Homework 7 (due: Mo, Dec. 4)

Problem 1*: a) Since we are working with $\mathcal{P}(\mathbb{N})$ every function $f : \mathbb{N} \to \mathbb{C}$ is integrable as for all $B \subset \mathbb{C}$ we have $f^{-1}(B) \subset \mathbb{N}$ so it is in $\mathcal{P}(\mathbb{N})$. If an arbitrary f preimages all subsets of $\mathbb{C}$ to measurable sets, it certainly preimages measurable sets to measurable and is therefore measure.

I claim that f is integrable if and only if

$$\sum_{n=1}^{\infty} |f(n)| < \infty.$$

To prove integrability we need to show that $\int |f| d\mu < \infty$ so it suffices to show that

$$\int |f| d\mu = \sum_{n=1}^{\infty} |f(n)|.$$

To this end, define

$$g_n = \sum_{i=1}^{n} |f(i)| \mathbb{1}_{\{i\}}.$$

It is clear that $g_n \nearrow |f|$ and each g_n is simple therefore measurable and has finite integral. By the monotone convergence theorem, it follows that

$$\int |f| d\mu = \int \left(\lim_{n\to\infty} g_n \right) d\mu = \lim_{n\to\infty} \int g_n d\mu = \lim_{n\to\infty} \sum_{i=1}^{n} |f(i)| = \sum_{n=1}^{\infty} |f(n)|$$

as required.

b) A complex valued function f has integral equal to

$$\int Re(f) + i \int Im(f).$$

and each of these can be split further into the positive negative parts as

$$\int Re(f)^+ - \int Re(f)^- + i \int Im(f)^+ - i \int Im(f)^-.$$

From part a) we can rewrite positive integrals as sums so we get

$$\sum_{n=1}^{\infty} Re(f)^+ - \sum_{n=1}^{\infty} Re(f)^- + i \sum_{n=1}^{\infty} Im(f)^+ - i \sum_{n=1}^{\infty} Im(f)^-.$$

c) For $n \geqslant 2$ consider the functions $f_n : \mathbb{N} \to \mathbb{C}$ such that $f_n(k) = k^{-n}$ for $k \geqslant 2$ and $f_n(1) = 0$. Then from part b)

$$\int f_n d\mu = \sum_{k=2}^{\infty} k^{-n}.$$

They are all measurable from part a) and are dominated by $g = f_2(k) = k^{-2}$. The function g has finite integral as

$$\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6} < \infty.$$

Also, the pointwise limit of f_n exists and is 0 as for $k \geqslant 2$, $\lim_{n\to\infty} k^{-n} = 0$. By Lebesgue Dominated Convergence it follows that

$$\lim_{n\to\infty} \sum_{k=2}^{\infty} k^{-n} = \lim_{n\to\infty} \int f_n d\mu = \int \left(\lim_{n\to\infty} f_n\right) d\mu = \int 0 d\mu = 0$$

as required.

Problem 3*: Let $(X, \mathcal{A}, \mu)$ be a measure space.

a) By definition of the infimum, each of the sets

$$A_n := \{x \in X : |f(x)| > \|f\|_\infty + 1/n\}$$

has measure zero. Then the set $A = \bigcup_{n\in\mathbb{N}} A_n$ also has measure 0 by countable subadditivity or continuity from below. In fact

$$A = \{x \in X : |f(x)| > \|f\|_\infty\}$$

and so the infimum is achieved as a minimum for $\lambda = \|f\|_\infty$. Since f is measurable, all A_n are measurable as they are preimages of Borel (in fact open) sets and so A is measurable. By definition, $|f(x)| \leqslant \|f\|_\infty$ on A^c and since $\mu(A) = 0$ this means that $|f(x)| \leqslant \|f\|_\infty$ a.e.

b) For measurable functions $f, g \colon X \to \mathbb{C}$ we write $f \sim g$ if $f = g$ a.e.

