Final Exam (due by noon on Wednesday, December 13)

(a) To prove that ν is a measure on $\mathcal{B}$, we will show that ν takes values in $[0, \infty]$, $\nu(\varnothing) = 0$ and ν is countably additive on $\mathcal{B}$.

(1) Let $B \in \mathcal{B}$. Then

$$\nu(B) = \mu\left(T^{-1}(B)\right) \in [0, \infty]$$

as $T^{-1}(B) \in \mathcal{A}$ since T is measurable and μ is a measure on $\mathcal{A}$ so it takes values in $[0, \infty]$.

(2) Since T is a map, $T^{-1}(\varnothing) = \varnothing$. From this it follows that

$$\nu(\varnothing) = \mu\left(T^{-1}(\varnothing)\right) = 0$$

where the last equality follows because μ is a measure.

(3) Let $\{B_n\}_{n\in\mathbb{N}}$ be a collections of pairwise disjoint sets in $\mathcal{B}$. First, note that

$$\begin{aligned} T^{-1}\left(\bigcup_{n\in\mathbb{N}} B_n\right) &= \left\{x \in X : T(x) \in \bigcup_{n\in\mathbb{N}} B_n\right\} \\ &= \{x \in X : (\exists n \in \mathbb{N} : T(x) \in B_n)\} \\ &= \{x \in X : (\exists n \in \mathbb{N} : x \in T^{-1}(B_n))\} \\ &= \left\{x \in X : x \in \bigcup_{n\in\mathbb{N}} T^{-1}(B_n)\right\} \\ &= \bigcup_{n\in\mathbb{N}} T^{-1}(B_n). \end{aligned}$$

Further, $\{T^{-1}(B_n)\}_{n\in\mathbb{N}}$ is a collection of pairwise disjoint sets. To see why, assume for the sake of contradiction that $\exists x \in X$ and $\exists i, j \in \mathbb{N}$ such that $x \in T^{-1}(B_i) \cap T^{-1}(B_j)$. But then $T(x) \in B_i \cap B_j$ which contradicts the pairwise disjointness of $\{B_n\}_{n\in\mathbb{N}}$. Now, we proceed with the proof.

$$\begin{aligned} \nu\left(\bigcup_{n\in\mathbb{N}} B_n\right) &= \mu\left(T^{-1}\left(\bigcup_{n\in\mathbb{N}} B_n\right)\right) \\ &= \mu\left(\bigcup_{n\in\mathbb{N}} T^{-1}(B_n)\right) \\ &= \sum_{n=1}^{\infty} \mu\left(T^{-1}(B_n)\right) \\ &= \sum_{n=1}^{\infty} \nu(B_n) \end{aligned}$$

where the second last equality follows from the countable additivity of μ since each $T^{-1}(B_n) \in \mathcal{A}$ by the measurability of T and the sets are disjoint by the previous argument.

(b) Let $C \subset [0, \infty]$ be Borel. Since f is measurable, $B := f^{-1}(C) \in \mathcal{B}$. Also, $T^{-1}(B) \in \mathcal{A}$ since T is measurable. Therefore

$$(f \circ T)^{-1}(C) = T^{-1}(f^{-1}(C)) \in \mathcal{A}$$

which proves that $f \circ T$ is measurable on $\mathcal{A}$.

To prove the desired equality, we will first prove it for characteristic functions, then simple functions and then non-negative measurable functions using the Monotone Convergence Theorem. Let $B \in \mathcal{B}$ and consider the characteristic function on B which I will denote as $\mathbb{1}_B : Y \to \{0, 1\}$. Note that $\mathbb{1}_B \circ T = \mathbb{1}_{T^{-1}(B)}$ which is also measurable for T measurable. This is because $(\mathbb{1}_B \circ T)(x) = 1$ iff $T(x) \in B$ iff $x \in T^{-1}(B)$ iff $(\mathbb{1}_{T^{-1}(B)})(x) = 1$. Now we show that the equality holds for $f = \mathbb{1}_B$.

