Homework 6 (due: Fr, Nov. 17)

Problem 2*: a) We will prove something stronger: If $\mathcal{A}$ is an algebra on X then for $S \subset X$ the collection $\mathcal{A}|_S := \{A \cap S : A \in \mathcal{A}\}$ is an algebra on S.

(1) Clearly $S = S \cap X \in \mathcal{A}|_S$ and $\varnothing = \varnothing \cap S \in \mathcal{A}|_S$.
(2) Let $A \cap S \in \mathcal{A}|_S$ then $S \backslash (A \cap S) = A^c \cap S \in \mathcal{A}|_S$.
(3) Let $A \cap S, B \cap S \in \mathcal{A}|_S$. Then $(A \cap S) \cup (B \cap S) = (A \cup B) \cap S \in \mathcal{A}|_S$.

This implies part a) as $\widetilde{\mathcal{A}} = \mathcal{A}|_{\mathbb{Q}}$ and $\mathbb{Q} \subset \mathbb{R}$ so $\widetilde{\mathcal{A}}$ is an algebra on $\mathbb{Q}$.

b) Let $q \in \mathbb{Q}$. Then

$$\{q\} = \bigcap_{n \in \mathbb{N}} (q - 1/n, q] \in \sigma(\widetilde{\mathcal{A}}).$$

We can write any arbitrary $Q \in \mathcal{P}(\mathbb{Q})$ as

$$Q = \bigcup \{q \in Q\} \in \sigma(\widetilde{\mathcal{A}})$$

since $\mathbb{Q}$ is countable so any subset of it is countable as well. Therefore, $\sigma(\widetilde{\mathcal{A}}) = \mathcal{P}(\mathbb{Q})$.

c) By definition $\nu(\varnothing) = 0$ and ν takes values in $[0, \infty]$. Let $A_n \in \widetilde{\mathcal{A}}$ such that $\bigcup_{n \in \mathbb{N}} A_n \in \widetilde{\mathcal{A}}$. If all $A_n = \varnothing$ then $\bigcup_{n \in \mathbb{N}} A_n = \varnothing$ so

$$\nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \nu(\varnothing) = 0 = \sum_{n=1}^{\infty} \nu(\varnothing) = \sum_{n=1}^{\infty} \nu(A_n).$$

On the other hand, if at least one of the $A_n \neq \varnothing$ then $\bigcup_{n \in \mathbb{N}} A_n \neq \varnothing$ so

$$\nu\left(\bigcup_{n \in \mathbb{N}} A_n\right) = \infty = \sum_{n=1}^{\infty} \nu(A_n)$$

as required.

d) We set $\mu_1(Q) = |Q|$ and $\mu_2(Q) = 2|Q|$ i.e. μ_1 is the counting measure and μ_2 is two times the counting measure. They are clearly measures as the cardinality of the empty set is 0, the cardinality takes non-negative values and for disjoint sets $A_n \in \mathcal{P}(\mathbb{Q})$

$$|\bigcup_{n \in \mathbb{N}} A_n| = \sum_{n=1}^{\infty} |A_n|$$

and similarly

$$2|\bigcup_{n \in \mathbb{N}} A_n| = 2\sum_{n=1}^{\infty} |A_n| = \sum_{n=1}^{\infty} 2|A_n|.$$

They are distinct as $\mu_1(\{1\}) = 1$ while $\mu_2(\{1\}) = 2$. It remains to show that they extend ν. We need to show that for $A \in \widetilde{\mathcal{A}}, A \neq \varnothing$ we get $\mu_1(A) = |A| = \infty = \nu(A)$. It follows that $\mu_2(A) = \infty$. Let $A \in \widetilde{\mathcal{A}}$ nonempty so it contains a set of the form $(a, b] \cap \mathbb{Q}$ where $a < b$ and $a, b \in \mathbb{R}$. But by the density of rationals in the real numbers, there exist countably infinite rationals in $(a, b] \cap \mathbb{Q}$. Therefore, by monotonicity of μ_1 we get

$$\mu_1(A) \geqslant \mu_1\left((a, b] \cap \mathbb{Q}\right) = |(a, b] \cap \mathbb{Q}| = \infty$$

as required.