First we show that addition and scalar mulitplication are well defined i.e. To show that $[f] + [g] = [f + g]$ is well-defined, let $f \sim f'$ and $g \sim g'$. We need to show that $f + g \sim f' + g'$. By assumption, there exist null sets N and M such that $f = f'$ on N^c and $g = g'$ on M^c. It is clear that $f + g = f' + g'$ on $(N \cup M)^c$ and $N \cup M$ has measure zero by finite subadditivity so $f + g = f' + g'$ a.e. which means that $f + g \sim f' + g'$ as required. Similarly we need to show that $c[f] = [cf]$. We show that if $f \sim f'$ then $cf \sim cf'$. Let $f \sim f'$ so $f = f'$ on N^c where $\mu(N) = 0$. Then clearly $cf = cf'$ on N^c so $cf \sim cf'$.

Next we simultaneously show that L^∞ is a vector space and $\|f\|_\infty$ is a norm on this space.

(1) Let $f, g \in L^\infty$. We will show that $\|f + g\|_\infty \leqslant \|f\|_\infty + \|g\|_\infty < \infty$ so in particular $f + g \in L^\infty$ which shows triangle inequality for the norm and closure under addition for the vector space. For all $x \in X$ it is true that

$$|f(x) + g(x)| \leqslant |f(x)| + |g(x)| \leqslant \|f\|_\infty + \|g\|_\infty$$

where the first inequality is just the triangle inequality for absolute value and the second inequality is from part a). From this it follows that

$$\|f + g\|_\infty \leqslant \|f\|_\infty + \|g\|_\infty < \infty$$

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(2) Let $f \in L^\infty$ and $c \in \mathbb{C}$. We show that $\|cf\|_\infty = |c|\|f\|_\infty < \infty$ so $cf \in L^\infty$ which shows that the norm is absolutely homgenous and the vector space is closed under scalar multiplication.

$$\begin{aligned}\|cf\|_\infty &= \inf\{\lambda : \mu\{cf > \lambda\} = 0\} \\ &= \inf\{|c|\lambda : \mu\{cf > |c|\lambda\} = 0\} \\ &= |c| \inf\{\lambda : \mu\{f > \lambda\} = 0\} \\ &= |c|\|f\|_\infty.\end{aligned}$$

(3) It is clear from the definition that $\|f\|_\infty \geqslant 0$ so we show that $\|f\|_\infty = 0$ implies that $f = 0$ a.e. From part a) this means that

$$0 \leqslant |f(x)| \leqslant \|f\|_\infty = 0 \quad \text{a.e.}$$

so $f = 0$ a.e.

c) Show that L^∞ with this norm is complete: if $\{f_n\}$ is a Cauchy sequence in L^∞, then there exists a function $f \in L^\infty$ such that $\|f_n - f\|_\infty \to 0$ as $n \to \infty$.

First we will show that $\|f_n - f\|_\infty \to 0$ if and only if there exists a measurable set E with $\mu(E) = 0$ and $f_n \to f$ uniformly on E^c. Assume that the LHS is true. Then for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geqslant N$

$$|f_n(x) - f(x)| \leqslant \|f_n - f\|_\infty < \epsilon \quad a.e.$$

Let M_n denote $\|f_n - f\|_\infty$ and set

$$A_n = \{x \in X : |f_n(x) - f(x)| > M_n\}$$

so that $\mu(A_n) = 0$. Setting $E = \bigcup_{n \geqslant N} A_n$ gives us what is required. Clearly $\mu(E) = 0$ by countably subadditivity and on E^c we have

$$|f_n(x) - f(x)| \leqslant \|f_n - f\|_\infty < \epsilon \quad \forall n \geqslant N$$

so $f_n \to f$ uniformly on E^c. Conversely, assume there exists E such that $\mu(E) = 0$ and $f_n \to f$ uniformly on E^c. Then for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for $n \geqslant N$ and on E^c

$$|f_n(x) - f(x)| < \epsilon$$

which means that it holds almost everywhere. Then, by definition, $\|f_n - f\| < \epsilon$.

