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Homework 3 (due: Fr, Oct. 20)

Problem 2*: a) By monotonicity we know that for each $n \in \mathbb{N}$, $\mu^*(T \cap U_n) \leqslant \mu^*(T \cap U)$ so the sequence $\{\mu^*(T \cap U_n)\}_{n\in\mathbb{N}}$ is bounded above. Further since $U_n \subset U_{n+1}$ this sequence is also non-decreasing, again by monotonicity. Since real-valued bounded monotone sequences convergence, the sequence convergences and its limit is bounded above by $\mu^*(U \cap T) < \infty$.

b) Note that $(T \cap U_{n+2} \backslash (T \cap U_{n+1})) \subset T \cap U^c_{n+1}$ and we estbalished in Problem 1 that dist $\left(U_n, U^c_{n+1}\right) > 0$ so by the condition given to us (which we shall call separated additivity) we get $\mu^* \left((T \cap U_n) \cup (T \cap U_{n+2} \backslash (T \cap U_{n+1}))\right) = \mu^*(T \cap U_n) + \mu^* \left(T \cap U_{n+2} \backslash (T \cap U_{n+1})\right)$. Observe also that $(T \cap U_n) \cup (T \cap U_{n+2} \backslash (T \cap U_{n+1})) \subset T \cap U_{n+2}$ so by monotonicity $\mu^*(T \cap U_n) + \mu^* \left(T \cap U_{n+2} \backslash (T \cap U_{n+1})\right) \leqslant \mu^*(T \cap U_{n+2})$ as required.

c) First note that $\bigcup_{n\in\mathbb{N}} C_n = T \cap U$. We then rewrite $\sum_{n\in\mathbb{N}} \mu^*(C_n) = \sum_{k\in\mathbb{N}} \mu^*(C_{2k-1}) + \sum_{k\in\mathbb{N}} \mu^*(C_{2k})$. Note that $\text{dist}(C_{n-1}, C_{n+1}) > 0$ as $C_{n-1} \subset U_n$ and $C_{n+1} \subset U^c_{n+1}$. Then by monotonicity and separated additivity, for all $N \in \mathbb{N}$ we can write $\sum_{k=1}^N \mu^*(C_{2k-1}) + \sum_{k=1}^N \mu^*(C_{2k}) = \mu^*(\bigcup_{k=1}^N C_{2k-1})\mu^*(\bigcup_{k=1}^N C_{2k}) \leqslant \mu^*(T \cup U) + \mu^*(T \cup U)$. Taking the limit $N \to \infty$ gives us that $\sum_{n\in\mathbb{N}} \mu^*(C_n) \leqslant 2\mu^*(T \cap U) < \infty$.

d) By definition, for all $n \in \mathbb{N}$, $T \cap U = (T \cap U_n) \cup (\bigcup_{k\geqslant n} C_k)$. Then, by countable subadditivity of μ^* it follows that $\mu^*(T \cap U) \leqslant \mu^*(T \cap U_n) + \sum_{k\geqslant n} \mu^*(C_k)$.

e) One equality follows from parts c) and d). Since the sum of $\mu^*(C_k)$ converges, it must be that $\lim_{n\to\infty} \sum_{k=n}^{\infty} \mu^*(C_k) = 0$. Then taking the limit $n \to \infty$ in part d) gives us that $\mu^*(T \cap U) \leqslant \lim_{n\to\infty} \mu^*(T \cap U_n)$. The other inequality is seen in part a) which gives us the desired equality.

f) For each $n \in \mathbb{N}$, construct U_n as in Problem 1. So clearly, $\text{dist}(U_n, U^c) > 0$. Also, $(T \cap U_n) \cup (T \cap U^c) \subset T$. Then by monotonicity and separated additivity of μ^*, we get the inequality $\mu^*(T \cap U_n) + \mu^*(T \cap U^c) \leqslant \mu^*(T)$. Since $U_n \nearrow U$, by lower semicontinuity of the outer measure we get that $\lim_{n\to\infty} \mu^*(T \cap U_n) + \mu^*(T \cap U^c) \leqslant \mu^*(T)$ which by part e) is the same $\mu^*(T) \geqslant \mu^*(T \cap U) + \mu^*(T \cap U^c)$.

