Math 131AH: Homework #4

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Problem 1

Let F be a non-empty set, $E = \mathcal{P}(F)$. We have already seen that this is a poset via $\subset$. We are asked to show that every $A \subset \mathcal{P}(F)$ i.e. every collection of subsets of F admits an infimum and supremum. First, we consider nonempty collections.

Let $A \neq \emptyset$. We show that $\sup(A) = \bigcup A$. By definition, $\forall X \in A : X \subset \bigcup A$. Now, let B be another upper bound for A. Then, $\forall X \in A : X \subset B \Rightarrow \bigcup A \subset B$. This shows that $\bigcup A$ is the least upper bound for A i.e. $\bigcup A = \sup(A)$.

Next, we show $\inf(A) = \bigcap A$. By definition, $\forall X \in A : \bigcap A \subset X$ so it is a lower bound. Let C be another lower bound for A i.e. $\forall X \in A : C \subset A \Rightarrow C \subset \bigcap A$. So, $\bigcap A$ is the greatest lower bound i.e. $\inf(A) = \bigcap A$.

Finally, we consider $A = \emptyset$. To prove that $\sup(\emptyset) = \emptyset$ it suffices to show that $\emptyset$ is an upper bound. It will definitely be the smallest one as it is the minimal element of this poset. Similarly, to show $\inf(\emptyset) = F$ it suffices to show that F is a lower bound. To show that $\sup(\emptyset) = \emptyset$, we require: $\forall X \in \emptyset : X \subset \emptyset$. Since $\emptyset$ is the emptyset, there exists no such X that can be compared. As a result, the statement is vacuously true. Similarly, $\forall X \in \emptyset : F \subset X$ is vacuously true so $\inf(\emptyset) = F$.

Problem 2

We define $m|n := (\exists k \in \mathbb{N}' : n = mk)$. We need to prove that this a partial order.

1. $\forall m : m = 1m$ which implies that $m|m$
2. Let $m|n$ and $n|m$. Then, $\exists k, l \in \mathbb{N}' : m = kn, n = lm$. Therefore, $n = lkn$. Since $n \neq 0, 1 = lk$. In $\mathbb{N}'$ this implies that $l = k = 1$. Therefore, $m = n$.
3. Let $m|n$ and $n|k$. So, $n = c_1 m$, $k = c_2 n$ for some $c_1, c_2 \in \mathbb{N}'$. Therefore, $k = c_1 c_1 m$ so $m|k$.

Therefore this is a partial order. Now we show that $\inf(A)$ exists and $\sup(A)$ exists if A is bounded.
Let $A \subset \mathbb{N}'$ such that $A \neq \emptyset$. It is clear that this set has a lower bound of 1 since $\forall m \in \mathbb{N}' : m = 1m$ so $1|m$. For each $a \in A$ we can consider its prime factorization into some form $a = p_1^{k_1} \cdots p_n^{k_n}$. We can do this for all elements in A and then for each prime p_i listed, we look at the powers k_i that it corresponds to for various $a \in A$. This gives us a set of naturals indexed by $A \subset \mathbb{N}$. Therefore, we can consider the minimal element (ordered by $\leq$) of each such set, call it m_i. If all primes listed are $p_1, \ldots, p_s$ then $\inf(A) = p_1^{m_1} \cdots p_s^{m_s}$. This is because, by construction, each $p_i^{m_i}$ divides all $a \in A$. Also, if some l divides all $a \in A$. Then, it must divide each of the prime factors $p_i^{m_i}$ listed earlier. In other words, $l|p_i^{m_i}$ for each i so $l|p_1^{m_1} \cdots p_s^{m_s}$. This concludes our proof as we have shown a greatest lower bound.

Similarly, let $A \subset \mathbb{N}'$ be a bounded subset. Then, we denote $u = \prod\limits_{x \in A} x$. Since A is bounded, this is a finite product and we can write $u = x_1 x_2 \cdots x_n$. This is an upper bound on A as $\forall x \in A : x|u$. Therefore, we can now consider the set U of upper bounds (ordered by $|$) on A which is nonempty. This is a subset of $\mathbb{N}$ and therefore has a minimal element when ordered by $\leq$. Let's call this $\min(U)$. We claim that $\sup(A) = \min(U)$ when A is ordered by $|$. We will prove this by contradiction. This is because $\sup(A)$ must lie in U. Therefore, if it is not the minimal element of U, there is an element that is a upper bound on A, call it u_0 such that $u_0 \leq \sup(A)$. This would in turn imply that $u_0 | \sup(A)$ which would contradict its minimality.