We now proceed with the proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be Cauchy in L^∞ so for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\forall m, n \geqslant N \quad \|f_m - f_n\|_\infty < \epsilon.$$

For each $m, n \in \mathbb{N}$ define

$$E_{m,n} = \{x \in X : |f_m(x) - f_n(x) > \|f_m - f_n\|_\infty|\}$$

so that $\mu(E_{m,n}) = 0$. Taking $E = \bigcup_{m,n\in\mathbb{N}} E_{m,n}$ we get that $\mu(E) = 0$ by countable subadditivity and

$$E^c = \{x \in X : \forall m, n \in \mathbb{N} |f_m(x) - f_n(x)| \leqslant \|f_m - f_n\|_\infty\}.$$

Note that on E^c and for $m, n \geqslant N$

$$|f_m(x) - f_n(x)| \leqslant \|f_n - f_m\|_\infty < \epsilon$$

so on E^c, $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{C}$ and by completeness, it's pointwise limit exists call it $f(x)$ and is measurable. Recall that

$$|f_m(x) - f_n(x)| < \epsilon$$

and taking $n \to \infty$ we get

$$|f_m(x) - f(x)| \leqslant \epsilon$$

so for $m \geqslant N$

$$\|f_m(x) - f(x)\|_\infty \leqslant \epsilon$$

which shows that $f_m \to f$ in L^∞. Finally, by the triangle inequality for $m \geqslant N$

$$\|f\|_\infty \leqslant \|f_m\|_\infty + \|f_m - f\|_\infty \leqslant \|f_m\|_\infty + \epsilon < \infty$$

so $f \in L^\infty$.

Problem 4*: a) Let $1 \leqslant p < r < q < \infty$ and let $f \in L^p \cap L^q$ so $\|f\|_p < \infty$ and $\|f\|_q < \infty$. We want to show that $\|f\|_r < \infty$.

$$\begin{aligned} \|f\|_r &= \left(\int |f|^r\right)^{1/r} \\ &= \left(\int |f|^{\lambda r} |f|^{(1-\lambda)r}\right)^{1/r} \end{aligned}$$

where $\lambda \in [0, 1]$ is such that

$$\frac{\lambda r}{p} + \frac{(1-\lambda)r}{q} = 1$$

so we can apply Holder's inequality to get

$$\begin{aligned} \|f\|_r &\leqslant \left(\|f^{\lambda r}\|_p \cdot \|f^{(1-\lambda)r}\|_q\right)^{1/r} \\ &\leqslant \|f\|_p^\lambda \cdot \|f\|_q^{1-\lambda} < \infty \end{aligned}$$

as required.

For $q = \infty$ we simply write

$$\int |f|^r = \int |f|^p |f|^{r-p} \leqslant \|f\|_\infty^{r-p} \int |f|^p < \infty$$

so $f \in L^p \cap L^\infty \Rightarrow f \in L^r$.

b) Let $f \in L^q$ where $q < \infty$. We will write f^p as $\mathbb{1}_X \cdot f^p$ and apply Holder's inequality with

$$\frac{q-p}{q} + \frac{p}{q} = 1.$$

This gives us

$$\|\mathbb{1}_X \cdot f^p\| \leqslant \|\mathbb{1}_X\|_{\frac{q}{q-p}} \cdot \|f^p\|_{q/p}$$

which can be rewritten as

$$\int f^p d\mu \leqslant \mu(X)^{\frac{q-p}{q}} \left(\int f^q\right)^{p/q}.$$

Taking the p^{th} root on both sides gives us

$$\|f\|_p \leqslant \mu(X)^{\frac{q-p}{pq}} \|f\|_q < \infty$$

as $f \in L^p$ and $\mu(X) < \infty$ so $f \in L^p$.

