CS 556 Homework - Vector Spaces and Subspaces Solutions

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Question 1 (10 points)

Write the complete solution of the following linear system:

$$x + 2y - z = 1 \tag{1}$$
$$3x + 5y + 2z = 3 \tag{2}$$
$$2x + y + 13z = 2 \tag{3}$$

Solution:
We'll use Gaussian elimination. First, form the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 3 & 5 & 2 & 3 \\ 2 & 1 & 13 & 2 \end{array}\right]$$

Row operations:

$$R_2 \leftarrow R_2 - 3R_1$$
$$R_3 \leftarrow R_3 - 2R_1$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 5 & 0 \\ 0 & -3 & 15 & 0 \end{array}\right]$$

$$R_3 \leftarrow R_3 - 3R_2 :$$
$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From the second row: $-y + 5z = 0 \Rightarrow y = 5z$
From the first row: $x + 2y - z = 1 \Rightarrow x + 10z - z = 1 \Rightarrow x = 1 - 9z$
Let $z = t$ (free parameter), then:

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -9 \\ 5 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}}$$

Question 2 (10 points)

Find the rank of the matrix:

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix}$$

Solution:
We'll reduce the matrix to row echelon form:

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 3 & 1 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3 - 3R_1} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -8 & -3 \end{pmatrix}$$

$$\xrightarrow{R_3 + 5R_2} \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 2 \end{pmatrix}$$

The matrix has 3 non-zero rows in row echelon form, therefore $\boxed{\text{rank}(A) = 3}$

Question 3 (10 points)

Construct a matrix A whose column space contains vectors $\begin{pmatrix} 3 \\ 6 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix}$, and whose null space contains $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$.

Solution:
We need a 3×3 matrix where: - First two columns are the given vectors (for column space) - The third column satisfies $A \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Let $A = \begin{pmatrix} 3 & 4 & a \\ 6 & 0 & b \\ 2 & 1 & c \end{pmatrix}$

For the null space condition:

$$A \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3(2) + 4(2) + a(1) \\ 6(2) + 0(2) + b(1) \\ 2(2) + 1(2) + c(1) \end{pmatrix} = \begin{pmatrix} 14 + a \\ 12 + b \\ 6 + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This gives us $a = -14$, $b = -12$, $c = -6$

$$\boxed{A = \begin{pmatrix} 3 & 4 & -14 \\ 6 & 0 & -12 \\ 2 & 1 & -6 \end{pmatrix}}$$

Question 4 (10 points)

Compute $\begin{pmatrix} 2 & 1 & 3 \\ 7 & 1 & 0 \\ 3 & 5 & 9 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}$ as:

a) Linear combination of columns:

$$2\begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix} + 4\begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ 14 \\ 6 \end{pmatrix} + \begin{pmatrix} 4 \\ 4 \\ 20 \end{pmatrix} + \begin{pmatrix} 3 \\ 0 \\ 9 \end{pmatrix} = \boxed{\begin{pmatrix} 11 \\ 18 \\ 35 \end{pmatrix}}$$

b) Dot product of rows:

$$\begin{pmatrix} (2,1,3)\cdot(2,4,1) \\ (7,1,0)\cdot(2,4,1) \\ (3,5,9)\cdot(2,4,1) \end{pmatrix} = \begin{pmatrix} 4+4+3 \\ 14+4+0 \\ 6+20+9 \end{pmatrix} = \boxed{\begin{pmatrix} 11 \\ 18 \\ 35 \end{pmatrix}}$$

Question 5 (10 points)

Find the value of k for which the matrix has:

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 3 & 1 \\ 4 & 8 & k \end{pmatrix}$$

Solution:

The columns are dependent if and only if $\det(A) = 0$.

$$\begin{aligned} \det(A) &= 1(3k-8) - 3(2k-4) + 2(16-12) \\ &= 3k - 8 - 6k + 12 + 8 \\ &= -3k + 12 \end{aligned}$$

a) Dependent columns: $\det(A) = 0 \Rightarrow -3k + 12 = 0 \Rightarrow \boxed{k = 4}$

b) Independent columns: $\det(A) \neq 0 \Rightarrow \boxed{k \neq 4}$

Question 6 (20 points)

Find a basis for the four fundamental subspaces of:

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

Solution:

First, identify pivot columns: columns 2 and 4 are pivot columns.

1. Column Space $C(A)$: Span of pivot columns of A

$$\boxed{C(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} \right\}}$$

2. Null Space $N(A)$: Solve $Ax = 0$

$$\begin{cases} x_2 + 2x_3 + 3x_4 + 4x_5 = 0 \\ x_2 + 2x_3 + 4x_4 + 6x_5 = 0 \\ x_4 + 2x_5 = 0 \end{cases}$$

From the third equation: $x_4 = -2x_5$

Subtracting first from second: $x_4 + 2x_5 = 0$ (consistent)

From first: $x_2 = -2x_3 - 3x_4 - 4x_5 = -2x_3 + 6x_5 - 4x_5 = -2x_3 + 2x_5$

Setting free variables: - $x_1 = s, x_3 = t, x_5 = r$: gives $x_2 = -2t + 2r, x_4 = -2r$

$$\boxed{N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}}$$

3. Row Space $C(A^T)$: Span of nonzero rows of the row echelon form

$$\boxed{C(A^T) = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix} \right\}}$$

