1 a) i) $\rho_\infty(x,y)=0 \iff \max_{i=1,\dots k} |x_i - y_i| = 0$

$\iff$ for max $i \in \{1,\dots,k\}$

$\therefore \forall_{i\in\{1,\dots,k\}} \ |x_i - y_i| = 0$

$\therefore \forall i \quad x_i = y_i$

$\therefore x = y$

ii) $|x_i - y_i| = |y_i - x_i| \quad \forall i$

$\therefore \max_i |x_i - y_i| = \max_i |y_i - x_i|$

$\therefore \rho_\infty(x,y) = \rho_\infty(y,x)$

iii) $\forall i \quad |x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$

$\therefore \max_i \{|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|\}$
$\qquad\qquad\qquad\qquad \geq$

$\max_i \{|x_i - y_i|\} \leq \max_i \{|x_i - z_i|\} + \max_i \{|z_i - y_i|\}$

$\therefore \rho_\infty(x,y) \leq \rho_\infty(x,z) + \rho_\infty(z,y)$ $\square$

b) $\Leftarrow$ easy as each $x_i^{(n)}$ conv to L_i

$\therefore$ let $\varepsilon > 0$

$\exists \ x_{n_i}^{(n)}$ s.t. $|L_i - x_{n_i}^{(n)}| < \varepsilon$

$\therefore \rho_\infty(\max(L_i), x^{(n)}) < \varepsilon$

$\Rightarrow$ assume that $\{x^{(n)}\}_{n\in\mathbb{N}}$ converges to L

Then let $\varepsilon > 0$.

$\exists \ x^{(m)}$ s.t

$\rho_\infty(x^{(m)}, L) < \varepsilon$

i.e. $\max_i |x_i^{(m)} - L_i| < \varepsilon$

$\therefore$ each $\{x_i^{(n)}\}$ converges to L_i $\square$

2) a) $B'(x,r) = \{y \in X : \rho(x,y) \leq r\}$.

$X \setminus B'(x,r)$ needs to be open

let $z \in X \setminus B'(x,r)$

$\therefore \rho(x,z) > r$

$\therefore \rho(x,z) - r > 0$

set $\epsilon = \rho(x,z) - r$.

consider $B(z,\epsilon)$

then $w \in B(z,\epsilon) \Rightarrow \rho(w,z) < \epsilon$

$\therefore \rho(w,z) < \rho(x,z) - r$

$\therefore \rho(x,z) > r + \rho(w,z)$

$\rho(w,x) = \rho(w,z) + \rho(z,x)$

$\therefore \rho(w,x) > r + 2\rho(w,z)$ $\square$

$\therefore w \notin B'(x,r)$ $\square$

$|X| \geq 2$.

b) Ex: let X be a set, with ρ discrete metric.

consider $B(x,1) = \{x\}$. The closure of this set is $\{x\}$ since singletons are closed. However $B'(x,1) = X$.

Q3] $\rho(x,y) \leq \max\{\rho(x,y), \rho(y,z)\}$ $\forall x,y,z$.

consider the open ball $B(x,r) := \{y \in X : \rho(x,y) < r\}$

we need to show its complement is open.

let $z \in X \setminus B(x,r)$ s.t. $\rho(x,z) \geq r$

claim: the ball $B(z,r)$ is disjoint from $B(x,r)$

let $y \in B(z,r) \cap B(x,r)$

$d(x,z) \leq \max\{d(x,y), d(y,z)\} < r$.

But this contradicts $z \in X \setminus B(x,r)$ $\therefore$ no such y exists.

$\therefore B(x,r)$ is closed.

Now consider the closed ball $B'(x,r) = \{y \in X : \rho(x,y) \leq r\}$.

let $y \in B'(x,r)$. Claim: $B(y,r) \subseteq B'(x,r)$. Then we are done as $B'(x,r)$ is open let $z \in B(y,r)$ $\therefore d(y,z) < r$

also $d(x,y) \leq r$

$\therefore d(x,z) \leq \max\{d(y,z), d(x,y)\} \leq r$

$\therefore z \in B'(x,r)$ ■

$\therefore \overline{B(x,r)} = B(x,r)$

so $\partial B(x,r) = \emptyset$.

eg) let $X = \{$set of binary sequences$\}$

$\rho(\sigma,\sigma') := 2^{-(\text{first index where they differ})}$

also $\rho(\sigma,\sigma) := 0$

$\therefore$ clear that (M1) holds and (M2) holds directly by defn.

for (M3) assume σ_x and σ_z differ for the first time at k.

Then if either σ_x, σ_y or σ_z, σ_y must differ for the first time before or at k.

