ENPM 667 PROJECT 2
Final Project

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Note: For all plots the simulation time is 100 seconds with a time step of 0.01 seconds. For all plots the initial state condition is taken to be

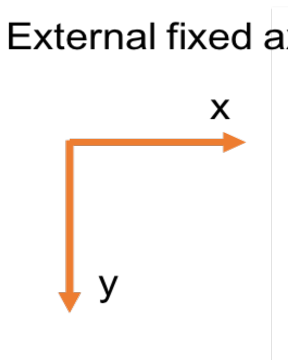
$$X(0) = 10^{-5}\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}$$

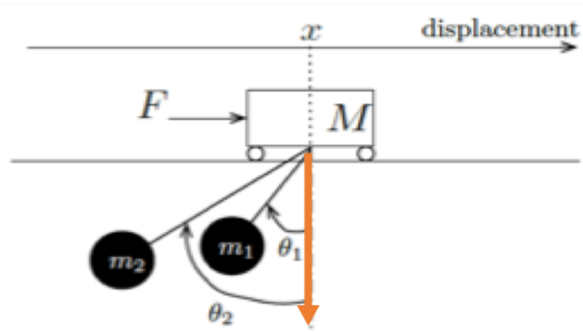
1 1st component Part A

Assumptions:

- There is no friction. It can possibly be considered as a disturbance in control design.
- The rods by which masses are hung are considered massless and rigid
- Cart is linearly moving along X-axis only. 1 dimensional motion of cart.
- Motion of pendulum is 2-D.
- All masses are of uniform density.
- The center of mass for all bodies is at their geometric centers.
- Coordinate axe are fixed externally with X axis at the height of point of suspension, which coincides with the center of mass of cart
- Clockwise rotation is positive.

The following pictures represents the axes that were taken into connsideration.





The positions of masses are given by the following expressions. (Notation: $\cos\theta_1 = C_{\theta_1}$ and $\sin\theta_1 = S_{\theta_1}$). x_c is the x position of cart. x_1 is position of pendulum 1 and x_2 is the position of pendulum 2. Similaryly y_c is the y position of cart and y_1 and the y_2 are the y positions of pendulums 1 and 2 respectively.

$$x_1 = x_c - l_1 S_{\theta_1} \quad (1)$$

$$y_1 = l_1 C_{\theta_1} \quad (2)$$

$$x_2 = x_c - l_2 S_{\theta_2} \quad (3)$$

$$y_2 = l_2 C_{\theta_2} \quad (4)$$

The velocities of masses in the respsective axes can be obtained by differentiating above expressions w.r.t time.

$$\dot{x_1} = \dot{x_c} - l_1 C_{\theta_1} \dot{\theta_1} \quad (5)$$

$$\dot{y_1} = -l_1 S_{\theta_1} \dot{\theta_1} \quad (6)$$

$$\dot{x_2} = \dot{x_c} - l_2 C_{\theta_2} \dot{\theta_2} \quad (7)$$

$$\dot{y_2} = -l_2 S_{\theta_2} \dot{\theta_2} \quad (8)$$

From these the expression for kinetic energy of the whole system is ,

$$KE = \frac{1}{2} M \dot{x_c}^2 + \frac{1}{2} m_1 (\dot{x_c} - l_1 C_{\theta_1} \dot{\theta_1})^2 + \frac{1}{2} m_1 (-l_1 S_{\theta_1} \dot{\theta_1})^2 + \frac{1}{2} m_2 (\dot{x_c} - l_2 C_{\theta_2} \dot{\theta_2})^2 + \frac{1}{2} m_2 (-l_2 S_{\theta_2} \dot{\theta_2})^2 \quad (9)$$

Simplifying this we get,

$$KE = \frac{1}{2}[M + m_1 + m_2]\dot{x_c}^2 + \frac{1}{2} m_1 l_1^2 \dot{\theta_1}^2 + \frac{1}{2} m_2 l_2^2 \dot{\theta_2}^2 - \dot{x_c}[m_1 l_1 C_{\theta_1} \dot{\theta_1} + m_2 l_2 C_{\theta_2} \dot{\theta_2}] \quad (10)$$

The expression for potential energy is,

$$PE = m_1 g l_1 C_{\theta_1} + m_2 g l_2 C_{\theta_2} \quad (11)$$

Writing the lagrangian from these,

$$L = KE - PE \quad (12)$$

Partial Differentiating L w.r.t $\dot{x_c}$,

$$\frac{\partial L}{\partial \dot{x_c}} = \dot{x_c}(M + m_1 + m_2) - (m_1 l_1 C_{\theta_1} \dot{\theta_1} + m_2 l_2 C_{\theta_2} \dot{\theta_2}) \quad (13)$$

Differentiating lagrangian w.r.t x_c,

$$\frac{\partial L}{\partial x_c} = 0$$

Differentiating 13 w.r.t time,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x_c}} = \ddot{x_c}(M + m_1 + m_2) - m_1 l_1 (C_{\theta_1} \ddot{\theta_1} - S_{\theta_1} \dot{\theta_1}^2) - m_2 l_2 (C_{\theta_2} \ddot{\theta_2} - S_{\theta_2} \dot{\theta_2}^2) = F \quad (14)$$

Differentiating lagrangian w.r.t $\dot{\theta_1}$

$$\frac{\partial L}{\partial \dot{\theta_1}} = -\dot{x_c} l_1 m_1 C_{\theta_1} + m_1 l_1^2 \dot{\theta_1} \quad (15)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_1}} = m_1 l_1^2 \ddot{\theta_1} - m_1 l_1 (\ddot{x_c} C_{\theta_1} - S_{\theta_1} \dot{\theta_1} \dot{x_c}) \quad (16)$$

Differentiating largrangian w.r.t. θ_1

$$\frac{\partial L}{\partial \theta_1} = m_1 l_1 g S_{\theta_1} + m_1 l_1 S_{\theta_1} \dot{x_c} \dot{\theta_1} \quad (17)$$

Writing equation for $\ddot{\theta_1}$

$$m_1 l_1^2 \ddot{\theta_1} - m_1 l_1 (\ddot{x_c} C_{\theta_1} - g S_{\theta_1}) = 0 \quad (18)$$

Taking $\ddot{\theta_1}$ on one side,

$$\ddot{\theta_1} = \frac{\ddot{x_c} C_{\theta_1} - g S_{\theta_1}}{l_1} \quad (19)$$

Similarly for $\ddot{x_c}$,

$$\ddot{x_c} = \frac{F + m_1 l_1 (C_{\theta_1}\ddot{\theta_1} - S_{\theta_1}\dot{\theta_1}^2) + m_2 l_2 (C_{\theta_2}\ddot{\theta_2} - S_{\theta_2}\dot{\theta_2}^2)}{M + m_1 + m_2} \qquad (20)$$

Differentiating lagrangian w.r.t $\dot{\theta_2}$

$$\frac{\partial L}{\partial \dot{\theta_2}} = -\dot{x_c} l_2 m_2 C_{\theta_2} + m_2 l_2^2 \dot{\theta_2} \qquad (21)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta_2}} = m_2 l_2^2 \ddot{\theta_2} - m_2 l_2 (\ddot{x_c} C_{\theta_2} - S_{\theta_2}\dot{\theta_2}\dot{x_c}) \qquad (22)$$

Differentiating largrangian w.r.t. θ_2

$$\frac{\partial L}{\partial \theta_2} = m_2 l_2 g S_{\theta_2} + m_2 l_2 S_{\theta_2} \dot{x_c} \dot{\theta_2} \qquad (23)$$

Writing equation for $\ddot{\theta_2}$

$$m_2 l_2^2 \ddot{\theta_2} - m_2 l_2 (\ddot{x_c} C_{\theta_2} - g S_{\theta_2}) = 0 \qquad (24)$$

Taking $\ddot{\theta_2}$ on one side,

$$\ddot{\theta_2} = \frac{\ddot{x_c} C_{\theta_2} - g S_{\theta_2}}{l_2} \qquad (25)$$

Expressing all double differential terms only in 1st order and 0 order terms, Substituting equations 19 and 25, into equation 20, and simplifying we get.,

$$\ddot{x_c} = \frac{F - \frac{1}{2} m_1 g S_{2*\theta_1} - m_1 l_1 S_{\theta_1}\dot{\theta_1}^2 - \frac{1}{2} m_2 g S_{2*\theta_2} - m_2 l_2 S_{\theta_2}\dot{\theta_2}^2}{M + m_1 S_{\theta_1}^2 + m_2 S_{\theta_2}^2} \qquad (26)$$

Substituting 25 in 20,

$$\ddot{x_c} = \frac{F + m_1 l_1 C_{\theta_1}\ddot{\theta_1} - m_1 l_1 S_{\theta_1}\dot{\theta_1}^2 - m_2 l_2 S_{\theta_2}\dot{\theta_2}^2 - m_2 g S_{\theta_2} C_{\theta_2}}{M + m_1 + m_2 S_{\theta_2}^2} \qquad (27)$$

Substituting 27 in 19,

$$\ddot{\theta_1} = \frac{[F - m_1 l_1 S_{\theta_1}\dot{\theta_1}^2 - m_2 l_2 S_{\theta_2}\dot{\theta_2}^2 - m_2 g S_{\theta_2} C_{\theta_2}] C_{\theta_1} - [M + m_1 + m_2 S_{\theta_2}^2] g S_{\theta_1}}{l_1 [M + m_1 S_{\theta_1}^2 + m_2 S_{\theta_2}^2]} \qquad (28)$$

Substituting equation 19 in equation 20,

$$\ddot{x_c} = \frac{F + m_2 l_2 C_{\theta_2}\ddot{\theta_2} - m_2 l_2 S_{\theta_2}\dot{\theta_2}^2 - m_1 l_1 S_{\theta_1}\dot{\theta_1}^2 - m_1 g S_{\theta_1} C_{\theta_1}}{M + m_2 + m_1 S_{\theta_1}^2} \qquad (29)$$

Substiuting equation 29 in 25,

$$\ddot{\theta_2} = \frac{[F - m_2 l_2 S_{\theta_2}\dot{\theta_2}^2 - m_1 l_1 S_{\theta_1}\dot{\theta_1}^2 - m_1 g S_{\theta_1} C_{\theta_1}] C_{\theta_2} - [M + m_2 + m_1 S_{\theta_1}^2] g S_{\theta_2}}{l_2 [M + m_2 S_{\theta_2}^2 + m_1 S_{\theta_1}^2]} \qquad (29.1)$$

writing above equations in non-linear state space form. Consider the following state vector,

$$X = \begin{bmatrix} \dot{x_c} \\ \dot{\theta_1} \\ \dot{\theta_2} \\ x_c \\ \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \qquad (30)$$

$$\dot{X} = \begin{bmatrix} \ddot{x_c} \\ \ddot{\theta_1} \\ \ddot{\theta_2} \\ \dot{x_c} \\ \dot{\theta_1} \\ \dot{\theta_2} \end{bmatrix} \qquad (31)$$

$$\dot{X} = \begin{bmatrix} \frac{F-\frac{1}{2}m_1 g S_{2x_5} - m_1 l_1 S_{x_5} x_2^2 - \frac{1}{2}m_2 g S_{2x_6} - m_2 l_2 S_{x_6} x_3^2}{M + m_1 S_{x_5}^2 + m_2 S_{x_6}^2} = \\ \frac{[F - m_1 l_1 S_{x_5} x_2^2 - m_2 l_2 S_{x_6} x_3^2 - m_2 g S_{x_6} C_{x_6}] C_{x_5} - [M + m_1 + m_2 S_{x_6}^2] g S_{x_5}}{l_1 [M + m_1 S_{x_5}^2 + m_2 S_{x_6}^2]} \\ \frac{[F - m_1 l_1 S_{x_5} x_2^2 - m_2 l_2 S_{x_6} x_3^2 - m_1 g S_{x_5} C_{x_5}] C_{x_6} - [M + m_2 + m_1 S_{x_5}^2] g S_{x_6}}{l_2 [M + m_1 S_{x_5}^2 + m_2 S_{x_6}^2]} \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \end{bmatrix} \qquad (31.1)$$

And

$$Y = CX$$

Since there is a full state access in this paritcular case. where

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad (31.2)$$

In the above equation, F is the input force, and the equations are written in terms of six states.

