Q4] Let $\{E_n\}$ be a sequence of closed non-empty subsets. with $E_n \supset E_{n+1}$ and $\lim_{n\to\infty} \text{diam } E_n = 0$.

since each E_n is nonempty, we can constructed a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that we let $x_n \in E_n$.

We are given then $E_n \supset E_m \quad \forall\, m>n$

$\therefore x_n, x_m \in E_n \quad \forall\, m>n$

$\therefore \rho(x_n, x_m) \leq \text{diam}(E_n) \quad \forall\, m>n$

we are given that $\lim_{n\to\infty} \text{diam}(E_n) = 0$

let $\epsilon>0$. $\therefore \exists\, n_0 \in \mathbb{N} \; \forall\, n>n_0 \quad \text{diam}(E_n) < \epsilon$

i.e. $\forall\, m,n > n_0 \quad \rho(x_n, x_m) < \epsilon$

$\therefore \{x_n\}_{n\in\mathbb{N}}$ is Cauchy.

Since X is complete $\{x_n\}_{n\in\mathbb{N}}$ converges to a point say x in X.

$\forall\, n_0 \in \mathbb{N}: \forall\, n > n_0 : x_n \leq E_{n_0}$

$\therefore \lim_{n\to\infty} \{x_n\} = x \in E$

$\therefore x \in E$. If $y \neq x, y \in E$, then $\text{diam}(E) \geq \rho(x,y) \neq 0$ which contradicts $\lim_{n\to\infty} \text{diam}(E_n) = 0$.

so $E = \{x\}$. If we remove the restriction on diameter, $E_n = [n, \infty)$ provides a nested, closed sequence of sets with empty intersection.

Q5] A set G is dense in X if $\forall\, x \in X$, every open set containing x, O_x intersects G.

Fix O an open set

then $G_1 \cap O$ is non empty open

$\therefore$ we can find an open ball $B(x_1, r_1)$ such that $\bar{B}(x_1, r_1) \subset G_1 \cap O$.

if we call this E_1, we can recursively construct a sequence of closed, non-empty and bounded sets and we can choose r_n s.t $\lim_{n\to\infty} \text{diam}(E_n) = 0$.

This reduces to the previous case so $\bigcap_{n=1}^{\infty} E_n$ is a singleton and $\bigcap_{n=1}^{\infty} E_n \subset \bigcap_{n=1}^{\infty} G_n$

so $\bigcap_{n=1}^{\infty} G_n$ is non-empty.

Q6]

a) $\{p_n\} \sim \{q_n\}$ if $\lim_{n \to \infty} d(p_n, q_n) = 0$.

i.e. $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N}$ s.t $\forall n \geq n_0$: $d(p_n, q_n) < \varepsilon$.

① clearly reflexive as $d(p_n, p_n) = 0 <$ any positive ε $\forall n$:

② symmetric as $d(p_n, q_n) = d(q_n, p_n)$

③ assume $\{p_n\} \sim \{q_n\}$ and $\{q_n\} \sim \{r_n\}$

let $\varepsilon > 0$

$\exists n_0 : \forall n > n_0 : d(p_n, q_n) < \varepsilon/2$

$\exists n_1 : \forall n > n_0 \ d(q_n, r_n) < \varepsilon/2$

let $N = \max\{n_0, n_1\}$

then $\forall n > N$: $d(p_n, q_n) + d(q_n, r_n) < \varepsilon$

$\therefore d(p_n, r_n) < \varepsilon$ (Δ ineq).

$\therefore \{p_n\} \sim \{r_n\}$

so it is transitive.

b) let X^* set of eq classes.

define $\Delta(P, Q) = \lim_{n \to \infty} d(p_n, q_n)$

assume $\{p_n\} \sim \{p_n'\}$ and $\{q_n\} \sim \{q_n'\}$

$\lim_{n \to \infty} d(p_n', p_n) = 0$

and $\lim_{n \to \infty} d(q_n', q_n) = 0$.

let $\lim_{n\to\infty} d(p_n, q_n) = L$

i.e. $\forall \epsilon, \exists n_0 : \forall n > n_0 : |d(p_n, q_n) - L| < \epsilon$

let $\epsilon > 0$.

then $\exists n_1 : \forall n > n_1 \quad d(p_n, p_n') < \epsilon/2$

$\exists n_2 : \forall n > n_2 \quad d(q_n, q_n') < \epsilon/2$

let $N = \max\{n_1, n_2\}$

$\forall n > N : \quad d(p_n', q_n')$

$\leq d(p_n', p_n) + d(p_n, q_n) + d(q_n, q_n')$

$< d(p_n, q_n) + \epsilon$

$\therefore \forall n > N : |d(p_n', q_n') - d(p_n, q_n)| < \epsilon$

$\therefore \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p_n', q_n')$