Math 131AH: Homework #1

Due on January 18, 2022

Professor Marek Biskup

Nakul Khambhati

Problem 1

We will construct the truth tables by evaluating intermediate expressions. For conciseness, we will abbreviate `TRUE` as 1 and `FALSE` as 0.

1. $(P \vee Q) \wedge \neg(P \wedge Q)$

P	Q	$\neg(P \wedge Q)$	$P \vee Q$	$(P \vee Q) \wedge \neg(P \wedge Q)$
0	0	1	0	0
0	1	1	1	1
1	0	1	1	1
1	1	0	1	0

2. $P \Rightarrow (Q \Rightarrow \neg P)$

P	Q	$Q \Rightarrow \neg P$	$P \Rightarrow (Q \Rightarrow \neg P)$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

We can verify this by checking the truth table of $(P \wedge \neg Q) \vee (P \wedge Q)$

P	Q	$P \wedge \neg Q$	$P \wedge Q$	$(P \wedge \neg Q) \vee (P \wedge Q)$
0	0	0	0	0
0	1	0	0	0
1	0	1	0	1
1	1	0	1	1

Since P and $(P \wedge \neg Q) \vee (P \wedge Q)$ have the same truth values, we conclude that $P \iff (P \wedge \neg Q) \vee (P \wedge Q)$ and the expression is a `TAUTOLOGY`.

Problem 2

First, we define the proposition $m|n$ as $\exists k \in \mathbb{Z} : n = km$. We now transcribe the english sentences to propositional logic:

1. $\forall n \in \mathbb{N} : 3|n \implies (7|n \implies 2|n)$
2. $(\exists n \in \mathbb{N} : 6|n \wedge 4|n) \wedge (\exists m \in \mathbb{N} : 6|m \wedge \neg(4|m))$
3. $(\forall n \in \mathbb{N} : (6|n \implies 5|n) \implies 20|n) \wedge (\exists m \in \mathbb{N} : 6|m \wedge \neg(5|m))$
4. $\exists n \in \mathbb{N} : 3|n \wedge 7|n \wedge \neg(2|n)$
5. $(\forall n \in \mathbb{N} : \neg(6|n) \vee \neg(4|n)) \vee (\forall m \in \mathbb{N} \neg(6|m) \vee (4|m))$
6. $(\exists n \in \mathbb{N} : (6|n \implies 5|n) \wedge \neg(20|n)) \vee (\forall m \in \mathbb{N} : \neg(6|m) \vee 5|m)$

Problem 3

We are working within the universal set $\mathbb{R}$ of real numbers.

1. $\forall A \subset \mathbb{R}(\exists x \in A : (\forall y \in A : y = x \iff y^2 = 1))$
2. $\forall A \subset \mathbb{R}(\exists x \in A : (\forall y \in A : (y \neq x \Rightarrow y < x)))$
3. $\forall x \in \mathbb{R}\ \exists A \subset \mathbb{R} : A \neq \emptyset \wedge x \notin A$

Problem 4

We are asked to consider the relation $A \subset B := (\forall x \in A : x \in B)$

(reflexive) It is clear that $\forall x \in A : x \in A$ so $A \subset A$.

(antisymmetric) Let $A \subset B$ and $B \subset A$. Then $\forall x \in A : x \in B$ and $\forall y \in B : y \in A$ so $\forall x \in C : x \in A \iff x \in B$, therefore $A = B$.

(transitive) Let $A \subset B$ and $B \subset D$. Let $x \in A$. Therefore, $x \in B$, so $x \in D$. Since x was arbitrarily chosen, we have $\forall x \in A : x \in D$ so $A \subset D$.

This proves that the relation is a partial order.

Problem 5

(a) We are asked to show that $\bigcup\limits_{\alpha \in I} A_\alpha^c = (\bigcap\limits_{\alpha \in I} A_\alpha)^c$.

Proof. $x \in \bigcup\limits_{\alpha \in I} A_\alpha^c \iff \exists i \in I : x \in Y \setminus A_i \iff \exists i \in I : (x \in Y \wedge x \notin A_i) \iff x \in Y \wedge x \notin \bigcap\limits_{\alpha \in I} A_\alpha$ $\iff x \in (\bigcap\limits_{\alpha \in I} A_\alpha)^c$ i.e. $x \in Y \setminus \bigcap\limits_{\alpha \in I} A_\alpha$. Since $x \in \texttt{LHS} \iff x \in \texttt{RHS}$, we say that `LHS = RHS`.

