Q1] let $a_0, b_0 > 0$

$$a_{n+1} = \frac{a_n + b_n}{2} \quad , \quad b_{n+1} = \sqrt{a_n \cdot b_n}$$

using AM-GM inequality:

$$\forall n : n \geq 1 \Rightarrow b_n \leq a_n$$

$$\therefore a_{n+1} = \frac{a_n + b_n}{2} \leq \frac{a_n + a_n}{2} = a_n$$

$\therefore \{a_n\}_{n \in \mathbb{N}}$ is non-increasing

since $a_0, b_0 > 0 \quad \therefore a_n, b_n > 0 \quad \forall n \in \mathbb{N}$

$\therefore \{a_n\}$ is non-increasing and bounded below $\Rightarrow \lim\limits_{n \to \infty} a_n$ exists

call it $a_\infty$.

similarly, $b_{n+1} = \sqrt{a_n \cdot b_n} \geq \sqrt{b_n^2} = b_n$

$\therefore \{b_n\}$ is non-decreasing

but since $\forall n : b_n \leq a_n \wedge a_n < a_1$,

$\{b_n\}$ is bounded above by $a_1$ and is non-decreasing.

$\therefore \{b_n\}$ also converges

It suffices to show that $\sup b_n = \inf a_n$.

consider $L = \sup b_n$ i.e. $\forall n \in \mathbb{N} : b_n \leq L$

and we know $b_n \leq a_n$

assume that $\exists m \in \mathbb{N} : a_m < L$

then $a_m$ is another upper bound on $\{b_n\}$.

But this is a contradiction since $L$ is the least upper bound

$\therefore \forall n \in \mathbb{N} : b_n \leq L \leq a_n$

assume we have a smaller lower bound $L'$ on $\{a_n\}$ s.t

⋮

$\therefore L = \inf \{a_n\}$.

Q2] We need to prove that $f$ is a contraction map.

let $x, y \in \mathbb{R}^+$

$$|f(x)-f(y)| = \left| \frac{1}{1+x^n} - \frac{1}{1+y^n} \right| = \left| \frac{1+y^n-(1+x^n)}{(1+x^n)(1+y^n)} \right| = \left| \frac{y^n-x^n}{(1+x^n)(1+y^n)} \right|$$

WLOG, let $x \leq y$

$\therefore \; y^n - x^n \leq n(y-x)y^{n-1}$

$$\therefore \; \leq |y-x| \frac{ny^{n-1}}{(1+x^n)(1+y^n)}$$

$$\leq |y-x| \cdot c$$

where $c = \frac{ny^{n-1}}{(1+x^n)(1+y^n} \leq \frac{ny^{n-1}}{(1+y^n)^2} \leq \frac{ny^{n-1}}{1+2y^n+}$

. . . .

NTS $0 \leq c < 1$ (?)

*do we need countable as in infinite ??

Q3] We need $A \subseteq \mathbb{R}$ totally bounded and closed such that its limit points form a countable set.

let $A = \{ 1/n+1 , n \in \mathbb{N} \} \cup \{0\}$ which is closed.

it is totally bounded since $\forall r > 0$, $B(0,r)$ contains all but finitely many elements of $x$. Then $\{ a \in A \setminus B(0,r) \}$ is finite so it can be easily covered by $B(a,r)$.

This set is finite and covers $A$. $\therefore$ $A$ is compact and has $\{0\}$ is a limit point ■

Q4] let $A_n := ( 1/n+2 , 1/n )$ for $n \geq 1$

let $x \in (0,1)$. Then $\exists m \in \mathbb{N} : \frac{1}{m+2} < x < \frac{1}{m}$ by archimedean principle.

therefore $\{A_n\}_{n \geq 1}$ covers $(0,1)$.

let $\{A_n\}_{n \in I}$ be a finite subcollection i.e. $|I| < \infty$. $\therefore$ $\sup(I)$ exists and call it $k$. But then $\frac{1}{k+3}$ is not contained in the subcollection.

$\therefore$ it is not a subcover of $(0,1)$.

