Homework 8

Q1] (1) $\Rightarrow$ (2)

Assume $\lim_{n\to\infty}\{x_n\} = L$

It is easily checked that every subsequence $\{x_{n_k}\}$ converges to L.

Let $\varepsilon > 0$.

Then $\exists n_0$ s.t $\forall n > n_0$: $|x_n - L| < \varepsilon$

then pick k_0 s.t $n_{k_0} > n_0$. Then $\forall k > k_0$ (i.e $n_k > n_{k_0}$)

$$|x_{n_k} - L| < \varepsilon$$

$\therefore \lim_{n\to\infty}\{x_{n_k}\} = L$

Then, $z = L$ since every subsequence of $\{x_n\}$ has a subsequence (itself) that converges to z.

(2) $\Rightarrow$ (1)

Assume $\exists z \in X$ s.t every subsequence of $\{x_n\}$ has a subsequence that conv. to z.

Assume, in search of contradiction $\{x_n\}$ doesn't converge to z.

Then $\exists \varepsilon > 0$ s.t $\forall k$ $\exists n_k > k$ $|x_{n_k} - z| \geq \varepsilon$.

This is because, if $\exists$ some k without such an n_k, we could pick $n_0 = k$ and the x_n would converge to z.

$\therefore$ we have a subsequence x_{n_k} which does not have any subsequence converging to z. which is a contradiction.

$\therefore$ we must have $\lim_{n\to\infty}\{x_n\} = z$ ■

Q2] Let $A \subseteq B$

Let $O \in A$ be an open set

$\therefore O \in B$ is also an open set.

$\therefore \{O \in A : O \text{ open}\} \subseteq \{O \in B : O \text{ open}\}$

$\therefore \operatorname{int}(A) = \bigcup_{\substack{O \in A \\ \text{open}}} O \subseteq \bigcup_{\substack{O' \in B \\ \text{open}}} O' = \operatorname{int}(B)$ □

Let C be closed such that $B \subseteq C$

$\therefore A \subseteq C$.

$\therefore \{C \text{ closed s.t. } B \subseteq C\} \subseteq \{C' \text{ closed s.t. } A \subseteq C\}$

$\therefore \bigcap_{\substack{A \subseteq C' \\ \text{closed}}} C' \subseteq \bigcap_{\substack{B \subseteq C \\ \text{closed}}} C$

$\therefore \quad \overline{A} \subseteq \overline{B}$

Now we show $X \setminus \text{int}(A) = \overline{X \setminus A}$

we simplify notation $(\text{int}(A))^c = \overline{(A^c)}$

$$\text{L.H.S} = \Big[\bigcup_{\substack{O \subseteq A \\ \text{open}}} O\Big]^c \underset{\text{De-Morgan}}{=} \bigcap_{\substack{O \subseteq A \\ \text{open}}} O^c .$$

* Next, note that $\{O \subseteq A \text{ open}\} \longleftrightarrow \{A^c \subseteq C \text{ closed}\}$

$O \longmapsto O^c$

i.e. we have a bijection between open sets in A and closed sets containing A^c

since if $O \subseteq A$ is open then $A^c \subseteq O^c$ and O^c is closed $\square$

$$\therefore \bigcap_{O \subseteq A} O^c = \bigcap_{A^c \subseteq C} C = \overline{A^c}$$

Similarly,

WTS $\overline{A}^c = \text{int}(A^c)$

using the same bijection as above

$$\overline{A}^c = \Big(\bigcap_{\substack{A \subset C \\ C \text{ closed}}} C\Big)^c = \bigcup_{\substack{A \subset C \\ C \text{ closed}}} C^c = \bigcup_{O \subseteq A^c} O = \text{int}(A^c) \quad \square$$

In particular
$$\begin{aligned}
\partial A &= \overline{A} \setminus \text{int}(A) \\
&= \overline{A} \cap (X \setminus \text{int}(A)) \\
&= \overline{A} \cap \overline{X \setminus A} \\
&= \overline{X \setminus A} \cap \overline{A} \\
&= \overline{A^c} \cap \overline{X \setminus A^c} \\
&= \partial(A^c) \quad \blacksquare
\end{aligned}$$

Q3] 1) Consider $A = (0,1) \cap \mathbb{Q}$

i.e the set of rationals q: $0 < q < 1$.

Recall that $A \subseteq \mathbb{Q}$ is relatively open iff $A = O \cap \mathbb{Q}$ for some O open in $\mathbb{R}$

it is clearly relatively open because $A = (0,1) \cap \mathbb{Q}$ and $(0,1)$ is open in $\mathbb{R}$

However, it is not open in $\mathbb{R}$.

Let $x \in A$. By density of irrationals, every open ball around x $B(x,r)$ contains irrationals so we cannot find any open ball $B(x,r)$ s.t $B(x,r) \subseteq A$.

2) Assume A is finite. Let $m = \min(A)$
$n = \max(A)$.

Then $A = (m-1, n+1) \cap \mathbb{N} = [m,n] \cap \mathbb{N}$

so it is both relatively open and relatively closed.

