The Hitchhiker's Guide to Probability and Optimization

MAT 2233

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Abstract

This course covers probability and optimization from a practical lens. We will first cover the basics of counting, before moving on the probability, where we will learn varied topics such as Bayes' Theorem and Random variables. Finally, we will move on to optimization, which covers some of the most common algorithms and methods for function optimization.

Syllabus[1]

1. **Permutation and Combination** — **8 Hours**
   (a) Basic Methods
   (b) Generating Functions
   (c) Distributions
   (d) Partition and Composition
2. **Probability** — **15 Hours**
   (a) Bayes' Theorem — 4 Hours
   (b) Random Variables — 5 Hours
   (c) Distributions and Functions of Random Variables — 6 Hours
3. **Optimization** — **7 Hours**
   (a) Vector Valued Functions — 3 Hours
   (b) Back Propagation — 4 Hours

Grading[2]

Internal Assessment — 50 Marks

1. Quiz/Assignment — 5 Marks
2. Quiz/Assignment — 5 Marks
3. Quiz/Assignment — 5 Marks
4. Quiz/Assignment — 5 Marks
5. Mid Semester Examination — 30 Marks

External Assessment — 50 Marks

[1] This syllabus needs to be updated as it only adds up to 30 hours of course material
[2] More details need to be added for grading.

Contents

Chapter 1

Combinatorics

1.1 Permutation

The permutation of a set can be defined as an arrangement of the members of the set; or the act of creating or changing such an arrangement. Common examples are Anagrams, which consist of different ways of arranging letters in a word.

$$^{n}P_{r} = \frac{n!}{(n-k)!} \tag{1.1}$$

1.1.1 Basic Permutations

Description	**Formula**
No. of arrangements of k distinct objects out of n total objects without repetition	$^{n}P_{r}$
No. of arrangements of k distinct objects out of n total objects **with repetition**	n^{k}
No. of arrangements of k distinct objects out of n total objects **with restricted repetition**	$\frac{n!}{n_1!\, n_2! \ldots n_k!}$
Circular permutation where direction is important	$(n-1)!$
Circular permutation where direction is **not important**	$\frac{(n-1)!}{2}$

Assuming k sets each having a certain number of repetitions with n_k items

Table 1.1: Basic Permutation Formulae

1.2 Combination

$$\begin{aligned} ^{n}C_{k} \times k! &= {}^{n}P_{k} \\ ^{n}C_{k} &= \frac{^{n}P_{k}}{k!} \\ ^{n}C_{k} &= \frac{n!}{k!\,(n-k)!} \end{aligned} \tag{1.2}$$

Alternatively, $^{n}C_{k}$ can be considered in terms of binomial coefficient $\binom{n}{k}$. In that case, by using the binomial expansion formula, we can derive $^{n}C_{k}$. The formula for a binomial coefficient is

$$\binom{n}{k} = \frac{n \times (n-1) \times (n-2) \cdots \times (n-k+1)}{k \times (k-1) \times (k-2) \cdots \times 1}$$

We can multiply the fraction with $(n-k)!$, which allows us to write

$$\binom{n}{k} = \frac{\overbrace{n \times (n-1) \times (n-2) \cdots \times (n-k+1) \times \overbrace{(n-k) \times (n-k-1) \cdots \times 1}^{(n-k)!}}^{n!}}{\underbrace{k \times (k-1) \times (k-2) \cdots \times 1}_{k!} \times \underbrace{(n-k) \times (n-k-1) \cdots \times 1}_{(n-k)!}}$$

and we can simplify to

$$\binom{n}{k} = \frac{n!}{k! \times (n-k)!} = {}^{n}C_{k} \tag{1.3}$$

1.2.1 Symmetry of Combinations

Combinations inherently have a symmetric property. Suppose you have a set of five objects, out of which two need to be selected. This is equivalent to the case where you select three objects to leave behind. Formulaically, we can derive the relation as

$$\binom{n}{k} = \frac{n!}{k! \times (n-k)!}$$

now substitute the case where $k = n - k \Rightarrow 2k = n \Rightarrow k = \frac{n}{2}$