In other words: $\rho(\sigma_x,\sigma_z) \leq \rho(\sigma_x,\sigma_y) + \rho(\sigma_y,\sigma_z)$ so (M3) holds.

Next, we prove it is an ultrametric.

assume x,y differ at k for the first time.

$\therefore$ at k, either x differs from z

or y differs from z

$\therefore \inf\{l : \sigma_x(l) \neq \sigma_z(l)\} \leq k$

or $\inf\{l : \sigma_y(l) \neq \sigma_z(l)\} \leq k$

$\therefore \rho(\sigma_x,\sigma_y) \leq \rho(\sigma_x,\sigma_z)$

or $\rho(\sigma_x,\sigma_y) \leq \rho(\sigma_y,\sigma_z)$

$\therefore \rho(\sigma_x,\sigma_y) \leq \max\{\rho(\sigma_x,\sigma_z), \rho(\sigma_y,\sigma_z)\}$ ■

Q4] let $\{x_n\}, \{y_n\}$ be cauchy s.t they get arbitrarily close after a point.

consider the sequence $\{\rho(x_n,y_n)\}_{n\in\mathbb{N}}$.

let $\varepsilon > 0$.

let $N \in \mathbb{N}$ s.t $\forall m,n \geq N$ (exists since cauchy)

$\rho(x_m,x_n) < \varepsilon/2$

$\rho(y_m,y_n) < \varepsilon/2$

then $|\rho(x_m,y_m)-\rho(x_n,y_n)| \leq |\rho(x_m,x_n)| + |\rho(y_m,y_n)| < \varepsilon$

Q5) a) $E^\circ = \bigcup$ open sets $\therefore$ it is open

b) if E° is open, $E \subset E^\circ$

and each $E^\circ \subset E$ always

$\therefore E = E^\circ$

if $E = E^\circ$ then E is open from a)

c) If $G \subset E$, G open

then $G \in \{$ open sets in $E\}$

$\therefore G \subset \bigcup$ open sets in $E = E^\circ$

d) We need to prove that that $X \setminus E^\circ = \bigcap_{\substack{E^c \subset C,\\ C \text{ closed}}} C$

$$x \in X \setminus E^\circ$$

$$\Leftrightarrow x \in X \setminus \bigcup_{O \subset E} O$$

$$\Leftrightarrow \forall_{O \subset E} \; x \notin O$$

$$\therefore \forall_{O \subset E} \; x \in O^c$$

$x \in$ open set in E

$\Leftrightarrow x \in$ closed set containing E^c

$$\therefore \forall_{E^c \subset C} \; x \in C$$

$$\Leftrightarrow x \in \bigcap_{E^c \subset C} C \qquad \blacksquare$$

e) No. Consider $E = (0,1) \cup (1,2) \subset \mathbb{R}$ with metric topology.

$\text{int}(E) = E$

$\bar{E} = [0,1] \cup [1,2] = [0,2]$

and $\text{int}(\bar{E}) = (0,2)$

$\therefore$ they are not equal

f) No. Consider

$E = \mathbb{N} \subset \mathbb{R}$ $\text{int}(\mathbb{N}) = \emptyset$, $\bar{\mathbb{N}} = \mathbb{N}$

$\therefore \overline{\text{int}(\mathbb{N})} = \emptyset$

Q6) the set of rationals is dense in $\mathbb{R}$

ie let $x \in \mathbb{R}$, $\epsilon > 0$

then $\exists q \in \mathbb{Q}$ s.t $|x-q| < \epsilon$

consider the set of points with rational co-ords $(q_1, \dots, q_k) \in \mathbb{R}^k$

which are countable. this set is also dense since

let $x \in \mathbb{R}^k$, for each x_i, can find q_i s.t $|q_i - x_i| < \epsilon/n$

$\therefore \rho(q,x) < \epsilon$.

$\therefore \mathbb{R}^k$ is separable.

Q7) let O be a nonempty subset of $\mathbb{R}$, define relation $x \sim y$ on O s.t.

$x \sim y$ iff $x \leq y \wedge [x,y] \subseteq O$

OR $y \leq x \wedge [y,x] \subseteq O$

which can be checked is an eq. relation

claim: each $[x]$ is open. suppose $y \in [x]$ and (WLOG) $x < y$.

then $y \in (u,v) \subseteq O$ for some $u, v \in \mathbb{R}$

pick $w \in (y,v)$

then $[x,w] = [x,y] \cup [y,w] \subseteq O$ so $y \in (x,w) \subseteq [x]$.

$\therefore \{[x] : x \in O\}$ partitions O into disjoint open sets. There are countably many such as each must contain a rational number (by density) and rational numbers are countable. ∎