g) Recall that for an outer measure μ^*, the set of μ^*-measurable sets $\mathcal{M}$ is a σ-algebra. Therefore, to show $\mathcal{B}_{\mathbb{R}} = \sigma(\mathcal{O}) \subset \mathcal{M}$ it suffices to show that $\mathcal{O} \subset \mathcal{M}$

i.e. every open set is μ^*-measurable. But this is exactly what part f) shows as the other inequality is always true by finite subadditivity.

Problem 3*: a) We will show (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (i) is obvious as cubes are rectangles.

(1) Let $M \subset \mathbb{R}^n$ and let $\epsilon > 0$. Assume that we can cover M with a countable collection of rectangles $\{R_k\}_{k\in\mathbb{N}}$ such that the sum of their content is less than $\epsilon/2$. Furthermore, by density of rationals and continuity of the finite product, we can expand the end points of each interval of each rectangle R_k by some amount less than $\epsilon/(n2^{k+1})$ so we get a new rectangle R'_k with rational endpoints, with $R_k \subset R'_k$ and $|R'_k| < |R_k| + \epsilon/2^{k+1}$. Each such rectangle R'_k can be covered exactly with finitely many cubes $Q_{k1}, Q_{k2}, \ldots, Q_{km_k}$ so the sum of their contents equals $|R'_k|$. Then $M \subset \bigcup_{k\in\mathbb{N}} \bigcup_{i\in[m_k]} Q_{ki}$ and $\sum_{k\in\mathbb{N}} \sum_{i\in[m_k]} |Q_{ki}| = \sum_{k\in\mathbb{N}} |R'_k| \leqslant \sum_{k\in\mathbb{N}} |R_k| + \epsilon/2 < \epsilon$.

(2) For each cube Q_k, we can cover it with a ball at the center of the cube and radius $\sqrt{n}(b_1 - a_1)$. Its "volume" is $(n^{1/2})^n (b_1 - a_1)^n$ which is a factor of $n^{n/2}$ greater than the cube. Let $\epsilon > 0$ and $M \subset \mathbb{R}^n$ arbitrary that can be covered with cubes whose sum of content is less than $\epsilon/n^{n/2}$. Cover each cube Q_k with a ball with center at the center of the cube and radius $r_k = b_1^{(k)} - a_1^{(k)}$. Clearly the balls also cover M and $\sum_{k\in\mathbb{N}} r_k^n < n^{n/2} \sum_{k\in\mathbb{N}} |Q_k| < \epsilon$.

(3) We can repeat a similar argument but instead, here we cover balls with cubes. We use a side length of $2r$ so that the "volume" scales up by a factor of 2^n but we can repeat the same trick by first getting (by assumption) a cover with balls of total volume less than $\epsilon/2^n$. Then we can expand the balls to cubes with total volume still less than ϵ.

b) $\Rightarrow$ Let $\epsilon > 0$ and assume that M is a set of measure zero so it has a cover of open balls $M \subset \bigcup_{k\in\mathbb{N}} B_k$ such that $\sum_{k\in\mathbb{N}} r_k^n < \epsilon$. Set $B = \bigcup_{k\in\mathbb{N}} B_k$ itself, which is a Borel set as it is a countable union of open sets. This set has measure zero as it is covered by itself. $\Leftarrow$ Now assume that M is such that there exists a borel set B of measure zero such that $M \subset B$. For each $\epsilon > 0$, we simply use the cover by balls of B for M which shows that M is also of measure zero.

c) Let $\epsilon > 0$ so for each M_k there exists a collection of open balls $\{B_{kj}\}_{j\in\mathbb{N}}$ such that $\sum_{j=1}^{\infty} r_{kj}^n < \epsilon/2^k$. Then $\bigcup_{k\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} B_{kj}$ is a countable cover (countable union of countable sets is countable) of M and $\sum_{k\in\mathbb{N}} \sum_{j\in\mathbb{N}} r_{jk}^n < \sum_{k\in\mathbb{N}} \epsilon/2^k = \epsilon$. Therefore, M is a set of measure zero.

