Homework 10

1) let $\lim_{n\to\infty} \sum_{k=0}^{n} a_k = L$

$\{b_n\}$ is monotonic and bounded.

WLOG, let $\{b_n\}$ be non-decreasing and $\forall n \in \mathbb{N} : b_n \leq B$

consider $\{\sum_{k=0}^{n} a_k b_k\}_{n\in\mathbb{N}}$

$$\sum_{k=0}^{n} a_k b_k \leq \sum_{k=0}^{n} B a_k$$

$$\leq B \sum_{k=0}^{n} a_k$$

by properties of limits, we know that $\lim_{n\to\infty} \left\{ B \sum_{k=0}^{n} a_k \right\} = BL$

$\therefore \sum_{n=0}^{\infty} a_n b_n \leq BL$ so it converges.

2) let $\sum_{n=0}^{\infty} a_n = A$

assume $\{b_n\}_{n\geq 0}$ is positive nondecreasing and unbounded.

let $s_n = \frac{1}{b_n} \sum_{k=1}^{n} a_k b_k$

since $\{b_n\}$ is non-decreasing.

$$s_n \leq \frac{1}{b_n} \sum_{k=1}^{n} a_k b_n$$

$$\therefore s_n \leq \frac{1}{b_n} \cdot b_n \sum_{k=1}^{n} a_k$$

$$\therefore s_n \leq \sum_{k=1}^{n} a_k$$

which shows s_n bounded but we need to show its limit is 0.

i.e we want to show that $\forall \varepsilon > 0.$

$\exists n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$

$$\frac{1}{b_n} \sum_{k=1}^{n} a_k b_k < \varepsilon$$

Let A_n denote partial sums $\sum_{k=0}^{n} a_k$

then we can write

$$\frac{1}{b_n} \sum_{k=1}^{n} b_k a_k = A_n - \frac{1}{b_n} \sum_{k=1}^{n-1} (b_{k+1} - b_k) A_k$$

now let $\varepsilon > 0$. Chose n_0 so that A_k is ε-close to A $\forall$ $k > n_0$

and we rewrite the R.H.S as:

$$A_n - \frac{1}{b_n} \sum_{k=1}^{n_0-1} (b_{k+1} - b_k) A_k - \frac{1}{b_n} \sum_{k=n_0}^{n-1} (b_{k+1} - b_k) A_k$$

$$- \frac{1}{b_n} \sum_{k=n_0}^{n-1} (b_{k+1} - b_k) A - \frac{1}{b_n} \sum_{k=n_0}^{n-1} (b_{k+1} - b_k)(A_k - A)$$

$$- \frac{b_n - b_{n_0}}{b_n} A - \frac{1}{b_n} \sum_{k=n_0}^{n-1} (b_{k+1} - b_k)(A_k - A)$$

Taking the limit as $n \to \infty$, we get:

$$\cancel{A} - 0 - \cancel{A} - \frac{1}{b_n} \sum_{k=N}^{n-1} (b_{k+1} - b_k)(S_k - S)$$

$$\leq \varepsilon \frac{(b_n - b_N)}{b_n} \leq \varepsilon \quad \blacksquare$$

3 a) $\alpha = \limsup_{n\to\infty} |n^3|^{1/n} = \left(\lim_{n\to\infty} n^{1/n}\right)^3 = 1$

$\therefore R = 1$

b) using the ratio test $\alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{2}{n+1}\right| = 0$

$\therefore R = \infty$

c) using the ratio test $\alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2n^2}{(n+1)^2} = 2$

$\therefore R = 1/2$

d) using the ratio test $\alpha = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)^3}{n^3 \cdot 3} = \frac{1}{3}$

$\therefore R = 3$

4) let $a_n > 0$ and assume $\sum_{n=0}^{\infty} a_n < \infty$.

if $m < n$ then $r_n < r_m$ since $a_n > 0$.

$$r_m - r_n = a_m + a_{m+1} + \cdots + a_n$$

$$\therefore \frac{r_m - r_n}{r_m} = \frac{a_m}{r_m} + \frac{a_{m+1}}{r_m} + \cdots + \frac{a_n}{r_m}$$

$$\therefore 1 - \frac{r_n}{r_m} < \frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} = \sum_{k=m}^{n} \frac{a_k}{r_k}$$

$\therefore \sum_{k=1}^{n} \frac{a_k}{r_k}$ is not Cauchy $\therefore$ it diverges.

b) since $r_n > r_{n+1}$,

$$1 + \sqrt{\frac{r_{n+1}}{r_n}} < 2$$

since $r_n = a_n + r_{n+1}$, we get $\frac{a_n}{\sqrt{r_n}} < 2\left(\sqrt{r_n} - \sqrt{r_{n+1}}\right)$

Note that $\sum a_n$ converges and $r_n \to 0$

$$\therefore \sum_{k=1}^{n} \frac{a_k}{\sqrt{r_k}} < 2\left(\sqrt{r_1} - \sqrt{r_{n+1}}\right)$$

$$\lim_{n\to\infty} 2\left(\sqrt{r_1} - \sqrt{r_{n+1}}\right) = 2r_1$$

$\therefore$ the series converges.