$$\int \mathbb{1}_B d\nu = \nu(B) = \mu(T^{-1}(B)) = \int \mathbb{1}_{T^{-1}(B)} d\mu = \int (\mathbb{1}_B \circ T) d\mu.$$

The last equality follows from the previous argument. Now, let $s : (Y, \mathcal{B}) \to [0, \infty)$ be a simple function. Then, it has a standard representation as

$$s = \sum_{i=1}^{n} \beta_i \mathbb{1}_{B_i}$$

where $\beta_i \geqslant 0$ and $B_i \in \mathcal{B}$ are pairwise disjoint for $i \in [n]$. By linearity of the integral and of the composite operator

$$\begin{aligned}
\int s d\nu &= \int \left(\sum_{i=1}^{n} \mathbb{1}_{B_i} \right) d\nu \\
&= \sum_{i=1}^{n} \beta_i \int \mathbb{1}_{B_i} d\nu \\
&= \sum_{i=1}^{n} \beta_i \int (\mathbb{1}_{B_i} \circ T) d\mu \\
&= \int \sum_{i=1}^{n} \beta_i (\mathbb{1}_{B_i} \circ T) d\mu \\
&= \int \left(\sum_{i=1}^{n} \beta_i \mathbb{1}_{B_i} \right) \circ T d\mu \\
&= \int s \circ T d\mu.
\end{aligned}$$

The third equality follows from the argument for characteristic functions. Now, let $f : (Y, \mathcal{B}) \to [0, \infty]$ be an arbitrary measurable function. From Theorem 6.9 of the lecture notes, there exist simple functions $s_n : (Y, \mathcal{B}) \to [0, \infty)$ for $n \in \mathbb{N}$ such that $s_n \nearrow f$. Then, by the Monotone Convergence Theorem (MCT) and the fact that composition preserves limits

$$\begin{aligned}
\int f d\nu &= \int \left(\lim_{n\to\infty} s_n \right) d\nu \\
&= \lim_{n\to\infty} \int s_n d\nu \\
&= \lim_{n\to\infty} \int s_n \circ T d\mu \\
&= \int \lim_{n\to\infty} \left(s_n \circ T \right) d\mu \\
&= \int \left(\lim_{n\to\infty} s_n \right) \circ T d\mu \\
&= \int f \circ T d\mu
\end{aligned}$$

as required. The second and fourth equality use MCT, the third equality follows from our previous argument for non-negative simple functions and the fifth equality uses that composition preserves limits.

Problem 2: (a) I claim that the sufficient and necessary condition is that

$$\lambda\left(\limsup_{n\to\infty} A_n\right) = 0$$

where

$$\limsup_{n\to\infty} A_n := \bigcap_{n\in\mathbb{N}} \bigcup_{m\geqslant n} A_m.$$

This set is measurable as it is the intersection of unions of measurable sets. From now on, I will denote this set as A^* for convenience so the condition is that $\lambda(A^*) = 0$. Intuitively, this is true because A^* is the set of all $x \in \mathbb{R}$ such that $x \in A_n$ for infinitely many $n \in \mathbb{N}$. I am not proving this as I will work with the original definition for the proof.

$\Rightarrow$ We prove that the condition is sufficient. Assume that $\lambda(A^*) = 0$. Let $x \in (A^*)^c$. Then

$$x \in \bigcup_{n\in\mathbb{N}} \bigcap_{m\geqslant n} A_m^c$$

by de Morgan's laws. So there exists some n_0 such that

$$x \in \bigcap_{m\geqslant n_0} A_m^c$$

i.e. for all $m \geqslant n_0$ we have $x \notin A_m$. This means that

$$\exists n_0 \in \mathbb{N}\ \forall m \geqslant n_0 : \mathbb{1}_{A_m}(x) = 0$$

which proves that

$$\lim_{n\to\infty} \mathbb{1}_{A_n}(x) = 0.$$

This is true for all $x \in (A^*)^c$ and $\lambda(A^*) = 0$ by assumption so

$$\lim_{n\to\infty} \mathbb{1}_{A_n} = 0 \quad a.e.$$

as required. Now we prove the other implication.

