Homework 4 (due: Fr, Oct. 27)

Problem 4*:

a) We assume that μ is a measure on the σ-algebra $\mathcal{L}$ and $\mu((0,1]^n) = c_0 < \infty$. Since μ is translation invariant it suffices to show that

$$\mu((0,1/k]^n) = c_0\lambda((0,1/k]^n) = c_0(1/k)^n.$$

We will do this by writing $(0,1]^n$ as a finite disjoint union of k^n translated versions of $(0,1/k]^n$ and then conclude by finite additivity and invariance under translation. Observe that

$$(0,1]^n = \bigsqcup_{i_1,\ldots,i_n \in \{0,\ldots,k-1\}^n} \left(\frac{i_1}{k}, \frac{i_1+1}{k}\right] \times \cdots \times \left(\frac{i_n}{k}, \frac{i_n+1}{k}\right].$$

Then, by finite additivity and translation invariance of μ we have that

$$\begin{aligned}
\mu((0,1]^n) &= \mu\left(\bigsqcup_{i_1,\ldots,i_n \in \{0,\ldots,k-1\}^n} \left(\frac{i_1}{k}, \frac{i_1+1}{k}\right] \times \cdots \times \left(\frac{i_n}{k}, \frac{i_n+1}{k}\right]\right) \\
&= \sum_{i_1,\ldots,i_n \in \{0,\ldots,k-1\}^n} \mu\left(\left(\frac{i_1}{k}, \frac{i_1+1}{k}\right] \times \cdots \times \left(\frac{i_n}{k}, \frac{i_n+1}{k}\right]\right) \\
&= \sum_{i_1,\ldots,i_n \in \{0,\ldots,k-1\}^n} \mu\left((0,1/k]^n\right) \\
&= k^n \mu((0,1/k]^n).
\end{aligned}$$

Rearranging then gives us that $\mu((0,1/k]^n) = \mu((0,1]^n)(1/k)^n = c_0(1/k)^n$ as required.

b) Without loss of generality, it suffices to consider arbitrary rectangles with one corner at the origin because μ is translation invariant. Let R have rational coefficients so

$$R = \left(0, \frac{p_1}{q_1}\right] \times \cdots \times \left(0, \frac{p_n}{q_n}\right]$$

where $p_i, q_i \in \mathbb{N}$. Then, as above, we can express R as a disjoint union of $p_1 \cdots p_n$ translated versions of $(0, 1/q_1 \cdots q_n]$. Then by finite additivity and part a) we get that

$$\mu(R) = c_0 \frac{p_1 \cdots p_n}{q_1 \cdots q_n} = c_0\lambda(R).$$

c) Let $R = (0, x_1] \times \cdots \times [0, x_n)$ where $x_i \in \mathbb{R}$. By density of rationals, there exist (non-decreasing) sequence of rationals $\{q_{im}\}_{m\in\mathbb{N}} \to x_i$ for each $i \in [n]$. Then,

$$R = \bigcup_{m\in\mathbb{N}} (0, q_{1m}] \times \cdots \times (0, q_{nm}]$$

and by continuity from below since the union is of nested sets (by construction) we get

$$\begin{aligned}
\mu(R) &= \lim_{m\to\infty} \mu\left((0, q_{1m}] \times \cdots \times (0, q_{nm}]\right) \\
&= \lim_{m\to\infty} c_0\lambda\left((0, q_{1m}] \times \cdots \times (0, q_{nm}]\right) \\
&= c_0\lambda\left((0, x_1] \times \cdots \times (0, x_n]\right) \\
&= c_0\lambda(R)
\end{aligned}$$

d) The set of h-rectangles is a π-system that generates the Borel σ-algebra on $\mathbb{R}^n$ so by HW 2 Problem 3, equality of a measure on h-rectangles from part c) implies equality of the meausre on the entire Borel σ-algebra.

e) Let $L \in \mathcal{L}$ so we can write $L = B \cup C$ where B is a Borel set and C is a subset of a Borel null set D. Then $\mu(B) \leqslant \mu(E) \leqslant \mu(B) + \mu(D) = \mu(B)$ which implies that $\mu(E) = \mu(B) = c_0\lambda(B) = c_0\lambda(E)$.

f) If $\mu((0,1]^n) = \mu([0,1]^n) = 1$, then $c_0 = 1$ so $\mu = \lambda$ as they are equal on all sets.