5) let $\lim_{n\to\infty} s_n = s$

let $\varepsilon > 0$, $\exists n_0 : \forall n \geq n_0$

$|s - s_n| < \varepsilon$

$$\therefore |\sigma_n - s| = \frac{1}{n+1} \left| (s_0 - s) + \cdots + (s_{n_0} - s) + \cdots + (s_n - s) \right|$$

$$< \frac{1}{n+1} \left| \left(\underbrace{(s_0 - s) + \cdots + (s_{n_0} - s)}_{\text{call this } K} , + (n+1-n_0)\varepsilon \right) \right.$$

then for $n \geq N$

$$|\sigma_n - s| < \frac{K + (n+1-n_0)\varepsilon}{n+1}$$

as $n \to \infty$, R.H.S $\to 0$

$\therefore \lim \sigma_n \to s$

b) let $s_n = (-1)^n$, which we know diverges.

however, $\sigma_n = \dfrac{\sum_{k=0}^{n} (-1)^k}{n+1}$ has an even subsequence $\left\{\dfrac{1}{n+1}\right\}$

and odd subsequence $\left\{\dfrac{-1}{n+1}\right\}$

both of which converge to 0.

c) consider $\{\log 1, 0, \log 3, 0, 0, \log 6, 0, 0, 0, \log 10, \cdots\}$

then $\limsup a_n = \lim_{n\to\infty} \log n = \infty$

but $\sigma_n = \dfrac{\log 1 + \log 3 + \cdots + \log k}{n+1}$ where $k = j(j+1)/2$ for greatest $j < n$

$$\sigma_n \le \frac{\sqrt{2(n+1)}\ \log(n+1)}{n+1} \to 0$$

as $n \to \infty$

d) let $a_k = s_k - s_{k-1}$

$$\sum_{k=1}^{n} k a_k = n s_n - s_{n-1} - \cdots - a_0$$

$$= (n+1) s_n - (s_0 + \cdots + s_n)$$

$$\therefore \frac{1}{n+1} \sum_{k=1}^{n} k a_k = s_n - \sigma_n$$

By assumptions,

let $\epsilon > 0$, $\exists n_0$ s.t. $\forall n > n_0$

$$|n a_n| < \epsilon$$

$$\therefore \left| \frac{1}{n+1} \sum_{k=1}^{n} k a_k \right| < \frac{1}{n+1} \Big(\underbrace{\sum_{k=1}^{n_0 - 1} |k a_k|}_{\text{call this } K} + (n - n_0)\epsilon \Big)$$

choosing $\epsilon = 1/n$

R.H.S converges to 0

$$\therefore \left\{ \frac{1}{n+1} \sum_{k=1}^{n} k a_k \right\} \to 0$$

since σ_n converges, so does s_n.

e) $$\frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m} \sum_{j=m+1}^{n} (s_n - s_j)$$

$$= \frac{m+1}{n-m}\sigma_n - \frac{m+1}{n-m}\sigma_m + s_n - \frac{1}{n-m} \sum_{j=m+1}^{n} s_j$$

$$= s_n + \frac{m+1}{n-m}\sigma_n - \frac{\sum_{j=1}^{n} s_j}{n-m}$$

$$= s_n + \frac{1}{n-m}\left((m+1) \sum_{j=1}^{n} s_j - (n+1) \sum_{j=1}^{n} s_j \right) = s_n - \sigma_n$$

$$|s_n - s_i| = |a_{i+1} + \cdots + a_n|$$
$$= \frac{1}{i+1} \left(|(i+1)a_i| + \cdots + |(i+1)a_n| \right)$$
$$\le \frac{(n-i)M}{i+1}$$

since $m<i$:

$$|s_n - s_i| \le \frac{(n-i)M}{i+1} \le \frac{(n-m-1)M}{m+1}$$

for fixed $\varepsilon>0$

pick n s.t

$$m \le \frac{n-\varepsilon}{1+\varepsilon} < m+1$$

to get $\quad \frac{m+1}{n-m} \le \frac{1}{\varepsilon} \le \frac{m+2}{n-m-1}$

$\therefore \lim_{n\to\infty} \sup|s_n - \sigma_n| \le M\varepsilon.$

$\therefore \lim s_n = \sigma.$

6) let $\sum_{n=1}^{\infty} |a_n| < \infty.$

$\forall n \in \mathbb{N}$ from previous H.W.

$$|a_1 + \cdots + a_n| \le |a_1| + |a_2| + \cdots + |a_n| \le \sum_{n=1}^{\infty} |a_n|.$$

$\therefore \forall m \in \mathbb{N} \qquad \left|\sum_{j=1}^{n} a_j\right| \le \sum_{m=1}^{\infty} |a_m|$

$\therefore$ take the limit $n\to\infty$,

we get that $\quad \left|\sum_{n=1}^{\infty} a_n\right| \le \sum_{n=1}^{\infty} |a_n|$

If $\sum_{n=1}^{\infty} |a_n| = \infty$, then inequality is true.