$\Leftarrow$ We prove that the condition is necessary. Assume that there exists a measurable set N such that $\lambda(N) = 0$ and

$$\lim_{n\to\infty} \mathbb{1}_{A_n} = 0$$

on N^c. Let $x \in N^c$. Then

$$\lim_{n\to\infty} \mathbb{1}_{A_n}(x) = 0$$

so by the definition of a limit, there exists some $n_0 \in \mathbb{N}$ such that for all $m \geqslant n_0$

$$\mathbb{1}_{A_m}(x) < 1/2.$$

But since each $\mathbb{1}_{A_m}$ is a characteristic function, it only takes values in $\{0,1\}$ so $\mathbb{1}_{A_m}(x) < 1/2 \Rightarrow \mathbb{1}_{A_m}(x) = 0$ which is the same as $x \notin A_m$. Therefore,

$$\exists n_0 \in \mathbb{N}\ \forall m \geqslant n_0 : x \notin A_m.$$

The argument that follows is the reverse of what was provided earlier. This means that

$$x \in \bigcap_{m \geqslant n_0} A_m^c$$

so

$$x \in \bigcap_{n \in \mathbb{N}} \bigcup_{m \geqslant n} A_m = (A^*)^c.$$

While n_0 depended on our choice for x, the expression above is independent of our choice of x as long as $x \in N^c$. Therefore, we have showed that $N^c \subset (A^*)^c$ so $A^* \subset N$. It follows from montonicity that

$$\lambda(A^*) \leqslant \lambda(N) = 0$$

so $\lambda(A^*) = 0$ as required.

(b) I claim it suffices to show that f takes values in $\{0,1\}$ almost everywhere. This is because, by assumption, $f \in L^1$ so it is measurable. Therefore, $A := f^{-1}(\{1\})$ is measurable so we can write $f = \mathbb{1}_A$ almost everywhere. Also $\mu(A) = \int f d\lambda < \infty$ since $f \in L^1$.

To prove the statement, for $k \in \mathbb{N}$ consider the sets

$$E_k := \{x : |f(x)| > 1/k, |f(x) - 1| > 1/k\}.$$

This is a measurable set as

$$E_k = |f|^{-1}(1/k, \infty) \cap |f-1|^{-1}(1/k, \infty)$$

where each of the sets is measurable since $f \in L^1$ so measurable and taking absolute value and adding constants keeps it measurable. By construction note that $|f(x) - 0| > 1/k$ and $|f(x) - 1| > 1/k$ for $x \in E_k$ so it follows that for any A_n, $|f(x) - \mathbb{1}_{A_n}(x)| > 1/k$ for $x \in E_k$ as the characteristic function only takes values in $\{0,1\}$. Finally, note that

$$E := \bigcup_{k \in \mathbb{N}} E_k = \{x : |f(x)| > 0, |f(x) - 1| > 0\} = \{x : f(x) \neq 0\} \cap \{x : f(x) \neq 1\}$$

i.e. $f(x) \in \{0,1\}$ for $x \in E^c$. Now, we proceed with the proof. By monotonicity of the integral, for all $n \in \mathbb{N}$

$$\int |f - \mathbb{1}_{A_n}| d\lambda \geqslant \int_{E_k} |f - \mathbb{1}_{A_n}| d\lambda > \int_{E_k} \frac{1}{k} d\lambda = \frac{\lambda(E_k)}{k}.$$