This happens because there are factors which are common to both the denominator and numerator and get canceled out.

$$\binom{n}{k} = \frac{n!}{(n-k)! \times (n-(n-k))!}$$
$$\binom{n}{k} = \frac{n!}{(n-k)! \times (\cancel{n-n}+k)!}$$
$$\binom{n}{k} = \frac{n!}{(n-k)! \times k!} = \binom{n}{n-k}$$

1.2.2 Basic Combinations

Table 1.2 gives a list of basic combination formulae. The number of combination formulae is significantly less than permutation formulae as order is irrelevant for this case.

Description	**Formula**
Combination without repetition	${}^{n}C_{k}$
Combination **with repetition**	${}^{n+k-1}C_{k}$

Table 1.2: Basic Combination Formulae

1.3 Distributions

- Distribution
  - Distinct Objects (Permutation)
  - Identical Objects (Combination)

- Distribution
  - Any number of objects per box (Repetition)
  - At most one obj per box (No Repetition)

Example 1.1: Finding Roots of an Equation

Find the total number of non-negative solutions to
$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 4$.
Ans: First, we start by taking random solutions. By trial and error we can see that $(1, 0, 1, 1, 0, 1)$ and $(4, 0, 0, 0, 0, 0)$ are solutions. Now we can see that this is a problem for distributions. We have to distribute 4 '1's into 6 places, and we can have any number of '1's in each box. i.e. identical objects (combination) and any number of objects per box (repetition). So, we use the formula ${}^{n+r-1}C_r$, with $r = 4$ $n = 6$.

$${}^{6+4-1}C_4 = {}^{9}C_4 \text{ ways.}$$

Example 1.2: Scheduling Exams

Find the number of ways in which 3 exams can be schedules in 5 days such that

1. No two exams are on the same day.
2. No restrictions on the number of exams per day.

Ans: This is a very typical distribution question. It consists of distint objects, so it is a permutation. We can also see that the subquestions are based on a case with repetition and a case without repetition.

1. ${}^{5}P_3$
2. 5^3

Example 1.3: Odd Numbers With Distint Digits

How many odd integers exist between 100 and 999 and have distinct digits?
Ans: This is a little trickier to answer. It is not clear as to which category this question belongs to. So we will take the normal route and try to come to an answer. First we list out all the restrictions that we need to adhere to.

- Three digit number - no 0 in the first box.
- Odd number - restricted third box to odd numbers only.
- Distinct digits - all boxes.

Here, we have an example of a situation with multiple restrictions. In such cases, we deal with the digit or position with the most restrictions.

Now, we start building out number based on these restrictions. We will take the last box, since it has the most restriction (0, 2, 4, 6, 8 cannot take that place). The numbers here represent the number of choices for a particular position.

$$\square\square\boxed{5}$$

Then we move on to the next most restrictied position, position 1. Here, we cannot use a 0 and the number used in the last box, that leaves us with 8 choices.

$$\boxed{8}\square\boxed{5}$$

In continuation of the previous note, suppose we filled 0 in the second box. In that case, we would not need the 0 restriction in the first box.

Finally we come to the box with the least number of restrictions, box 2, which has to have digits not already used in box 1 and 3.

$$\boxed{8}\boxed{8}\boxed{5}$$
$$8 \times 8 \times 5 \text{ ways}$$

Example 1.4: Number of Integers With Distinct Digits

How many of the first 1000 integers have distinct digits?
Ans: This is a classic example of a multi-case question. In multi-case questions, there will not be any possible generalisation that you can apply to all the objects in question, in this instance, to numbers. So, we consider three cases. One digit, Two digit, and three digit. The specifics of arriving to the answer is left to the reader as an exercise.

Why is it (9,9) and not (9,8)?

- One digit: $\boxed{9}$ ways.
- Two digits: $\boxed{9}\boxed{9}$ ways.
- Three digits: $\boxed{9}\boxed{9}\boxed{8}$ ways.