Q5) we use diagonalization to show it's not countable.

Assume there exist some enumerations $\{x_n\}_{n \in \mathbb{N}}$, $x_n = 0.d_{n1} d_{n2} \cdots$ where $d_{ij} \in \{4,7\}$.

Construct another sequence $d_n = \begin{cases} 4 & \text{if } d_{nn} = 7 \\ 7 & \text{otherwise} \end{cases}$

now consider $x = d_1 d_2 d_3 \cdots$ , $x \in E$

however we cannot have $x \notin \{x_n : n \in \mathbb{N}\}$ as it differs from $x_n$ at $n^{th}$ digit.
so we get a contradiction i.e $E$ is not countable.

ii) if $E$ dense in $[0,1]$ then $\exists$ a sequence that converges to $0$.
But for $\varepsilon = 0.4$
we cannot find any $x \in E$ s.t $|x-0| < \varepsilon$
i.e $\forall x \in E$ $|x-0| \geq \varepsilon$.
$\therefore E$ is not dense.

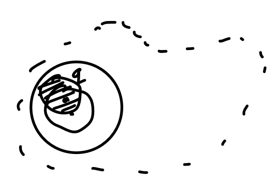
iii) let $E' \subset E$ be infinite. Since $E'$ is bounded, it has a limit point $x$. $\therefore$
every ball $B(x,r)$ has infinitely many points of $E'$.
let $r_n = \frac{1}{10^n}$, $\forall n \in \mathbb{N}$ $\exists x_n \in E'$ s.t $|x-x_n| < r_n$.

$\therefore x$ must have 4 or 7 in its first $n$ decimal places
$\therefore x \in E$, so $E$ is compact.

iv) we need to show that $E$ is closed with no isolated points.
However it is clear that $0.4 \in E$ is isolated so it is not perfect.

Q6) let $X$ be separable i.e it has a countable dense subset $A$ s.t $\overline{A} = X$
the natural next step is to consider $\{V_\alpha\}$ the collection of open balls in $X$ with rational radius and centers in $D$. It is countable as $A$ countable and $\mathbb{Q}$ countable. $\therefore D \times \mathbb{Q}$ is countable.

let $O \in X$ be open; let $x \in X$. Consider an open ball $B(x,r)$ s.t $B \subset U$.
Also consider $B' \subset B$ with radius $r/2$.



By density of $A$, $B' \cap A \neq \emptyset$. so let $z \in D \cap B'$.
we can find rational $q$ : $0 < \rho(x,z) < q < r/2$.
consider $B''(\rho(u,r), q)$. then $u \in B'' \subset O$ and $B'' \in \{V_\alpha\}$ so we are done ■

Q7) Let $X$ be a compact metric space.

Using the hint: Fix $n \in \mathbb{N}$ and consider the cover $\{U_n\}_{n\geq 1}$ where $U_n := B(x_n, 1/n)$ for some $x_n \in X$.

By compactness, for each $n \in \mathbb{N}$, we can also find a finite subcover $V_n$ of $X$.

Consider the collection $B = \{V_n : n \in \mathbb{N}\}$ which is countable since each $V_n$ finite and there are countably many.

We are done if we show $B$ is a basis.

Let $x \in X$ and $O \subseteq X$ open s.t. $x \in O$. Then, we can find some $n \in \mathbb{N}$, such that $B(x, 1/n) \subseteq G$ and since $\forall m \geq n$: $V_m$ is an open covering $\therefore \exists y$ s.t. $x' \in B(y, 1/m) \subseteq V_m$. Then, $x \in B(y, 1/m) \subset G$ so $B$ is a basis for $X$.

It is clear that if $X$ is has a countable basis, it is separable.

Let $B$ be a countable basis $B = \{O_i : i \in \mathbb{N}\}$

Set $A = \{a_i : a_i \in O_i\}$ which is countable subset and dense since $\forall x \in X$, $\exists O_x \in B$ s.t. $x \in O_x$ by defn of a basis and then $\exists a_x \in O_x$ so.