Problem 3

$\Rightarrow$ Assume that A is Dedekind infinite. Let $f : A \to A$ be an injective map such that $\exists a \in A : a \notin Im(f)$. We can then construct a map $i : \mathbb{N} \to A$ by recursive construction which is injective. This proves that A is unbounded as $\mathbb{N}$ can be identified as a subset of A. For the construction, we first set $i(0) = a$. We then set $i(n) = f(i(n-1))$. We just need to verify that this is injective. Fix $n \in \mathbb{N}$. We will show inductively that $\forall m \neq n$ or WLOG $\forall m < n : i(m) \neq i(n)$. The base case is true since $i(0) = x \notin Im(f), i(n) \in Im(f)$. For the inductive step, we use the fact that f is injective. Assume $f(m+1) = f(n)$ for $m+1 < n$. But then $f(i(m)) = f(i(n-1))$ so $i(m) = i(n-1)$ which contradicts the the inductive hypothesis. This proves that i is injective and A is unbounded.

$\Leftarrow$ Assume that A is unbounded. We first show that we can construct $f : \mathbb{N} \to A$ injective. Since A is unbounded, it is nonempty so choose $a \in A$. Then, set $f(0) = a$. We can recursively construct the map by choosing for $f(n)$ some element in A that hasn't been chosen before i.e. $f(n) \notin \{f(0), f(1), \dots f(n-1)\}$. This is permitted by the axiom of choice and since A is unbounded. This map, by construction, is injective. We can easily modify this map to get another map $b : A \to A$ where b is injective but not surjective. We construct b by simply shifting the values in the image of f. Explicitly, $b(f(0)) = f(1)$ and $b(f(n)) = b(f(n+1))$. This map is injective since f is and is not surjective since $f(0) \notin Im(b)$.

Problem 4

Let $A, B \subset \mathbb{Q}$ such that $\sup(A), \sup(B)$ exist and $A \subset B$. Since $\sup(B)$ exists, $\forall b \in B : b \leq \sup(B)$. In particular, $\forall a \in A : a \leq \sup(B)$. Therefore, $\sup(B)$ is an upper bound for A as well. By definition of $\sup(A)$, this must mean that $\sup(A) \leq \sup(B)$

Under these conditions, we can consider $\sup(A \cup B)$. We show that this exists. We know that $\sup(A)$ is an upper bound for A and $\sup(B)$ is an upper bound for B. Therefore, $\forall x \in A \cup B : x \leq \sup(A) \vee x \leq \sup(B) \Rightarrow x \leq \max\{\sup(A), \sup(B)\}$. Clearly, $\max\{\sup(A), \sup(B)\}$ is an upper bound for $A \cup B$
Now, let u be another upper bound on $A \cup B$. Then, $\forall x \in A \cup B : x \leq u$. Since $A \subset A \cup B$, $\sup(A) \leq u$. This follows from the previous part: if $A \subset B$, then $\sup(A) \leq \sup(B) \leq$ any upper bound on B. Similarly, $B \subset A \cup B \Rightarrow \sup(B) \leq u$. Therefore, $\max\{\sup(A), \sup(B)\} \leq u$. This proves that $\sup(A \cup B) = \max\{\sup(A), \sup(B)\}$.

Problem 5

Consider $A + B := \{a + b, a \in A \wedge b \in B\}$. Assume that $\sup(A)$ and $\sup(B)$ exist. Let $a \in A, b \in B$. Then $a \leq \sup(A), b \leq \sup(B)$ so $a + b \leq \sup(A) + \sup(B)$. So, $\sup(A) + \sup(B)$ is an upper bound on $A + B$. Now, let u be another upper bound on $A + B$. So, $\forall a, b : a + b \leq u$. Then we can write $u \geq l + k$ where l, k are upper bounds on A and B respectively. Then, $l + k$ is an upper bound on $A + B$. Therefore, $\sup(A) + \sup(B) \leq l + k \leq u$. By transitivity, $\sup(A) + \sup(B) \leq u$ so $\sup(A + B) = \sup(A) + \sup(B)$.

Problem 6

Let A have a lower bound l i.e. $\forall a \in A : l \leq a$. Now consider the set $-A = \{-a : a \in A\}$. Then, $\forall(-a) \in -A : -a \leq -l$. Therefore, $-l$ is an upper bound on the set $-A$. By the completeness of $\mathbb{R}$, $\sup(-A)$ exists. This means that for any upper bound u of $-A$, $\sup(-A) \leq u$. Therefore, for any lower bound l of A, $l \leq -\sup(-A)$. Therefore, $\inf(A) = -\sup(-A)$.

Problem 7

We need to show that we cannot define an order on $\mathbb{C}$. First note that $0 \neq i$ Assume that in some ordering we have $0 < i$. Then since i is positive, multiplication by i preserves the ordering relation: $0 < i^2 \Rightarrow 0 < -1$ which is a contradiction. Instead, assume that $i < 0$. Then $i^2 > 0 \Rightarrow -1 > 0$ which is again a contradiction. Therefore, it is impossible to make C an ordered field. As the hint suggests, this is because -1 is a square and by our axioms of of an ordered field, a square must always be positive.