Example 1.5: Selecting Integers to Sum To a Multiple of 3

In how many ways can we select 3 integers from $3n$ consecutive integers such that their sum is a multiple of 3?
Ans: Admittedly, this is a tricky question. It would be advantageous to first understand what is being asked here. The question takes a set of consecutive integers, from 1 to $3n$, with n being an arbitrary natural number. Then, we have to count the number of subsets, containing three elements, that sum up to a multiple of three. In such cases, we first have to identify any possible simplifications in the question.
We can see that the set of integers from 1 to $3n$, can be split into three sets each of which has $\frac{3n}{3} = n$ elements,

It would be warranted to think that the last element of this series should be $3n + 2$, but we have to note that $3n + 2 > 3n$ and so $3n + 2 - 3$ would give the previous (and last) element in the series.

$$\begin{aligned} x\%3 == 0 &\rightarrow\ 0, 3, 6 \ldots 3n \\ x\%3 == 1 &\rightarrow\ 1, 4, 7 \ldots 3n - 1 \\ x\%3 == 2 &\rightarrow\ 2, 5, 8 \ldots 3n - 2 \end{aligned}$$

Now, we can simplify the question. We not that we can choose any three numbers from the group $3n$ and we would still get a subset divisible by three. Similarly, we can choose three numbers from the other two groups and get the same result. Additionally, we can select one number from $3n$,$3n - 2$ and $3n - 1$ to give

$$\sum \{3n, 3n - 1, 3n - 2\} = 3 \times 3n - 3$$

which is still divisible by three. We have to repeat the same procedure to identify any other combinations that fit our question, which is none in this case. Finally, we sum up all these combinations to get our answer.

$$\underbrace{{}^nC_3}_{3n} + \underbrace{{}^nC_3}_{3n-1} + \underbrace{{}^nC_3}_{3n-2} + \underbrace{{}^nC_1{}^nC_1{}^nC_1}_{\text{one from each}} \text{ ways}$$

Example 1.6: Permutations of INSTITUTION

Find the number of permutations of the word INSTITUTION.

- Find the number of them that begin with I and end with N.
- How many permutations are possbile if three Ts should not be together?

Ans: This is a straight-forward question. The total number of permuations of n objects, with partial repetition is given in table 1.1. Comparing with the formula, we see that there are 11 letters ($n = 11$), with 3 Is and Ts ($n_1 = 3, n_2 = 3$). So, the total number of permutations will be

$$\frac{11!}{3! \times 3! \times 2!}$$

Next, we move onto the subquestions. For the first one, we fix I and N in their respective places. So,

$$\boxed{I}\square\square\square\square\square\square\square\square\square\boxed{N}$$

Basically simplifying the permutation to 9 objects. So the answer to the second part is

$$\frac{9!}{2! \times 3!}$$

For the third part, we need to find the total number of permutations which are having 3 Ts together. Consider these three Ts as one object, alongside the other letters. Now, we can find the total number of permutations in this case.

$$\frac{9!}{3! \times 2!}$$

Now finding the complement,

$$\begin{aligned}\{\text{All possible permutations}\} - &\{\text{All possible permutations with TTT}\}\\ &= \{\text{All possible permutations without TTT}\}\\ &\frac{11!}{3! \times 3! \times 2!} - \frac{9!}{3! \times 2!} \text{ ways}\end{aligned}$$

Example 1.7: A New National Flag

A new national flag needs to be designed with blue, green, yellow and red. It has 6 strips. In how many ways can we colour the strips such that no two adjacent strips have the same colour?
Ans: This is easy. For the first strip, we can choose one out of 4 colours. For the next strip, we choose one of 3, as one colour is already taken by the previous strip. Similarly, we have 3 choices for the third strip (not 2 choices) and so on.

$$4 \times 3 \times 3 \times 3 \times 3 \times 3 \text{ ways}$$

Example 1.8: Choosing Days of a Week

In how many ways can you choose three days out of a week with repetition allowed?
Ans: $n = 7, r = 3; \quad {}^{7+3-1}C_3 = {}^9C_3 \; ways$

Example 1.9: Sum of Permutations

Find the sum of all 4 digit numbers that can be obtained by using the digits 1, 2, 3, 4 once in each.

