Math 210B: Homework #4

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Problem 1

Let M be a left module that is generated by some $a \in M$ i.e. $M = Ra$. Consider the R-linear map $f : R \to M$ where $1 \mapsto a$. (Recall that 1 generates R when R is viewed as a module over itself.) This map is surjective onto M since we can write each $m = ra = f(r)$ for some $r \in R$. Therefore, by the first isomorphism theorem, $M \cong R/\ker(I)$. Since $\ker(f) \subset R$ submodule, it is an ideal. Therefore, $M \cong R/I$.

Problem 2

We define a relation on $M \times S$ as $(m, s) \sim (m', s') \iff \exists u \in S : u(ms' - m's) = 0$. Addition and scalar multiplication are defined in the usual way. It is clear that this makes $S^{-1}M$ an $S^{-1}R$-module. Let $L : RMod \to S^{-1}RMod$ be the functor that takes an R-module to its localization (which is a module over the localized ring). We need to show this is functor is exact. Consider the following exact sequence. $0 \longrightarrow L \xrightarrow{u} M \xrightarrow{v} N \longrightarrow 0$. Now consider the sequence after the localization functor is applied. $0 \longrightarrow S^{-1}L \xrightarrow{u} S^{-1}M \xrightarrow{v} S^{-1}N \longrightarrow 0$. Let $x/s \in S^{-1}M : x/s \in \ker(v)$ i.e. $\exists t \in S : v(xt) = v(x)t = 0 \in M$ i.e. $xt \in \ker(v)$. By exactness, $xt = u(y)$ for $y \in L$. Then x/s is the image of y/st which proves exactness of this sequence as well.

Problem 3

Let $J \subset R$ and consider $JM \subset M$. Then the quotient module M/JM is annihilated by J. We define the R/J action on M/MJ as $\overline{rm} = rm$. This map is well-defined if and only if J is contained in the annihilator of M/JM which is the case here.

Problem 4

Let F be a free module with basis $(f_i)_{i \in I}$ i.e. we can write every $x \in F$ as $x = \sum_I r_i f_i$ for $r_i \in R$. Let $R \subset J$. From the previous problem, we saw that we can give a natural structure to F/JF as an R/J-module. It follows that the set $(f_i + JF)_{i \in I}$ is a basis for F/JF since $\sum_I \overline{r_i}\overline{f_i} = \sum_I r_i f_i$.

Problem 5

We will solve this using the previous parts. Assume that $R^n \cong R^m$. We know that $\exists M \subset R$ a maximal ideal. As a result, we can view $R^n/MR^n \cong R^m/MR^m$ as R/M-modules i.e. vector spaces over a field. This reduces to $(R/MR)^m \cong (R/MR)^n$ as vector spaces so $m = n$.

Problem 6

Assume A is a nonzero abelian group where $A \oplus A \cong A$. Let $R = End(A)$. We have an isomorphism $\alpha : A^2 \to A$. We need to extend this to an isomorphism $\alpha' : R^2 \to R$. Then the result follows inductively. We construct the map as follows: $(f, g)(x) \mapsto \alpha(f, g)(x)$. This map has domain R^2 and codomain R. It is a ring homomorphism as $\alpha((f_1, g_1) + (f_2, g_2)) = \alpha(f_1 + f_2, g_1 + g_2) = \alpha(f_1, g_1) + \alpha(f_2, g_2)$. It follows that α' is bijective since α is bijective. As a result, $R^2 \cong R$ and by induction $R^m \cong R \cong R^n$.

Problem 7

We prove that the arbitrary product of injective modules is injective. Recall that a module N is injective if every diagram of the form below can be extended as shown to a map from a larger module.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] \arrow[d] & B \arrow[dl, dashed, "\exists"] \\ & N & \end{tikzcd}$$

We are given a family of injective modules $(N_i)_{i\in I}$ such that the above property holds for all N_i. Fix some A, B.
Now consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] \arrow[d] & B \\ & \prod N_i & \end{tikzcd}$$

By composing the canonical projection map $p_i : \prod N_i \to N_i$, we get a diagram of the form below.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] \arrow[d] & B \arrow[dl, dashed, "\exists\beta_i"] \\ & N_i & \end{tikzcd}$$

Let's call the induced map β_i. Finally, by universality, the maps β_i factor through $\prod N_i$ via some map $\beta : B \to \prod N_i$ which makes the previous diagram commute. Therefore, $\prod N_i$ is injective. By duality, this argument can be extended to coproducts of projective modules being projective (using the same universal properties).

Problem 8

A module P is projective if every short exact sequence with P at the end splits. A module N is injective if every short exact sequence with N at the front splits. Therefore, every module is projective $\iff$ every short exact sequence of modules splits $\iff$ every module is injective.

Problem 9

We will prove that P (the module described) is projective by showing that it is the direct summand of a free module. To do this, we need to a construct an exact sequence that splits. A natural map that gives rise to our module is $f : R^n \to R$ where $(x_1, \dots, x_n) \mapsto a_1x_1 + \cdots + a_nx_n$ which is an R-linear map. The kernel of the map is $\ker(f) = P$. Further, this map is surjective as $(a_1, \dots, a_n)$ generate R. This gives us the exact sequence

$$0 \longrightarrow P \longrightarrow R^n \xrightarrow{f} R \longrightarrow 0$$

which splits since R is free. Therefore, $R^n = R \oplus P$ and P is projective.

Problem 10

In lecture, we saw that the functor $- \otimes_R N$ has right-adjoint $Hom_{\mathbb{Z}}(N, -)$ since $Hom(M \otimes_R N, P) \cong Hom_R(M, Hom_{\mathbb{Z}}(N, P))$ for all $P \in AbGroups$. Since it has a right adjoint, this functor commutes with colimits (which we proved last quarter).