2 Part B

Now, we linearize the above obtained non-linear system using jacobian linearization, around the equilibrium point, $x = 0$, $\theta_1 = 0$ and $\theta_2 = 0$. A system is said to be linearizable around an equilibrium point , if we can express the system in terms of linear matrices A, B, C, D , such that :

$$A_F = \bigtriangledown_x F(x, u) \qquad (32)$$

$$B_F = \bigtriangledown_u F(x, u) \qquad (33)$$

$$C_H = \bigtriangledown_x H(x, u) \qquad (34)$$

$$D_H = \bigtriangledown_u H(x, u) \qquad (35)$$

where , the derivatives are calculated at equilibrium point.
So, we can write the linearized A matrix as:

$$A_{lin} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} & \frac{\partial f_1}{\partial x_5} & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_2}{\partial x_4} & \frac{\partial f_2}{\partial x_5} & \frac{\partial f_2}{\partial x_6} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} & \frac{\partial f_3}{\partial x_4} & \frac{\partial f_3}{\partial x_5} & \frac{\partial f_3}{\partial x_6} \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \frac{\partial f_4}{\partial x_3} & \frac{\partial f_4}{\partial x_4} & \frac{\partial f_4}{\partial x_5} & \frac{\partial f_4}{\partial x_6} \\ \frac{\partial f_5}{\partial x_1} & \frac{\partial f_5}{\partial x_2} & \frac{\partial f_5}{\partial x_3} & \frac{\partial f_5}{\partial x_4} & \frac{\partial f_5}{\partial x_5} & \frac{\partial f_5}{\partial x_6} \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \frac{\partial f_6}{\partial x_3} & \frac{\partial f_6}{\partial x_4} & \frac{\partial f_6}{\partial x_5} & \frac{\partial f_6}{\partial x_6} \end{bmatrix} \qquad (36)$$

$$B_{lin} = \begin{bmatrix} \frac{\partial f1}{\partial u} \\ \frac{\partial f2}{\partial u} \\ \frac{\partial f3}{\partial u} \\ \frac{\partial f4}{\partial u} \\ \frac{\partial f5}{\partial u} \\ \frac{\partial f6}{\partial u} \end{bmatrix} \quad (37)$$

Finding the derivatives and evaluating them at equilibrium point , the following is obtined. (Refer code script linearimation.m).

$$A_{lin} = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{-(gm_1)}{M} & \frac{-(gm_2)}{M} \\ 0 & 0 & 0 & 0 & \frac{-(g(M+m_1)}{Ml_1} & \frac{-(gm_2)}{Ml_1} \\ 0 & 0 & 0 & 0 & \frac{-(gm_1)}{Ml_2} & \frac{-g(M+m_2)}{Ml_2} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad (38)$$

$$B_{lin} = \begin{bmatrix} \frac{1}{M} \\ \frac{1}{Ml_1} \\ \frac{1}{Ml_2} \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (39)$$

The. linearized state space representation will be,

$$\dot{X} = A_{lin}X + B_{lin}U \quad (40)$$

$$Y = CX \quad (41)$$

Where A_{lin} and B_{lin} are given by matrices in equations 38 and 39, and $U = F$ and matrix C is given by,

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (42)$$

3 Part C

Refer linearimation.m script First it is checked if the system is controllable. Since the linearized system is a linear time invariant system, we use the rank condition for controllability matrix to check for the controllability. The controllability matrix in this case will be given by,

$$controllability\_matrix = \begin{bmatrix} B_{lin} & A_{lin}B & A_{lin}^2B & A_{lin}^3B & A_{lin}^4B & A_{lin}^5B \end{bmatrix} \quad (43)$$

It is observed that the controllability matrix is a 6x6 matrix which. For a matrix to be full rank, it must have linearly independent row and column vectors. The linear independence can be tested by checking the determinant of the controllability matrix. If the determinant is not equal to zero, then the matrix will be full rank and thus controllable. So, first computing the determinant of controllability matrix,

$$|controllability\_matrix| = \frac{-(g^6l_1^2 - 2g^6l_1l_2 + g^6l_2^2)}{(M^6l_1^6l_2^6)} \quad (45)$$

The above expression can be simplified as,

$$|controllability\_matrix| = \frac{-g^6(l_1 - l_2)^2}{(M^6 l_1^6 l_2^6)} \qquad (46)$$

For this expression to not be equal to 0 , $l_1 \neq l_2$. Which is the condition for matrix to be full rank. Note that there is no dependency of condition on masses, and they can be anything, but if M is too high (powers of 10) then though the determinant might not actually be 0 but might go out of precision limits in certain frameworks and thus needs to be kept in mind. $l_1 \neq l_2$ is the final condition.

4 Part D

Substiuting the given values in $A_l in$, and $B_l in$, the following are obtained,

$$A_{lin} = \begin{bmatrix} 0 & 0 & 0 & 0 & -0.9800 & -0.9800 \\ 0 & 0 & 0 & 0 & -0.5390 & -0.0490 \\ 0 & 0 & 0 & 0-0.0980 & -1.0780 & \\ 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \end{bmatrix} \qquad (47)$$

$$B_{lin} = 1.0e-03 \begin{bmatrix} 1.0000 \\ 0.0500 \\ 0.1000 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (47.1)$$

Substiuting these matrices in the controllability matrix the following controllability matrix was obtained,

$$controllability\_matrix = 1.0e-17 * \begin{bmatrix} 1.0000 & 0 & 0.0000 & 0 & 0.0000 & 0 \\ 0.0500 & 0 & -0.0250 & 0 & 0.0125 & 0 \\ 0.1000 & 0 & 0.1000 & 0 & 0.1000 & 0 \\ 0 & 1.0000 & 0 & 0.0000 & 0 & 0.0000 \\ 0 & 0.0500 & 0 & -0.0250 & 0 & 0.0125 \\ 0 & 0.1000 & 0 & -0.1000 & 0 & 0.1000 \end{bmatrix} \qquad (48)$$

The rank of this matrix is found to be 6. Since, its a full rank matrix the system for substiuted values is controllable. To determine control input from the LQR controller, the matlab function 'lqr' was used (Refer script). For different Q and R matrices different types of responses were obtained, as shown below.

$$Q = 10 \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 \end{bmatrix} \qquad (49)$$

$$R = 0.00001 \qquad (50)$$

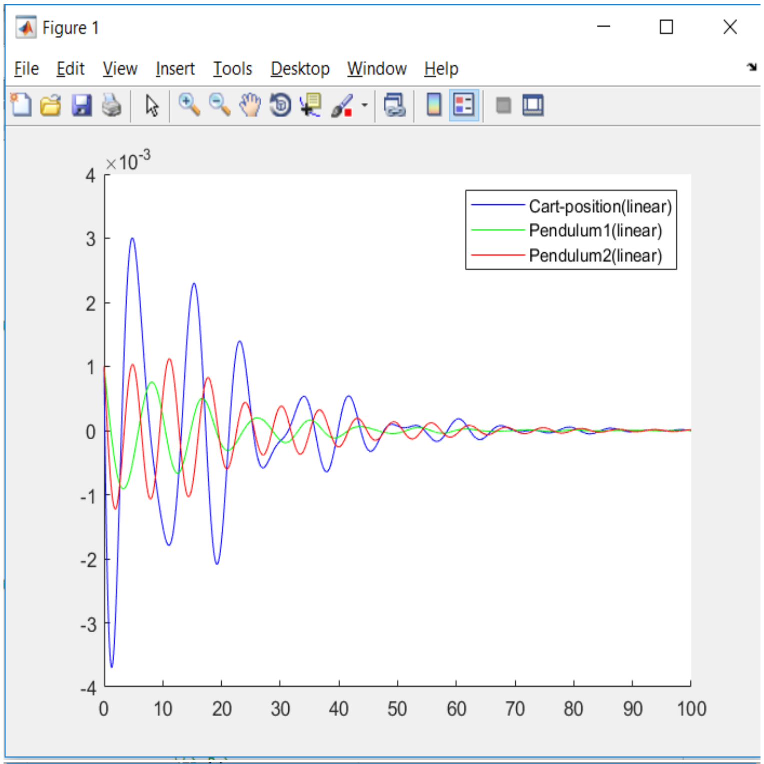


Figure 1
File Edit View Insert Tools Desktop Window Help
×10^-3
Cart-position(linear)
Pendulum1(linear)
Pendulum2(linear)
4
3
2
1
0
-1
-2
-3
-4
0 10 20 30 40 50 60 70 80 90 100

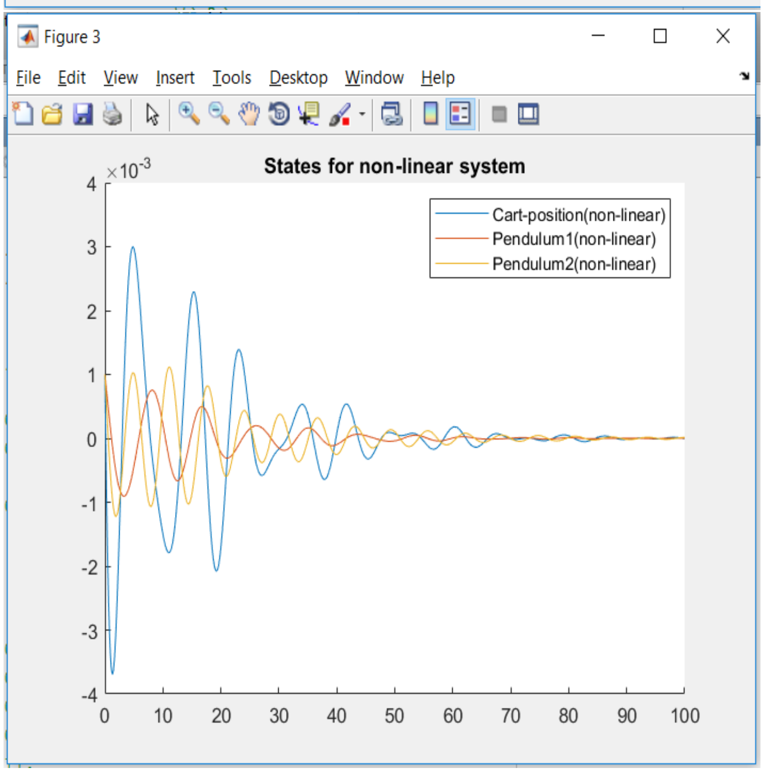
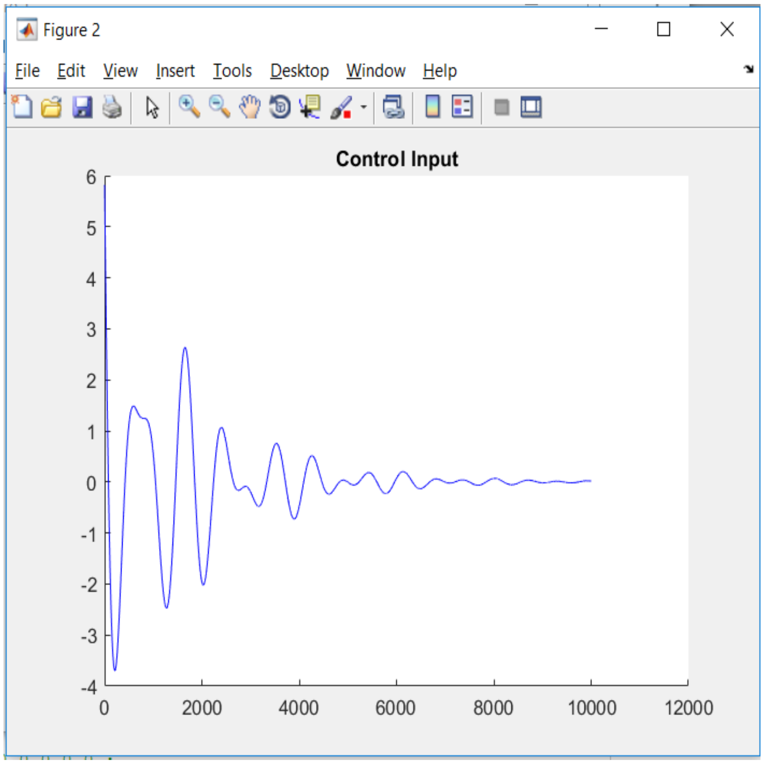


Figure 3
File Edit View Insert Tools Desktop Window Help
States for non-linear system
×10^-3
Cart-position(non-linear)
Pendulum1(non-linear)
Pendulum2(non-linear)
4
3
2
1
0
-1
-2
-3
-4
0 10 20 30 40 50 60 70 80 90 100



$$Q = 10\begin{bmatrix} 20 & 0 & 0 & 0 & 0 & 0 \\ 0 & 200 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 0 & 0 & 200 & 0 \\ 0 & 0 & 0 & 0 & 0 & 20 \end{bmatrix} \tag{51}$$

$$R = 0.00002 \tag{52}$$

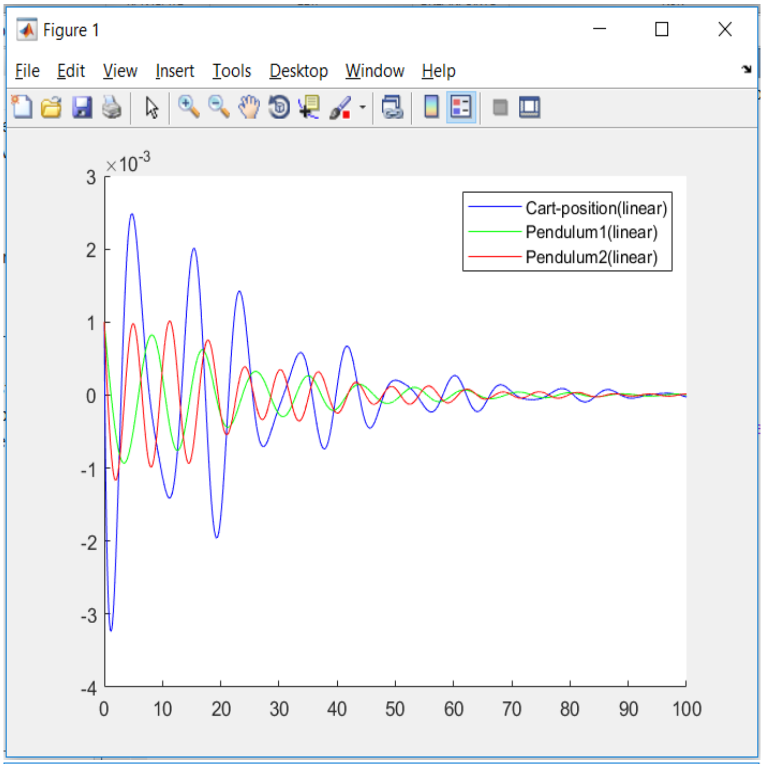


Figure 1
File Edit View Insert Tools Desktop Window Help
×10^-3
Cart-position(linear)
Pendulum1(linear)
Pendulum2(linear)
3
2
1
0
-1
-2
-3
-4
0 10 20 30 40 50 60 70 80 90 100