Ans: The answer to this may not be obvious at first. So, we try permuting these numbers. We already know that there will be 4! permutations, from this set

$$\begin{Bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \\ 1 & 4 & 2 & 3 \\ \vdots & \vdots & \vdots & \vdots \end{Bmatrix}$$

Since there are 24 permutations and 4 digits to choose from, we know that in each column, we will have 6 copies of each digit. Thus, the sum of each column is

$$6 \times (1 + 2 + 3 + 4) = 60$$

Now, we add the requisite weights to each column,

$$\left.\begin{array}{ll} \text{Ones:} & 60 \cdot 1 = 60 \\ \text{Tens:} & 60 \cdot 10 = 600 \\ \text{Hundreds:} & 60 \cdot 100 = 6000 \\ \text{Thousands:} & 60 \cdot 1000 = 60000 \end{array}\right\} \sum = 66660 \text{ ways}$$

Example 1.10: Preparing a Question Paper

In how many ways can an examiner assign 30 marks to 8 questions such that no question receives less than 2 marks?
Ans: This sounds like a distribution question with extra steps. Firstly, we allot all the minimum marks, i.e. give 2 marks to all questions, leaving us with 14 marks. Now, these marks can be distributed as necessary to the 8 questions. So,

$$^{14+8-1}C_{14} = {}^{20}C_{14} \text{ ways}$$

Example 1.11: Transmission Signals

A signal consisting of 6 symbols and 12 blank spaces needs to be transmitted. There must be at least two blanks between each symbol. In how many ways can the signal be transmitted?
Ans: This is similar to example 1.10. Here we can show the signal as

$$\mathbf{S_1}\ G_1\ \mathbf{S_2}\ G_2\ \mathbf{S_3}\ G_3\ \mathbf{S_4}\ G_4\ \mathbf{S_5}\ G_5\ \mathbf{S_6}$$

and we have to distribute 12 spaces into 5 boxes (G_1 to G_5) with at least two blanks each. So,

$$^{2+5-1}C_2 = {}^{6}C_2$$

Example 1.12: Diagonal Division

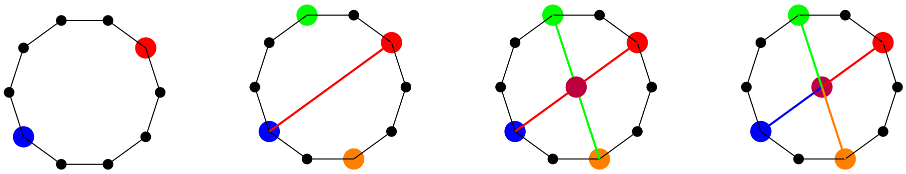
What are the number of line-segments in the diagonals of a convex decagon?
Ans: This is a really hard problem, and I will do my best to guide you to the answer. But it requires that you pick up a pen and try to solve this problem as we go. We will first find a way to count the number of diagonals that are present in a decagon, then we will move on to counting the number of intersections, and then we will look at how many line segments these intersections create.
Suppose we are give a n-sided polygon. The number of diagonals of that polygon, is relatively easy to find. A diagonal is defined as a line segment connecting two vertices of a polygon. So, the number of diagonals in a polygon is the number of ways we can pick two points in the polygon. In a

decagon, we have

$$ {}^{10}C_2 - 10 \text{ diagonals} $$

The subtracted 10 accounts for the sides of the decagon as they are also incorrectly counted as diagonals.
Now, we can focus on counting the number of intersections between these diagonals. We can note that not all diagonals intersect. So, we cannot simply count the total number of pairs of diagonals. On observation however, we can note that if you pick 4 pairs of points, there is guaranteed to be exactly one intersection point for that unordered quadruplet. So, we try to count the number of intersections with

$$ {}^{10}C_4 \text{ intersections} $$



Once this is done, we can note that of the existing diagonals, each intersection doubles the number of line segments. Illustrated in the figure above, we can see the existence of the green and red diagonals, but, the purple intersection splits them into 4 diagonal line segments, namely green, red, blue and orange.
So, the total number of diagonal line segments is

$$ \text{Total number of line segments} = \underbrace{\text{Number of pairs} - 10}_{\text{Exclude Sides}} + 2 \times \text{Number of intersections.} $$

