# Math 245A - Real Analysis: Homework #1

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# Problem 1

We proceed by verifying the 3 conditions for $\mathcal{A}$ to be an algebra:

1. $\emptyset \in \mathcal{F} \subset \mathcal{A}$. This is sufficient to show $X \in \mathcal{A}$ once we show it is closed under taking complement.

2. Let $A \in \mathcal{A}$ so we can write $A = A_1 \cup \cdots \cup A_m$ where $A_i \in \mathcal{F}$. Then by condition (iii) for each $i \in [m]$ there exists $n_i \in \mathbb{N}$ and pairwise disjoint $A_{i1}, \ldots, A_{in_i} \in \mathcal{F}$ such that $A_i^c = A_{i1} \cup \ldots \cup A_{in_i}$. So, $A^c = A_1^c \cap \cdots \cap A_m^c = \bigcup_{j_1 \in [n_1], \ldots, j_m \in [n_m]} (A_{1j_1} \cap \cdots \cap A_{mj_m}) \in \mathcal{A}$ since we are taking a disjoint union of elements in $\mathcal{F}$ and the family is closed under finite intersection.

3. Let $A, B \in \mathcal{A}$. It suffices to show that $A \cap B \in \mathcal{A}$ as then $A \cup B = (A^c \cap B^c)^c$ and we have already checked that $\mathcal{A}$ is closed under taking complement. By assumption, we can write $A = A_1 \cup \ldots \cup A_n$ and $B = B_1 \cup \ldots \cup B_m$ where $A_i, B_j \in \mathcal{F}$. Then $A \cap B = \bigcup_{i,j=1}^{n} (A_i \cap B_j) \in \mathcal{A}$ since $\mathcal{F}$ is closed under intersections so we have expressed $A \cap B$ as a finite disjoint union of sets in $\mathcal{F}$.

# Problem 2

(a) First we verify the conditions for $\mathcal{A}$ to be an algebra on $X$.

1. By considering $I = \emptyset$ and $I = [n]$ we see that $\emptyset = \bigcup_{i \in \emptyset} M_i \in \mathcal{A}$ and $X = \bigcup_{i \in [n]} M_i \in \mathcal{A}$.
2. Let $M \in \mathcal{A}$ so we can write $M = \bigcup_{i \in I} M_i$ for some $I \subset [n]$. Since the $\{M_i\}_{i \in [n]}$ form a partition of $X$, we can write $M^c = (\bigcup_{i \in I} M_i)^c = \bigcup_{i \in I^c} M_i \in \mathcal{A}$.
3. Let $M_1 = \bigcup_{i \in I_1} M_i, M_2 = \bigcup_{i \in I_2} M_i$ be elements of $\mathcal{A}$. Then, $M_1 \cup M_2 = \bigcup_{i \in I_1 \cup I_2} M_i \in \mathcal{A}$.

Elements of $\mathcal{A}$ are in bijection with subsets of $[n]$ so there are $2^n$ elements in the algebra. Yes, every finite algebra is a $\sigma$-algebra since countable unions and finite unions are the same.

(b) *Need to explicitly construct this set somehow by partitioning based on disjoint parts. Just need to formalize this construction.*

# Problem 3

# Problem 4

# Problem 5

(a) We are given that $\mathcal{A}$ is an algebra on $X$. First assume that it is a $\sigma$-algebra and $A_n \nearrow$. We know that a $\sigma$-algebra is closed under countable unions so $\bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$. Conversely assume that the property holds and we need to show that $\mathcal{A}$ is closed under countable unions. Let $\{B_n\}_{n \in \mathbb{N}}$ be an arbitrary collection of elements in $\mathcal{A}$. Define $A_n = \bigcup_{i \in [n]} B_i$ so it is the union of the first $n$ elements in the collection. Since $\mathcal{A}$ is an algebra, each $B_i \in \mathcal{A}$ by closure under finite unions. Clearly $A_n \subset A_{n+1}$ so $A_n \nearrow$. But also note that for all $n$ we have $\bigcup_{i \in [n]} A_i = A_n = \bigcup_{i \in [n]} B_i$ so in particular $\bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} A_n \in \mathcal{A}$ by the property we assumed and so we are done.

