Math 245A - Real Analysis: Homework #2

Due on October 13, 2023

Professor Mario Bonk

Nakul Khambhati

Problem 1

(a) We are required to show that for all $m < \mu(M)$ there exists $U \in \mathcal{A}$ such that $m < \mu(U) \leq \mu(M)$ and $\overline{U} \subset M$. Let $M \in \mathcal{A}$ so we can write it as $M = \bigsqcup_{i\in[n]}(a_i, b_i]$. Let $m = \mu(M) = \sum_{i=1}^{n}(b_i - a_i)$. Set $U = \bigsqcup_{i\in[n]}(a_i - \epsilon, b_i] \in \mathcal{A}$ so that $\overline{U} = \bigsqcup_{i\in[n]}[a_i - \epsilon, b_i] \subset M$. Then $\mu(U) = \sum_{i=1}^{n}(b_i - a_i) - \epsilon n$. By setting $\epsilon < \frac{\mu(M)-m}{n}$ we get that $\mu(M) > m$ as required.
Next we show that for each $\epsilon > 0$ there exists $V \in \mathcal{A}$ such that $\mu(V) \leq \mu(M) + \epsilon$ and $M \subset int\,(V)$. Again we let $M \in \mathcal{A}$ arbitrary so we can write it as above. We set $V = \bigsqcup_{i\in[n]}(a_i, b_i + \epsilon/2^n] \in \mathcal{A}$ so clearly $M \subset int(V) = \bigsqcup_{i\in[n]}(a_i, b_i + \epsilon/2^n)$. We calculate $\mu(V) = \sum_{i=1}^{n}(b_i - a_i) + \epsilon \sum_{i=1}^{n} 2^{-n} < \mu(M) + \sum_{i=1}^{\infty} 2^{-n} = \mu(M) + \epsilon$ as required. (In hindsight, any $\epsilon' < \epsilon/n$ would have worked.)

(b) We use the facts from part (a) along with a covering argument to show that μ is countably additive, proving that it is a premeasure. We will show the equality by proving two inequalities. We start with the harder one i.e. $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i)$.
Let $\epsilon > 0$. By part (a), we can find U_ϵ such that $\overline{U_\epsilon} \subset A$ and $\mu(A) \leq \frac{\epsilon}{2} + \mu(U_\epsilon)$. Similarly, for each $n \in \mathbb{N}$, we can find $V_n \in \mathcal{A}$ such that $A_n \subset int\,(V_n)$ and $\mu(V_n) \leq \mu(A_n) + \frac{\epsilon}{2}2^{-n}$. By the transitivity of inclusion note that $\overline{U_\epsilon} \subset \bigcup_{n\in\mathbb{N}} int(V_n)$. We have an open covering of a compact set so we can find a finite subcovering $int(V_{n_1}), \ldots, int(V_{n_k})$ so that $U_\epsilon \subset \overline{U_\epsilon} \subset \bigcup_{i\in[k]} int\,(V_{n_i}) \subset \bigcup_{i\in[k]} V_{n_i}$. Finite additivity and monotonicty from HW1 Q4 gives us subadditivity of μ therefore $\mu(U_\epsilon) \leq \sum_{i=1}^{k} \mu(V_{n_i}) \leq \sum_{i=1}^{\infty} \mu(V_i) \leq \sum_{i=1}^{\infty} \mu(A_i) + \frac{\epsilon}{2}$. Combining this with the inequality on the first line of this paragraph, we get $\mu(A) \leq \sum_{i=1}^{\infty} \mu(A_i) + \epsilon$. Let $\epsilon \to 0$ gives us the required inequality.
The other side is easy to see. By finite additivity and monotonicity of μ, for all $n \in \mathbb{N}$ we have $\sum_{i=1}^{n} \mu(A_i) = \mu\left(\bigsqcup_{i=1}^{n} A_i\right) \leq \mu\left(\bigsqcup_{i=1}^{\infty} A_i\right)$. Then taking $n \to \infty$ gives us the desired inequality.

Problem 2

(a) Define $\lambda(\mathcal{S}) = \bigcap_{\mathcal{F}\supset\mathcal{S}} \mathcal{F}$ where $\mathcal{F}$ is a λ-system. If this is a λ-system then clearly it is the smallest such as it contains every other λ-system that contains $\mathcal{S}$. Checking that it is a λ-system is straightforward:

(i) Since $X \in \mathcal{F}$ for each λ-system $\mathcal{F}$ containing $\mathcal{S}$, we get that $X \in \bigcap_{\mathcal{F}\supset\mathcal{S}} \mathcal{F}$.

(ii) Let $A, B \in \bigcap_{\mathcal{F}\supset\mathcal{S}} \mathcal{F}$ such that $B \subset A$ so $A \backslash B \in \mathcal{F}$ for each λ-system $\mathcal{F}$ containing $\mathcal{S}$ so $A \backslash B \in \bigcap_{\mathcal{F}\supset\mathcal{S}} \mathcal{F}$.