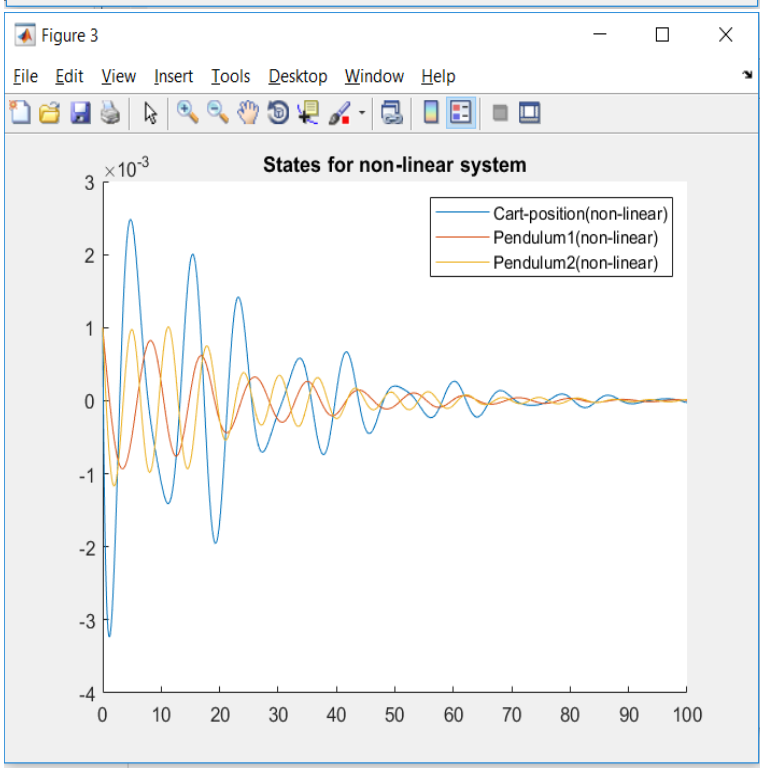
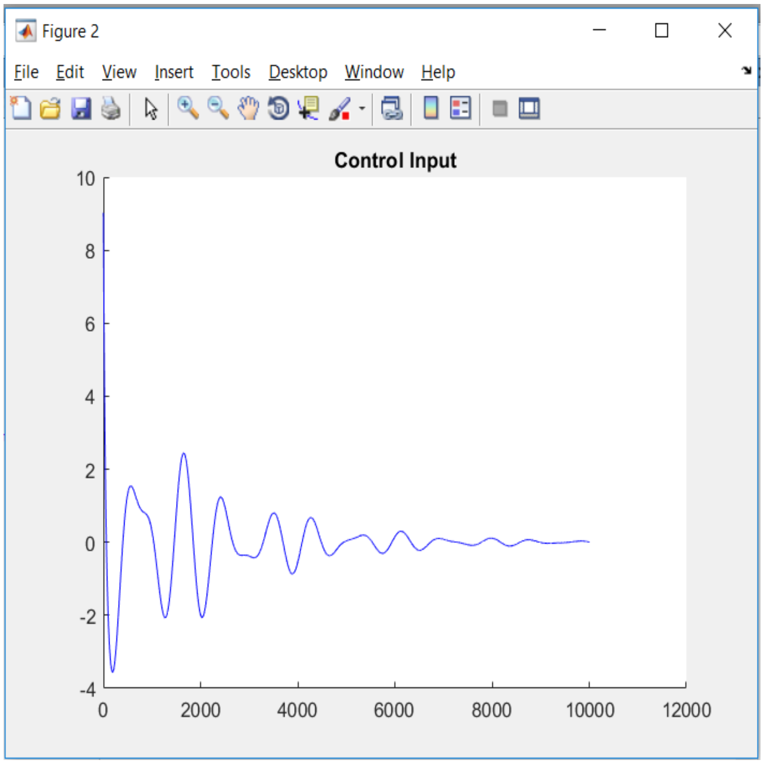


Figure 3
File Edit View Insert Tools Desktop Window Help
States for non-linear system
×10^-3
Cart-position(non-linear)
Pendulum1(non-linear)
Pendulum2(non-linear)
3
2
1
0
-1
-2
-3
-4
0 10 20 30 40 50 60 70 80 90 100



$$Q = 10\begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\0 & 1 & 0 & 0 & 0 & 0\\0 & 0 & 1 & 0 & 0 & 0\\0 & 0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 1 & 0\\0 & 0 & 0 & 0 & 0 & 1\end{bmatrix} \tag{53}$$

$$R = 1 \tag{54}$$

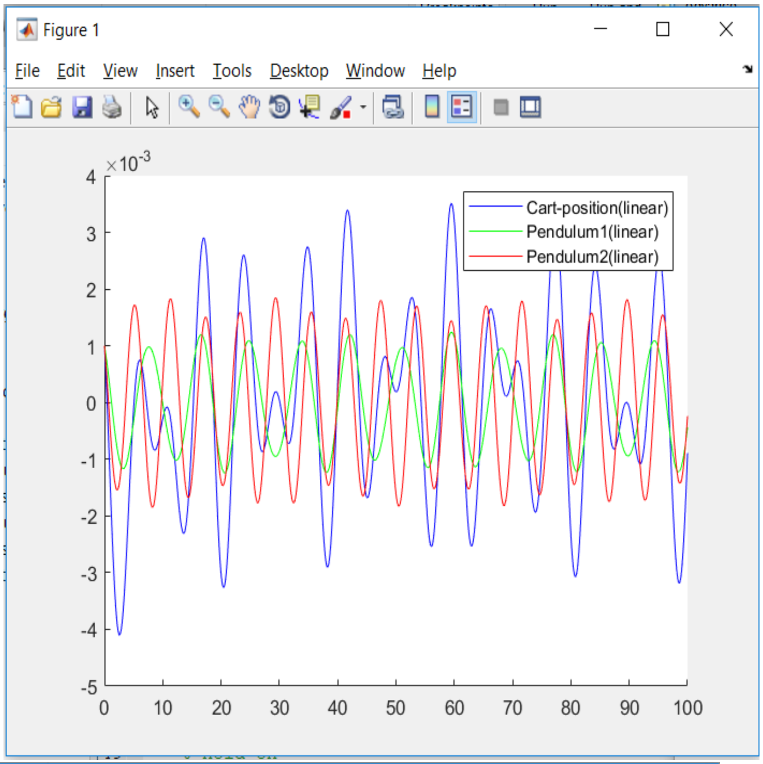


Figure 1
File Edit View Insert Tools Desktop Window Help
Cart-position(linear)
Pendulum1(linear)
Pendulum2(linear)
×10-3

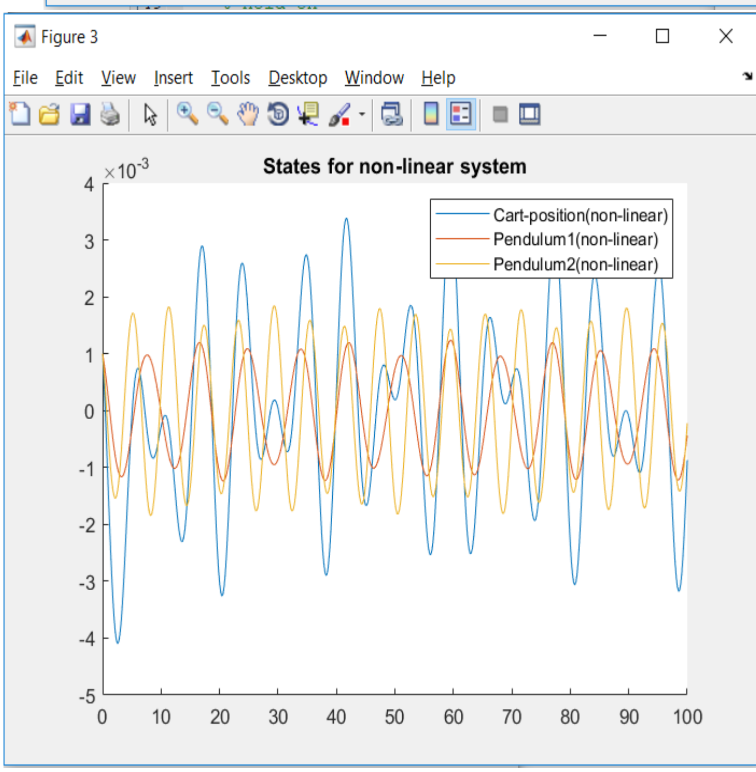
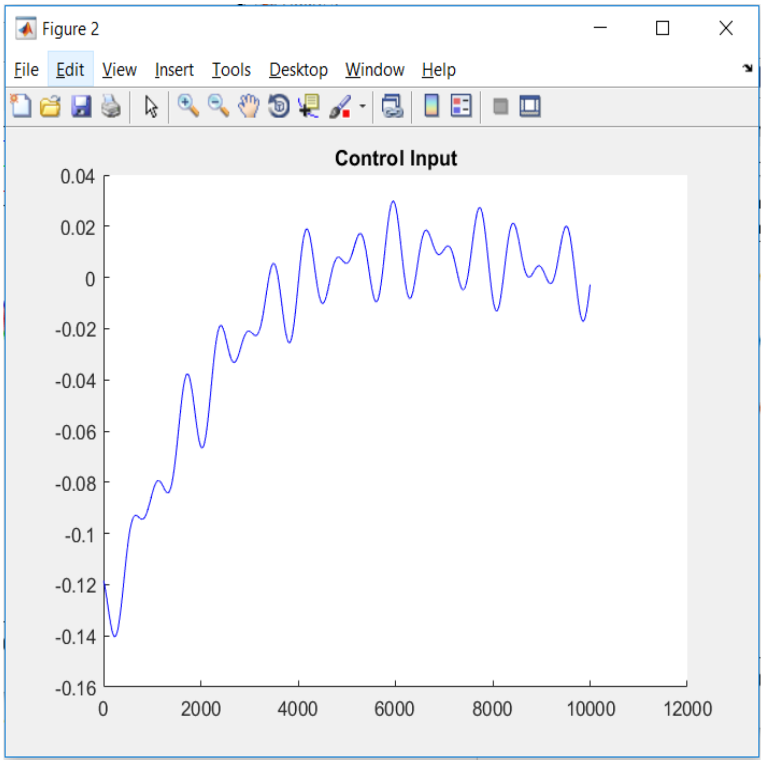


Figure 3
File Edit View Insert Tools Desktop Window Help
States for non-linear system
Cart-position(non-linear)
Pendulum1(non-linear)
Pendulum2(non-linear)
×10-3



$$Q = \begin{bmatrix} 10 & 5 & 80 & 2 & 6 & 10 \\ 10 & 20 & 100 & 50 & 20 & 10 \\ 30 & 50 & 30 & 10 & 10 & 10 \\ 20 & 60 & 10 & 30 & 20 & 40 \\ 30 & 10 & 10 & 10 & 20 & 80 \\ 60 & 10 & 20 & 60 & 50 & 10 \end{bmatrix} \tag{57}$$

$$R = 0.001 \tag{58}$$

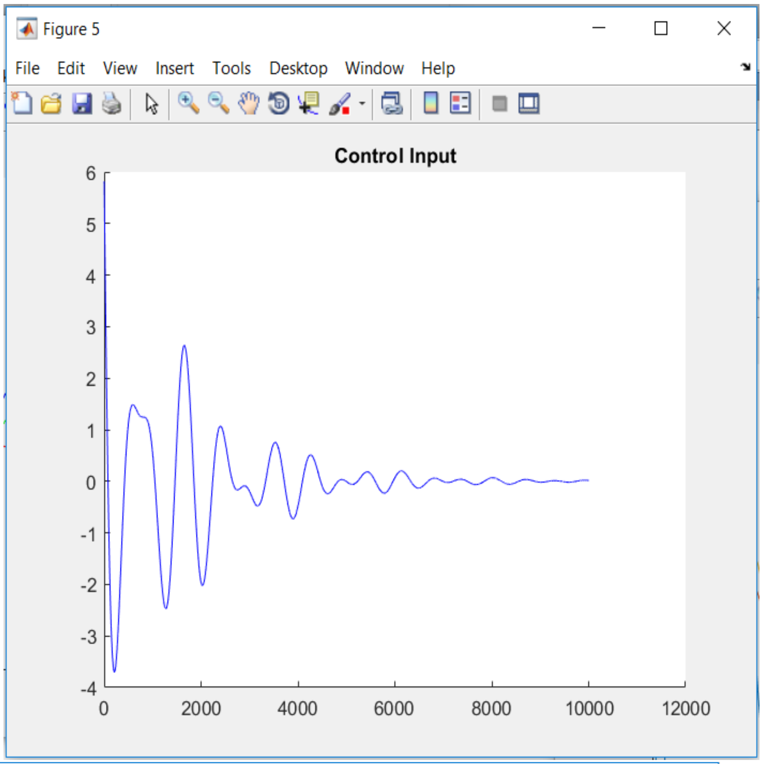


Figure 5
File Edit View Insert Tools Desktop Window Help
Control Input
6
5
4
3
2
1
0
-1
-2
-3
-4
0 2000 4000 6000 8000 10000 12000