4. Left Null Space $N(A^T)$: Solve $A^T y = 0$

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 0 \\ 3 & 4 & 1 \\ 4 & 6 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

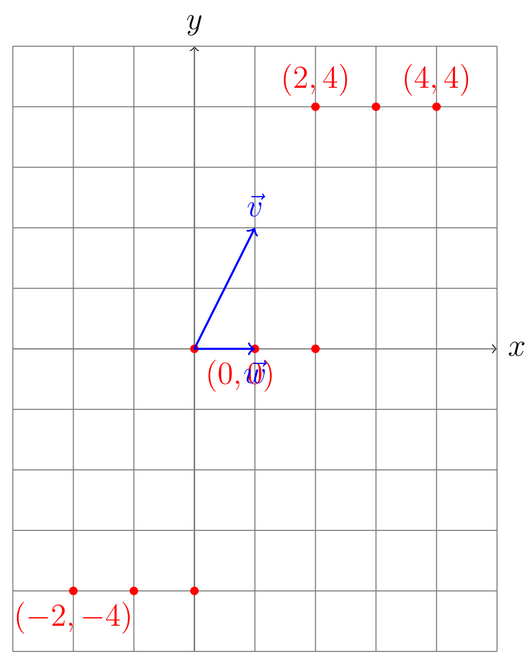
From the equations: $y_1 + y_2 = 0$ and $3y_1 + 4y_2 + y_3 = 0$
Let $y_1 = t$, then $y_2 = -t$ and $y_3 = -3t + 4t = t$

$$\boxed{N(A^T) = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\right\}}$$

Question 7 (10 points)

Consider $\vec{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\vec{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

a) Mark all nine linear combinations:
The nine combinations with $c \in \{-2, 0, 2\}$ and $d \in \{0, 1, 2\}$ are:



b) All linear combinations $c\vec{v} + d\vec{w}$ fill $\boxed{\text{the entire plane}}$. The vectors $\vec{v}$ and $\vec{w}$ are $\boxed{\text{linearly independent}}$ because they are not parallel (one is not a scalar multiple of the other).

Question 8 (10 points)

Consider $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$

a) Can you solve $x\vec{u} + y\vec{v} + z\vec{w} = \vec{b}$ if $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$?

Setting up the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$$

Row reduction gives:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The last row gives $0 = 1$, which is inconsistent. $\boxed{\text{No solution exists.}}$

b) What if $\vec{b} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$**?**

The augmented matrix becomes:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From row 2: $y = z$
From row 1: $x = -2z$
Let $z = t$, then $(x, y, z) = (-2t, t, t)$

$\boxed{\text{Infinitely many solutions: } (x, y, z) = t(-2, 1, 1),\ t \in \mathbb{R}}$

c) The vectors are $\boxed{\text{linearly dependent}}$ (since the homogeneous system has non-trivial solutions).

d) Since the columns of A are linearly dependent, $\boxed{A \text{ is not invertible}}$.

Question 9 (10 points)

Consider the system:

$$x - 2y + 3z = 3 \tag{4}$$
$$2x + y + bz = -4 \tag{5}$$
$$x + 0y + z = g \tag{6}$$

a) What constant b makes the system singular?

Form the coefficient matrix and find when $\det(A) = 0$:

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 2 & 1 & b \\ 1 & 0 & 1 \end{pmatrix}$$

$$\det(A) = 1(1 - 0) + 2(2 - b) + 3(0 - 1) = 1 + 4 - 2b - 3 = 2 - 2b$$

For singular matrix: $2 - 2b = 0 \Rightarrow \boxed{b = 1}$

b) For $b = 1$, for which values of g does the system have infinitely many solutions?

With $b = 1$, the augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 2 & 1 & 1 & -4 \\ 1 & 0 & 1 & g \end{array}\right]$$

Row operations:

$$\xrightarrow{R_2-2R_1, R_3-R_1} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 5 & -5 & -10 \\ 0 & 2 & -2 & g-3 \end{array}\right]$$

$$\xrightarrow{R_3-\frac{2}{5}R_2} \left[\begin{array}{ccc|c} 1 & -2 & 3 & 3 \\ 0 & 5 & -5 & -10 \\ 0 & 0 & 0 & g-3+4 \end{array}\right]$$

For infinitely many solutions, we need $g + 1 = 0$, so $\boxed{g = -1}$

c) Find two distinct solutions for $g = -1$:

From row 2: $5y - 5z = -10 \Rightarrow y = z - 2$

From row 1: $x - 2y + 3z = 3 \Rightarrow x = 3 + 2y - 3z = 3 + 2(z-2) - 3z = -1 - z$

Let $z = t$, then $(x, y, z) = (-1 - t, t - 2, t)$

Two distinct solutions:

- For $t = 0$: $\boxed{(x, y, z) = (-1, -2, 0)}$
- For $t = 1$: $\boxed{(x, y, z) = (-2, -1, 1)}$