(iii) Let $\{A_n\}_{n\in\mathbb{N}} \in \bigcap_{\mathcal{F}\supset\mathcal{S}} \mathcal{F}$ with $A_n \nearrow$ so $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$ for each λ-system F containing $\mathcal{S}$ so $\bigcup_{n\in\mathbb{N}} A_n \in \bigcap_{\mathcal{F}\supset\mathcal{S}} \mathcal{F}$.

(b) We have to show that $\lambda(\mathcal{S})$ is a π-system i.e. it is closed under intersections. For any $T \in \lambda(\mathcal{S})$ we define $\mathcal{D}_T = \{A \in \lambda(S) : A \cap T \in \lambda(\mathcal{S})\}$ so it is the subset of elements whose intersection with T stays in $\lambda(\mathcal{S})$. Clearly if for all $T \in \lambda(\mathcal{S})$ we get that $\lambda(\mathcal{S}) \subset \mathcal{D}_T$ then $\lambda(S)$ is a π-system. Note that is suffices to check that $\mathcal{D}_T$ is a λ-system that contains $\mathcal{S}$ as then, by definition, it contains $\lambda(\mathcal{S})$ the smallest λ-system containing $\mathcal{S}$. Fix $T \in \lambda(\mathcal{S})$. Checking that $\mathcal{D}_T$ is a λ-system:

(i) Then $X \in \mathcal{D}_T$ as $X \cap T = T \in \lambda(S)$.

(ii) Let $A, B \in \mathcal{D}_T$ so $A \cap T, B \cap T \in \lambda(\mathcal{S})$. Since $\lambda(\mathcal{S})$ is a λ-system, $(A \cap T)\backslash(B \cap T) = (A\backslash B) \cap T \in \lambda(\mathcal{S})$ so $A \backslash B \in \mathcal{D}_T$.

(iii) Let $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{D}_T$ so for each $n \in \mathbb{N}$ $A_n \cap T \in \lambda(\mathcal{S})$. Then by $\lambda(\mathcal{S})$ being a λ-system, $\bigcup_{n\in\mathbb{N}}(A_n \cap T) = (\bigcup_{n\in\mathbb{N}} A_n) \cap T \in \lambda(\mathcal{S})$ so $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{D}_T$.

We are left to show that each $\mathcal{D}_T$ contains $\mathcal{S}$. It is clear that D_S contains $\mathcal{S}$ for each $S \in \mathcal{S}$ since $\mathcal{S}$ is a π-system. So $T \cap S \in \lambda(\mathcal{S})$ for $T \in \lambda(S), S \in \mathcal{S}$. Therefore, $S \in \mathcal{D}_T$ for $S \in \mathcal{S}$ so $\mathcal{S} \in \mathcal{D}_T$.

Problem 2 continued on next page. . .

(c) One inclusion is clear: $\lambda(\mathcal{S}) \subset \sigma(\mathcal{S})$ since the conditions for being a λ system are weaker than those required for being a σ-algebra. In particular, every σ-algebra containing $\mathcal{S}$ is also a λ-system so $\sigma(\mathcal{S})$ is a λ system and then the inclusion is clear. Given that $\mathcal{S}$ is a π-system, we need to show that $\lambda(\mathcal{S}) \supset \sigma(\mathcal{S})$. We will prove that for λ-systems, it is equivalent to be a π-system and a σ-algebra. Then, since we saw in part (a) that $\lambda(\mathcal{S})$ is a π-system and (by definition) it is a λ-system it follows that it is a σ-algebra containing $\mathcal{S}$ from which we conclude that it must contain the smallest σ-algebra containing $\mathcal{S}$. we proceed to show that if $\mathcal{A} \subset 2^X$ is a λ system, then $\mathcal{A}$ is a π-system $\iff$ $\mathcal{A}$ is a σ-algebra. $\Leftarrow$ is obvious because an algebra is closed under intersection. To check $\Rightarrow$, assume that $\mathcal{A}$ is both a π and λ-system and check the conditions for being a σ-algebra.

(i) Clearly $X \in \mathcal{A}$

(ii) Let $A \in \mathcal{A}$. Then $A^c = X \backslash A \in \mathcal{A}$ by closure under differences of subsets of a λ-system.

(iii) First note that the set is closed under finite unions since for $A, B \in \mathcal{A}$, $A \cup B = (A^c \cap B^c)^c$ and the system is closed under complements and intersections. Let $\{B_n\}_{n\in\mathbb{N}} \in \mathcal{A}$ and construct $A_n = \bigcup_{i\in[n]} B_i \in \mathcal{A}$. Clearly, $B_n \nearrow$ so $\bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Problem 3

From the previous question recall that $\mathcal{A} = \lambda(\mathcal{S})$. We will directly prove part (b) which subsumes part (a) since we can take $S_1 = X$ and $S_i = \emptyset$ for $i > 1$ which reduces part (b) to part (a). We proceed in 2 steps. First, we show that for all $A \in \mathcal{A}$ and for any $S \in \mathcal{S}$, $\mu(A \cap S) = \nu(A \cap S)$. Then, we lift this property to all $A \in \mathcal{A}$ using the $S_i \in \mathcal{S}$.
As we did in the previous problem, for a fixed $S \in \mathcal{S}$ consider the set $\mathcal{D}_S = \{A \in \mathcal{A} : \mu(A \cap S) = \nu(A \cap S)\}$. Then the goal is to show that for all $S \in \mathcal{S}$, $\mathcal{A} \subset \mathcal{D}_S$. Since $\mathcal{A} = \lambda(S)$ and $S \subset \mathcal{D}_S$, it suffices to show that $\mathcal{D}_S$ is a λ-system.