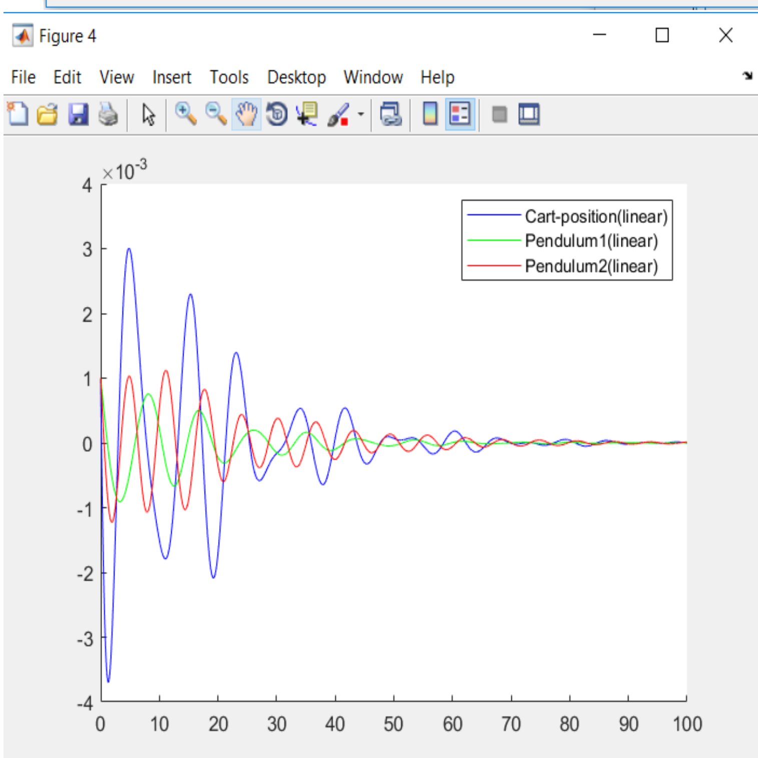
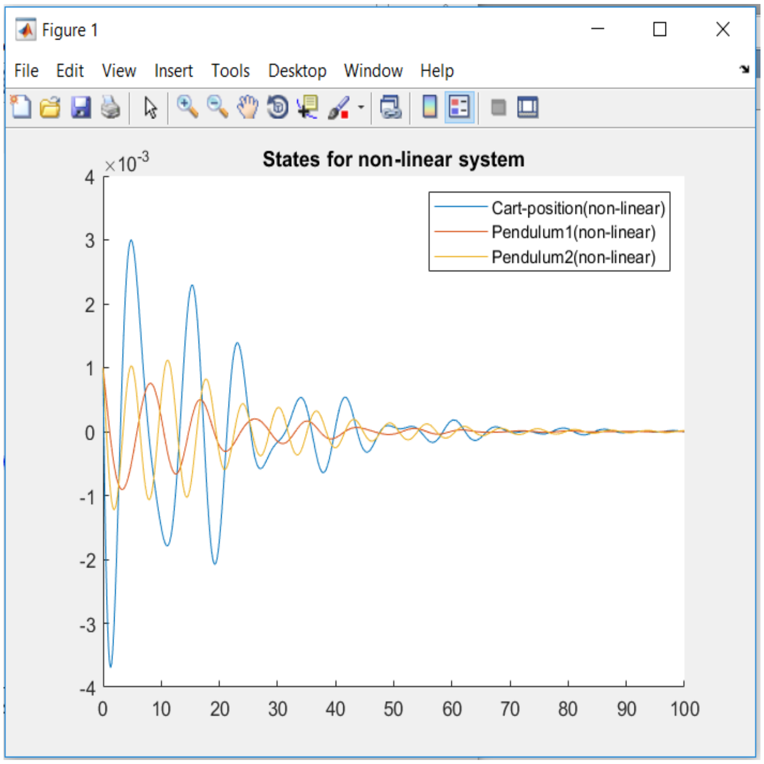


Figure 4
File Edit View Insert Tools Desktop Window Help
×10-3
4
3
2
1
0
-1
-2
-3
-4
0 10 20 30 40 50 60 70 80 90 100
Cart-position(linear)
Pendulum1(linear)
Pendulum2(linear)



Observations and tuning:

- It was observed that as Q and R were increased there were quite a lot of number of oscillations in the response of the system. This was expected since increasing Q and R would penalize heavily on the state change as well as the how much control input can be provided to the system.
- In Q specifically if the state change of velocity cart was penalized more heavily then the response of the system significantly changed, more oscillations were induced. The oscillations mainly exist, and system requires significant control effort to converge because theere is huge amount of difference between the masses of cart and pendulums (almost 10 times), because of this discrepencies between the cost elements of Q matrix and lower R (cost on control input) is needed to have a faster convergence with fewr oscillations which is observed in figure 1.
- If the other elements of the Q matrix apart from the diagonal matrix are changed then, the 'mutual' state changes were observed to be penalized. This caused changes in the temporary characterstics of response of the system, in between oscillations.
- The tuning for the system was done by observing the eigen values of the closed loop state $A_{lin} - B_{lin}K$ matrix. By lyaponav's indirect stability method the eigen values that matrix were in the left half plance, and the system is atleast locally stable. The 1st set of Q and R matrices (which is considered as a more suitable response) as shown.

5 Part E

Refer script Luenberger Observer. For each of the output vectors given in the question, the corresponding C matrices will be will be.

- $x(t)$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (59)$$

- $(\theta_1(t), \theta_2(t))$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (60)$$

- $(x(t), \theta_2(t))$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (61)$$

- $(x(t), \theta_1(t) m \theta_2(t))$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (62)$$

For each of these the Observability rank condition was checked for the linearized system. The observability matrix in this case for the linearized system will be given by will be given by,

$$observability\_matrix = \begin{bmatrix} C \\ A_{lin}C \\ A_{lin}^2 C \\ A_{lin}^3 C \\ A_{lin}^4 C \\ A_{lin}^5 C \end{bmatrix} \quad (63)$$

The rank for of obsvervability matrix for for $x(t)$ was found to be 6, the rank of observability matrix for $(\theta_1(t), \theta_2(t))$ was found to be 4, the rank of observability matrix for $(x(t), \theta_2(t)$ was found to be 6 and the rank of observability matrix for $(x(t), \theta_1(t), \theta_2(t))$ was found to be 6. The vectors for which observability matrix is full rank, will only be observable, hence the vectors $x(t)$, $(x(t), \theta_2(t))$ and $(x(t), \theta_1(t), \theta_2(t))$ are observable.

6 Part F

Step Function is defined as,

$$U = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad (64)$$

Taking this as control input, and substituting it in the solution of the state space equation, and assuming that initial time is 0.

$$X(t) = e^{At}X(0) + \int_0^t e^{A_{lin}(t-\tau)} B_{lin} u(\tau) d\tau \quad (65)$$

Subsituting $u(\tau) = 1$,

$$X(t) = e^{At}X(0) + \int_0^t e^{A_{lin}(t-\tau)}Bd\tau \quad (66)$$

$$X(t) = e^{At}X(0) + \int_0^t e^{A_{lin}t}e^{-A\tau}Bd\tau \quad (67)$$

Taking $e^{A_{lin}t}$ and B out since they are not dependent on τ,

$$X(t) = e^{At}X(0) + e^{A_{lin}t}\int_0^t e^{-A\tau}d\tau B \quad (67.1)$$

It is not possible to evaluate the integral further since A_{lin} is not invertible (Determinant of A_{lin} is 0). Hence the integral is evaluated as it is in MATLAB. Simulating with initial conditions, this gives the origian state vector. For an L matrix, then the errors at different time steps are obtained. The solution of error is not dependent on state vector. It is only dependent on initial error, A, L and C matrices. The solution of error is given by.

$$\dot{e(t)} = (A - LC)e \quad (68)$$

The above is obtained by subtracting $\dot{X}$ and $\dot{\hat{X}}$ Solving the above equation,

$$error = e^{(A-LC)t}error(0) \quad (69)$$

Here $e(0) = X(0) - \hat{X(0)}$ It is assumed that $\hat{X(0)}$ is a zero vector. Subtracting, $X(t) - e(t)$ gives the state estimator. The 'best' Luenberger observer is obtained by tuning L matrix. The tuning of L matrix is done by checking the eigenvalues of $A - LC$ matrix. The eigen values must have high negative real part and low complex part for the luenberger observer to be best.

For the non-linear system, instead of considering, error the the state estimator equation is directly considered. Consider the following State estimator equation,

$$\dot{\hat{X}} = f(\hat{X}, U, t) + L(Y - \hat{Y}) \quad (70)$$

This can be written as,

$$\dot{\hat{X}} = f(\hat{X}, U, t) + LC(X - \hat{X}) \quad (71)$$

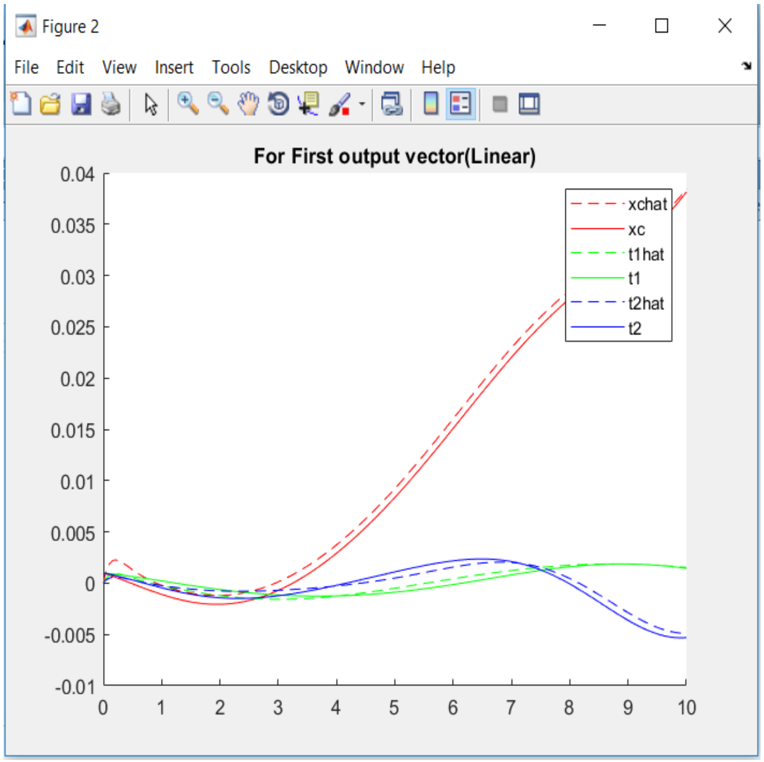
since $Y = CX$ and $\hat{Y} = C\hat{X}$. To this the original non linear state equation is appended, and new set of equations is obtained,

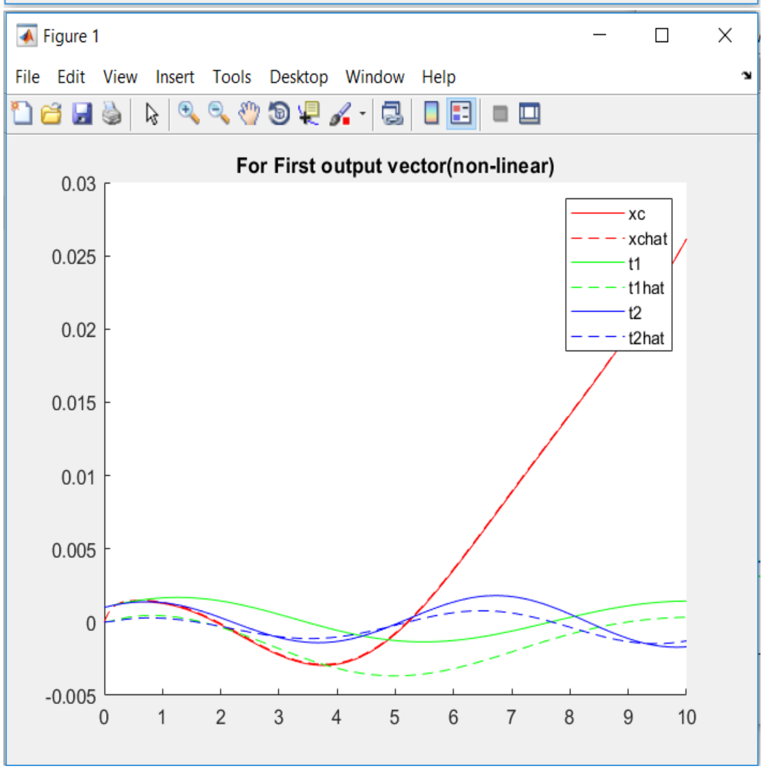
$$\begin{bmatrix} \dot{\hat{X}} \\ \dot{X} \end{bmatrix} = \begin{bmatrix} f(\hat{X}, U, t) + LC(X - \hat{X}) \\ f(X, U, t) \end{bmatrix} \quad (72)$$

These set of 12 differential equations are given t the ODE45 solver in matlab which uses analytical Runge Kutta(reference) methods to solve for differential equations. The following are the plots of states vs. their estimates, for each of the L matrices corresponding to the output vectors for linear and non-linear systems. The figures have been shown only for 1st 10 seconds so as to observe the output and if $\hat{X}$ and X are converging quickly or not.