The first inequality follows because $|f - \mathbb{1}_{A_n}| \geqslant |f - \mathbb{1}_{A_n}| \mathbb{1}_{E_k}$ and the second inequality follows from our earlier argument. Since $\mathbb{1}_{A_n} \to f$ in L^1 the first term above goes to 0 as $n \to \infty$. Taking the limit, we therefore have

$$\frac{\lambda(E_k)}{k} \leqslant \lim_{n \to \infty} \int |f - \mathbb{1}_{A_n}| d\lambda = 0$$

so $\lambda(E_k) = 0$ for all $k \in \mathbb{N}$. It follows from subadditivity that

$$\lambda(E) \leqslant \sum_{k=1}^{\infty} \lambda(E_k) = 0$$

and recall that we showed $f(x) \in \{0,1\}$ for $x \in E^c$. Therefore, f takes values in $\{0,1\}$ almost everywhere and we are done by the argument in the start.

Problem 3:(a) We first show that $f \in L^1$ implies that $|f|$ is finite almost everywhere. Define

$$E_n := |f|^{-1}\left([n, \infty]\right)$$

which is measurable since $f \in L^1 \Rightarrow f$ is measurable. It is clear that $|f(x)| \geqslant n$ for $x \in E_n$ and that

$$|f|^{-1}\{\infty\} = \bigcap_{n \in \mathbb{N}} E_n.$$

From monotonicity of the integral

$$\int_{E_n} |f| d\mu \geqslant \int_{E_n} n d\mu = n\mu(E_n).$$

Then again, by monotonicity of the integral,

$$\mu(E_N) \leqslant \frac{1}{n} \int_{E_n} |f| d\mu \leqslant \frac{1}{n} \int |f| d\mu.$$

Since $f \in L_1$, $\int |f| d\mu < \infty$ so $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \mu(E_n) = 0.$$

Also, by construction, $E_n \searrow \bigcap_{n \in \mathbb{N}} E_n$. By this and $\mu(E_1) < \infty$ we use continuity of the measure from above to conclude that

$$\mu\left(\bigcap_{n \in \mathbb{N}} E_n\right) = \mu\left(|f|^{-1}\{\infty\}\right) = \lim_{n \to \infty} \mu(E_n) = 0.$$

It is clear that $|f(x)| < \infty$ on the complement of $|f|^{-1}\{\infty\}$ and since this set has measure zero, $|f(x)|$ is finite almost everywhere.

Going back to our proof, we have shown that there exists some $N \in \mathcal{A}$ with $\mu(N) = 0$ and $|f(x)| < \infty$ for $x \in N^c$. Define

$$A_n := \{x \in X : |f(x)| > n\}$$

and $f_n := |f| \mathbb{1}_{A_n}$. Note that $|f_n| \leqslant |f|$ for $n \in \mathbb{N}$ and by assumption $f \in L^1$ so it serves as a dominating function and Lebesgue's Dominated Convergence (LDC) applies. Finally since $|f(x)| < \infty$ on N^c we have that for each $x \in N^c$

$$\exists n_0 \in \mathbb{N} : |f(x)| < n_0$$

and so for all $n \geqslant n_0, x \notin A_n$ or in other words

$$\lim_{n \to \infty} \mathbb{1}_{A_n}(x) = 0$$

for $x \in N^c$ so

$$\lim_{n \to \infty} f_n = 0$$

for $x \in N^c$. Define $g_n = f_n \mathbb{1}_{N^c}$ and then

$$\lim_{n \to \infty} g_n = 0$$

for all $x \in X$ by construction. Since $f_n = g_n$ almost everywhere, from HW 6 Problem 4 each g_n is also measurable and $\int f_n d\mu = \int g_n d\mu$ for all $n \in \mathbb{N}$. Then by LDC

$$\lim_{n\to\infty} \int f_n d\mu = \int \lim_{n\to\infty} f_n = \int \lim_{n\to\infty} g_n = \int 0 d\mu = 0.$$