$\therefore$ we get,

$$ {}^{10}C_2 - 10 + 2 \times {}^{10}C_4 \text{ line segments} $$

Chapter 2

Generating Functions

The general form of a generating function for a sequence $\langle a_1, a_2, \cdots a_n \rangle$ is given by

$$\sum_{n=0}^{\infty} a_n x^n$$

Generating functions are used as a way of representing sequences. Dealing directly with sequences is cumbersome, and a generating function makes operations like multiplication trivial. The variable used in a generating function is of no significance, as it is only a formal sum, and we do no worry about convergence as we would with a normal series.

Generating functions can be formed easily and manipulated, but the essence of a generating function lies in the sequence it encodes. For example,

$$\langle 1, 2, 3, 4 \cdots \rangle = \sum_{n=0}^{\infty} nx^n = 0x^0 + 1x^1 + 2x^2 \cdots$$

is a simple generating function. Suppose we have another generating function

$$\langle 1, 4, 9, 16 \cdots \rangle = \sum_{n=0}^{\infty} n^2 x^n = 0x^0 + 1x^1 + 4x^2 \cdots$$

Now, assume we have to find the sequence which combines them using 'And' or the multiplication operator. And then we have to find the r^{th} term of the new sequence. This would not be easy (but is still possible) to find by just multiplying each of the sequences. But it is made trivial by generating functions.

$$\langle 1, 2, 3, 4 \cdots \rangle \times \langle 1, 4, 9, 16 \cdots \rangle$$
$$(0x^0 + 1x^1 + 2x^2 \cdots) \times (0x^0 + 1x^1 + 4x^2 \cdots)$$
$$(\sum_{n=0}^{\infty} nx^n) \times (\sum_{n=0}^{\infty} n^2 x^n)$$
$$\sum_{n=0}^{\infty} (nx^n \times n^2 x^n) = \sum_{n=0}^{\infty} n^3 x^{2n}$$

Now, to find the coefficient of the r^{th} term, we substitute $r = 2n$,

$$a_r = \left(\frac{r}{2}\right)^3$$

Example 2.1: Choose one of three

Write the generating function for choosing one of three objects a, b, c.
Ans: The generating function can be written as

$$(\underbrace{1}_{\text{not choosing}} + \underbrace{ax}_{\text{choosing}})\underbrace{(1+bx)}_{\text{obj b}}\underbrace{(1+cx)}_{\text{obj c}}$$

Multiplying and taking $a = b = c = 1$ ways (of representing each obj) we get,

$$1 + 3x + 3x^2 + x^3$$

Which upon further evaluation gives

$$(1+x)^3$$

2.1 Combination Generating Functions

2.1.1 Combination without Repetition

The generating function for choosing r objects out of n objects is given by

$${}^nC_0x^0 + {}^nC_1x^1 + {}^nC_2x^2 + \cdots$$

Which basically says that there are ${}^nC_2$ ways of choosing two objects out of n objects etc. We can note that this series is equal to the binomial

$$(1+x)^n$$

2.1.2 Combination with Repetition

We can represent the possibility of choosing an object infinitely many times using

$$1 + x + x^2 + x^3 \cdots = \sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

We do not bother with the convergence of the series as it is only a formal sum.