(b) We need to show that $\mu$ has countable additivity assuming it has finite additivity and the property stated. Let $\{B_n\}_{n \in \mathbb{N}} \in \mathcal{A}$ be a collection of pairwise disjoint sets. We define $A_n = \bigcup_{i \in [n]} B_i$ as above so that $A_n \nearrow$. Then $\mu\left(\bigcup_{n \in \mathbb{N}} B_n\right) = \mu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \lim_{N \to \infty} \mu(A_n)$. But from finite additivity, we can write $\mu(A_n) = \sum_{i=1}^{n} \mu(B_i)$ so the term above simplifies to $\sum_{n=1}^{\infty} \mu(B_n)$.

# Problem 6

(a) Note that $x \in A$ if and only if for all $n \in \mathbb{N}$, there exists some $m \geq n$ such that $x \in A_m$. This is because if $x \in A$ i.e. it is in infinitely many $A_m$ then it is also in infinitely many $A_m$ if we exclude a finite number of sets (say the first $n$ sets). On the other hand, if it is not in infinitely many $A_n$ there is an $N$ such that for all for all $n > N$, $x \notin A_n$ and so the right hand side becomes false. Finally, we can write $\forall n \in \mathbb{N}, \exists m \geq n : x \in A_m \iff x \in \bigcap_{n\in\mathbb{N}} \bigcup_{m\geq n} A_m$. $A$ is a countable intersection of a countable union of sets in $\mathcal{A}$ and is therefore in $\mathcal{A}$.

(b) Let $B_n$ denote $\bigcup_{m\geq n} A_m$. Then $A = \bigcap_{n\in\mathbb{N}} B_n$ and $B_n \searrow$. Also $\mu(B_1) = \mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) \leq \sum\limits_{n=1}^{\infty} \mu(A_n) < \infty$ so we can apply continuity from above to get $\mu(A) = \lim_{n\to\infty} \mu(B_n)$. Then, by countable subadditivity, $\mu(B_n) \leq \sum_{m=n}^{\infty} \mu(A_m)$ which goes to 0 as $n \to \infty$ because $\sum\limits_{n=1}^{\infty} \mu(A_n) < \infty$.

# Problem 7

(a) Pick $A_1$ to be a set such that there are infinitely many sets in $\mathcal{A}$ that intersect with $A_1^c$. Such an $A_1$ always exists because $\mathcal{A}$ has infinite elements. Next, choose $A_2$ from $\{A_1^c \cap B : B \in \mathcal{A}\}$ such that there are infinitely many sets in $\mathcal{A}$ that intersect with both $A_1^c$ and $A_2^c$. This way, $A_1$ and $A_2$ are disjoint. We can repeat this process indefinitely. Formally, assume we have picked $A_1, A_2, \ldots, A_{n-1}$ that are disjoint and there are infinite sets in $\mathcal{A}$ that intersect with all $A_i^c$. Then, once again, we can pick $A_n$ disjoint from the others such that infinitely many sets in $\mathcal{A}$ intersect it. This gives us an infinite series of disjoint sets in $\mathcal{A}$.

(b) Call this disjoint family $\mathcal{F} = \{A_i\}_{i\in\mathbb{N}}$. Now consider the collection $\mathcal{G}$ of all sets that can be obtained by taking (disjoint) unions of sets in this family. Each element in $\mathcal{G}$ is also in $\mathcal{A}$ since it is closed under countable unions. Since the $A_i$ are disjoint, this collection $\mathcal{G}$ is in bijection with subsets of $\mathbb{N}$ which we know is uncountable by Cantor's diagonal argument. Since $\mathcal{G} \subset \mathcal{A}$, we also get that $\mathcal{A}$ is uncountable.