Problem 4*: a) First we assume that N is bounded so that f is Lipschitz continuous on V with constant L. Let $\epsilon > 0$. Since N is of measure zero, there exists a countable cover of cubes $\{Q_k\}_{k\in\mathbb{N}}$ of N such that $\sum_{k\in\mathbb{N}} |Q_k| < \epsilon/L^n$. Denote $Q_k = [a_1^{(k)}, b_1^{(k)}] \times \cdots \times [a_n^{(k)}, b_n^{(k)}]$. By Lipschitz continuity, for each $j \in [n]$,

$|f(b_j^{(k)}) - f(a_j^{(k)})| \leqslant L|b_j^{(k)} - a_j^{(k)}|$ and so $f(Q_k) \subset Q'_k$ where Q'_k is some cube with $|Q'_k| \leqslant L^n|Q_k|$. Then, $f(N) \subset \bigcup_{k\in\mathbb{N}} Q'_k$ and $\sum_{k\in\mathbb{N}} |Q'_k| \leqslant L^n \sum_{k\in\mathbb{N}} |Q_k| < \epsilon$ so $f(N)$ is of measure zero. If N is not bounded, we can cover it by a countable union of bounded sets N_i for $i \in \mathbb{N}$, each of measure zero and then $f(N) \subset \bigcup_{i\in\mathbb{N}} f(N_i)$ where each $f(N_i)$ has measure zero and therefore by part c) of the previous problem $f(N)$ has measure zero.

b) Let $\epsilon > 0$. By the previous part, it suffices to show that the set A of points with n-th coordinate equal to 0 is of measure zero as this is homeomorphic to the affine hyperplane being considered. Furthermore, it suffices to show that $B = A \cap [0,1]^n$ is of measure zero because A is a countable unions of translated copies of B and we can then use Problem 3 part c). But now this is easy as we can just cover A with the rectangle $R = [0,1]^{n-1} \times [-\epsilon/3, \epsilon/3]$ which has $|R| = 2\epsilon/3 < \epsilon$ and so we are done.

Problem 5*: Let $\mathcal{R}_k$ denote the family of k-dimensional open rectangles. Recall that $\mathcal{B}_k = \sigma(\mathcal{R}_k)$. It suffices to show that $\mathcal{R}_k \otimes \mathcal{R}_n = \mathcal{R}_{k+n}$. We will show this by two inclusions. Let $R_1 \in \mathcal{R}_k, R_2 \in \mathcal{R}_n$ so we can write $R_1 = (a_1, b_1) \times \cdots \times (a_k, b_k)$ and $R_2 = (a_{k+1}, b_{k+1}) \times \cdots \times (a_{k+n}, b_{k+n})$ so then $R_1 \times R_2 \in \mathcal{R}_k \otimes \mathcal{R}_n$ equals $(a_1, b_1) \times \cdots \times (a_{k+n}, b_{k+n}) \in \mathcal{R}_{k+n}$.Therefore, $\mathcal{R}_k \otimes \mathcal{R}_n \subset \mathcal{R}_{k+n}$. The other inclusion follows identically as we can write an arbitrary element $(a_1, b_1) \times \cdots \times (a_{k+n}, b_{k+n}) \in \mathcal{R}_{k+n}$ as a product of the two elements $(a_1, b_1) \times \cdots \times (a_k, b_k) \in \mathcal{R}_k$ and $(a_{k+1}, b_{k+1}) \times \cdots \times (a_{k+n}, b_{k+n}) \in \mathcal{R}_n$.