- For $x(t)$

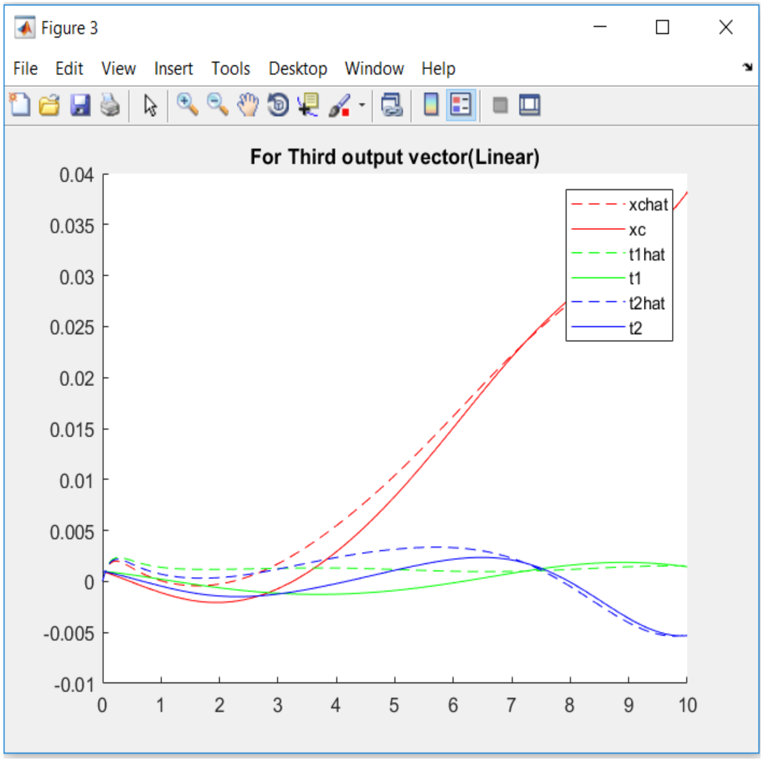
$$L = \begin{bmatrix} 30 \\ 12.5 \\ 8 \\ 11 \\ 0.2 \\ 0.08 \end{bmatrix}$$

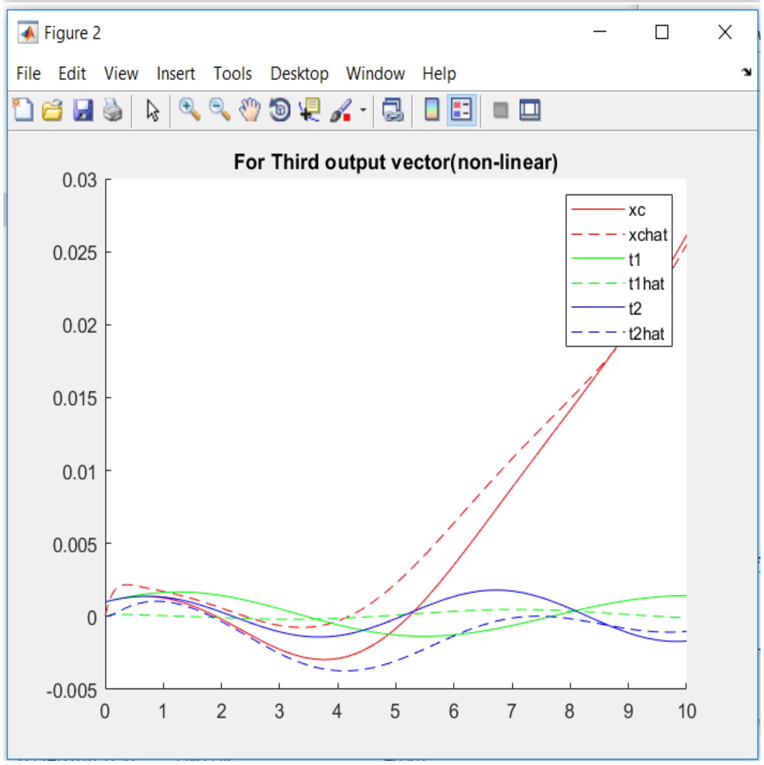




- For $(x(t), \theta_2(t))$

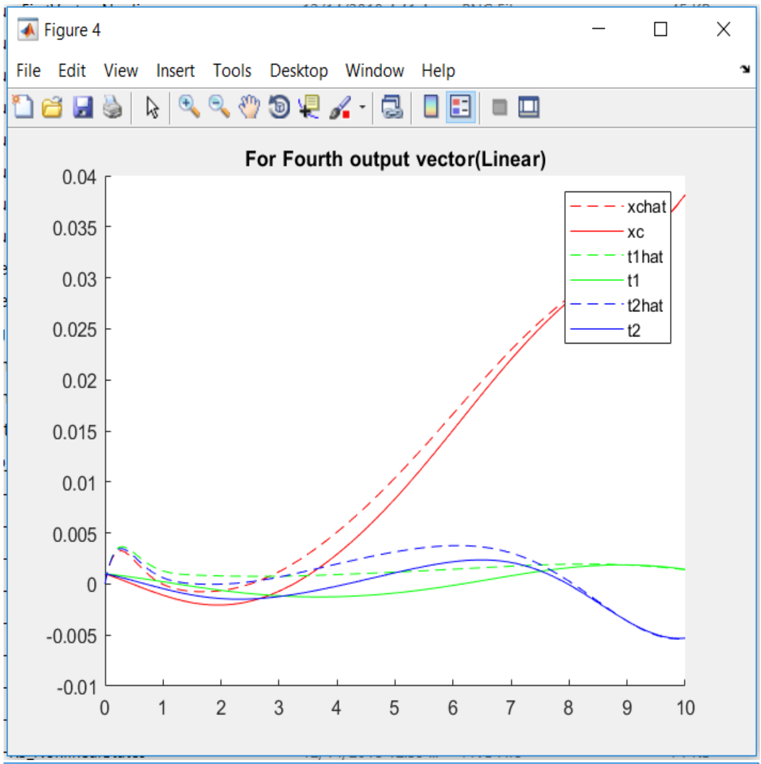
$$L = \begin{bmatrix} 12 & 12 \\ 0.55 & 0.55 \\ 12 & 12 \\ 10 & 10 \\ 0.75 & 0.75 \\ 0.05 & 0.05 \end{bmatrix}$$

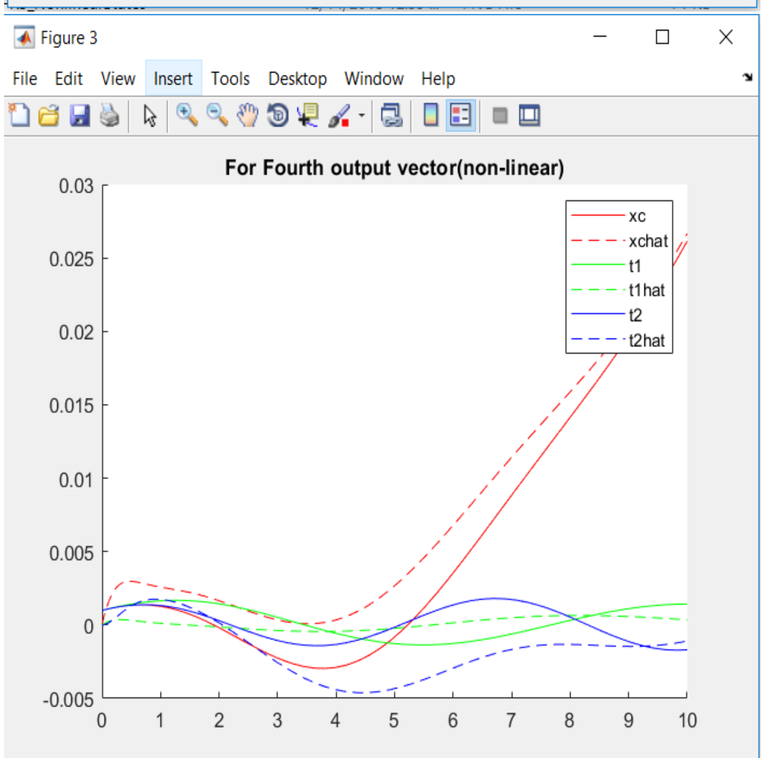




- For $(x(t), \theta_1(t), \theta_1(t))$

$$L = \begin{bmatrix} 10 & 10 & 10 \\ 0.5 & 0.5 & 0.5 \\ 10 & 10 & 10 \\ 6 & 6 & 6 \\ 1 & 1 & 1 \\ 0.01 & 0.01 & 0.01 \end{bmatrix}$$





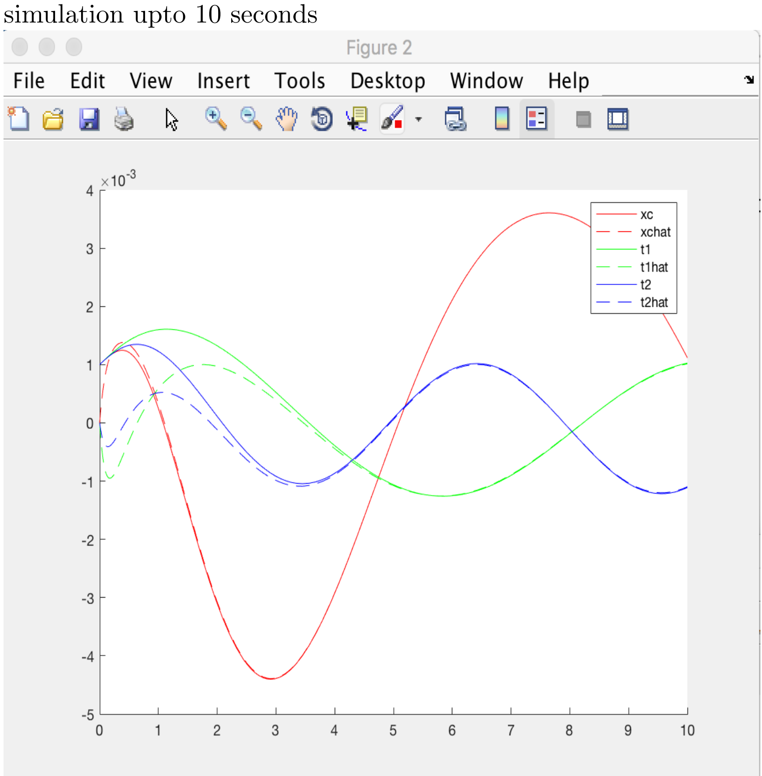
7 Part G

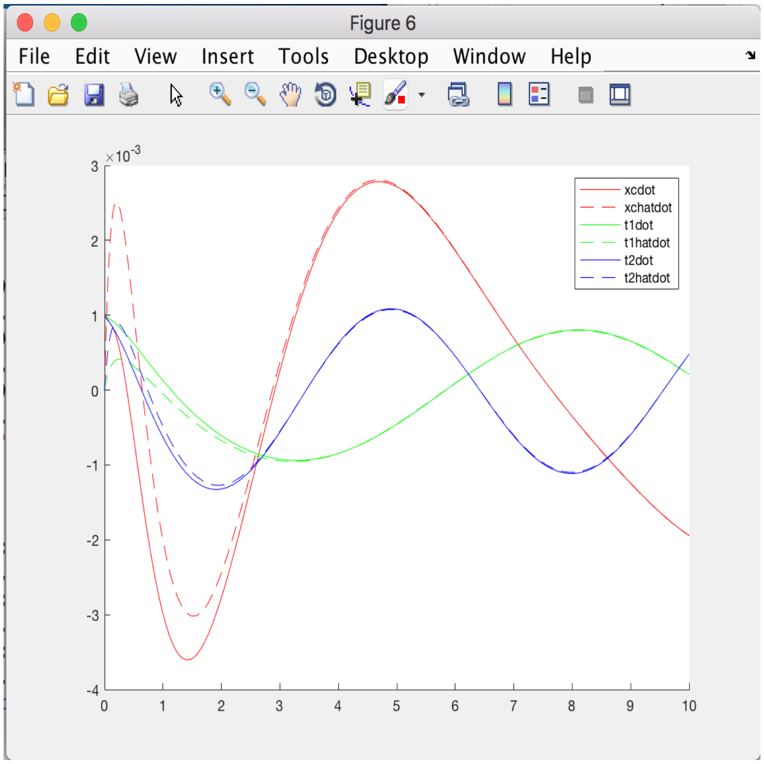
(Refer script LQG) Since the smallest observable output is just x(t), we design the LQG controller for the same. By seperation principle, it is possible to design the controller and observer separately. The controller has already been designed throught the LQR controller above. To design the observer Kalman filter function in MATLAB was used. The following values of Q_n and R_n were used.

