Q6 c) let $\{[y^{(k)}]\}_{k\in\mathbb{N}} \in X^{*\mathbb{N}}$ be Cauchy in (X^*, Δ)

Fix $n_0 := 0 \wedge n_{i+1} = \inf\{n > n_i : (\forall l, j \geq n : \rho(y_l^k, y_j^k) < 2^{-i-k})\}$
which exists since the set is non-empty since each $y^{(k)}$ is Cauchy.
note that by construction, $\{n_i\}$ is strictly increasing.

We can now consider a sequence in each class which is a "representative" of that class i.e. define $z_i^k := y_{n_i}^k$. Then $\{z_i^k\}_{i\in\mathbb{N}} \in [y^k]$.

We have that $\forall k, j \in \mathbb{N} : \forall m, n \geq j : \rho(z_m^k, z_n^k) < 2^{-j-k}$
$\forall l \geq k : \rho(z_i^k, z_i^l) > 2^{-k}$.

Next, define $x_k := z_k^k$.

$$\forall k \leq l : \rho(x_k, x_l) = \rho(z_k^k, z_l^l) \leq \underbrace{\rho(z_k^k, z_l^k)}_{\leq 2^{-l-k}} + \underbrace{\rho(z_l^k, z_l^l)}_{\substack{\leq \Delta([z^k][z^l]) \\ \leq 2^{-k}}}$$

which gives us that $\{x_k\}$ is Cauchy so $\{[y^{(k)}]\}_{k\in\mathbb{N}}$ converges.
$\Rightarrow (X^*, \Delta)$ is complete.

d) Consider the map
$$\varphi : X \longrightarrow X^*$$
$$p \longmapsto P_p$$
where $P_p = \{p\}_{n\in\mathbb{N}}$ i.e. the constant sequence which is clearly Cauchy.

Then $\Delta(P_p, P_q) = \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p,q) = d(p,q)$.

thus φ is an isometry which gives us an embedding $X \hookrightarrow X^*$

e) we need to show that for any $[x] \in X^*$, we can construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ $x_n \in X$ such that $\{\phi(x_n)\}_{n\in\mathbb{N}} \longrightarrow [x]$.

let $[x] \in X^*$ and choose a member of the equivalence class such that $[x] = [\{x_n\}_{n\in\mathbb{N}}] \in X^*$.

Then $\lim\limits_{m\to\infty} \Delta\left(\phi(x_m), [\{x_n\}_{n\in\mathbb{N}}]\right) = \lim\limits_{n\to\infty} \rho(x_m, x_n) = 0$.

since $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy.

$\Rightarrow \phi(x_m) \to [\{x_n\}_{n\in\mathbb{N}}]$ in (X^*, Δ) $\therefore$ $[\{x_n\}_{n\in\mathbb{N}}]$ is an adherent point of $\phi(X)$. $\therefore$ $\overline{\phi(X)} = X^*$. ∎