$O_x \cap A \neq \emptyset$ ■

Q8) Using the hint, $X$ has a countable basis.

Let $\{V_n\}_{n\in\mathbb{N}}$ be a countable basis. Let $\{U_\alpha\}_{\alpha\in J}$ be an open cover s.t. for all $x \in G_\alpha$, $\exists V_n$ s.t. $x \in V_n \subseteq G_\alpha$. $\therefore$ we can use this to show every cover has a countable subcover.

Let $\{G_n\}_{n\in\mathbb{N}}$ be the countable subcovering. Assume, by contr $\nexists$ finite subcover.

take $F_n = (G_1 \cup \cdots \cup G_n)^c$

then for each $n \in \mathbb{N}$, $F_n$ is non-empty and $F_1 \supset F_2 \supset \cdots$ is nested.

But since $\bigcup_{n=1}^{\infty} G_n$ covers $X$, $\bigcap_{n=1}^{\infty} F_n = \emptyset$.

Consider an infin subset $E \subseteq X$ s.t. $E = \{x_n \in F_n : n \in \mathbb{N}\}$. Let $x \in X$ be a limit point of $E$. Then $x \in G_n$ for some $n \in \mathbb{N}$. Since $G_n$ open, $\exists r$: $B(x, r) \subseteq G_n$.

Choose $m \geq n$, then $x_m \in F_m = (G_1 \cup \cdots \cup G_n \cup \cdots G_m)^c$ implies $x_m \notin B(x, \epsilon)$.

$\therefore B(x,\epsilon)$. has only finitely many points $\therefore$ we can choose $\epsilon' < \epsilon$ s.t
$B(x,\epsilon') \cap E = \emptyset$ which contradicts $x$ lim point of $E$ $\Leftarrow\Rightarrow$

$\therefore X$ has a finite subcover.

Q9] $s_1 = 0$, $s_2 = 1/2$
$s_3 = 1/4$, $s_4 = 3/4$
$s_5 = 3/8$, $s_6 = 7/8$

it seems as tho $\{s_{2m+1}\} \to 1/2$
and $\{s_{2m}\} \to 1$

By substitution $s_{2m} = 1/2\,(1/2 + s_{2(m-1)})$
claim: $s_{2m} = \frac{1}{2}\left(1 - \frac{1}{2^m}\right)$

clearly true for $m=0$
assume for $m$.
we show for $s_{2m+2} = \frac{1}{2}\left(\frac{1}{2} + s_{2m}\right) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{2^n}\right)\right)$

$= \frac{1}{2}\left(1 - \frac{1}{2^{n+1}}\right)$

$\therefore \liminf_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{2^n}\right) = \frac{1}{2}$

By substitution, also $s_{2m+1} = \frac{1}{2} + \frac{1}{2} s_{2m-1}$

claim: $s_{2m-1} = 1 - \frac{1}{2^m}$

clearly true for $m=1$
assume $m$.

$$S_{2m+1} = \frac{1}{2} + \frac{1}{2} S_{2n-1} = \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{2^n}\right)$$

$$= 1 - \frac{1}{2^{n+1}}$$

$$\therefore \lim_{n\to\infty} \sup s_n = \lim_{n\to\infty}\left(1 - \frac{1}{2^n}\right) = \underline{1}$$

$\therefore$ we have upper limit 1,
lower limit 1/2.

Q10] Recall that $a_n \le \sup\{a_n\}$
$b_n \le \sup\{b_n\}$

$$\therefore \forall n \quad a_n + b_n \le \underbrace{\sup\{a_n\} + \sup\{b_n\}}_{\text{upper bound}}$$

$$\therefore \sup\{a_n + b_n\} \le \sup\{a_n\} + \sup\{b_n\}$$

Then $\forall m \geq n$, we have

$$\sup\{a_m + b_m\} \le \sup\{a_m\} + \sup\{b_m\}$$

$$\therefore \lim_{n\to\infty} \sup\{a_n + b_n\} \le \limsup_{n\to\infty} a_n + \lim_{n\to\infty} \sup b_n .$$