Math 210B: Homework #2

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Problem 1

Proof. Let $f : R \to S$ be a surjective ring homomorphism, $I \subset S$ an ideal. In the previous HW, we saw that $f^{-1}(I) \subset R$ is an ideal that contains $\ker(f)$. In particular this yields a bijection between the set of ideals in R containing $ker(f)$ and the set of ideals in S. Let f be the canonical projection onto the quotient ring $f : R \to R/I$. This is a surjection with $ker(f) = I$. By the above result, this yields a bijection between ideals of R/I and ideals of R containing I. □

Problem 2

Proof. Let $X = R$ as a set. Consider the identity map $id : X \to R$. In the previous homework, we saw that this extends to a well-defined ring homomorphism $f : \mathbb{Z}[X] \to R$. By construction, this is surjective. We can simply quotient $\mathbb{Z}[X]$ with the ideal $ker(f)$ which is isomorphic to R by the First Isomorphism Theorem. □

Problem 3

Proof. We know that the coproduct exists in the category of rings as we can take the tensor product over $\mathbb{Z}$ i.e. the tensor product as $\mathbb{Z}$-modules which are equivalent to abelian groups, with induced ring structure. □

Problem 4

Proof. Let $P \subset R$ be prime ideal such that A/P is a domain i.e. $xy = 0 \Rightarrow x = 0 \vee y = 0$ in R/P. Now, we are given that every element in R is idempotent. So let $x \in R, x \notin P$ such that $\overline{x} \neq \overline{0}$. Then, $x^2 = x$ so $x(x-1) = 0$. Therefore, $\overline{x(x-1)} = 0 \in R/P$. Therefore, $\overline{x-1} = \overline{0}$, so $\overline{x} = \overline{1}$. Therefore, every nonzero in R/P is invertible so it is a field. Therefore, P is maximal in R. □

Problem 5

Let X be the set of prime ideals in R. It is non-empty by Zorn's lemma. This set can be partially ordered via inclusion. Let $C \subset X$ be a chain of ideals. I claim that $Q = \bigcap_{I \in C} I$ is a prime ideal. It is clearly an ideal since the intersection of ideals is always an ideal. Assume, by contradiction, it is not prime. Then, there exists $xy \in Q : x \notin Q \wedge y \notin Q$. But, if $xy \in Q$ then $xy \in I$ for all prime ideals in C. Since $x, y \notin Q$, we can find P_1, P_2 such that $x \notin P_1, y \notin P_2$. Since the ideals are ordered by inclusion, assume $x, y \notin P_1$. But then $xy \notin P_1$ since it is prime. This is a contradiction as $xy \in Q$. Therefore, Q is a prime ideal. Clearly, Q is a lower bound for the chain C. Since every chain has a lower bound, X has a minimal element.

Problem 6

Consider $R \subset \mathbb{Q}$ of all fractions $\dfrac{a}{b}$ where b is odd. Let $P \subset R$ be a prime ideal. Recall that $\mathbb{Z} \subset R$. Therefore, since P is closed under external multiplication. Then, we always have $2 \in P$. As a result, we can describe the spectrum as the set of all ideals generated by primes with even numerators.

Problem 7

To construct this bijection, it suffices to show that any prime ideal $P \subset A \times B$ is either of the form $P = P_A \times B$ for $P_A \subset A$ prime or $P = A \times P_B$ for $P_B \subset B$ prime. It is clear that prime ideals in $A \times B$ must be of the form $P_A \times P_B$. Then, $(A \times B)/P \cong A/P_A \times B/P_B$. This is a domain since we quotiented by a prime ideal. Also, $(0,1)(1,0) = (0,0)$ so one of the two must be zero in the quotient summand rings. So, either A/P_A or B/P_B is 0. So either $A = P_A$ or $B = P_B$.

