Math 210B H.W. 6

1) let P be a projective module over R a PID.
Then $\exists F$ free such that $F = P \oplus Q$. This allows us to identify
$P \subset F$ a submodule. Since F is f.g. over a PID, P is free as well.

2) We write $R = \mathbb{Z}[i]$ which is a free R-module over itself with basis $\{1\}$.
similarly, $5R$ is generated by $\{5\}$
and $5 = 5 \cdot 1$
$\therefore A = [5]$
$\therefore$ The invariant factor is $\{5R\}$ which is also its only elementary divisor.

3) let M be fin gen p-primary i.e. $p^n M = 0$ for some $n > 0$
$\therefore$ it is a field over R/P.

Consider M/pM and ${}_pM$ as vector spaces over R/P.
we need to show that they are isomorphic.
${}_pM = \{m \in M : pm = 0\}$
$M/pM = \{m + pM\}$
Consider the map ${}_pM \longrightarrow M/pM$
$m \longmapsto m + pM$.
let $m \in pM$
$\therefore m = pm'$
but $pm = 0$
$\therefore p^2 m' = 0$
$\therefore m' = 0$, $m = 0$ $\therefore$ trivial kernel.
The map is surjective as $p(m + pM) = pm + pM = pM = 0$.
$\therefore$ every such element $m + pM$ is killed by p.
as the two vector spaces are iso, $\dim_k(M/pM) = \dim_k({}_pM)$.

4) let $\{(1,0,0), (0,1,0), (0,0,1)\}$ be a basis for $\mathbb{Z}^3$.
The matrix A is given by: $\begin{pmatrix} -4 & 16 & 8 \\ 4 & -4 & 4 \\ 2 & -8 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -8 & 2 \\ 4 & -4 & 4 \\ -4 & 16 & 8 \end{pmatrix}$

$$\longrightarrow \begin{pmatrix} 2 & -8 & 2 \\ 4 & -4 & 4 \\ -4 & 16 & 8 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & -8 & 2 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 4 & 2 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix} \sim \begin{pmatrix} 2 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix}$$

$\therefore$ IF $= \{2\mathbb{Z}, 12\mathbb{Z}, 12\mathbb{Z}\}$

5) let $A = \begin{pmatrix} -2 & 0 & 0 \\ -1 & -4 & -1 \\ 2 & 4 & 0 \end{pmatrix}$

$$x \cdot I_3 - A = \begin{pmatrix} x+2 & 0 & 0 \\ 1 & x+4 & 1 \\ -2 & -4 & x \end{pmatrix} \Rightarrow \begin{pmatrix} x+2 & 0 & 0 \\ 0 & x+4 & 1 \\ -x-2 & -4 & x \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x+2 & 0 & 0 \\ 0 & x+4 & 1 \\ 0 & -4 & x \end{pmatrix} \Rightarrow \begin{pmatrix} x+2 & 0 & 0 \\ 0 & x+4 & 1 \\ 0 & -(x+2)^2 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & x+2 & 0 \\ 0 & 0 & (x+2)^2 \end{pmatrix}$$

$\therefore$ IF $= \{x+2, x^2+4x+4\}$

so $RCF(A) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & 1 & -4 \end{pmatrix}$

6) The issue here is that $\mathbb{Z}/n\mathbb{Z}$ is not a PID. However, we have the surjection $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z}$ so we can pullback each such module to a $\mathbb{Z}$-module. We can then apply its classification to all $\mathbb{Z}/n\mathbb{Z}$-modules

7) again, we can pull back via the localization homomorphism $\mathbb{Z} \rightarrow S^{-1}\mathbb{Z}$ and classify all such modules.

8) a. $\Rightarrow$ b $\Rightarrow$ c are all basic exercises in linear algebra. If $[A]$ is

diagonal it is in RCF, so each ED is a linear poly $\therefore$ all invariant factors are products of distinct linear polys. Also $m_A = f_k$. This shows (a) $\Rightarrow$ (d) $\Rightarrow$ (e) $\Rightarrow$ (f). Now, if m_A is a product of dist lin polys, then all invariant factors must be $\therefore$ elementary divisors must all be linear $\therefore \exists B$ s.t $[A]_B$ is diagonal $\square$

9) Let A s.t $A^N = 0$ for some $N > 0$. Then $m_A \mid x^N$

$\therefore f_k = m_A = x^k$ for some power of $k \leq N$.

$\therefore$ each $f_i \mid f_k$ = some x^{m_i}.

$\therefore$ The invariant factors of A are powers of x.

Further we cannot have $\deg(f_k = m_A) > n$ since $\sum \deg(f_i) = \dim(V) = n$.

$\therefore \deg(m_A) \leq n$, so $m_A \mid x^n$

$\therefore A^n = 0$.

10) We need to show that $A \sim J \sim J^T \sim A^T$

over algebraically closed field.

Then, $A \sim B$ over K

iff $A \sim B$ over K' (extension)

which is true since similar $\Leftrightarrow$ same RCF,

doesn't change with field K. (as we saw in lecture)