Suppose we have n objects, we can modify the above equation to

$$(1 + x + x^2 + x^3 \cdots)^n = (\sum_{i=0}^{\infty} x^i)^n = \left(\frac{1}{1-x}\right)^n$$

Which simplifies to

$$(1-x)^{-n}$$

And when we use the binomial expansion for negative powers,

$${}^{n+0-1}C_0x^0 + {}^{n+1-1}C_1x^1 \cdots = \sum_{i=1}^{\infty} {}^{n+i-1}C_ix^i$$

Example 2.2: Limited Repetition Choosing

Out of 3 objects, the first object can be chosen at most once, the second object can be chosen at most twice and the third object can be chosen at most thrice. Find the number of ways of selecting 4 objects which satisfy the above condition.
Ans: We get the following generating function based on the above condi-

tions.

$$(1+x)\times(1+x+x^2)\times(1+x+x^2+x^3)$$

Example 2.3: Repetition, Lower Bound

Obtain a generating function to select r objects with repetition from five distinct objects with at least two of each type.
Ans: The generating function is

$$\left(x^2+x^3\cdots\right)^5=\left(x^2\right)^5\left(1+x+x^2\cdots\right)^5=x^{10}(1-x)^{-5}$$

Expanded Form	**Summation Form**	**Generating Function**
$1+x+x^2+\cdots$	$\sum_{n=0}^{\infty} x^n$	$\frac{1}{1-x}$
$1+\ {}^nC_1x+\ {}^nC_2x^2+\cdots$	$\sum_{r=0}^{n}\ {}^nC_rx^r$	$(1+x)^n$
$1+\ {}^nC_1x^m+\ {}^nC_2x^{2m}+\cdots$	$\sum_{r=0}^{n}\ {}^nC_rx^{rm}$	$(1+x^m)^n$
$1-\ {}^nC_1x+\ {}^nC_2x^2-\cdots$	$\sum_{r=0}^{n}(-1)^r\ {}^nC_rx^r$	$(1-x)^n$
$1-\ {}^nC_1x+\ {}^{n+1}C_2x^2-\cdots$	$\sum_{r=0}^{\infty}(-1)^r\ {}^{n+r-1}C_rx^r$	$(1+x)^{-n}$
$1+\ {}^nC_1x+\ {}^{n+1}C_2x^2+\cdots$	$\sum_{r=0}^{\infty}\ {}^{n+r-1}C_rx^r$	$(1-x)^{-n}$
$1+x+x^2+\cdots+x^m$	$\sum_{n=0}^{m} x^n$	$\frac{1-x^{m+1}}{1-x}$
$1+nx+n^2\frac{x^2}{2!}+n^3\frac{x^3}{3!}+\cdots$	$\sum_{r=0}^{\infty} n^r\frac{x^r}{r!}$	e^{nx}
$1+x+nx+\frac{n(n-1)}{2!}x^2+\cdots$	$\sum_{n=0}^{\infty}{}^nP_r\frac{x^r}{r!}$	$(1+x)^r$
$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots$	$\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}=\cos(x)$	$\frac{e^x+e^{-x}}{2}$
$x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots$	$\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}=\sin(x)$	$\frac{e^x+e^{-x}}{2}$

Table 2.1: Basic Generating Functions

Example 2.4: Marbles from a Pile

Using generating functions, find the number of ways to select 10 marbles from a large pile of blue, red and white marbles if

1. Selection has at least two marbles of each colour,
2. Solution has at most two red marbles,
3. The selection has an even number of blue marbles.

Ans:

1.

$$\left(x^2+x^3\cdots\right)^3$$
$$x^6\left(1+x+x^2\cdots\right)^3$$
$$x^6\left(\frac{1}{1-x}\right)^3$$
$$x^6(1-x)^{-3}$$

using the general term for $(1-x)^{-3}$, we get

$$x^6\quad {}^{3+r-1}C_r(-1)^rx^r$$

Now, to get the term for x^{10}, put $r=4$

$$x^6\quad {}^6C_4x^4$$

where we get the coefficient, ${}^6C_4$ ways.