Assume A infinite $\therefore$ countable.

Let $\{x_0, x_1, \cdots, x_n, \cdots\}$ be an enumeration of the elements of A.

Let $B_i = B(x_i, 1/2)$ s.t. B_i is open in $\mathbb{R}$
and $B_i \cap \mathbb{N} = \{x_i\}$

Then $O = \bigcup_{i \in \mathbb{N}} B_i$ is an open set $\subseteq \mathbb{R}$ such that $A = O \cap \mathbb{N}$

$\therefore A$ is relatively open.

Now, sort the elements of A is increasing order so we get
$x_0 = \inf(A)$, $x_1 = \inf(A \setminus x_0)$, $\cdots$
$x_0 < x_1 < \cdots < x_n < \cdots$

let $O = (-\infty, x_0) \cup (x_0, x_1) \cup \cdots (x_{n-1}, x_n) \cup (x_n, x_{n+1}) \cup \cdots$ which is open

Note that $O = \mathbb{R} \setminus A$

$\therefore A$ is closed.

$\therefore A$ is relatively closed

3) If $A \subseteq \mathbb{N}$ is finite it is the finite union of closed sets (singletons) so it is closed in $\mathbb{R}$.

If A is countable, sort the elements of A is increasing order so we get:

$x_0 = \inf(A)$, $x_1 = \inf(A \setminus x_0)$, $\cdots$

$x_0 < x_1 < \cdots < x_n < \cdots$

let $O = (-\infty, x_0) \cup (x_0, x_1) \cup \cdots (x_{n-1}, x_n) \cup (x_n, x_{n+1}) \cup \cdots$ which is open

Note that $O = \mathbb{R} \setminus A$

$\therefore$ A is closed.

4) let $A \subseteq \mathbb{Q}$ rel open and rel closed.

$\therefore$ it is of the form $O \cap \mathbb{Q}$ and $C \cap \mathbb{Q}$ where O open, C closed.

Recall that every open set O is a finite or countable union of disjoint intervals. If we want $O \cap \mathbb{Q} = C \cap \mathbb{Q}$. this is only possible if the symmetric difference $\Delta(C, O) \subset \mathbb{R} \setminus \mathbb{Q}$. This is only possible if we have all disjoint intervals having irrational endpoints such that $[q_1, q_2] \cap \mathbb{Q} = (q_1, q_2) \cap \mathbb{Q}$ where $q_1, q_2 \in \mathbb{R} \setminus \mathbb{Q}$.

Q4] Let $\{E_n\}$ be a sequence of closed non-empty subsets. with $E_n \supset E_{n+1}$ and $\lim_{n\to\infty} \text{diam } E_n = 0$.

since each E_n is nonempty, we can constructed a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that we let $x_n \in E_n$.

We are given then $E_n \supset E_m$ $\forall\, m>n$

$\therefore x_n, x_m \in E_n \quad \forall\, m>n$

$\therefore \rho(x_n, x_m) \leq \text{diam}(E_n) \quad \forall\, m>n$

we are given that $\lim_{n\to\infty} \text{diam}(E_n) = 0$

Let $\epsilon>0$. $\therefore \exists\, n_0 \in \mathbb{N}$ $\forall\, n>n_0$ $\text{diam}(E_n) < \epsilon$

i.e. $\forall\, m,n > n_0$ $\rho(x_n, x_m) < \epsilon$

$\therefore \{x_n\}_{n\in\mathbb{N}}$ is Cauchy.

Since X is complete $\{x_n\}_{n\in\mathbb{N}}$ converges to a point say x in X.

$\forall\, n_0 \in \mathbb{N} : \forall\, n > n_0 : x_n \leq E_{n_0}$

$\therefore \lim_{n\to\infty} \{x_n\} = x \in E$

$\therefore$ $x \in E$. If $y \neq x$, $y \in E$, then $\text{diam}(E) \geq \rho(x,y) \neq 0$ which contradicts $\lim_{n\to\infty} \text{diam}(E_n) = 0$.

so $E = \{x\}$. If we remove the restriction on diameter, $E_n = [n, \infty)$ provides a nested, closed sequence of sets with empty intersection.

Q5] A set G is dense in X if $\forall\, x \in X$, every open set containing x, O_x intersects G.

Fix O an open set

then $G_1 \cap O$ is non empty open

$\therefore$ we can find an open ball $B(x_1, r_1)$ such that $\overline{B}(x_1, r_1) \subset G_1 \cap O$.

if we call this E_1, we can recursively construct a sequence of closed, non-empty and bounded sets and we can choose r_n s.t $\lim_{n\to\infty} \text{diam}(E_n) = 0$.

This reduces to the previous case so $\bigcap_{n=1}^{\infty} E_n$ is a singleton and $\bigcap_{n=1}^{\infty} E_n \subset \bigcap_{n=1}^{\infty} G_n$

so $\bigcap_{n=1}^{\infty} G_n$ is non-empty.