Summarizing what we have so far

$$\lim_{n\to\infty} \int_{\{x:|f(x)|>n\}} |f| d\mu = 0.$$

Now let $\epsilon > 0$. Using the definition of the limit, there exists some $n_1 \in \mathbb{N}$ such that for all $n \geqslant n_1$

$$\int_{\{x:|f(x)|>n\}} |f| d\mu < \frac{\epsilon}{2}.$$

In particular, this is true for $n = n_1$. Now let $A \in \mathcal{A}$ be such that $\mu(A) < \delta = \frac{\epsilon}{2n_1}$. We will bound $\int_A |f| d\mu$ from above. Note that we can split the integral into two by observing that

$$|f| = |f|\mathbb{1}_X = |f|\mathbb{1}_{A_{n_1}} + |f|\mathbb{1}_{A^c_{n_1}}.$$

Therefore,

$$\begin{aligned}
\int_A |f| d\mu &= \int_A |f|\mathbb{1}_{A_{n_1}} d\mu + \int_A |f|\mathbb{1}_{A^c_{n_1}} d\mu \\
&= \int_{A\cap\{x:|f(x)|>n_1\}} |f| d\mu + \int_{A\cap\{x:|f(x)|\leqslant n_1\}} |f| d\mu \\
&\leqslant \int_{\{x:|f(x)|>n_1\}} f d\mu + \int_A N d\mu \\
&\leqslant \frac{\epsilon}{2} + \delta n_1 \\
&< \epsilon.
\end{aligned}$$

The first inequality (third line) follows from the monotonicity of the integral and the fact that $|f| \leqslant n_1$ on $A^c_{n_1}$ by definition, the second inequality is by our previous argument and $\int_A n_1 d\mu = \mu(A) n_1 = \delta n_1$ and the final inequality by our choice of δ. Finally, from the triangle inequality for integrals, it follows that

$$\left| \int_A f d\mu \right| \leqslant \int_A |f| d\mu.$$

Since $\epsilon > 0$ was arbitrary and $A \in \mathcal{A}$ such that $\mu(A) < \delta$ was arbitrary, we have shown that for all $\epsilon > 0$, there exists a $\delta > 0$ such that for $A \in \mathcal{A}$

$$\mu(A) < \delta \Rightarrow \left| \int_A f d\mu \right| < \epsilon$$

as required.

(b) Since $f \in L^2(\mu)$ we know that $\|f\|_2 < \infty$ so we can apply Cauchy–Schwarz. Let $\epsilon > 0$ and let $A \in \mathcal{A}$ such that

$$\mu(A) < \delta = \frac{\epsilon^2}{(\|f\|_2 + 1)^2}$$

so we have picked $c = \frac{1}{(\|f\|_2+1)^2} > 0$. Then by triangle inequality for integrals and Cauchy-Schwarz

$$\left|\int_A f d\mu\right| \leqslant \int_A |f| d\mu = \int |f| \mathbb{1}_A d\mu \leqslant \|f\|_2 \cdot \|\mathbb{1}_A\|_2.$$

Note that $\mathbb{1}_A^2 = \mathbb{1}_A$ so

$$\|\mathbb{1}_A\|_2 = \left(\int \mathbb{1}_A^2 d\mu\right)^{1/2} = \left(\int \mathbb{1}_A d\mu\right)^{1/2} = \mu(A)^{1/2} < \delta^{1/2}.$$

Therefore,

$$\left|\int_A f d\mu\right| < \|f\|_2 \delta^{1/2} = \frac{\|f\|_2}{\|f\|_2 + 1} \epsilon < \epsilon$$

as required.

Problem 4: As per the hint, we first deal with the case $\lambda(A) < \infty$, and consider the outer measure and covers by intervals. Assume, for the sake of contradiction, that for all nonempty closed intervals $I = [a, b]$, $\lambda(A \cap I) \leqslant 0.99\lambda(I)$.