$$Q_n = 10 \qquad (73)$$

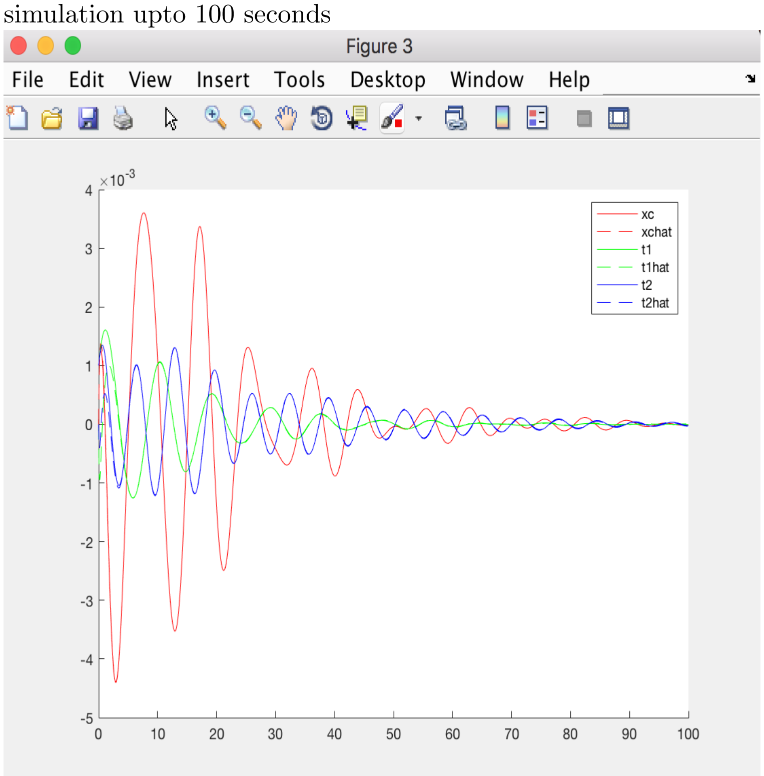
$$R_n = 0.1 \qquad (74)$$

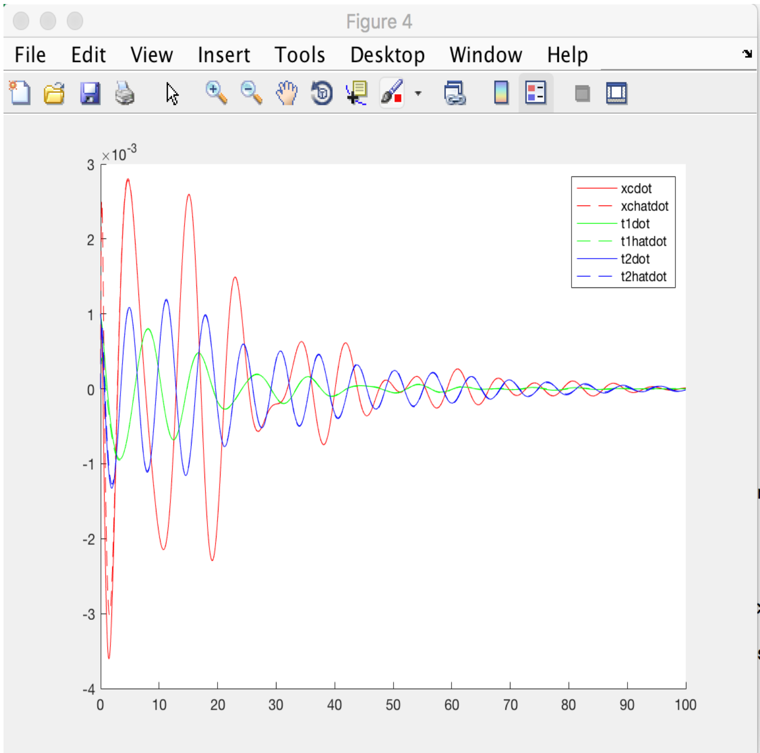
We use the same frame work as above with the only difference that the L (Kalman Gain), will be computed using the Kalman filter. The following was the simulation obtained for the linear and non linear system. Here we use the same framework (code structure) as in previous part with the only difference that the L matrix (Kalman Gain) will be obtained from the Kalman Filter function.
simulation upto 10 seconds





simulation upto 100 seconds





To get the system to track a constant reference error, a strategy similar to the that of lqr can be used. Let x_d and u_∞ be the desired states and the control input required to hold the state be known before hand.It is defined that,

$$\tilde{X} = X - X_d \qquad (75)$$

$$X = \tilde{X} + X_d \qquad (76)$$

And,

$$\tilde{U} = U - U_\infty \qquad (77)$$

$$U = \tilde{U} + U_\infty \qquad (78)$$

Substituing the above values in the state equation below,

$$\dot{X} = A_{lin}X + B_{lin}U + D \qquad (78.1)$$

Then,

$$\dot{\tilde{X}} + X_d = A_{lin}(X_d + \tilde{X}) + B_{lin}(\tilde{U} + U_\infty) + D \qquad (79)$$

Considering, $AX_d + BU_\infty$ as the equilibrium point (equal to 0) and the reference does not change with time.

$$\dot{\tilde{X}} = A_{lin}\tilde{X} + B_l in\tilde{U} + D \qquad (80)$$

The above process does not effect the error term in any way since the error term does not contiain X.

$$err\dot{o}r = (A - LC)error + D - LN \qquad (81)$$

where N is measurement noise and D is the process noise. L is the kalman gain. The above frame work is similar to that of the standard LQG only with changed variable of $\tilde{X}$. The separation principle allows us to modify LQR and use a strategy similar to asymptotically track a constant reference in LQR and incorporate in the LQG system. In this case the solution would be given by,

$$\tilde{U} = -K\hat{\tilde{X}} \qquad (82)$$

where,

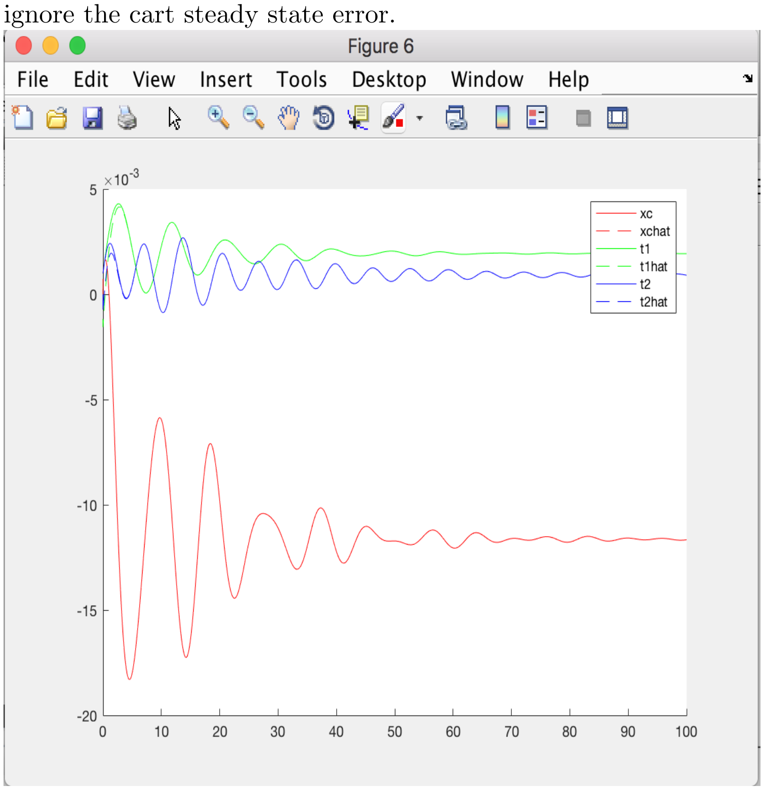
$$\hat{\tilde{X}} = \hat{X} - X_d \qquad (83)$$

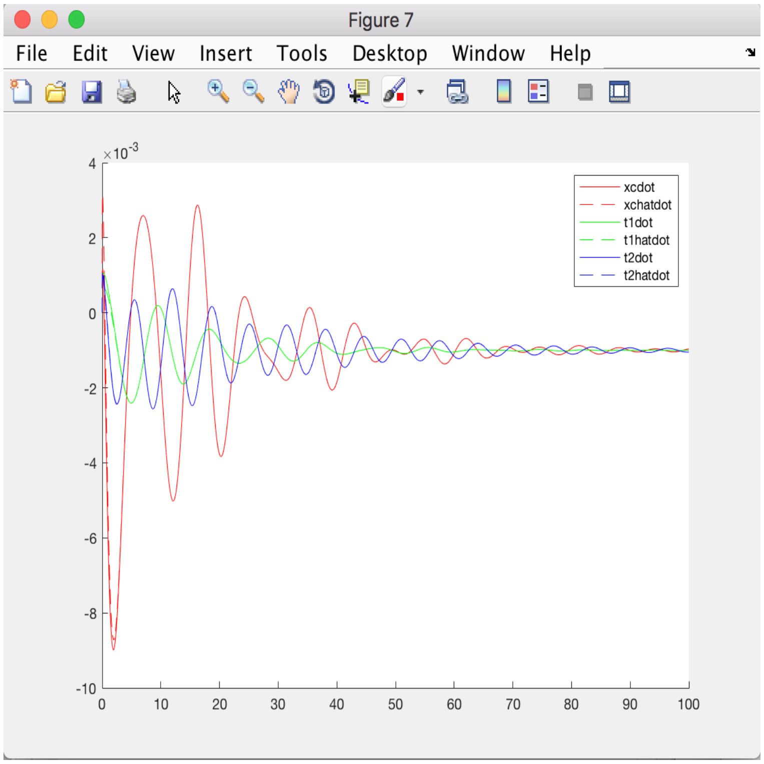
$$U = -K(\hat{X} - X_d) + U_\infty \qquad (84)$$

Note that,

$$\hat{\tilde{X}} = \tilde{X} + error \qquad (85)$$

A simillar change of variable from X to $\tilde{X}$ in the non-linear system can be used, by replacing X with $X + X_d$. A constant disturbance would be a process noise which would have a covariance of 0. LQG does not solve for covariance of 0. Hence, LQG cannot solve for a constant disturbance. A possible work around for this could be by considering the process noise covariance to be significantly low 10^-5 . However, this results in large kalman gains. Following is the graph obtained in case of non-linear system when a constant disturbance of $10^{-3}\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}$ although the system is seen to be stable it does not converge to the desired equilibrium point, there is a steady state error. It is like an offset. To overcome this an integral component of $\tilde{X}$ needs to be appended to the state so that steady state needs to be appended so that the steady state error is removed. However for the system to remain controllable we cannot append all states error, hence certain state errors need to prioritized l(maybe pendulums for crane) and ignore the cart steady state error.





Run the scripts in following order:

- llinearization.m
- non_linear_sol.m
- luenberger_observed.m
- non-linear_luenberger.m
- lqg.m

```
%close all
time_step = 0.01
sim_time = 100
syms M m1 m2 l1 l2 x1 x2 x3 x4 x5 x6 g
syms F real
syms M m1 m2 l1 l2 x1 x2 x3 x4 x5 x6 g real
syms k1 k2 k3 k4 k5 k6 real

%Representing the non-linear state-space equations
f1 = (F - 0.5* m1*g*sin(2*x5) - m1*l1*sin(x5)*x2^2-0.5*m2*g*sin(2*x6)-m2*l2*sin(x6)
*x3^2)/(M+m1*sin(x5)^2+m2*sin(x6)^2);
f2 = ((F - m1*l1*sin(x5)*x2^2 - m2*l2*sin(x6)*x3^2 - m2*g*sin(x6)*cos(x6))*cos(x5) -
(M+m1+m2*sin(x6)^2)*g*sin(x5))/(l1*(M+m1*sin(x5)^2+m2*sin(x6)^2));
f3 = ((F - m1*l1*sin(x5)*x2^2 - m2*l2*sin(x6)*x3^2 - m1*g*sin(x5)*cos(x5))*cos(x6) -
(M+m2+m1*sin(x5)^2)*g*sin(x6))/(l2*(M+m1*sin(x5)^2+m2*sin(x6)^2));
f4 = x1;
f5 = x2;
f6 = x3;
K = [ k1 ;k2; k3; k4; k5; k6];

%Linearizing the state-dynamics matrix A, B

%Linearize the matrix A at the equilibrium point
Jacob_A = [ diff(f1,x1) diff(f1,x2) diff(f1,x3) diff(f1,x4) diff(f1,x5) diff(f1,x6);
    diff(f2,x1) diff(f2,x2) diff(f2,x3) diff(f2,x4) diff(f2,x5) diff(f2,x6);
    diff(f3,x1) diff(f3,x2) diff(f3,x3) diff(f3,x4) diff(f3,x5) diff(f3,x6);
    diff(f4,x1) diff(f4,x2) diff(f4,x3) diff(f4,x4) diff(f4,x5) diff(f4,x6);
    diff(f5,x1) diff(f5,x2) diff(f5,x3) diff(f5,x4) diff(f5,x5) diff(f5,x6);
    diff(f6,x1) diff(f6,x2) diff(f6,x3) diff(f6,x4) diff(f6,x5) diff(f6,x6) ];
%Substituting the values of equilibrium point
Jacob_A_sub=subs(Jacob_A, [x1 x2 x3 x4 x5 x6],[ 0 0 0 0 0 0]);
D = M + m1*sin(x5)^2 + m2*sin(x6)^2;


%Linearize the matrix B at the equilibrium point
Jacob_B = [ diff(f1,F); diff(f2,F); diff(f3,F); diff(f4,F); diff(f5,F); diff(f6,F)];
%Substituting the values of equilibrium point
Jacob_B_sub = subs(Jacob_B, [x1 x2 x3 x4 x5 x6],[ 0 0 0 0 0 0]);
closed_loop = Jacob_B_sub*K'
controllability_test = [ Jacob_B_sub Jacob_A_sub*Jacob_B_sub
Jacob_A_sub^2*Jacob_B_sub Jacob_A_sub^3*Jacob_B_sub Jacob_A_sub^4*Jacob_B_sub
Jacob_A_sub^5*Jacob_B_sub];
controllability_test_sub = double(subs(controllability_test,[l1 l2 M m1 m2 g],[20 10
1000 100 100 10]));
check_rank = rank(controllability_test_sub)

%
A_lin1 = subs(Jacob_A, [l1 l2 M m1 m2 g],[20 10 1000 100 100 9.8]);
A_lin = double(subs(A_lin1,[x1 x2 x3 x4 x5 x6],[0 0 0 0 0 0]));
```

```
B_lin1 = subs(Jacob_B,[l1 l2 M m1 m2 g ],[20 10 1000 100 100 9.8]);
B_lin = double(subs(B_lin1,[x1 x2 x3 x4 x5 x6],[0 0 0 0 0 0]));

% Q_cost = 0.1*[1 5 5 5 5 5
%             5 1 5 5 5 5
%             5 5 1 5 5 5
%             5 5 5 1 5 5
%             5 5 5 5 1 5
%             5 5 5 5 5 1];
%Q_cost = 1000*eye(6,6);

%  Q_cost = 10*[10 3 2 5 2 3;
%                 3 100 5 2 2 3;
%                 2 4 5 2 2 4;
%                 5 7 6 5 6 6;
%                 2 5 2 5 100 6;
%                 3 7 3 5 4 10];
%   R_cost = 0.0001;          %Q1_R1

% Q_cost = 10*[20 0 0 0 0 0;
%                 0 100 0 0 0 0;
%                 0 0 10 0 0 0;
%                 0 0 0 10 0 0;
%                 0 0 0 0 100 0;
%                 0 0 0 0 0 20];
%
% R_cost = 0.00002;          %Q2_R2

% Q_cost = 10*[1 0 0 0 0 0;
%                  0 1 0 0 0 0;
%                  0 0 1 0 0 0;
%                  0 0 0 1 0 0;
%                  0 0 0 0 1 0;
%                  0 0 0 0 0 1];
% R_cost = 1;                %Q3_R3
% Q_cost = 1*[0.5 0 0 0 0 0 ;
%               0 0.5 0 0 0 0;
%               0 0 0.5 0 0 0;
%               0 0 0 0.5 0 0;
%               0 0 0 0 0.5 0;
%               0 0 0 0 0 0.5];
% R_cost = 0.5;              %Q4_R4

%Defining the positive-definite weight matrices Q and R
Q_cost = 10*[10 3 2 5 2 3;
              3 100 5 2 2 3;
              2 4 5 2 2 4;
              5 7 6 5 6 6;
              2 5 2 5 100 6;
```

```
              3 7 3 5 4 10];
R_cost = 0.0001;          %Q5_R5

% Q_cost = 1*[10 5 80 2 6 10 ;
%               10 20 100 50 20 10;
%               30 50 30 10 10 10;
%               20 60 10 30 20 40;
%              30 10 10 10 20 80;
%               60 10 20 60 50 10];
% R_cost = 0.001;


%Designing the LQR controller, by defining the Q and R matrices above,
%and giving A,B,Q,R as the parameters of lqr command.

[K,s,e]= lqr(A_lin,B_lin,Q_cost,R_cost,0);
syms t real
x_init = 0.001*ones(6,1);
x_fin=[];
test= A_lin-B_lin*K;
eig(test)
for t=0:time_step:sim_time
    x_fin_temp = expm((A_lin-B_lin*K).*t)*x_init;
    x_fin= [x_fin x_fin_temp];
end

%figures for states-position, theta1 and theta2
figure
title('States of the system')
hold on
plot([0:time_step:sim_time],x_fin(1,:),'b');
plot([0:time_step:sim_time],x_fin(2,:),'g');
plot([0:time_step:sim_time],x_fin(3,:),'r');
legend('Cart-position(linear)','Pendulum1(linear)','Pendulum2(linear)')



% Defining the control input
u=K*x_fin;
figure
hold on
title('Control Input')
plot(u,'b')
```

```
%Declaring the control input, for linear feedback control
control_input = -K*[x1; x2; x3; x4; x5; x6];
f1_sub = simplify(vpa(subs(f1, [F l1 l2 M m1 m2 g],[ control_input 20 10 1000 100 100 
9.8])));
f2_sub = simplify(vpa(subs(f2, [F l1 l2 M m1 m2 g],[ control_input 20 10 1000 100 100 
9.8])));
f3_sub = simplify(vpa(subs(f3, [F l1 l2 M m1 m2 g],[ control_input 20 10 1000 100 100 
9.8])));
%assuming that the oscillations are really small.
%f1_sub1 = subs(f1_sub,[sin(x5) cos(x5) sin(x6) cos(x6)],[ x5 1 x6 1]);
%f2_sub1 = subs(f2_sub,[sin(x5) cos(x5) sin(x6) cos(x6)],[ x5 1 x6 1]);
%f3_sub1 = subs(f3_sub,[sin(x5) cos(x5) sin(x6) cos(x6)],[ x5 1 x6 1]);
 syms x1t(t) x2t(t) x3t(t) x4t(t) x5t(t) x6t(t) real
 f1_sub1_t = subs(f1_sub,[ x1 x2 x3 x4 x5 x6],[x1t x2t x3t x4t x5t x6t]);
 f2_sub1_t = subs(f2_sub,[ x1 x2 x3 x4 x5 x6],[x1t x2t x3t x4t x5t x6t]);
 f3_sub1_t = subs(f3_sub,[ x1 x2 x3 x4 x5 x6],[x1t x2t x3t x4t x5t x6t]);
 state_vec = [x1t; x2t; x3t; x4t; x5t; x6t];
 diff_eqn1 = diff(x1t,t,1)==  f1_sub1_t;
 diff_eqn2 = diff(x2t,t,1)== f2_sub1_t;
 diff_eqn3 = diff(x3t,t,1) == f3_sub1_t;
 diff_eqn4= diff(x4t,t,1) == x1t;
 diff_eqn5 = diff(x5t,t,1) == x2t;
 diff_eqn6 = diff(x6t,t,1) == x3t;
 eqns = [ diff_eqn1; diff_eqn2; diff_eqn3; diff_eqn4; diff_eqn5; diff_eqn6];

[M1,F1] = massMatrixForm(eqns,state_vec);
f = M1\F1;
ode_fun = odeFunction(f,state_vec);
[t,x_out] = ode45(ode_fun,[0 sim_time],x_init);
figure;
hold on;
title('States for non-linear system')
plot(t,x_out(:,1));
plot(t,x_out(:,2));
plot(t,x_out(:,3));
legend('Cart-position(non-linear)','Pendulum1(non-linear)','Pendulum2(non-linear)')

% % For the phase-plots(position)
% hold on
% title('Phase-plot for position and velocity')
% xout_t=x_out';
% plot(xout_t(1,:),xout_t(4,:),'b');
%
% % For the phase-plot(theta1)
% figure
% hold on
% title('Phase-plot for theta1 and theta1dot')
% xout_t=x_out';
```

```
% plot(xout_t(2,:),xout_t(5,:),'b');
%
% % For the phase-plot(theta2)
% figure
% hold on
% title('Phase-plot for theta2 and theta2dot')
% xout_t=x_out';
% plot(xout_t(3,:),xout_t(6,:),'b');
```

```
%Second Part--Observability
close all
sim_time=10;

%Declaration of C matrices/vector for corresponding output vectors
Cx=[0 0 0 1 0 0]  %----observable;
Ct1t2=[0 0 0 0 0 0;0 0 0 0 0 0;0 0 0 0 0 0;0 0 0 0 0 0;0 0 0 0 1 0;0 0 0 0 0 1];
Cxt2=[0 0 0 1 0 0;0 0 0 0 0 1] %----observable;
Cxt1t2=[0 0 0 1 0 0;0 0 0 0 1 0;0 0 0 0 0 1] %----observable;


%Checking the condition for observability
O1=rank([Cx; Cx*A_lin; Cx*A_lin^2; Cx*A_lin^3; Cx*A_lin^4; Cx*A_lin^5]);
O2=rank([Ct1t2; Ct1t2*A_lin; Ct1t2*A_lin^2; Ct1t2*A_lin^3; Ct1t2*A_lin^4;
Ct1t2*A_lin^5]);
O3=rank([Cxt2; Cxt2*A_lin; Cxt2*A_lin^2; Cxt2*A_lin^3; Cxt2*A_lin^4; Cxt2*A_lin^5]);
O4=rank([Cxt1t2; Cxt1t2*A_lin; Cxt1t2*A_lin^2; Cxt1t2*A_lin^3; Cxt1t2*A_lin^4;
Cxt1t2*A_lin^5]);

%Step input(for simulation)
x_fin2=[];
syms tau
integral=vpa(int(expm(-A_lin).*tau));
for t=0:time_step:sim_time
    x_fin_temp3 = expm((A_lin).*t)*x_init + expm((A_lin).*t)*subs(integral,tau ,t)
*B_lin ;
    x_fin2= [x_fin2 x_fin_temp3];
    t
end
figure
hold on
title('')
plot([0:time_step:sim_time],x_fin2(1,:),'b');
plot([0:time_step:sim_time],x_fin2(2,:),'g');
plot([0:time_step:sim_time],x_fin2(3,:),'r');

%Leunberger Observer

%For the first controllability matrix(first vector of output)
x_error=x_init;
L=[ 30;
    12.5;
    8 ;
    11;
    0.2 ;
    0.08];

% L=[ 30;
%     0.7;
```

```
%     10 ;
%     6;
%     1 ;
%     0.01];
x_error_dot=(A_lin-L*Cx)*x_error;

% Solution for first output vector
x_fin3=[];
for t=0:time_step:sim_time
    x_fin_temp = expm((A_lin-L*Cx).*t)*x_error;
    x_fin3= [x_fin3 x_fin_temp];

end

figure
title('For First output vector(Linear)')
hold on
plot([0:time_step:sim_time],x_fin2(1,:)-x_fin3(1,:),'--','Color',[1,0,0]);
plot([0:time_step:sim_time],x_fin2(1,:),'r');

plot([0:time_step:sim_time],x_fin2(2,:)-x_fin3(2,:),'--','Color',[0,1,0]);
plot([0:time_step:sim_time],x_fin2(2,:),'g');

plot([0:time_step:sim_time],x_fin2(3,:)-x_fin3(3,:),'--','Color',[0,0,1]);
plot([0:time_step:sim_time],x_fin2(3,:),'b');

legend('xchat','xc','t1hat','t1','t2hat','t2')

%plot([0:time_step:sim_time],x_fin2(3,:)-x_fin3(3,:),'r');

eig(A_lin-L*Cx)


% For the third controllability matrix(third vector of output)

L_3=[ 12 12;
     0.55 0.55;
     12 12;
     10 10;
     0.75 0.75;
     0.05  0.05];

% L_3=[ 7.5 7.5;
%      0.3 0.3;
%      8  8;
%      6  6;
%      1  1;
%      0.2  0.2];
x_error_dot3=(A_lin-L_3*Cxt2)*x_error;
```

```
% Solution for third output vector
x_fin4=[];
for t=0:time_step:sim_time
    x_fin_temp = expm((A_lin-L_3*Cxt2).*t)*x_error;
    x_fin4= [x_fin4 x_fin_temp];

end

figure
title('For Third output vector(Linear)')
hold on
plot([0:time_step:sim_time],x_fin2(1,:)-x_fin4(1,:),'--','Color',[1,0,0]);
plot([0:time_step:sim_time],x_fin2(1,:),'r');

plot([0:time_step:sim_time],x_fin2(2,:)-x_fin4(1,:),'--','Color',[0,1,0]);
plot([0:time_step:sim_time],x_fin2(2,:),'g');

plot([0:time_step:sim_time],x_fin2(3,:)-x_fin4(1,:),'--','Color',[0,0,1]);
plot([0:time_step:sim_time],x_fin2(3,:),'b');
legend('xchat','xc','t1hat','t1','t2hat','t2')