We again deal with $q = \infty$ separately. Assume $\|f\|_\infty < \infty$. Recall from 3a that

$$|f(x)| \leqslant \|f\|_\infty \quad a.e.$$

so we write

$$\int |f|^p d\mu \leqslant \|f\|_\infty^p \int d\mu = \|f\|_\infty \mu(X) < \infty.$$

c) Consider the Lebesgue σ-algebra and Lebesgue measure on $[1, \infty]$ induced by the Lebesgue measure on $\mathbb{R}$. Denote $f_p(x) = x^{-1/p}$. We show that it belongs to $L^q \backslash L^p$ given $p < q$. First we show that $f_p \in L^q$ i.e. $\int f^q d\mu < \infty$. By the equivalence of the Riemann and Lebesgue integral we get

$$\int f_p^q d\mu = \int x^{-q/p} d\mu = \int_1^\infty x^{-q/p} dx = \frac{p}{q-p} < \infty$$

assuming that $p < q$. On the other hand $f \notin L^p$ as $\int_1^\infty \frac{1}{x} dx = \infty$. In more detail, using the Monotone Convergence Theorem and equivalence of Riemann and Lebesgue integrals

$$\int f^p d\mu = \int \frac{1}{x} d\mu = \int \lim_{n\to\infty} \left(\frac{1}{x} \cdot \mathbb{1}_{[1,n]}\right) d\mu = \lim_{n\to\infty} \int \frac{1}{x} \cdot \mathbb{1}_{[1,n]} d\mu$$
$$= \lim_{n\to\infty} \int_1^n \frac{1}{x} dx = \lim_{n\to\infty} \ln(n) = \infty.$$

Problem 5*: Let $\mathcal{I}$ denote the set of all intervals (a, b) on $\mathbb{R}$ with rational endpoints i.e. $a < b$ and $a, b \in \mathbb{Q}$. This set is countable. Now consider the vector space V over $\mathbb{Q}$ generated by $\mathbb{1}_I, I \in \mathcal{I}$. In other words, elements of V are linear combinations of characteristic functions on intervals in $\mathcal{I}$ with rational scalars. This set is also countable and I claim that it is dense in $L^p(\mathbb{R})$. Let $f \in L^p(\mathbb{R})$ and recall that $C_c(\mathbb{R})$ the set of condtinuous functions with compact support is dense in $L^p(\mathbb{R})$. Let $\epsilon > 0$. Then there exists $f_1 \in C_c(\mathbb{R})$ with $\|f - f_1\|_p < \epsilon/2$. Since f_1 is bounded, let $I \in \mathcal{I}$ be such that it contains the support of f_1. For any $\delta > 0$ we can pick $f_2 \in V$ such that $\|f_2 - f_1\|_\infty < \delta$ which follows from density of rationals and continuity of f_1. This is done as follows: by continuity of f_1 we can choose a partition of $I = I_1 \cup \ldots \cup I_n$ such that the oscillation of f_1 on each of the subintervals $\sup_{I_i} f - \inf_{I_i} f < \delta$. Then pick $q_i \in (\sup_{I_i} f, \inf_{I_i}) \cap \mathbb{Q}$ and define $f_2 = \sum_{i=1}^n q_i \mathbb{1}_{I_1} \in V$. By construction, it is clear that in each interval I_i, the distance between q_i and f_1 is less than δ so $\|f_1 - f_2\|_\infty < \delta$. We can set δ such that $|I|^{1/p}\delta < \epsilon/2$ Then

$$\begin{aligned}
\|f_1 - f_2\|_p &= \left(\int_I |f_1 - f_2|^p\right)^{1/p} \\
&\leqslant \left(\int_I \|f_1 - f_2\|_\infty^p\right)^{1/p} \\
&\leqslant |I|^{1/p} \cdot \|f_1 - f_2\|_\infty \\
&\leqslant |I|^{1/p}\delta < \epsilon/2.
\end{aligned}$$

Finally, from the triangle inequality in $L^p(\mathbb{R})$ it follows that $\|f - f_2\| \leqslant \|f - f_1\| + \|f_1 - f_2\| < \epsilon/2 + \epsilon/2 = \epsilon$ so V is a countable dense subset and $L^p(\mathbb{R})$ is separable.