(a) Clearly $X \in \mathcal{D}_S$.

(b) Let $A, B \in \mathcal{D}_S$ such that $B \subset A$. Then $\mu((A\backslash B) \cap S) = \mu(A \cap S) - \mu(B \cap S) = \nu(A \cap S) - \nu(B \cap S) = \nu((A\backslash B) \cap S)$ so $A\backslash B \in \mathcal{D}_S$.

(c) Let $A_n \in \mathcal{D}_S, A_n \nearrow A$ so that $A_n \cap S \nearrow A \cap S$. Then $\mu(A\cap S) = \sum_{n=1}^{\infty} \mu(A_n \cap S) = \sum_{n=1}^{\infty} \nu(A_n \cap S) = \nu(A \cap S)$.

Next, we show that this equality holds on all $A \in \mathcal{A}$. Define $T_n = \bigcup_{i\in[n]} S_n$. Then we can write $T_n = \bigsqcup_{i\in[n]} S_i \backslash T_{i-1} = \bigsqcup_{i\in[n]} T_{i-1}^c \cap S_i$. Therefore $\mu(A \cap T_n) = \sum_{i=1}^{n} \mu((A \cap T_{i-1}^c) \cap S_i) = \sum_{i=1}^{n} \nu((A \cap T_{i-1}^c) \cap S_i) = \nu(A \cap T_n)$. Finally since $T_n \nearrow X$ and by lower semicontinuity of μ and ν, by taking $n \to \infty$ we get that $\mu(A) = \nu(A)$.

Problem 4

(a) We verify the 3 conditions for $\mathcal{A}$ to be an algebra:

(i) Let $\epsilon > 0$. Let $U = \mathbb{R}^n$ and let $K = \overline{B}_N(0)$ so $K \subset X \subset U$. Then for large enough N, since $\mu(\mathbb{R}^n) < \infty$, we get that $\mu(U\backslash K) = \mu(U) - \mu(K) < \epsilon$. This shows that $X \in \mathcal{A}$.

(ii) Next we show closure under complement. Let $A \in \mathcal{A}$ so there exists a compact K and an open U such that $K \subset A \subset U$ and $\mu(U\backslash K) < \epsilon$. Then $U^c \subset A^c \subset K^c$ and $\mu(K^c \backslash U^c) = \mu(U\backslash K) < \epsilon$. This satisfies the conditions as U^c is closed (and bounded) so it is compact and K^c is open.

(iii) Finally, we show closure under union. Let $A_1, A_2 \in \mathcal{A}$ so there exists U_1, U_2 open and K_1, K_2 compact such that $K_i \subset A_i \subset U_i$ and $\mu\left(U_i \backslash K_i\right) < \epsilon/2$ for $i = 1, 2$. Then $K_1 \cup K_2 \subset A_1 \cup A_2 \subset U_1 \cup U_2$ and $\mu\left((U_1 \cup U_2) \backslash (K_1 \cup K_2)\right) = \mu((U_1 \backslash K_1) \cup (U_2 \backslash K_2)) < \epsilon/2 + \epsilon/2 = \epsilon$.

(b) Let $A_n \nearrow A$ and $A_n \in \mathcal{A}$. Therefore for each n we have $K_n \subset A_n \subset U_n$ as above with $\mu(U_n \backslash K_n) < \epsilon/2^n$. Then $U = \bigcup_{n \in \mathbb{N}} U_n$ is an open set and $K = \bigcap_{n \in \mathbb{N}} K_n$ is compact and we can write $K \subset A \subset U$ with $\mu(U \backslash K) = \mu\left(\bigcup_{n \in \mathbb{N}} U_n \backslash K_n\right) \leq \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon$. Therefore, $A \in \mathcal{A}$.

(c) Recall that the Borel algebra on $\mathbb{R}^n$ is generated by R tbe set of rectangles in $\mathbb{R}^n$. Since we have already shown that $\mathcal{A}$ is a σ-algebra, in order to show that $\sigma(R) \subset \mathcal{A}$, it suffices to show $R \subset \mathcal{A}$. Let $A = [a, b]$ be an arbitrary rectangle where $a_i < b_i$ for all $i \in [n]$. Let $\epsilon > 0$. Consider $K = A$ which is compact and $U = (a - \epsilon/3n, b + \epsilon/3n)$ where this notation means that the buffer $\epsilon/3n$ is added to each coordinate. Then $K \subset A \subset U$ with U open and $\mu(U \backslash K) = \frac{2\epsilon}{3} < \epsilon$.