2.

$$(1+x+x^2)(1+x+x^2\cdots)^2$$
$$(1+x+x^2)(1-x)^{-2}$$
$$(1+x+x^2)\left({}^{2+r-1}C_r x^r\right)$$

multiplying with each term on the left, we get three terms where x^{10} is possible

$$^{2+10-1}C_{10}x^{10} + {}^{2+9-1}C_9 x^{10} + {}^{2+8-1}C_8 x^{10}$$

now we get ${}^{11}C_{10} + {}^{10}C_9 + {}^{9}C_8 = 30$ ways

3.

$$(1+x^2+x^4+x^6+x^8+x^{10})(1+x+x^2\cdots)^2$$
$$(1+x^2+x^4+x^6+x^8+x^{10})(1-x)^{-2}$$
$$(1+x^2+x^4+x^6+x^8+x^{10})^{2+r-1}C_r x^r$$

Now, finding coefficients of x^{10},

$$^{11}C_{10} + {}^{9}C_8 + {}^{7}C_6 + {}^{5}C_4 + {}^{3}C_2 + {}^{1}C_0 = 36 \text{ ways}$$

Example 2.5: Examiners' Problem

In how many ways can an examiner distribute 30 marks among 8 questions such that all questions get at least 2 marks?
Ans:

$$(x^2+x^3\cdots)^8$$
$$x^{16} \quad (1-x)^{-8}$$
$$x^{16} \quad {}^{8+r-1}C_r x^r \qquad \text{r} = 14,$$
$$^{21}C_{14}x^{30}$$

Example 2.6: Ice Cream Selection

How many ways are there to select 12 ice creams from 5 types of sundaes with at most 4 of each type? **Ans:**

$$(1+x+x^2+x^3+x^4)^5$$
$$\left(\frac{1-x^5}{1-x}\right)^5$$
$$(1-x^5)^5(1-x)^{-5}$$
$$(-1)^r\, {}^5C_r x^5 r \times {}^{5+r-1}C_r x^r$$
$$(-1)^r\, {}^5C_r\, {}^{5+r-1}C_r\, x^{6r} \qquad \text{r} = 2,$$
$$^5C_2\, {}^6C_2 x^{12}$$

Example 2.7: Collecting Money from People

Find the number of ways to collect \$15 from 20 different people if 19 people can give \$1 or nothing and the last person can give either \$1 or \$5 or nothing.

Ans:

$$(1+x)^{19}(1+x+x^5)$$
$$^{19}C_r x^r(1+x+x^5) \qquad \text{r} = 15,14,10$$
$$(^{19}C_{15} + {}^{19}C_{14} + {}^{19}C_{10})x^{20}$$

Example 2.8: Distributing Oranges

While Shopping on saturday, Mary bought 12 oranges for her children, Grace, George and Frank. In how many ways can she distribute the oranges so that Grace gets at least 4, George and Frank get at least 2 and Frank gets no more than 5.

Ans:

$$(x^4 + x^5 \cdots)(x^2 + x^3 \cdots)(x^2 + x^3 + x^4 + x^5)$$
$$x^4(1-x)^{-1} \quad x^2(1-x)^{-1} \quad x^2\frac{1-x^4}{1-x}$$
$$x^8(1-x)^{-3}(1-x^4)$$
$$x^8 \quad {}^{3+r-1}C_r x^r(1-x^4) \qquad \text{r} = 4,0$$
$$^6C_4 - 1$$

Example 2.9: Partial Restriction

Find the number of ways to distribute 25 identical balls into 7 distinct boxex if the first box can have not more than 10 balls, but the other boxes have no restriction.

Ans:

$$(1+x+x^2 \cdots + x^{10})(1+x+x^2\cdots)^6$$
$$\left(\frac{1-x^{11}}{1-x}\right)(1-x)^{-6}$$
$$(1-x^{11})(1-x)^{-7}$$
$$(1-x^{11})^{6+r-1}C_r x^r \qquad \text{r} = 25,14$$
$$\left(^{30}C_{25} - {}^{19}C_{14}\right)x^{25}$$

2.2 Permutation Genetating Functions

2.2.1 Permutation Without Repetition

We know that

$$^nP_r = \frac{^nC_r}{r!}$$

And we can see in the binomial distribution,

$$\sum_{r=0}^{n} {}^nC_r x^r = \sum_{r=0}^{n} {}^nP_r \frac{x^r}{r!}$$

So, to find the possible permutations, we take the coefficient of $\frac{x^r}{r!}$ instead of x^r.