%plot([0:time_step:sim_time],x_fin2(3,:)-x_fin3(3,:),'r');

eig(A_lin-L_3*Cxt2)



% For the fourth controllability matrix(fourth vector of output)
L_4=[ 10 10 10;
      0.5 0.5 0.5;
      10 10 10;
      6 6 6;
      1 1 1;
      0.01 0.01 0.01];
x_error_dot4=(A_lin-L_4*Cxt1t2)*x_error;

% Solution for fourth output vector
x_fin5=[];
for t=0:time_step:sim_time
    x_fin_temp = expm((A_lin-L_4*Cxt1t2).*t)*x_error;
    x_fin5= [x_fin5 x_fin_temp];

end

figure
title('For Fourth output vector(Linear)')
hold on
plot([0:time_step:sim_time],x_fin2(1,:)-x_fin5(1,:),'--','Color',[1,0,0]);
```

```
plot([0:time_step:sim_time],x_fin2(1,:),'r');
plot([0:time_step:sim_time],x_fin2(2,:)-x_fin5(1,:),'--','Color',[0,1,0]);
plot([0:time_step:sim_time],x_fin2(2,:),'g');
plot([0:time_step:sim_time],x_fin2(3,:)-x_fin5(1,:),'--','Color',[0,0,1]);
plot([0:time_step:sim_time],x_fin2(3,:),'b');
legend('xchat','xc','t1hat','t1','t2hat','t2')

%plot([0:time_step:sim_time],x_fin2(3,:)-x_fin3(3,:),'r');

eig(A_lin-L_4*Cxt1t2)



%L=[0 0 0  10 0 0;
%   0 0 0 0.5 0 0;
%  0 0 0 10 0 0;
%   0 0 0 5 0 0;
%   0 0 0 1 0 0;
%   0 0 0 0.01 0 0];

%L_3=[0 0 0  10 0 10;
%   0 0 0 0.5 0 0.5;
%   0 0 0 10 0 10;
%    0 0 0 5 0 5;
%    0 0 0 1 0 1;
%   0 0 0 0.01 0 0.01];

%L_4=[0 0 0  10 10 10;
%    0 0 0 0.5 0.5 0.5;
%    0 0 0 10 10 10;
%    0 0 0 5 5 5;
%    0 0 0 1 1 1;
%    0 0 0 0.01 0.01 0.01];
```

```
%Implementing the Luenberger observer for non-linear system
control_input2 = 1;
f1_sub2 = simplify(vpa(subs(f1, [F l1 l2 M m1 m2 g],[ 1 20 10 1000 100 100 9.8])),5);
f2_sub2 = simplify(vpa(subs(f2, [F l1 l2 M m1 m2 g],[ 1 20 10 1000 100 100 9.8])),5);
f3_sub2 = simplify(vpa(subs(f3, [F l1 l2 M m1 m2 g],[ 1 20 10 1000 100 100 9.8])),5);
f1_sub2_t = subs(f1_sub2,[ x1 x2 x3 x4 x5 x6],[x1t x2t x3t x4t x5t x6t]);
f2_sub2_t = subs(f2_sub2,[ x1 x2 x3 x4 x5 x6],[x1t x2t x3t x4t x5t x6t]);
 f3_sub2_t = subs(f3_sub2,[ x1 x2 x3 x4 x5 x6],[x1t x2t x3t x4t x5t x6t]);
syms x1hat(t) x2hat(t) x3hat(t) x4hat(t) x5hat(t) x6hat(t) real
f1_sub2_hat = subs(f1_sub2,[ x1 x2 x3 x4 x5 x6],[x1hat x2hat x3hat x4hat x5hat ↙
x6hat]);
f2_sub2_hat = subs(f2_sub2,[ x1 x2 x3 x4 x5 x6],[x1hat x2hat x3hat x4hat x5hat ↙
x6hat]);
f3_sub2_hat = subs(f3_sub2,[ x1 x2 x3 x4 x5 x6],[x1hat x2hat x3hat x4hat x5hat ↙
x6hat]);
LnC = L*Cx;
vec_xt= [ x1t; x2t; x3t; x4t;x5t;x6t];
vec_xhat =[ x1hat; x2hat; x3hat; x4hat; x5hat; x6hat];
diff_eqn1 = diff(x1hat,t,1)==f1_sub2_hat+ LnC(1,:)*(vec_xt-vec_xhat);
diff_eqn2 = diff(x2hat,t,1)==f2_sub2_hat+ LnC(2,:)*(vec_xt-vec_xhat);
diff_eqn3 = diff(x3hat,t,1)==f3_sub2_hat+ LnC(3,:)*(vec_xt-vec_xhat);
diff_eqn4= diff(x4hat,t,1)==x1hat + LnC(4,:)*(vec_xt-vec_xhat);
diff_eqn5 = diff(x5hat,t,1)==x2hat + LnC(5,:)*(vec_xt-vec_xhat);
diff_eqn6 = diff(x6hat,t,1)==x3hat + LnC(6,:)*(vec_xt-vec_xhat);
diff_eqn7 = diff(x1t,t,1)==  f1_sub2_t;
diff_eqn8 = diff(x2t,t,1)== f2_sub2_t;
diff_eqn9 = diff(x3t,t,1) == f3_sub2_t;
diff_eqn10= diff(x4t,t,1) == x1t;
diff_eqn11 = diff(x5t,t,1) == x2t;
diff_eqn12 = diff(x6t,t,1) == x3t;
state_vec1= [vec_xhat; vec_xt];
eqns2 = [ diff_eqn1; diff_eqn2; diff_eqn3; diff_eqn4; diff_eqn5; diff_eqn6; ↙
diff_eqn7; diff_eqn8; diff_eqn9; diff_eqn10; diff_eqn11;diff_eqn12];
[M3,F3] = massMatrixForm(eqns2,state_vec1);
f = M3\F3;
ode_fun = odeFunction(f,state_vec1);
x_init2 =[zeros(6,1); x_init];
[t,x_out] = ode45(ode_fun,[0 sim_time],x_init2);
figure;
title('For First output vector(non-linear)')
hold on;
plot(t,x_out(:,10),'r');
plot(t,x_out(:,4),'--','Color',[1,0,0]);
plot(t,x_out(:,11),'g');
plot(t,x_out(:,5),'--','Color',[0,1,0]);
plot(t,x_out(:,12),'b');
plot(t,x_out(:,6),'--','Color',[0,0,1]);
legend('xc','xchat','t1','t1hat','t2','t2hat')
```

```
%plot(t,x_out(:,12),'r');


% (For third output Vector)

syms x1hat(t) x2hat(t) x3hat(t) x4hat(t) x5hat(t) x6hat(t) real

LnC_3 = L_3*Cxt2;
diff_eqn1 = diff(x1hat,t,1)==f1_sub2_hat+ LnC_3(1,:)*(vec_xt-vec_xhat);
diff_eqn2 = diff(x2hat,t,1)==f2_sub2_hat+ LnC_3(2,:)*(vec_xt-vec_xhat);
diff_eqn3 = diff(x3hat,t,1)==f3_sub2_hat+ LnC_3(3,:)*(vec_xt-vec_xhat);
diff_eqn4= diff(x4hat,t,1)==x1hat + LnC_3(4,:)*(vec_xt-vec_xhat);
diff_eqn5 = diff(x5hat,t,1)==x2hat + LnC_3(5,:)*(vec_xt-vec_xhat);
diff_eqn6 = diff(x6hat,t,1)==x3hat + LnC_3(6,:)*(vec_xt-vec_xhat);
diff_eqn7 = diff(x1t,t,1)==  f1_sub2_t;
diff_eqn8 = diff(x2t,t,1)== f2_sub2_t;
diff_eqn9 = diff(x3t,t,1) == f3_sub2_t;
diff_eqn10= diff(x4t,t,1) == x1t;
diff_eqn11 = diff(x5t,t,1) == x2t;
diff_eqn12 = diff(x6t,t,1) == x3t;

eqns2 = [ diff_eqn1; diff_eqn2; diff_eqn3; diff_eqn4; diff_eqn5; diff_eqn6; ↙
diff_eqn7; diff_eqn8; diff_eqn9; diff_eqn10; diff_eqn11;diff_eqn12];
[M3,F3] = massMatrixForm(eqns2,state_vec1);
f = M3\F3;
ode_fun = odeFunction(f,state_vec1);
x_init3 =[zeros(6,1); x_init];
[t,x_out] = ode45(ode_fun,[0 sim_time],x_init3);
figure;
title('For Third output vector(non-linear)')
hold on;
plot(t,x_out(:,10),'r');
plot(t,x_out(:,4),'--','Color',[1,0,0]);
plot(t,x_out(:,11),'g');
plot(t,x_out(:,5),'--','Color',[0,1,0]);
plot(t,x_out(:,12),'b');
plot(t,x_out(:,6),'--','Color',[0,0,1]);
legend('xc','xchat','t1','t1hat','t2','t2hat')
%plot(t,x_out(:,12),'r');


% (For fourth output Vector)

syms x1hat(t) x2hat(t) x3hat(t) x4hat(t) x5hat(t) x6hat(t) real

LnC_4 = L_4*Cxt1t2;
diff_eqn1 = diff(x1hat,t,1)==f1_sub2_hat+ LnC_4(1,:)*(vec_xt-vec_xhat);
diff_eqn2 = diff(x2hat,t,1)==f2_sub2_hat+ LnC_4(2,:)*(vec_xt-vec_xhat);
diff_eqn3 = diff(x3hat,t,1)==f3_sub2_hat+ LnC_4(3,:)*(vec_xt-vec_xhat);
```

```
diff_eqn4= diff(x4hat,t,1)==x1hat + LnC_4(4,:)*(vec_xt-vec_xhat);
diff_eqn5 = diff(x5hat,t,1)==x2hat + LnC_4(5,:)*(vec_xt-vec_xhat);
diff_eqn6 = diff(x6hat,t,1)==x3hat + LnC_4(6,:)*(vec_xt-vec_xhat);
diff_eqn7 = diff(x1t,t,1)==  f1_sub2_t;
diff_eqn8 = diff(x2t,t,1)== f2_sub2_t;
diff_eqn9 = diff(x3t,t,1) == f3_sub2_t;
diff_eqn10= diff(x4t,t,1) == x1t;
diff_eqn11 = diff(x5t,t,1) == x2t;
diff_eqn12 = diff(x6t,t,1) == x3t;

eqns2 = [ diff_eqn1; diff_eqn2; diff_eqn3; diff_eqn4; diff_eqn5; diff_eqn6;
diff_eqn7; diff_eqn8; diff_eqn9; diff_eqn10; diff_eqn11;diff_eqn12];
[M3,F3] = massMatrixForm(eqns2,state_vec1);
f = M3\F3;
ode_fun = odeFunction(f,state_vec1);
x_init4 =[zeros(6,1); x_init];
[t,x_out] = ode45(ode_fun,[0 sim_time],x_init4);
figure;
title('For Fourth output vector(non-linear)')
hold on;
plot(t,x_out(:,10),'r');
plot(t,x_out(:,4),'--','Color',[1,0,0]);
plot(t,x_out(:,11),'g');
plot(t,x_out(:,5),'--','Color',[0,1,0]);
plot(t,x_out(:,12),'b');
plot(t,x_out(:,6),'--','Color',[0,0,1]);
legend('xc','xchat','t1','t1hat','t2','t2hat')

%plot(t,x_out(:,12),'r');
```

```
sys1 = ss(A_lin,[B_lin ones(6,1)],Cx,0); %converting into state space
Qn = (10); % setting covraince matrices
Rn = (0.000001);
[kest,L_kal] = kalman(sys1,Qn,Rn,0);  %using kalman filter to estimate kalman gain
reg = lqgreg(kest, K);
syms x1t(t) x2t(t) x3t(t) x4t(t) x5t(t) x6t(t)
syms x1hat(t) x2hat(t) x3hat(t) x4hat(t) x5hat(t) x6hat(t);
disturbance_vec = [0.001 0.001 0.001 0.001 0.001 0.001]'; % reference and
control_input3= vpa(-K*(vec_xhat)); % setting control input from LQR
%solving for non linear model by appending states with hat and using ode45
%to solve.
f1_sub4 = simplify(vpa(subs(f1, [F l1 l2 M m1 m2 g],[ control_input3 20 10 1000 100
100 9.8])));
f2_sub4 = simplify(vpa(subs(f2, [F l1 l2 M m1 m2 g],[ control_input3 20 10 1000 100
100 9.8])));
f3_sub4 = simplify(vpa(subs(f3, [F l1 l2 M m1 m2 g],[ control_input3 20 10 1000 100
100 9.8])));
f1_sub4_t = subs(f1_sub4,[ x1 x2 x3 x4 x5 x6],[x1t x2t x3t x4t x5t x6t]);
f2_sub4_t = subs(f2_sub4,[ x1 x2 x3 x4 x5 x6],[x1t x2t x3t x4t x5t x6t]);
f3_sub4_t = subs(f3_sub4,[ x1 x2 x3 x4 x5 x6],[x1t x2t x3t x4t x5t x6t]);
f1_sub4_hat = subs(f1_sub4,[ x1 x2 x3 x4 x5 x6],[x1hat x2hat x3hat x4hat x5hat
x6hat]);
f2_sub4_hat = subs(f2_sub4,[ x1 x2 x3 x4 x5 x6],[x1hat x2hat x3hat x4hat x5hat
x6hat]);
f3_sub4_hat = subs(f3_sub4,[ x1 x2 x3 x4 x5 x6],[x1hat x2hat x3hat x4hat x5hat
x6hat]);
LnC_kal = L_kal*Cx;
diff_eqn11 = diff(x1hat,t,1)==f1_sub4_hat+ LnC_kal(1,:)*(vec_xt-vec_xhat)
+disturbance_vec(1,1);
diff_eqn21 = diff(x2hat,t,1)==f2_sub4_hat+ LnC_kal(2,:)*(vec_xt-vec_xhat)
+disturbance_vec(2,1);
diff_eqn31 = diff(x3hat,t,1)==f3_sub4_hat+ LnC_kal(3,:)*(vec_xt-vec_xhat)
+disturbance_vec(3,1);
diff_eqn41= diff(x4hat,t,1)==x1hat + LnC_kal(4,:)*(vec_xt-vec_xhat)+disturbance_vec
(4,1);
diff_eqn51 = diff(x5hat,t,1)==x2hat + LnC_kal(5,:)*(vec_xt-vec_xhat)+disturbance_vec
(5,1);
diff_eqn61 = diff(x6hat,t,1)==x3hat+ LnC_kal(6,:)*(vec_xt-vec_xhat)+disturbance_vec
(6,1);
diff_eqn71 = diff(x1t,t,1)==  f1_sub4_t + disturbance_vec(1,1);
diff_eqn81 = diff(x2t,t,1)== f2_sub4_t+ disturbance_vec(2,1);
diff_eqn91 = diff(x3t,t,1) == f3_sub4_t+disturbance_vec(3,1);
diff_eqn101= diff(x4t,t,1) == x1t+disturbance_vec(4,1);
diff_eqn111 = diff(x5t,t,1) == x2t+disturbance_vec(5,1);
diff_eqn121 = diff(x6t,t,1) == x3t+disturbance_vec(6,1);
state_vec1= [vec_xhat; vec_xt];
eqns3 = [ diff_eqn11; diff_eqn21; diff_eqn31; diff_eqn41; diff_eqn51; diff_eqn61;
diff_eqn71; diff_eqn81; diff_eqn91; diff_eqn101; diff_eqn111;diff_eqn121];
```

```
[M4,F4] = massMatrixForm(eqns3,state_vec1);
f = M4\F4;
ode_fun2 = odeFunction(f,state_vec1);
x_init2 =[zeros(6,1); x_init];
[t,x_out3] = ode45(ode_fun2,[0 sim_time],x_init2);
figure;
hold on;
%plots
plot(t,x_out3(:,10),'r');
plot(t,x_out3(:,4),'--','Color',[1,0,0]);
plot(t,x_out3(:,11),'g');
plot(t,x_out3(:,5),'--','Color',[0,1,0]);
plot(t,x_out3(:,12),'b');
plot(t,x_out3(:,6),'--','Color',[0,0,1]);
legend('xc','xchat','t1','t1hat','t2','t2hat')
%eig(A_lin-L_kal*Cx) % eigen value check of error
figure;
hold on;
plot(t,x_out3(:,7),'r');
plot(t,x_out3(:,1),'--','Color',[1,0,0]);
plot(t,x_out3(:,8),'g');
plot(t,x_out3(:,2),'--','Color',[0,1,0]);
plot(t,x_out3(:,9),'b');
plot(t,x_out3(:,3),'--','Color',[0,0,1]);
legend('xcdot','xchatdot','t1dot','t1hatdot','t2dot','t2hatdot')
```