Disjunctive Domination on Interval Graphs

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May 8, 2017

Overview

Motivation

We consider a city as a graph viz. places as nodes and roads connecting them as edges. If a situation of emergency arises in a place, then the response time for emergency system should be minimal; However we cannot afford to put emergency systems at every place. So we want to find minimum number of emergency systems that can "cover" the whole city. This type of situations are common in communication systems, locating facilities, vigilance systems etc., and are modeled by domination problem in graph theory.

Domination Problem

Introduction

Dominating Set

A subset,D of Vertices,V is Dominating Set of a Graph,G(V,E) if every other vertex in V other than D is adjacent to some vertex in D.

Disjunctive Dominating Set

A subset, D of vertices, V is Disjunctive Dominating Set of Graph,G(V,E) if every other vertex in V other than D is either adjacent to some vertex in D (or) there exist atleast two vertices in D at a distance of two.

Disjunctive Domination Problem

Given a Graph,G(V,E) finding the minimum cardinality Disjunctive Dominating Set is called *Disjunctive Domination Problem,DDP*.

It cannot be solved on General graphs,however DDP have polynomial time algorithm for Proper Interval Graphs.

Special Classes Of Graphs

Introduction

A graph which shows the intersection patterns of a family of sets is called Intersection Graph.

Intersection Graph

A graph in which vertices represent a set in the family of sets and an edge exists between two vertices iff the sets represented by them have non-empty intersection.

If the sets are constrained by conditions then we get special classes of Graphs.

Example: If we restrict the sets to contain vertices of subtrees of a tree then we get Chordal Graphs. Perfect Elimination Ordering of vertices is one of the characterization of Chordal Graphs.

Interval Graphs

Interval Graph

A Graph is said to be Interval Graph iff it is an Intersection graph of family of Intervals on Real Line.

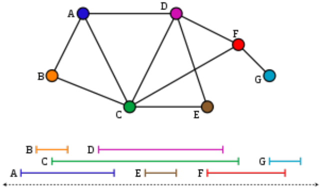

Figure: Interval Graph and corresponding Intervals on Real Line

Proper Interval Graph

An Interval Graph in which no interval is contained in another interval is called Proper Interval Graph.

LVO vs BCO

Proper Interval Graph

Characterization of Proper Interval Graph is Bi-Compatible Elimination Ordering.

Perfect Elimination Ordering(PEO)

An Ordering of vertices in a Graph,G(V,E) such that every vertex, v_i is simplicial vertex of the induced graph, $G[v_i, v_{i+1}, \cdots, v_n]$.

Bi-Compatible Elimination Ordering(BCO)

A PEO of Chordal Graph is Bi-Compatible Elimination Ordering, $(v_1, v_2, \cdots, v_n)$ iff the reverse ordering i.e $(v_n, v_{n-1}, \cdots, v_1)$ is also a PEO of the Graph,G(V,E).

LVO/RVO

Left endpoint Vertex Ordering/ Right endpoint Vertex Ordering is ordering of vertices,V in G(V,E) in ascending order of their Left or Right endpoints of Intervals represented by them.

Result

Theorem

In case of PIG, Left Vertex Ordering(LVO) of vertices is same as Right Vertex Ordering(RVO) is same as Bi-compatible Elimination Ordering(BCO). $LVO \equiv RVO \equiv BCO$

- LVO and RVO can be computed in $O(n \log n)$
- BCO can be computed in $O(n+m)$
- Minimum number of edges,m an interval Graph can have is k cliques with $\frac{n}{k}$ vertices in each clique.
- Total number of edges, $m = k * \frac{n}{k} * \frac{n}{k} = \frac{n^2}{k}$
- If number of cliques(k) are less when compared to n, then m is large in which case $O(n \log n)$ is better than $O(n+m)$. And so computing LVO is faster than BCO.

Approach to DDP on Interval Graph

Labeling Method

A polynomial time Algorithm for Disjunctive Total Domination Problem on Trees is given, using Labeling approach by [Lin and Sheng, 2016]. In which every leaf of Tree should be dominated by internal vertices. We model domination as flow and reduce the problem to Min Cost-Max Flow Problem to find partial dominating set. And repeat this method until all vertices are dominated.

Theorem

For any Interval Graph G, there is a Disjunctive Dominating set D such that D does not contain any vertices belonging to single clique.

