Math 131AH: Homework #3

Due on February 1, 2022

Professor Marek Biskup

Nakul Khambhati

Problem 1

Recall $\forall m, n \in \mathbb{N}$ we define $m^0 = 1$ and $m^{S(n)} = mm^n$.

1. Fix $s \in \mathbb{N}$. We will prove the statement by induction on r. $P_0 : m^{0+s} = m^s = 1m^s = m^0 m^s$. Assume P_r. Let's prove P_{r+1}: $m^{S(r)+s} = m^{S(r+s)} = mm^{r+s} = mm^r m^s = m^{S(r)} m^s$. By induction, the statement follows for all $r \in \mathbb{N}$.

2. Now, fix r and proceed by induction on s. $P_0 : m^{r0} = m^0 = 1 = (m^r)^0$. Assume P_s and prove $P_{s+1} : m^{rS(s)} = m^{rs+r} = m^{rs}m^r = (m^r)^s m^r = (m^r)^{S(s)}$. By induction, the result follows for all $s \in \mathbb{N}$.

Problem 2

We will prove this by proposing some fact about A which we will prove for all $n \in \mathbb{N}$. Let P_n be the statement: If A contains an integer k where $0 \le k \le n$ then, A has a minimal element. P_0 is clear as if $0 \in A$ then 0 is the minimal element. Now assume P_n. We show P_{n+1}. Assume A contains some k such that $0 \le k \le n+1$. We now deal with cases: If there does not exist any $a \in A$ such that $0 \le a \le n$ then we must have $k = n+1$ which would then be the minimal element in A. On the other hand, if $\exists a : 0 \le a \le n$ then by the inductive hypothesis, A has a minimal element. By induction we have proved P_n for all $n \in \mathbb{N}$. Since we are given that A is nonempty, we can say that A contains some integer $k : 0 \le k \le m$ for some $m \in \mathbb{N}$. As a result, A has a minimal element.

Problem 3

We will use slightly different notation to prove that this relation is well defined. Assume that $[(m,n)] \sim [(m',n')]$ and $[(k,l)] \sim [(k',l')]$. Further assume that $[(m,n)] \le [(k,l)]$. We are required to show that $[(m',n')] \le [(k',l')]$.

First let's summarize our assumptions: $m + n' = m' + n, k + l' = k' + l$ and $m + l \le n + k$. We need to show $m' + l' \le n' + k'$. In other word, there exists some $a \in \mathbb{N}$ such that $n + k = m + l + a$. We need to construct some $a' \in \mathbb{N} : n' + k' = m' + l' + a'$. By adding some of the above equations, we get $n + k + m + n' + k' + l = m + l + a + m' + n + k + l'$. Injectivity of addition allows us to cancel k, l, m, n from both sides leaving us with $n' + k' = m' + l' + a$. So in fact, the required a' is a and this makes the relation well defined.

First we show that the relation is a partial order:

1. By commutativity, $m + n = n + m = n + m + 0$ so $[(m,n)] \le [(m,n)]$.

2. Assume that $[(m,n)] \le [(m',n')]$ and $[(m',n')] \le [(m,n)]$. Then, $m + n' = m' + n + k$ and $m' + n = m' + n + l$ for some $k, l \in \mathbb{N}$. So, $m + n' = m + n' + k + l$. By injectivity of addition, $k + l = 0$ and $k = l = 0$. So, $m + n' = n + m' \Rightarrow [(m,n)] \sim [(m',n')]$. Therefore, the relation is antisymmetric.

3. Assume that $[(m,n)] \le [(m',n')]$ and $[(m',n')] \le [(m'',n'')]$. Then, $m + n' = m' + n + k$ and $m' + n'' = m'' + n' + l$ for some $k, l \in \mathbb{N}$. When we add the two, we get $m + n' + m' + n'' = m' + n + k + m'' + n' + l$. By cancellation, we cancel out m', n' from both sides and by commutativity we rearrange to get $m + n'' = m'' + n + k + l$. So $[(m,n)] \le [(m'',n'')]$ and the relation is transitive.

Now we need to show that for any two $[(m,n)], [(k,l)]$ either $[(m,n)] \le [(k,l)]$ or $[(k,l)] \le [(m,n)]$.
We will prove this by using the total ordering of naturals. Given $[(m,n)]$ and $[(k,l)]$, either $m + l \le n + k \Rightarrow [(m,n)] \le [(k,l)]$ or $n + k \le m + l \overset{comm}{\Rightarrow} k + n \le l + m \Rightarrow [(k,l)] \le [(m,n)]$.