Since A is measurable, we have that $\lambda(A) = \lambda^*(A)$. From pg 64 of the lecture notes (specialized to $n = 1$), we have the following characterization for the outer meausure:

$$\lambda^*(A) = \inf\left\{\sum_{k=1}^{\infty} \lambda(I_k) : I_k \text{ is an interval and } A \subset \bigcup_{k\in\mathbb{N}} I_k\right\}.$$

By definition of the infimum, there exists a cover by intervals $I = \bigcup_{k\in\mathbb{N}} I_k$ of A where $A \subset I$ such that

$$\sum_{k=1}^{\infty} \lambda(I_k) < \frac{1}{0.99}\lambda(A) < \infty.$$

Therefore

$$0.99\sum_{k=1}^{\infty} \lambda(I_k) < \lambda(A) = \lambda(A \cap I) \leqslant \sum_{k=1}^{\infty} \lambda(A \cap I_k) \leqslant 0.99\sum_{k=1}^{\infty} \lambda(I_k)$$

where the second last inequality follows from countable subadditivity of λ and the last inequality follows from the assumption that $\lambda(A \cap I_k) \leqslant 0.99\lambda(I_k)$ for all intervals I_k. This finally gives us $\sum_{k=1}^{\infty} \lambda(I_k) < \sum_{k=1}^{\infty} \lambda(I_k)$ which is a contradiction because this expression is finite. Therefore, our assumption must have been false and there exists some nonempty interval $I = [a, b]$ such that $\lambda(A \cap I) > 0.99\lambda(I)$ if $\lambda(A) < \infty$.

Now we extend the result to the general case. Let A be measurable with $\lambda(A) > 0$ so it is nonempty and pick $M \in \mathbb{N}$ such that $A_M := A \cap [-M, M]$ is nonempty and $\lambda(A_M) > 0$. Then $\lambda(A_M) \leqslant \lambda([-M, M]) = 2M < \infty$ so from our argument for the finite case, there exists some nonempty interval $I = [a, b]$ such that $\lambda(A_M \cap I) > 0.99\lambda(I)$ so $A_M \cap I$ is nonempty. Set $J = I \cap [-M, M] \supset A_M \cap I$ so it is nonempty and it is an interval as the intersection of two intervals is also an interval (set of intervals are a π-system). Note that $A \cap J = A_M \cap I$ by construction. By monotonicity, $\lambda(J) \leqslant \lambda(I)$ so

$$\lambda(A \cap J) = \lambda(A_M \cap I) > 0.99\lambda(I) \geqslant 0.99\lambda(J)$$

so J is the required interval and we are done.

Problem 5: (a) We will prove the statement for $C = 1$ i.e.

$$\frac{|e^{-ihx} - 1|}{|h|} \leqslant |x|$$

for all $x \in \mathbb{R}$ and all $h \in \mathbb{R}\backslash\{0\}$. As the hint suggests, we consider $|e^{-it} - 1| = |(\cos(t) - 1) - i\sin(t)|$. For the sake of calculation, it helps to consider the square of this. We will use two trigonometric identities for this

$$1 - \cos(t) = 2\sin^2\left(\frac{t}{2}\right) \tag{1}$$

and

$$\sin^2(t) \leqslant t^2. \tag{2}$$

Equation (2) follows from the fact that $|\sin(t)| \leqslant |t|$ for all $t \in \mathbb{R}$ and Equation (1) is true because $\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta) = 1 - 2\sin^2(\theta)$ so the identity follows from rearranging and substituting $\theta = t/2$. Now we compute

$$\begin{aligned} |e^{-it} - 1|^2 &= |(\cos(t) - 1) - i\sin(t)|^2 = (\cos(t) - 1)^2 + \sin^2(t) \\ &= 2 - 2\cos(t) = 4\sin^2\left(\frac{t}{2}\right) \leqslant 4\left(\frac{t}{2}\right)^2 = t^2. \end{aligned}$$

From here, it follows that

$$|e^{-it} - 1| \leqslant |t|$$

and substituting $t = hx$ for $h \neq 0$ gives us

$$\frac{|e^{-ihx} - 1|}{|h|} \leqslant |x|$$

as required.