Problem 8

The bijection is $f : V(I) \to Spec(R/I)$ given by $f(P) = P/I$. By the correspondence theorem, we know that there is a bijection between ideals of R containing I and ideals of R/I. We need to show that if an ideal is prime in R and contains I then it is prime in R/I. Let $I \subset P \subset R$ be prime. Consider $\overline{xy} \in P/I$. Assume that $\overline{x} \notin P/I$. Then, $x \notin P$. Therefore $y \in P$ since P is prime and $xy \in P$. So, $\overline{y} \in P/I$ so it is prime.

Problem 9

1. We define $\sim$ on $R \times S$ as follows: $(r_1, s_1) \sim (r_2, s_2)$ if $\exists s \in S$ such that $s(r_1s_2 - r_2s_1) = 0$. This is clearly reflexive since $s(r_1r_2 - r_2r_1) = s(0) = 0$ for all $s \in S$. Assume that $(r_1, s_1) \sim (r_2, s_2)$ i.e. $\exists s \in S : s(r_1s_2 - r_2s_1) = 0$. Then $(-s)(r_2s_1 - r_1s_2) = 0$. Therefore, $(r_2, s_2) \sim (r_1, s_1)$ and $\sim$ is transitive. Assume $(r_1, s_1) \sim (r_2, s_2)$ and $(r_2, s_2) \sim (r_3, s_3)$. Then $\exists s, s' \in S$ such that $s(r_1s_2 - r_2s_1) = 0, s'(r_2s_3 - r_3s_2) = 0$. $sr_1s_2 = sr_2s_1$ and $s'r_2s_3 = s'r_3s_2$. We need to show $r_1s_3 - r_3s_1$ multiplied by some element in S gives us 0. This proves transitivity.

2. Assume $\dfrac{r_1}{s_1} \sim \dfrac{r_1'}{s_1'}$ and $\dfrac{r_2}{s_2} \sim \dfrac{r_2'}{s_2'}$. Then $m(r_1s_1' - r_1's_1) = n(r_2s_2' - r_2's_2) = 0$. We want to show that $\dfrac{r_1s_2 + r_2s_1}{s_1s_2} \sim \dfrac{r_1's_2' + r_2's_1'}{s_1's_2'}$. That is $l((r_1s_2 + r_2s_1)(s_1's_2') - (r_1's_2' + r_2's_1')(s_1s_2) = 0)$ for some $l \in S$. We can rearrange $m, n, s_1, s_2, s_1', s_2'$ to create l so that this holds. Multiplication is checked in a similar way. This is a commutative ring with $0 = \dfrac{0}{1}$ and $1 = \dfrac{1}{1}$.

3. It is clear that $f(0) = 0$ and $f(1) = 1$ from above. Let $r, q \in R$. Then, $f(rq) = \dfrac{rq}{1} = \dfrac{r}{1} \cdot \dfrac{q}{1} = f(r)f(q)$. Similarly, $f(r+q) = \dfrac{r+q}{1} = \dfrac{r}{1} + \dfrac{q}{1} = f(r) + f(q)$. Therefore, f is a ring homomorphism.

Problem 10

We define $h\left(\dfrac{r}{s}\right) = g(s)^{-1}g(r)$. Then, clearly, $h(f(r)) = h\left(\dfrac{r}{1}\right) = g(1)^{-1}g(r) = g(r)$. To show this is a ring homomorphism, we observe that $h\left(\dfrac{r_1}{s_1} + \dfrac{r_2}{s_2}\right) = h\left(\dfrac{r_1s_2 + r_2s_1}{s_1s_2}\right) = g(s_1s_2)^{-1}g(r_1s_2 + r_2s_1) = g(s_1)^{-1}g(r_1) + g(s_2)^{-1}g(r_1) = h\left(\dfrac{r_1}{s_1}\right) + \left(\dfrac{r_2}{s_2}\right)$. Similarly, $h\left(\dfrac{r_1r_2}{s_1s_2}\right) = g(s_1)^{-1}g(s_2)^{-1}r_1r_2 = h\left(\dfrac{r_1}{s_1}\right)h\left(\dfrac{r_2}{s_2}\right)$. We have shown a homomorphism and given an explicit formula for it. The uniqueness follows from this and the fact that S is a multiplicative subset.