Problem 3*: a) $h(x)$ is non-decreasing and continuous while x is strictly increasing and continuous. Therefore, $g(x) = h(x) + x$ is strictly increasing and continuous. Since it is a bijection, it has an inverse g^{-1}. Moreover, since increasing continuous functions map open intervals to open intervals, g^{-1} preimages open intervals to open intervals and is therefore continuous. In summary, g is continuous with a continuous inverse so it is a homeomorphism.

b) Since g is continuous, it maps compact sets to compact sets. Since J_n is compact, so is $g(J_n)$ therefore it is closed so Borel. Note $g(0) = 0$ and $g(1) = 2$. Since g is continuous and $[0, 1]$ is compact, by the intermediate value theorem $[0, 2] \subset g([0, 1])$. But since g is monotonous, $g([0, 1]) = [0, 2]$. By finite additivity and since $g(x) = x$ on $[0, 1]\backslash J_n$,

$$\lambda(g(J_n)) = 2 - \lambda(g([0, 1]\backslash J_n)) = 2 - \lambda\left([0, 1]\backslash J_n\right) = 2 - (1 - \lambda(J_n)) = 1 + \lambda(J_n) \geqslant 1$$

as required.

c) As in part b), since C is compact from the previous HW, $g(C)$ is compact as well because g is continuous and therefore measurable. Also $g(0) = 0$ and $g(1) = 2$ so by the IVT $g(C) = [0, 2]$ as in part b) so $\lambda(g(C)) = 2 > 0$.

d) From part a) $f = g^{-1}$ is continuous so it preimages open intervals to open intervals and therefore to Borel sets. Since open intervals generate $\mathcal{B}_{\mathbb{R}}$, f preimages Borel sets to Borel sets and is therefore Borel measurable. By part c) $g(C)$ is measurable and $\lambda(g(C)) > 0$. Then by Problem 1 there exists a set $A \subset g(C)$ such that A is not measurable. Consider $f(A) \subset f(g(C)) = C$. I claim that $f(A)$ is the required M. Clearly $f^{-1}(M) = A$ is not measurable by construction. Also $M = f(A)$ is measurable by completeness of λ since it is a subset of C a λ-null set.

Problem 4*: a) Assume $f\colon X \to \overline{\mathbb{R}}$ is measurable and $g\colon X \to \overline{\mathbb{R}}$ is a function such that $f = g$ μ-almost everywhere. Let $B \subset \overline{\mathbb{R}}$ be Borel. We need to show that $g^{-1}(B) \in \mathcal{A}$. Since $f = g$ a.e. we know there exists $N \in \mathcal{A}$ such that $\mu(N) = 0$ and $f = g$ on N^c i.e. $f \cdot \mathbb{1}_{N^c} = g \cdot \mathbb{1}_{N^c}$. Note that as a result we get $f^{-1}(B) \cap N^c = g^{-1}(B) \cap N^c$. In more detail, $x \in f^{-1}(B) \cap N^c$ iff $x \in N^c$ and

$f(x) \in B$ iff $x \in N^c$ and $g(x) \in B$ (by $f(x) = g(x)$ on N^c) iff $x \in g^{-1}(B) \cap N^c$. We are now almost done as

$$g^{-1}(B) = \left(g^{-1}(B) \cap N\right) \cup \left(g^{-1}(B) \cap N^c\right) = \left(g^{-1}(B) \cap N\right) \cup \left(f^{-1}(B) \cap N^c\right).$$

Since f is measurable by assumption, $f^{-1}(B) \in \mathcal{A}$ so $f^{-1}(B) \cap N^c \in \mathcal{A}$. Further, $\mathcal{A}$ is complete and N is a μ-null set so $g^{-1}(B) \cap N \subset N$ is also measurable. Finally, by closure under union, $g^{-1}(B) \in \mathcal{A}$ as required.