(b) First note that

$$\begin{aligned}\lim_{n\to\infty} n\left(e^{-ix/n}-1\right) &= \lim_{h\to 0}\frac{e^{-ixh}-1}{h} = \lim_{h\to 0}\frac{e^{-ix(0+h)}-e^{-ix\cdot 0}}{h}\\ &= \frac{d}{dy}e^{-ixy}|_{y=0} = -ixe^{-ixy}|_{y=0} = -ix.\end{aligned}$$

Defining $f_n := n(e^{-ix/n}-1)$ we see from above that $\lim_{n\to\infty} f_n = -ix$. Also each $|f_n| \leqslant |x|$ from part a) for each $n \in \mathbb{N}$ (with $h = 1/n$ so $h \neq 0$) so the f_n are dominated by x and by assumption $x \in L^1$ as $\int |x| d\mu(x) < \infty$. Therefore, we can apply Lebesgue Dominated Convergence (LDC) to get

$$\begin{aligned}f'(u_0) := \lim_{h\to 0}&\frac{\int e^{-i(u_0+h)x}d\mu(x) - \int e^{-ihx}d\mu(x)}{h}\\ &= \lim_{h\to 0}\frac{\int e^{-i(u_0+h)x} - e^{-ihx}d\mu(x)}{h}\\ &= \lim_{h\to 0}\frac{\int e^{-iu_0x}(e^{-ihx}-1)d\mu(x)}{h}\\ &= \lim_{h\to 0}\int e^{-iu_0x}\frac{(e^{-ihx}-1)}{h}d\mu(x)\\ &= \lim_{n\to\infty}\int e^{-iu_0x}n(e^{-ix/n}-1)d\mu(x)\\ &= \int e^{-iu_0x}\left(\lim_{n\to\infty} n(e^{-ix/n}-1)\right)d\mu(x)\\ &= \int e^{-iu_0x}(-ix)d\mu(x)\\ &= -i\int xe^{-iu_0x}d\mu(x).\end{aligned}$$

The second, fourth and eighth equality follow from linearity of the integral, the sixth one from LDC and the seventh from the previous argument. This shows that f is differentiable for arbitrary $u_0 \in \mathbb{R}$ and the derivative is as required.

(c) To show that $f'(u)$ is continuous we show that for any arbitrary sequence $\{u_n\}_{n\in\mathbb{N}}$ such that $u_n \to u_0$ we get $f'(u_n) \to f'(u_0)$ as $n \to \infty$. Let $\{u_n\}_{n\in\mathbb{N}}$ be such that $u_n \to u_0$. Define $g_n(x) = -ixe^{-iu_n x}$ for $n \in \mathbb{N}$. Then by the continuity of $g(u) = -ixe^{-ux}$ for fixed x it follows that

$$\lim_{n\to\infty} g_n(x) = g_0(x)$$

as $u_n \to u_0$. In other words, the functions g_n converge to g_0 pointwise. Also note that $|g_n| = |x|$ so, as before, the functions g_n are dominated by x which has finite integral with respect to μ by assumption. By LDC

$$\begin{aligned}
\lim_{n\to\infty} f'(u_n) &:= \lim_{n\to\infty} \int -ixe^{-iu_n x} d\mu(x) \\
&= \lim_{n\to\infty} \int g_n(x) d\mu(x) \\
&= \int \lim_{n\to\infty} g_n(x) d\mu(x) \\
&= \int g_0(x) d\mu(x) \\
&= \int -ixe^{-iu_0 x} d\mu(x) \\
&= f'(u_0).
\end{aligned}$$

The third equality follows from LDC, the fourth from the previous argument and the rest from definition. This shows the continuity of $f'(u)$ which proves that f is C^1-smooth.