Homework 9

1) let K/F be purely insep with char$(F)=p$.
let $\alpha \in K$ has min poly $m_\alpha(x) \in F[x]$. we can write $m_\alpha(x) = \overline{m_\alpha}(x^{p^m})$ with m maximal so $p^m \mid \deg m_\alpha$. Then $\overline{m_\alpha}(x)$ sep and irred in $F[x]$ so α^{p^m} is sep over F. $\therefore \alpha^{p^m} \in F$ by inseparability. set $a = \alpha^{p^m}$ then α is a root of $x^{p^m} - a \in F[x]$
$\Rightarrow m_\alpha(x) \mid x^{p^m} - a \Rightarrow m_\alpha(x) = x^{p^m} - a$ ■

2) Assume that the extension E^G/F is not purely inseparable i.e $\exists$ some $a \in E^G$ that is inseparable over F. This means that $\forall \sigma : E \xrightarrow{\sim} E$ over F, $\sigma(a) = a$
It suffices to show E/E^G is not normal as that $\Rightarrow E/F$ not normal.
However, inseparability of a shows E/E^G is not normal $\Leftrightarrow$ we are done by contraposition.

$$\begin{array}{c} E \\ | \\ E^G \\ | \\ F \end{array}$$

3) Recall, from the definition that a finite f.e is purely insep if F is the maximal separable subextension in E/F. we need to show that

$$\begin{array}{c} K \\ | \\ L \\ | \\ F \end{array}$$

if L/F is the max sep extension in K/F, then L is also the max sep extension in K/L.

Assume L is not i.e there is a large separable extension L'

$$\begin{array}{c} K \\ | \\ L' \\ | \\ L \\ | \\ F \end{array}$$

such that L'/L is separable. Then L'/F is also separable so we have found a larger sep extension for K/F $\Rightarrow\!\Leftarrow$

$\therefore L$ is the max sep ext of K/L
$\Rightarrow K/L$ is purely inseparable □

4) Let K/F be an algebraic field extension.

Let $f: K \to K$ be a F-homomorphism. It is clear that this homomorphism is injective as it is a field homo.

to show surjection, let $a \in F$. since K/F alg, $\exists\, p \in F[x]$ with root a. Let K_p denote the set of roots of $p(x)$ in K, which is a finite set. Also $f(K_p) \subset K_p$ (f maps a root of $p(x)$ to another root).

$\therefore f$ induces an injective map $K_p \to K_p$ and since K_p finite, this is a surjective so $\exists\, b \in K$ s.t. $f(b) = a$ ■

5) Let $\sigma: K \to K$ be a field iso over F.

consider K^σ the subfield of σ-invariant elements. $K^\sigma \subset K$.

consider

$$\begin{array}{c} K \\ | \\ L \\ | \\ K^\sigma \end{array}$$

i.e L is also invariant under σ

and L is finite i.e $L = K(\alpha_1, \dots, \alpha_n)$

$\therefore \sigma(\alpha_i) = \delta_i \quad \forall i$.

consider L/K^σ, we need to show that the extension is cyclic.

let $\alpha \in L/K^\sigma$, then $\sigma(\alpha) = \alpha^i$ for some i

$\therefore \forall\, \beta \in L/K \;\exists\, i$ s.t. $\sigma^i(\alpha) = \beta$. $\therefore$ we see that the extension has a cyclic group of automorphisms and is Galois so $Gal(L/K^\sigma)$ is cyclic.

6) Over a perfect field, every algebraic field extension is separable. Let F be perfect. We need to show every extension E/F is perfect. i.e every L/E is separable. Consider the tower.

$$\begin{tikzcd} L \arrow[d, dash] \\ E \arrow[d, dash] \\ F \end{tikzcd}$$

Then L/F is separable $\Rightarrow$ L/E is separable.
Therefore E is perfect as well $\blacksquare$

7) as functions $K \to K$

$m_{\alpha+\beta}(x) = (\alpha+\beta)x = \alpha x + \beta x = (m_\alpha + m_\beta)(x)$

$m_\alpha \circ m_\beta(x) = \alpha\beta x = m_{\alpha\beta}(x)$

Picking a basis for K/F, and passing to a matrix:

$Tr_{K/F}(x+x') = Tr([m_x + m_{x'}]) = Tr_{K/F}(x) + Tr_{K/F}(x')$

$N_{L/K}(xx') = \det([m_{xx'}]) = \det([m_x]\cdot[m_{x'}]) = N_{K/F}(x)\cdot N_{K/F}(x')$

8)

$$\begin{tikzcd} L \arrow[d, dash, "m"] \\ F \arrow[d, dash, "d"] \\ K \end{tikzcd}$$

Let $\{e_1, \dots, e_m\}$ be F-basis for L,
$\{f_1, \dots, f_d\}$ be K-basis of F.

$\therefore \{e_1f_1, \dots, e_1f_d, \dots, e_mf_1, \dots, e_mf_d\}$

let $\alpha \in L$. $\alpha e_j = \sum_{i=1}^{m} c_{ij} e_i$, $c_{ij} f_s = \sum_{r=1}^{d} b_{ijrs} f_r$,

for $c_{ij} \in F$, $b_{ijrs} \in K$.

$[m_\alpha]_{L/F} = (c_{ij})$, $[m_{c_{ij}}]_{F/K} = (b_{ijrs})$, $[m_\alpha]_{L/K} = ([m_{c_{ij}}]_{F/K})$.

$$\therefore Tr_{F/K}(Tr_{L/F}(\alpha)) = Tr_{F/K}\left(\sum_i c_{ii}\right)$$

$$= \sum_i Tr_{F/K}(c_{ii})$$

$$= \sum_i \sum_r b_{iirr}$$

$$= Tr_{L/K}(\alpha).$$

The argument above can be replicated for the norm as well, with some linear algebra manipulations

9) Assume $\exists$ a linear relation :

$$\sum_{k=1}^{n} c_k \sqrt{p_k} + c_0 = 0$$

let $L/\mathbb{Q}$ be smallest extension containing all $\sqrt{p_k}$.

if $d \in \mathbb{N}$ is not a perfect square $T(\sqrt{d}) = 0$.

also $T(q) = 0$ iff $q = 0$ for rational q.

$\therefore$ By taking trace of both sides, $c_0 = 0$. let $1 \le j \le n$ and multiply by $\sqrt{p_j}$

$$c_j p_j + \sum_{\substack{1 \le k \le n \\ k \ne j}} c_k \sqrt{p_k p_j} = 0.$$

taking trace again gives $T(c_j p_j) = 0 \Rightarrow c_j = 0$.

$\therefore$ linear independence ■

10) if $\sigma \in G$, then all automorphism in $\sigma K \sigma^{-1}$ fix $\sigma(B)$

$\therefore$ given $\sigma \in G$, galois correspondence associates $\sigma K \sigma^{-1}$ subgroup with the field $\sigma(B)$.

In other words, $Gal(E/\sigma(B)) = \sigma K \sigma^{-1}$ $\therefore$ the fixed field of $\sigma K \sigma^{-1}$ is equal to $\sigma(B)$.

$\therefore$ $\sigma(B) = B$ iff σ normalizes K

$$\therefore \; Gal(K^H/K) \simeq N_G(H)/H$$