

Homework 9

1) let K/F be purely insep with $\text{char}(F)=p$.

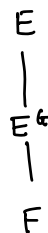
let $\alpha \in K$ has min poly $m_\alpha(x) \in F[x]$. we can write $m_\alpha(x) = \bar{m}_\alpha(x^{p^m})$ with m maximal so $p^m | \deg m_\alpha$. Then $\bar{m}_\alpha(x)$ sep and irred in $F[x]$ so α^{p^m} is sep over F . $\therefore \alpha^{p^m} \in F$ by inseparability. Set $a = \alpha^{p^m}$ then a is a root of $x^{p^m} - a \in F[x]$
 $\Rightarrow m_\alpha(x) | x^{p^m} - a \Rightarrow m_\alpha(x) = x^{p^m} - a$

2) Assume that the extension E^G/F is not purely inseparable i.e. \exists some $a \in E^G$ that is inseparable over F . This means that

$$\forall \sigma : E \xrightarrow{\sim} E \text{ over } F, \sigma(a) = a$$

It suffices to show E/E^G is not normal as that $\Rightarrow E/F$ not normal.

However, inseparability of a shows E/E^G is not normal \Rightarrow we are done by contraposition.

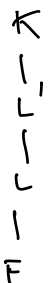


3) Recall, from the definition that a finite g is purely insep if F is the maximal separable subextension in E/F . we need to show that



if L/F is the max sep extension in K/F , then L is also the max sep extension in K/L .

Assume L is not i.e. there is a larger separable extension L'



such that L'/L is separable. Then L'/F is also separable so we have found a larger sep extension for $K/F \Rightarrow$

$\therefore L$ is the max sep ext of K/L
 $\Rightarrow K/L$ is purely inseparable \square

4) let K/F be an algebraic field extension.

let $f: K \rightarrow K$ be a F -homomorphism. It is clear that this homomorphism is injective as it is a field homo.

to show surjection, let $a \in F$. since K/F alg, $\exists p \in F[x]$ with root a . let K_p denote the set of roots of $p(x)$ in K , which is a finite set. Also $f(K_p) \subset K_p$ (f maps a root of $p(x)$ to another root).

$\therefore f$ induces an injective map $K_p \rightarrow K_p$ and since K_p finite, this is a surjection so $\exists b \in K$ st. $f(b) = a$ ■

5)

let $\sigma: K \rightarrow K$ be a field iso over F .

consider K^σ the subfield of σ -invariant elements.
 $K^\sigma \subset K$.

consider

$$\begin{array}{c} K \\ | \\ L \\ | \\ K^\sigma \end{array}$$

ie L is also invariant under σ
and L is finite ie $L = K(\alpha_1, \dots, \alpha_n)$
 $\therefore \sigma(\alpha_i) = \alpha_i \quad \forall i$.

consider L/K^σ , we need to show that the extension is cyclic.

let $\alpha \in L/K^\sigma$, then $\sigma(\alpha) = \alpha^i$ for some i
 $\therefore \forall \beta \in L/K^\sigma \exists i$ st. $\sigma^i(\alpha) = \beta$. \therefore

we see that the extension has a cyclic group of automorphisms and is Galois so $\text{Gal}(L/K^\sigma)$ is cyclic.

6) Over a perfect field, every algebraic field extension is separable. Let F be perfect. We need to show every extension E/F is perfect. i.e. every L/E is separable. Consider the tower.

$$\begin{array}{c} L \\ | \\ E \\ | \\ F \end{array}$$

Then L/F is separable $\Rightarrow L/E$ is separable.
Therefore E is perfect as well. ■

7) as functions $K \rightarrow K$

$$m_{\alpha+\beta}(x) = (\alpha+\beta)x = \alpha x + \beta x = (m_\alpha + m_\beta)(x)$$

$$m_{\alpha \circ \beta}(x) = \alpha \beta x = m_\alpha \beta(x)$$

Picking a basis for K/F , and passing to a matrix:

$$\text{Tr}_{K/F}(x+x') = \text{Tr}([m_x + m_{x'}]) = \text{Tr}_{K/F}(x) + \text{Tr}_{K/F}(x')$$

$$N_{L/K}(xx') = \det([m_{xx'}]) = \det([m_x] \cdot [m_{x'}]) = N_{K/F}(x) \cdot N_{K/F}(x')$$

8)

$$\begin{array}{c} L \\ | \quad m \\ F \\ | \quad d \\ K \end{array}$$

Let $\{e_1, \dots, e_m\}$ be F -basis for L ,
 $\{f_1, \dots, f_d\}$ be K -basis of F .

$$\therefore \{e_1 f_1, \dots, e_1 f_d, \dots, e_m f_1, \dots, e_m f_d\}$$

$$\text{let } \alpha \in L \quad \alpha e_j = \sum_{i=1}^m c_{ij} e_i, \quad c_{ij} f_1 = \sum_{r=1}^d b_{ijr} f_r,$$

for $c_{ij} \in F$, $b_{ijr} \in K$.

$$[m_d]_{L/F} = (c_{ij}), \quad [m_{c_{ij}}]_{F/K} = (b_{ijrs}), \quad [m_d]_{L/K} = ([m_{c_{ij}}]_{F/K}).$$

$$\therefore \text{Tr}_{F/K}(\text{Tr}_{L/F}(x)) = \text{Tr}_{F/K} \left(\sum_i c_{ii} \right)$$

$$= \sum_i \text{Tr}_{F/K}(c_{ii})$$

$$= \sum_i \sum_r b_{iirr}$$

$$= \text{Tr}_{L/K}(x).$$

The argument above can be replicated for the norm as well, with some linear algebra manipulations

9) Assume \exists a linear relation:

$$\sum_{k=1}^n c_k \sqrt{p_k} + c_0 = 0$$

let L/\mathbb{Q} be smallest extension containing all $\sqrt{p_k}$.

if $d \in \mathbb{N}$ is not a perfect square $T(\sqrt{d}) = 0$.

also $T(q) = 0$ iff $q = 0$ for rational q .

\therefore By taking trace of both sides, $c_0 = 0$. let $1 \leq j \leq n$ and multiply by $\sqrt{p_j}$

$$c_j p_j + \sum_{\substack{1 \leq k \leq n \\ k \neq j}} c_k \sqrt{p_k p_j} = 0.$$

taking trace again gives $T(c_j p_j) = 0 \Rightarrow c_j = 0$.

\therefore linear independence ■

10) if $\sigma \in G$, then all automorphism in $\sigma K \sigma^{-1}$ fix $\sigma(B)$

\therefore given $\sigma \in G$, galois correspondence associates $\sigma K \sigma^{-1}$ subgroup with the field $\sigma(B)$.

In other words, $\text{Gal}(E/\sigma(B)) = \sigma K \sigma^{-1}$ \therefore the fixed field of $\sigma K \sigma^{-1}$ is equal to $\sigma(B)$.

$\therefore \sigma(B) = B$ iff σ normalizes K

$$\therefore \text{Gal}(K^H/K) \cong N_G(H)/H$$