Alternatively, we can describe both sets using propositional logic:

$\texttt{LHS} = \{x \in Y : (\exists \alpha \in I : x \notin A_\alpha)\}$

$\texttt{RHS} = \{x \in Y : \neg(\forall \alpha \in I : x \in A_\alpha)\} = \{x \in Y : (\exists \alpha \in I : x \notin A_\alpha)\}$

Clearly, `LHS = RHS`. □

(b) We are asked to show that $\bigcap\limits_{\alpha \in I} A_\alpha^c = (\bigcup\limits_{\alpha \in I} A_\alpha)^c$.

Proof. $x \in \bigcap\limits_{\alpha \in I} A_\alpha^c \iff \forall i \in I : x \in Y \setminus A_i \iff \forall i \in I : x \in Y \wedge x \notin A_i$ $\iff x \in Y \wedge x \notin \bigcup\limits_{\alpha \in I} A_\alpha \iff x \in (\bigcup\limits_{\alpha \in I} A_\alpha)^c$. Since $x \in \texttt{LHS} \iff x \in \texttt{RHS}$, we say that `LHS = RHS`.

Again, we can also describe both sets using propositional logic:

$\texttt{LHS} = \{x \in Y : (\forall \alpha \in I : x \notin A_\alpha)\}$

$\texttt{RHS} = \{x \in Y : \neg(\exists \alpha \in I : x \in A_\alpha)\} = \{x \in Y : (\forall \alpha \in I : x \notin A_\alpha)\}$

Therefore, `LHS = RHS`.

□

Problem 6

We define $[x] := \{y \in A : x \sim y\}$ and we have to prove that $\forall x, y \in A : [x] = [y] \vee [x] \cap [y] = \emptyset$.

Proof. Assume that $[x] \cap [y] \neq \emptyset$ i.e. $\exists z \in A : z \in [x] \wedge z \in [y]$. Then, by definition, $x \sim z$ and $y \sim z$. By symmetry, $z \sim y$ and by transitivity, $x \sim y$. Then $x \in [y]$. Let $w \in [x]$ i.e. $x \sim w$. Again, by symmetry and transitivity, we can show $y \sim w$ so $w \in [y]$. This implies that $[x] \subset [y]$. Similarly, we can argue that $[y] \subset [x]$ so $[x] = [y]$. We have shown that if $[x]$ and $[y]$ are not disjoint then $[x] = [y]$. So, we have proved the claim that $[x] = [y] \vee [x] \cap [y] = \emptyset$. Since $x, y \in A$ were arbitrarily chosen, this holds $\forall x, y \in A$. $\square$

Problem 7

Proof. It is clear that $x = x' \wedge y = y' \implies \{x, \{x, y\}\} = \{x', \{x', y'\}\} \iff (x, y) = (x', y')$.
Now assume that $(x, y) = (x', y') \iff \{x, \{x, y\}\} = \{x', \{x', y'\}\}$. Then, by size considerations, $x = x' \wedge \{x, y\} = \{x', y'\}$ so $y = y'$. $\square$

Problem 8

1. $f^{-1}(\bigcup\limits_{\alpha \in I} Y_\alpha) = \left\{x \in X : f(x) \in \bigcup\limits_{\alpha \in I} Y_\alpha\right\} = \{x \in X : (\exists \alpha \in I : f(x) \in Y_\alpha)\}$

   $\bigcup\limits_{\alpha \in I} f^{-1}(Y_\alpha) = \{x \in X : (\exists \alpha \in I : f(x) \in Y_\alpha)\}$

   Since the two have identical expressions, $f^{-1}(\bigcup\limits_{\alpha \in I} Y_\alpha) = \bigcup\limits_{\alpha \in I} f^{-1}(Y_\alpha)$

2. $f^{-1}(\bigcap\limits_{\alpha \in I} Y_\alpha) = \left\{x \in X : f(x) \in \bigcap\limits_{\alpha \in I} Y_\alpha\right\} = \{x \in X : (\forall \alpha \in I : f(x) \in Y_\alpha)\}$

   $\bigcap\limits_{\alpha \in I} f^{-1}(Y_\alpha) = \{x \in X : (\forall \alpha \in I : f(x) \in Y_\alpha)\}$

   Since the two have identical expressions, $f^{-1}(\bigcap\limits_{\alpha \in I} Y_\alpha) = \bigcap\limits_{\alpha \in I} f^{-1}(Y_\alpha)$