Problem 4

Again, we switch notation a bit to show that addition is well-defined. Let $[(p,q)] \sim [(p',q')]$ and $[(r,s)] \sim [(r',s')]$. Then, $pq' = p'q$ and $rs' = r's$. We need to then show that $(ps+qr, qs) \sim (p's'+q'r', q's')$. LHS $= (ps+qr)(q's') = (psq's' + qrq's') = (p'qss' + r'sqq') = (p's'+q'r')(qs) =$ RHS.

Problem 5

1. Let $a, b \in F$. Let $0 \leq b$. Then by (O2) $0 + a = a \leq a + b$. Conversely, assume $a \leq a + b \Rightarrow a - a = 0 \leq a + b - a = b$.
2. Let $a, b \in F$. Then, $a \leq b \Rightarrow a - a - b \leq b - a - b \Rightarrow -b \leq -a$.
3. Let $a, b \in F$. Assume $0 < a \wedge a \leq b$. Therefore, $a \neq 0 \wedge b \neq 0$. Therefore a^{-1}, b^{-1} exist. Further, $0 < a^{-1}, 0 < b^{-1}$ because otherwise we would get $1 < 0$. Therefore, $aa^{-1}b^{-1} \leq ba^{-1}b^{-1} \Rightarrow b^{-1} \leq a^{-1}$.

Problem 6

These results follow from the proposition we proved in class: $0 \leq a \iff -a \leq 0$.

1. If $0 \leq a$, then $|a| = a$ so $0 \leq |a|$. Else, $a < 0, |a| = -a, 0 \leq |a|$.
2. Assume $0 \leq a$. Then $-a \leq a \leq a$ so $-|a| \leq a \leq |a|$. Else, $a < 0, |a| = -a$. Then, $a \leq a \leq -a$ so $-|a| \leq a \leq |a|$.
3. If both $a, b \geq 0$ then $|a+b| = |a| + |b|$. Similarly, if both $a, b < 0$ then $|a+b| = -(a+b) = (-a) + (-b) = |a| + |b|$. If one is negative and one is positive: Assume WLOG $0 \leq a$ and $b < 0$. Then, $|a+b| = |a - (-b)| \leq |a| + |b|$.
4. Assume both a, b are positive. Then ab is positive so $|ab| = ab = |a||b|$. Similarly, for both negative, ab is positive so $|ab| = ab = (-|a|)(-|b|) = |a||b|$. If only one is positive, assume WLOG $0 \leq a$ and $b < 0$. Then ab is negative so $|ab| = -(ab) = -(|a|(-|b|)) = |a||b|$.

Let's prove the claim by induction on n. P_0 is trivially true. Assume P_n. Then, $\left|\sum_{i=0}^{n+1} a_i\right| = \left|\sum_{i=0}^{n} a_i + a_{n+1}\right| \leq \left|\sum_{i=0}^{n} a_i\right| + |a_{n+1}| \leq \sum_{i=0}^{n} |a_i| + |a_{n+1}| = \sum_{i=0}^{n+1} |a_i|$. By induction, the claim is true for all $n \in \mathbb{N}$.

Problem 7

1. The radical expression $\sqrt[3]{5-\sqrt{3}}$ solves some polynomial equation. Let's use α to denote the radical expression. Then, $5 - \sqrt{3} = \alpha^3 \Rightarrow (5-\alpha^3)^2 = 3$. So α is a root of the polynomial $p(x) = x^6 - 10x^3 + 22$. By the rational root theorem, the only possible rational roots of $p(x)$ are $\pm 1, \pm 2, \pm 11, \pm 22$. Substituting each, we see that none of them make the polynomial equal zero. Therefore α cannot be expressed as a rational number.
2. We can rewrite this radical as $\sqrt{3+2\sqrt{2}} - \sqrt{2} = \sqrt{(1+\sqrt{2})^2} - \sqrt{2} = 1 + \sqrt{2} - \sqrt{2} = 1$ which is rational.
3. By the rational root theorem, the only possible roots for this polynomial are ± 1. We check that $p(1) = 4,\ p(-1) = 0$. Therefore, -1 is the only rational root of the polynomial.