Homework 8

Q1] (1) $\Rightarrow$ (2)

Assume $\{x_n\}_{n\to\infty} = L$

It is easily checked that every subsequence $\{x_{n_k}\}$ converges to L.

Let $\varepsilon > 0$.

Then $\exists n_0$ s.t $\forall n > n_0$ : $|x_n - L| < \varepsilon$

then pick $k_0$ s.t $n_{k_0} > n_0$. Then $\forall k > k_0$ (i.e $n_k > n_{k_0}$)

$|x_{n_k} - L| < \varepsilon$

$\therefore \{x_{n_k}\}_{n\to\infty} = L$

Then, $z = L$ since every subsequence of $\{x_n\}$ has a subsequence (itself) that converges to z.

(2) $\Rightarrow$ (1)

Assume $\exists z \in X$ s.t every subsequence of $\{x_n\}$ has a subsequence that conv. to z.

Assume, in search of contradiction $\{x_n\}$ doesn't converge to z.

Then $\exists \varepsilon > 0$ s.t $\forall k$ $\exists n_k > k$ $|x_{n_k} - z| \geq \varepsilon$.

This is because, if $\exists$ some k without such an $n_k$, we could pick $n_0 = k$ and the $x_n$ would converge to z.

$\therefore$ we have a subsequence $x_{n_k}$ which does not have any subsequence converging to z. which is a contradiction.

$\therefore$ we must have $\lim_{n\to\infty} \{x_n\} = z$ ■

Q2] Let $A \subseteq B$

Let $O \in A$ be an open set

$\therefore O \in B$ is also an open set.

$\therefore \{O \in A : O \text{ open}\} \subseteq \{O \in B : O \text{ open}\}$

$\therefore \text{int}(A) = \bigcup_{\substack{O \in A \\ \text{open}}} O \subseteq \bigcup_{\substack{O' \in B \\ \text{open}}} O' = \text{int}(B)$ □

Let C be closed such that $B \subseteq C$

$\therefore A \subseteq C$.

$\therefore \{C \text{ closed s.t. } B \subseteq C\} \subseteq \{C' \text{ closed s.t. } A \subseteq C\}$

$$\therefore \bigcap_{\substack{A \subseteq C' \\ \text{closed}}} C' \subseteq \bigcap_{\substack{B \subseteq C \\ \text{closed}}} C$$

$$\therefore \quad \overline{A} \subseteq \overline{B}$$

Now we show $X \setminus \operatorname{int}(A) = \overline{X \setminus A}$

we simplify notation $(\operatorname{int}(A))^c = \overline{(A^c)}$

$$\text{L.H.S} = \Big[\bigcup_{\substack{O \subseteq A \\ \text{open}}} O\Big]^c \underset{\text{De-Morgan}}{=} \bigcap_{\substack{O \subseteq A \\ \text{open}}} O^c .$$

* Next, note that $\{O \subseteq A \text{ open}\} \longleftrightarrow \{A^c \subseteq C \text{ closed}\}$

$$O \longmapsto O^c$$

i.e. we have a bijection between open sets in $A$ and closed sets containing $A^c$

since if $O \subseteq A$ is open then $A^c \subseteq O^c$ and $O^c$ is closed $\square$

$$\therefore \bigcap_{O \subseteq A} O^c = \bigcap_{A^c \subseteq C} C = \overline{A^c}$$

Similarly,

WTS $\overline{A}^c = \operatorname{int}(A^c)$

$$\overline{A}^c = \Big(\bigcap_{\substack{A \subset C \\ C \text{ closed}}} C\Big)^c = \bigcup_{\substack{A \subset C \\ C \text{ closed}}} C^c \overset{\substack{\text{using the same} \\ \text{bijection as above}}}{=} \bigcup_{O \subseteq A^c} O = \operatorname{int}(A^c) \quad \square$$

In particular
$$\begin{aligned} \partial A &= \overline{A} \setminus \operatorname{int}(A) \\ &= \overline{A} \cap (X \setminus \operatorname{int}(A)) \\ &= \overline{A} \cap \overline{X \setminus A} \\ &= \overline{X \setminus A} \cap \overline{A} \\ &= \overline{A^c} \cap \overline{X \setminus A^c} \\ &= \partial(A^c) \quad \blacksquare \end{aligned}$$

Q3] 1) Consider $A = (0,1) \cap \mathbb{Q}$

i.e the set of rationals $q$: $0<q<1$.