From above theorem we can see that all vertices in single clique have to be dominated by other vertices. Algorithm is as follows.

Input: Interval Graph,G(V,E)
Output: Disjunctive Dominating Set
Initialization: $D = \phi$, G'(V,E) is same as G(V,E) with all vertices labeled as 2 ;
while *(all vertices are not labeled zero)* **do**

- $Y = \{y | y \in$ Single Clique in the graph $G'\}$
- $X = \{x | x$ is at a distance ≤ 2 from y$\in Y\}$
- Construct a flow graph as given in [Lin and Sheng, 2016]
- Find a partial dominating set D_{part} using Min Cost-Max Flow
- $D = D \cup D_{part}$
- Update the labels of the vertices as per the new dominating set D
- Delete all the vertices from graph,G' whose labels are zero i.e completely dominated

end

Algorithm 1: Labeling method on Interval Graph

Working

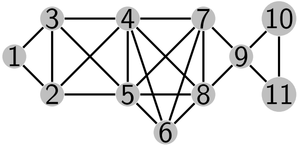

Figure: Interval Graph

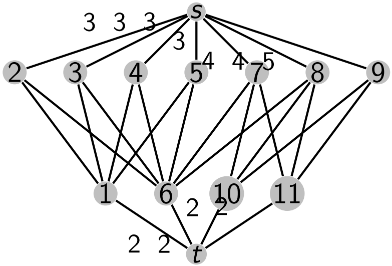

Figure: Flow graph constructed from graph fig:2 as per the algorithm

The flows from vertex, $u \in X$ to $v \in Y$ is $3 - d(u, v)$ where $d(u, v)$ is the shortest distance between vertices u and v. Applying Min Cost Max Flow on figure 3 gives a dominating set of $\{2, 9\}$ which dominates the whole graph fig:2.

Counter Example

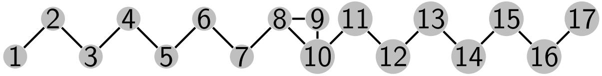

Figure: Interval Graph

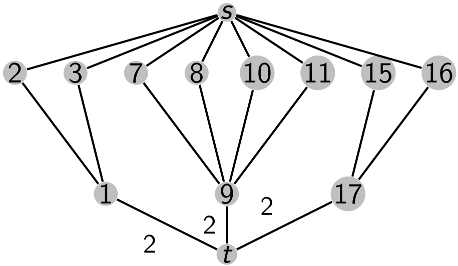

Figure: Flow graph constructed from graph fig:4 as per the algorithm

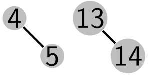

Figure: Reduced Interval Graph

We get disjunctive dominating set $\{2, 7, 11, 16\} \cup \{4, 13\}$ whose size is 6. However we have a smaller solution set $\{2, 6, 10, 14, 16\}$.

New Ordering for Interval Graphs

Clique Elimination Ordering

Every Interval Graph has a maximal clique ordering,$(C_1, C_2, \cdots, C_k)$ in which every vertex belong to a continuous set of cliques.

level(i)

$level(v_i)$ is the maximum index,I in the clique ordering of the Interval Graph such that C_I contains vertex,v_i.

Clique Elimination Ordering,CEO

Clique Elimination Ordering,CEO is the ordering of the vertices of the Interval Graph,G(V,E) in the ascending order of $level(v_i)$ and if two vertices have same $level(v_i)$ then order them by the left endpoint of the intervals represented by them.

Max(v_i)

$Max(v_i)$ is the largest vertex in the CEO,$(v_1, v_2, \cdots, v_n)$ of the Graph,G(V,E) which is adjacent to v_i.

Clique Elimination Ordering

Properties of Clique Elimination Ordering.

Theorem (Elimination Ordering)

Every vertex,v_i in CEO is a simplicial vertex of the induced graph, $G[v_i, v_{i+1}, \cdots, v_n]$.

Theorem

In CEO,$(v_1, v_2, \cdots, v_n)$ of Graph G(V,E) all vertices between v_i and $Max(v_i)$ are adjacent to $Max(v_i)$.

These properties are used to prove the correctness of the extended algorithm.

References

Lin and Sheng (2016)
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Combinatorial Optimization and Applications: 10th International Conference, COCOA 2016, Hong Kong, China, December 16–18, 2016, Proceedings

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Algorithmic aspects of disjunctive domination in graphs
CoRR

The End