Problem 5*:a) If the map is non-invertible then by by HW 3 Problem 4b the image has measure zero and is therefore Lebesgue measurable. If it is invertible, then T^{-1} exists, is linear and therefore measurable. T^{-1} then preimages measurable sets to measurable sets or in other words, T images measurable sets to measurable sets which is what we had to show.

b) μ_T is well-defined for each measurable set M by part a). It is also translation invariant as

$$\mu(t + M) = \lambda\left(T(M) + T(t)\right) = \lambda(T(M)) = \mu(M)$$

and by Problem 4 it must be a scalar multiple of the Lebesgue measure i.e. $\mu_T(M) = \Delta_T\lambda(M)$.

c) If $\det(T) = 0$ then the linear map has a non-trivial kernel which in turn means that the image of T is a subspace of a hyperplane of dimension $n-1$ which has measure zero by HW 3 Problem 4b. Therefore, $\mu_T(M) = 0$ for all $M \subset \mathbb{R}^n$ and $\Delta_T = 0$.

d) By definition, $\mu_{S\circ T}(M) = \lambda(T(S(M))) = \Delta_T(\lambda(S(M))) = \Delta_T{\cdot}\Delta_S(M)$ where the last two equalities follow from part b).

Problem 6*: a) Let

$$Q = \{(x_1, \ldots, x_n) \,|\, x_1, \ldots, x_n \in (0, 1]\} \subset \mathbb{R}.$$

Then the image under L is

$$L(Q) = \{(x_1 + x_2, \ldots, x_n) \,|\, x_1, \ldots, x_n \in (0, 1]\}.$$

Next, let

$$A = \{(a_1, \ldots, a_n) \in L(Q) : a_1 \leqslant 1\} = \{(a_1, \ldots, a_n) \in Q : a_2 < a_1\}$$

and finally let

$$B = \{(b_1, \ldots, b_n) \in L(Q) : b_1 > 1\}.$$

Notice that

$$B - e_1 = \{(b_1, \ldots, b_n) \in Q : b_2 \geqslant b_1\}$$

so $A \sqcup B = Q$.

b) We can reduce any linear transformation to a matrix by considering its action on the standard basis $B = \{e_1, \ldots, e_n\}$. Type 1 operations permute the columns/rows of a matrix. By combining Type 1,2,3 operations, we can reduce M to a matrix $M^{(1)}$ such that $M_{11}^{(1)} \neq 0$ and $M_{1i}^{(1)} = 0$ for $1 \leqslant i \leqslant 1$. Proceed inductively to obtain M_n which is lower triangular. We can add a multiple of columns to make M_n' which is diagonal and therefore Type 2. By reversing these operations (each is invertible) we can obtain M from M_n' as required.

c) We have already proved this for T non-invertible. Let $T = S_1 \times \cdots \times S_n$ be a product of elementary transformations as in the previous question. By 5d) $\mu_T(M) = \prod_{i=1}^n \Delta_{S_i} \lambda(M)$ for any measurable M. It is clear that each elementary operation has $\Delta_S = |\det(S)|$. Since the product of determinants is the determinant of their product

$$\Delta_T \lambda(M) = \prod_{i=1}^{n} |\det(S_i)| \lambda(M) = |\det(T)| \lambda(M).$$

Problem 7*: Assume that $M \subset \mathbb{R}^n$ is measurable so there exists a collection of open rectangles $\{R_n\}_{n \in \mathbb{N}}$ that covers M such that $\sum_{n \in \mathbb{N}} \lambda(R_n) < \lambda^*(M) + \epsilon$. Set $U = \bigcup_{n \in \mathbb{N}} R_n$ which is open and $M \subset U$. Then $M \Delta U = U \backslash M$ so $\mu\,(M \Delta U) < \epsilon$.

Conversely, assume M is a set such that the condition is met. We need to show it satisfies the Caratheodory criteria. Let $T \subset \mathbb{R}^n$ be arbitrary. Then

$$\begin{aligned} \lambda^*\left(T \cap M\right) + \lambda^*\left(T \cap M^c\right) \leqslant\; & \lambda^*\left(T \cap (M \cap A)\right) + \lambda^*\left(T \cap (M^c \cap A)\right) \\ & + \lambda^*\left(T \cap (M \cap A^c)\right) + \lambda^*\left(T \cap (M^c \cap A^c)\right) \end{aligned}$$

where $A \subset \mathbb{R}^n$ is an arbitrary an open set. Let $\epsilon > 0$ and choose A such that its symmetric difference with M is at most $\epsilon/2$. We call the expression on the right hand side above RHS. It follows that:

$RHS \leqslant \epsilon+\lambda^*(T\cap(M\cap A))+\lambda^*(T\cap(M^c\cap A^c)) \leqslant \epsilon+\lambda^*(T\cap A)+\lambda^*(T\cap A^c) = \epsilon+\lambda(T)$.
Taking $\epsilon \to 0$ we get the desired inequality.