Recall that $A \subseteq \mathbb{Q}$ is relatively open iff $A = O \cap \mathbb{Q}$ for some $O$ open in $\mathbb{R}$

it is clearly relatively open because $A = (0,1) \cap \mathbb{Q}$ and $(0,1)$ is open in $\mathbb{R}$

However, it is not open in $\mathbb{R}$.

Let $x \in A$. By density of irrationals, every open ball around $x$ $B(x,r)$ contains irrationals so we cannot find any open ball $B(x,r)$ s.t $B(x,r) \subseteq A$.

2) Assume $A$ is finite. Let $m = \min(A)$
$n = \max(A)$.

Then $A = (m-1, n+1) \cap \mathbb{N} = [m,n] \cap \mathbb{N}$

so it is both relatively open and relatively closed.

Assume $A$ infinite $\therefore$ countable.

Let $\{x_0, x_1, \cdots, x_n, \cdots\}$ be an enumeration of the elements of $A$.

Let $B_i = B(x_i, 1/2)$ s.t. $B_i$ is open in $\mathbb{R}$
and $B_i \cap \mathbb{N} = \{x_i\}$

Then $O = \bigcup_{i \in \mathbb{N}} B_i$ is an open set $\subseteq \mathbb{R}$ such that $A = O \cap \mathbb{N}$

$\therefore A$ is relatively open.

Now, sort the elements of $A$ is increasing order so we get
$x_0 = \inf(A)$, $x_1 = \inf(A \setminus x_0)$, $\cdots$
$x_0 < x_1 < \cdots < x_n < \cdots$

Let $O = (-\infty, x_0) \cup (x_0, x_1) \cup \cdots (x_{n-1}, x_n) \cup (x_n, x_{n+1}) \cup \cdots$ which is open

Note that $O = \mathbb{R} \setminus A$

$\therefore A$ is closed.

$\therefore A$ is relatively closed

3) If $A \subseteq \mathbb{N}$ is finite it is the finite union of closed sets (singletons) so it is closed in $\mathbb{R}$.

If $A$ is countable, sort the elements of $A$ is increasing order so we get:

$x_0 = \inf(A)$, $x_1 = \inf(A \setminus x_0)$, $\cdots$

$x_0 < x_1 < \cdots < x_n < \cdots$

let $O = (-\infty, x_0) \cup (x_0, x_1) \cup \cdots (x_{n-1}, x_n) \cup (x_n, x_{n+1}) \cup \cdots$ which is open

Note that $O = \mathbb{R} \setminus A$

$\therefore$ $A$ is closed.

4) let $A \subseteq \mathbb{Q}$ rel open and rel closed.

$\therefore$ it is of the form $O \cap \mathbb{Q}$ and $C \cap \mathbb{Q}$ where $O$ open, $C$ closed.

Recall that every open set $O$ is a finite or countable union of disjoint intervals. If we want $O \cap \mathbb{Q} = C \cap \mathbb{Q}$. this is only possible if the symmetric difference $\Delta(C, O) \subset \mathbb{R} \setminus \mathbb{Q}$. This is only possible if we have all disjoint intervals having irrational endpoints such that $[q_1, q_2] \cap \mathbb{Q} = (q_1, q_2) \cap \mathbb{Q}$ where $q_1, q_2 \in \mathbb{R} \setminus \mathbb{Q}$.