2.2.2 Permutation With Repetition

The sequence encoding permutations with repetion is given by

$$\langle 1, n, n^2, n^3 \cdots \rangle$$

And the series encoding this sequence,

$$1 + nx + n^2\frac{x^2}{2!} + n^3\frac{x^3}{3!}\cdots$$
$$\sum_{r=0}^{\infty} n^r\frac{x^r}{r!} = e^{nx}$$

Example 2.10: ENGINE Permutation

Find the number of arrangements of 4 letters from the word ENGINE
Ans: We can note that the letters N and E are repeating twice, while G and I are repeating once each. So,

$$\underbrace{(1+x)^2}_{\text{G and I}}\underbrace{\left(1+x+\frac{x^2}{2!}\right)^2}_{\text{N and E}}$$
$$(1+2x+x^2)\cdot\left(1+2x+\frac{3x^2}{2}+x^3+\frac{x^4}{4}\right) \qquad \text{finding } x^4$$
$$24\times\left(1+\frac{1}{4}+1+2\right)\frac{x^4}{4!}$$
$$102 \text{ ways}$$

Example 2.11: Distributing Toys

How many ways are there to distribute 8 toys among 4 children if the first child should get at least 2?
Ans:

A little bit of complicated algebra has happened here, but when you do it on your own it will be clearer.

$$\left(\frac{x^2}{2!}+\frac{x^3}{3!}\cdots\right)\left(1+x+\frac{x^2}{2!}\cdots\right)^3$$
$$(e^x-1-x)e^{3x}$$
$$e^{4x}-e^{3x}-xe^{3x}$$
$$(4^r-3^r-r3^{r-1})\frac{x^r}{r!} \qquad \text{r} = 8$$
$$4^8-3^8-8\times 3^7$$

Example 2.12: Ternery Sequence

How many 4 digit ternery sequences are there with at least one 0, one 1, one 2.
Ans: A ternery sequence is a sequence comprising of a three letter alphabet.

$$\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}\cdots\right)^3$$
$$(e^x-1)^3$$
$$e^{3x}+3e^{2x}+3e^x-1$$
$$3^r\frac{x^r}{r!}-3\times 2^r\frac{x^r}{r!}+3\times\frac{x^r}{r!}-1 \qquad \text{r} = 4$$
$$3^4-3\times 2^4+3 \text{ ways}$$

Example 2.13: Ship Singal Permutation

A ship carries 48 flags, 12 each of the colour white, red, black and blue. 12 of these flags are placed on a vertical pole, in order to communicate a signals to other ships. How many of these signals use

1. an even number of blue flags and an odd number of black flags.
2. at least three white flags or no white flags at all.

Ans:

1.

$$\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}\cdots\right)\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right)\left(1+x+\frac{x^2}{2!}\cdots\right)^2$$

$$(\cos x)(\sin x)(e^x)^2$$

$$\frac{e^x+e^{-x}}{2}\times\frac{e^x-e^{-x}}{2}\times e^2x$$

$$\frac{1}{4}(e^{2x}-e^{-2x})e^{2x}$$

$$\frac{1}{4}(e^{4x}-1) \qquad \text{r} = 12$$

$$\frac{1}{4}4^{12}\frac{x^{12}}{12!}=4^{11}\text{ ways}$$

2.

$$\left(1+\frac{x^3}{3!}\cdots\right)\left(1+x+\frac{x^2}{2!}\cdots\right)^3$$

$$\left(e^x-x-\frac{x^2}{2!}\right)e^{3x}$$

$$e^{4x}-xe^{3x}-\frac{x^2}{2!}e^{3x}$$

$$4^r\frac{x^r}{r!}-3^r\frac{x^{r+1}}{r!}-3^r\frac{x^{r+2}}{r!} \qquad \text{r=12,11,10}$$

Note that we have to find the coeff. of $\frac{x^r}{r!}$, not just x^r.

$$\left(4^{12}-3^{12}\cdot 12-3^{12}\cdot 12\cdot 11\right)\frac{x^{12}}{12!}$$

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