b) Define $f' = f \cdot \mathbb{1}_{N^c}$. We will show that f' is measurable and conclude that f is measurable by part a). Also define $f'_n = f_n \cdot \mathbb{1}_{N^c}$. Then $f' = \lim_{n\to\infty} f'_n$ pointwise. This is because for for $x \in N^c$, $f' = f$ and $f'_n = f_n$ and for $x \in N^c$ both the LHS and RHS are 0. Each f'_n is measurable since it is the product of two measurable maps and f' is measurable since it is the pointwise limit of measurable maps. By construction $f = f'$ a.e. so if f' is measurable then by part a) even f is measurable as required.

Problem 5*: We follow the hint and first show that $\chi_A \in \mathcal{F}$ for each $A \in \mathcal{A}$. For convenience, from now on I will denote the indicator function on the set A as $\mathbb{1}_A$ rather than χ_A.

Let $\mathcal{A}' \subset \mathcal{A}$ denote the collections of set A such that $\mathbb{1}_A \in \mathcal{F}$. By definition $\mathcal{P} \subset \mathcal{A}'$. Since $\sigma(\mathcal{P}) = \mathcal{A}$, to show that $\mathcal{A}' = \mathcal{A}$, it suffices to show that $\mathcal{A}'$ is a σ-algebra. But note that $\mathcal{P}$ is a π-system so by the Dynkin-π-λ Theorem, it suffices to show that $\mathcal{A}'$ is a λ-system.

(1) $X \in \mathcal{P} \subset \mathcal{A}'$ by assumption.
(2) Let $A, B \in \mathcal{A}'$ such that $A \subset B$. Then $\mathbb{1}_A$ and $\mathbb{1}_B$ are in $\mathcal{F}$. By (ii)

$$\mathbb{1}_{B\setminus A} = \mathbb{1}_B - \mathbb{1}_A \in \mathcal{F}$$

so $B\setminus A \in \mathcal{A}'$. The equality is true because $(\mathbb{1}_B - \mathbb{1}_A)(x) = 1$ if $x \in B$ and $x \notin A$ i.e. $x \in B\setminus A$. Else, it is equal to 0. We don't need to worry about this being negative as $x \in A \Rightarrow x \in B$ since $A \subset B$ by assumption.
(3) Let $A_n \in \mathcal{A}'$ such that $A_n \nearrow A$. Then $\mathbb{1}_{A_n} \in \mathcal{F}$ and $\mathbb{1}_{A_n} \nearrow \mathbb{1}_A$ since

$$x \in A \iff \exists N : \forall n \geqslant N \quad x \in A_n$$

by monotonicity. So if $x \in A$ then $\lim_{n\to\infty} \mathbb{1}_{A_n}(x) = 1$ else the limit is 0. But by (iii) this means that $\mathbb{1}_A \in \mathcal{F}$ so $A \in \mathcal{A}'$.

This proves that $\mathcal{A}'$ is a λ-system so $\mathcal{A}' = \mathcal{A}$ or in other words $\mathbb{1}_A \in \mathcal{F}$ for all $A \in \mathcal{A}$. We can take this further to show that any simple function $s = \sum_{i=1}^n \alpha_i \mathbb{1}_{A_i}$ for $A_i \in \mathcal{A}$ is in $\mathcal{F}$ by (ii) since it is closed under linear combinations.

Now let $f : (X, \mathcal{A}) \to \mathbb{R}$ be measurable. We write it as $f = f^+ - f^-$ where both $f^+ = \max(f, 0)$ and $f^- = -\min(f, 0)$ are positive measurable maps. Then from Theorem 6.9 of the lecture notes, there exist simple functions $s_n : X \to [0, \infty)$ and $t_n : X \to [0, \infty)$ such that $s_n \nearrow f^+$ and $t_n \nearrow f^-$. We just showed that simple functions are in $\mathcal{F}$ so by (iii) $f^+, f^- \in \mathcal{F}$. Then again by (ii) $f = f^+ - f^- \in \mathcal{F}$ as required.