\overline{A} + AB\overline{C} && \text{Distributive law} \\
 &= ABC + A\overline{B}A + AB\overline{C} && \text{Complement law} \\
 &= ACB + AA\overline{B} + AC\overline{B} && \text{Associative law} \\
 &= ACB + A\overline{B} + AC\overline{B} && \text{Idempotent law} \\
 &= AC(B + \overline{B}) + A\overline{B} && \text{Idempotent law} \\
 &= AC.(1) + A\overline{B} && \text{Complement law} \\
 &= AC + A\overline{B} && \text{Identity law} \\
 &= A(C + \overline{B}) && \text{Identity law}
 \end{aligned}$$

$$3. ABC + AB(\overline{AC}) = A(C + \overline{B}), \text{ hence the simplified circuit is :}$$

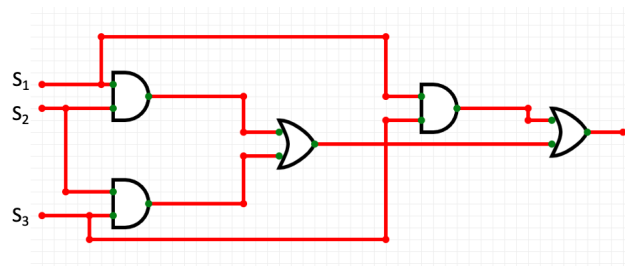


Question 17.

A set of three sensors in a factory detects whether the pollution level it is outputting from an incinerator exceeds the safety limit. In which case the incinerator is shut down. An alarm A goes off if at least two the three sensors s_1, s_2 and s_3 detect a pollution level above the limit. Draw a logic circuit for the system showing the inputs s_1, s_2 and s_3 and the output A .

Solution:

The output of this circuit should be $A = s_1.s_2 + s_1.s_3 + s_2.s_3$. The logical circuit for the output A is



End of questions