Q6]

a) $\{p_n\} \sim \{q_n\}$ if $\lim_{n\to\infty} d(p_n, q_n) = 0.$

i.e. $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$: $d(p_n, q_n) < \varepsilon$.

① clearly reflexive as $d(p_n, p_n) = 0 <$ any positive ε $\forall n$:

② symmetric as $d(p_n, q_n) = d(q_n, p_n)$

③ assume $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{r_n\}$

let $\varepsilon > 0$

$\exists n_0 : \forall n > n_0 : d(p_n, q_n) < \varepsilon/2$

$\exists n_1 : \forall n > n_0 \ d(q_n, r_n) < \varepsilon/2$

let $N = \max\{n_0, n_1\}$

then $\forall n > N : d(p_n, q_n) + d(q_n, r_n) < \varepsilon$

$\therefore d(p_n, r_n) < \varepsilon$ (Δ ineq).

$\therefore \{p_n\} \sim \{r_n\}$

so it is transitive.

b) let X^* set of eq classes.

define $\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$

assume $\{p_n\} \sim \{p_n'\}$ and $\{q_n\} \sim \{q_n'\}$

$\lim_{n\to\infty} d(p_n', p_n) = 0$

and $\lim_{n\to\infty} d(q_n', q_n) = 0.$

let $\lim_{n\to\infty} d(p_n, q_n) = L$

i.e. $\forall \epsilon, \exists n_0 : \forall n > n_0 : |d(p_n, q_n) - L| < \epsilon$

let $\epsilon > 0$.

then $\exists n_1 : \forall n > n_1 \quad d(p_n, p_n') < \epsilon/2$

$\exists n_2 : \forall n > n_2 \quad d(q_n, q_n') < \epsilon/2$

let $N = \max\{n_1, n_2\}$

$\forall n > N: \quad d(p_n', q_n')$

$\leq d(p_n', p_n) + d(p_n, q_n) + d(q_n, q_n')$

$< d(p_n, q_n) + \epsilon$

$\therefore \forall n > N : |d(p_n', q_n') - d(p_n, q_n)| < \epsilon$

$\therefore \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p_n', q_n')$

Q6 c) let $\{[y^{(k)}]\}_{k\in\mathbb{N}} \in X^{*\mathbb{N}}$ be Cauchy in (X^*, Δ)

Fix $n_0 := 0 \wedge n_{i+1} = \inf\{n > n_i : (\forall l, j \geq n : \rho(y_l^k, y_j^k) < 2^{-i-k})\}$
which exists since the set is non-empty since each $y^{(k)}$ is Cauchy.
note that by construction, $\{n_i\}$ is strictly increasing.

We can now consider a sequence in each class which is a "representative" of that class i.e. define $z_i^k := y_{n_i}^k$. Then $\{z_i^k\}_{i\in\mathbb{N}} \in [y^k]$.

We have that $\forall k, j \in \mathbb{N} : \forall m, n \geq j : \rho(z_m^k, z_n^k) < 2^{-j-k}$
$\forall l \geq k : \rho(z_l^k, z_l^l) > 2^{-k}$.

Next, define $x_k := z_k^k$.

$$\forall k \leq l : \rho(x_k, x_l) = \rho(z_k^k, z_l^l) \leq \underbrace{\rho(z_k^k, z_l^k)}_{\leq 2^{-l-k}} + \underbrace{\rho(z_l^k, z_l^l)}_{\substack{\leq \Delta([z^k][z^l]) \\ \leq 2^{-k}}}$$

which gives us that $\{x_k\}$ is Cauchy so $\{[y^{(k)}]\}_{k\in\mathbb{N}}$ converges.
$\Rightarrow (X^*, \Delta)$ is complete.

d) Consider the map
$$\varphi : X \longrightarrow X^*$$
$$p \longmapsto P_p$$
where $P_p = \{p\}_{n\in\mathbb{N}}$ i.e. the constant sequence which is clearly Cauchy.

Then $\Delta(P_p, P_q) = \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p, q) = d(p, q)$.

then φ is an isometry which gives us an embedding $X \hookrightarrow X^*$

e) we need to show that for any $[x] \in X^*$, we can construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ $x_n \in X$ such that $\{\phi(x_n)\}_{n\in\mathbb{N}} \longrightarrow [x]$.

let $[x] \in X^*$ and choose a member of the equivalence class such that $[x] = [\{x_n\}_{n\in\mathbb{N}}] \in X^*$.

Then $\lim\limits_{m\to\infty} \Delta(\phi(x_m), [\{x_n\}_{n\in\mathbb{N}}]) = \lim\limits_{n\to\infty} \rho(x_m, x_n) = 0$.

since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy.

$\Rightarrow \phi(x_m) \to [\{x_n\}_{n\in\mathbb{N}}]$ in (X^*, Δ) $\therefore$ $[\{x_n\}_{n\in\mathbb{N}}]$ is an adherent point of $\phi(X)$. $\therefore$ $\overline{\phi(X)} = X^*$. $\blacksquare$